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On perfectness of sums of graphs

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Abstract

We consider the operation of summation of two graphs G_1 and G_2. Necessary and sufficient conditions for $G_1 + G_2$ to be perfect are derived. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

All graphs considered here will be simple (no loops, no multiple edges).

Given two graphs $G=(X,U)$ and $H=(Y,V)$ their sum $G+H=(Z,W)$ is defined as follows (see [2]):

$$Z=\{(x,y): x\in X, y\in Y\},$$
$$W=\{[(x,y),(x',y')]: x=x',\ [y,y']\in V \text{ or } y=y',\ [x,x']\in U\}.$$

A special case of this construction is used for reducing problems of node coloring to problems of finding independent sets [3]. We have the following property: the nodes of G can be colored with q colors iff there is in $G+K_q$ (where K_q is a clique on q nodes) an independent set S with $|S|=|X|$. We notice indeed that there is a one-to-one correspondence between the colorings $(S_1,S_2,\dots,S_q)$ of $G=(X,U)$ and the independent sets S in $G+K_q$ with $|S|=|X|$ nodes: $(x,i)\in S$ if and only if $x\in S_i$. C_k will denote a chordless cycle on k nodes and K_p, $\overline{C_k}$ will be the complement of C_k.

A graph G is *perfect* [2] if for any induced subgraph G' of G the chromatic number $\chi(G')$ equals the maximum clique size $\omega(G')$. A Berge graph is a graph containing *neither* an induced C_{2k+1} nor an induced $\overline{C_{2k+1}}$ $(k\geqslant 2)$. The strong perfect graph conjecture (SPGC) states that a graph is perfect if and only if it is a Berge graph [1].

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In this note we will examine conditions for the sum of graphs to be perfect. Notice that if G and H are perfect, $G+H$ may not be perfect; for example, C_4+K_3 is not perfect. All graph-theoretical notations and definitions not given here are in [2].

Although most of the results in this paper are not new (they were obtained in different forms by different authors, see, for instance, [12]), we think that a unified presentation may be appropriate. An additional argument is that some of the proofs published earlier were incomplete. In this note we also give some refinements of earlier published results.

2. Diamonds and cliques

A graph isomorphic to K_4-e (a clique on four nodes with one edge removed) is called a *diamond*; if a graph contains no induced diamond, we call it *diamond-free*. A graph containing no induced C_k ($k\geqslant 4$) is *triangulated.* We call G a TDF graph if it is triangulated and diamond-free.

Theorem 1. *If $G_1=(X_1,Y_1,U_1)$ and $G_2=(X_2,Y_2,U_2)$ are bipartite, then G_1+G_2 is bipartite.*

Proof. Construct a partition X_{12},Y_{12} of the nodes of G_1+G_2 as follows:

$$X_{12}=\{(u,v)\colon u\in X_1, v\in X_2 \text{ or } u\in Y_1, v\in Y_2\},$$
$$Y_{12}=\{(u,v)\colon u\in X_1, v\in Y_2 \text{ or } u\in Y_1, v\in X_2\}.$$

One verifies easily that each edge in G_1+G_2 links a node of X_{12} to a node of Y_{12}. □

So the class of bipartite graphs is closed under summation. The next assertion states that it is also true for the class of diamond-free graphs.

Lemma 1. *If G_1 and G_2 are diamond-free, then G_1+G_2 is diamond-free.*

Proof. Assume G_1+G_2 has a diamond on nodes $a=(a_1,a_2)$, $b=(b_1,b_2)$, $c=(c_1,c_2)$, $d=(d_1,d_2)$ and let a,b,c be a K_3. Then it corresponds to a K_3 in G_1 or in G_2; so assume $a_2=b_2=c_2$, i.e., we have a K_3 on a_1,b_1,c_1 in G_1. Let d be linked to b and c in G_1+G_2. This implies that $d_2=b_2=c_2=a_2$, so G_1 contains a diamond. This is a contradiction. □

Lemma 2. *If G_1 is a TDF graph and $G_2=K_p$ is a clique on $p\geqslant 1$ nodes, then G_1+G_2 contains no induced C_{2k+1} ($k\geqslant 2$).*

Proof. Assume $G+K_p$ contains an induced C_{2k+1} ($k\geqslant 2$) on nodes $(x_1,i_1),\ldots,$ (x_{2k+1},i_{2k+1}). Edges of the form $[(x,i),(y,i)]$ will be called 1-edges (they correspond

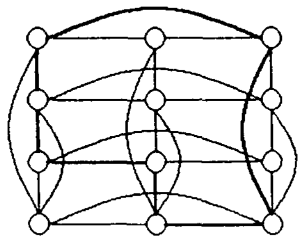

Fig. 1. A diamond + K_3 contains an induced C_7.

to edges $[x,y]$ of G_1); similarly, edges of the form $[(x,i),(x,j)]$ (corresponding to $[i,j]$ in G_2) are called 2-edges.

C_{2k+1} cannot contain two consecutive 2-edges (since G_2 is a clique, if $[(x,i),(x,j)]$, $[(x,j),(x,k)]$ are in C_{2k+1}, then there is also a chord $[(x,i),(x,k)]$). So there must necessarily be two consecutive 1-edges, say $[(x,i),(y,i)],[(y,i),(z,i)]$, in C_{2k+1}. Furthermore $[(x,i),(z,i)]$ is not an edge of G_1+G_2 since there is no chord. Traverse now C_{2k+1} from (x,i) back to (x,i) and consider the 1-edges. This traversal corresponds to a (not necessarily simple) cycle C' in G_1; this means that some edges may have been traversed more than once. Now, the presence of $[(x,i),(y,i)],[(y,i),(z,i)]$ and the non-existence of $[(x,i),(z,i)]$ imply that C' contains an elementary cycle containing edges $[x,y],[y,z]$ and x and z are not linked by any edge in G_1. So C' contains either a chordless cycle with at least four edges or a diamond. This is impossible. □

Theorem 2. *For a graph G the following statements are equivalent*:
(a) *G is a TDF graph,*
(b) *every cycle of length at least four has at least two chords,*
(c) *for each positive q, $G+K_q$ is perfect,*
(d) *for some $q\geqslant 3, G+K_q$ is perfect,*
(e) *$G+K_3$ is perfect.*

Proof. The equivalence of (a) and (b) is immediate. Also (c)$\Rightarrow$(d)$\Rightarrow$(e). In order to show that (e)$\Rightarrow$(a), assume that G is not a TDF graph. If G contains a diamond D, then the induced subgraph $D+K_3$ of $G+K_3$ contains an induced C_7 (see Fig. 1). So assume now that G contained an induced C_p with $p\geqslant 4$. If p is even, it is easy to verify that C_p+K_3 contains an induced C_{p+3} and if p is odd, then $G+K_1=G$ is not perfect.

Let us now show that (a)$\Rightarrow$(c). Assume G is a TDF graph, then from Lemma 1, it follows that $G+K_q$ is diamond-free. Furthermore, from Lemma 2, $G+K_q$ contains no induced C_{2k+1} ($k\geqslant 2$). Tucker [13] has shown that Berge diamond-free graphs are perfect.

The proof in [13] consists in showing that in any induced subgraph G' of such a graph a node x belonging to at most two maximal cliques can be found if all maximal cliques have size at least 3. Then by using induction on the number of nodes, one shows that if $G'-x$ has been optimally colored, it is still possible to color x (possibly

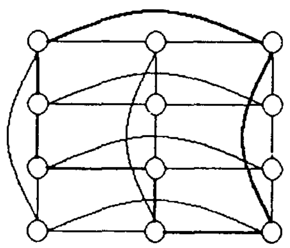

Fig. 2. $C_4 + K_3$ contains an induced C_7.

by performing a bichromatic exchange in the neighborhood of x to make an already used color available for x). The case where some maximal cliques have size 2 can also be handled with bichromatic exchanges.

In the case of $G+K_q$, where G is a TDF graph, such a node x is easy to find in G and in all its subgraphs. Since G is triangulated there exists an order $x_1 < x_2 < \cdots < x_n$ of the nodes such that x_i is *simplicial* (it belongs to exactly one maximal clique) in the subgraph induced by $x_i, x_{i+1}, \ldots, x_n$ [5].

Consider now any induced subgraph G' of $G + K_q$. For any $i \leqslant q$ the first node x in the order $<$ for which (x, i) is in G' belongs to at most two maximal cliques of G'. □

It is worth observing that the class P of perfect graphs obtained as subgraphs of the sum of a TDF graph and a clique are different from the known classes of perfect graphs. Since $\overline{C}_6 = K_3 + K_2$ is in P, we observe that such graphs are not strongly perfect.

Note that P contains also the line-graphs of bipartite graphs because the line-graph of a complete bipartite graph $K_{m,n}$ is $K_m + K_n$; so graphs in P are not locally perfect [11]. Although there is a very simple coloring algorithm for $G+K_q$ when G is a TDF graph, we do not think that there is a greedy-type algorithm for getting a maximum independent set in an arbitrary subgraph of $G+K_q$ (the reason is that such an algorithm would give a greedy algorithm for maximum matching in a bipartite graph).

Remark 1. It would be interesting to characterize graphs in P in terms of forbidden subgraphs. One should also observe that the sum of a graph in P and a clique may not be perfect (see $C_4 + K_3$ in Fig. 2).

Remark 2. As recalled in the introduction coloring the largest possible number of nodes with q colors in a TDF graph G reduces to finding a maximum independent set of nodes in $G + K_q$. A polynomial graph-theoretical algorithm for finding such an independent set would give a good algorithm for the coloring problem in G.

Since $G + K_q$ is perfect, there exists a polynomial algorithm for the independent set problem, so there is a polynomial algorithm for coloring the largest possible number of nodes with q colors in a TDF graph.

It is worth mentioning that a TDF graph is sometimes called a *block graph* (see [2]): it is characterized by the fact that its (inclusionwise) maximal 2-connected components are cliques.

3. Flags and trees

A K_3 with a pending edge is called a *flag*. A Berge graph containing no induced flag will be called a *Berge flag free* (for short BFF) graph.

Remark 3. It is worth observing that if $G_1 = P_3$ (a chain on three nodes a, b, c) and G_2 is a flag, then $G_1 + G_2$ contains an induced C_9 (see Fig. 3).

Remark 4. Flag-free graphs have been studied by Olariu in [9] where they are called *paw-free* graphs. It is shown that G is flag-free if and only if either (a) G is triangle-free or (b) G is the join of several independent sets.

One could use the above to derive the results in this section. We, nevertheless, give a direct derivation to keep the paper as self-contained as possible.

Lemma 3. *Let G_1 be a BFF graph containing a diamond D and G_2 a tree; then $G_1 + G_2$ has no induced C_{2k+1} ($k \geqslant 2$).*

Proof. (A) It is immediate to observe that G_1 cannot contain any induced P_4.

(B) Let C be the shortest induced C_{2k+1} ($k \geqslant 2$) in $G_1 + G_2$. Let H_2 be the subgraph of G_2 induced by the edges corresponding to the 2-edges traversed in C. H_2 has a pendent node i; there exists a 2-edge in C with endpoint $(x, j), (x, i)$ where x is a node of G_1. This 2-edge is followed in C by two 1-edges $[(x, i), (y, i)], [(y, i), (z, i)]$ (if there were only one, we would go next to (y, j) from (y, i) and this would give a chord $[(x, j), (y, j)]$ in C); note that we cannot go to (z, j) instead of (y, j) because i was a pendent node in H_2.

(C) Since G_1 has no induced P_4, the next edge in C (after $[(y, i), (z, i)]$) is a 2-edge; the next 1-edge will be of the form $[(z, t), (w, t)]$. We shall construct a shorter odd cycle C'; let us first associate with the chain I of C going through nodes $(z, i_1), \ldots, (z, i_p)$, $(z, t), (w, t)$ after (z, i) another chain $Q = (w, i_1), \ldots, (w, i_p), (w, t)$. If $w = y$, we now go through $(x, i), (y, i), Q$ instead of $(x, i), (y, i), (z, i), I$ (see Fig. 4). If $w \neq y$, then there

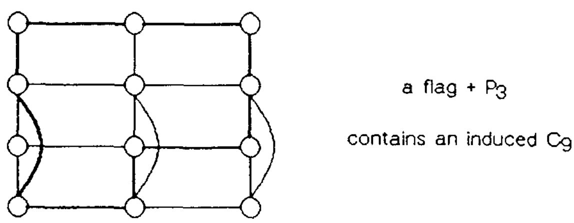

Fig. 3. A flag + P_3 contains an induced C_9.

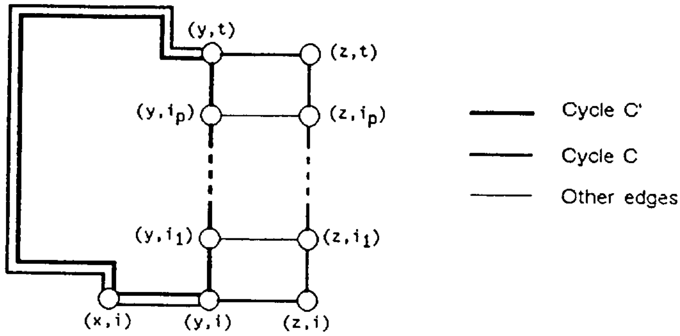

Fig. 4. (——) Cycle C'; (——) cycle C; (——) other edges.

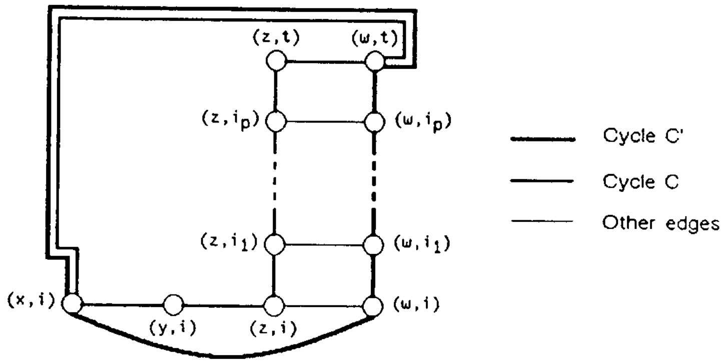

Fig. 5. (——) Cycle C'; (——) cycle C; (——) other edges.

must exist an edge $[(x,i),(w,i)]$ in G_1+G_2 (otherwise there would be a P_4 or a flag on nodes x,y,z,w in G_1). We now go through $(x,i),(w,i)Q$ (see Fig. 5).

In both cases we get a shorter odd cycle (it has exactly two nodes less) which may have chords, but it has no short chords (i.e. chords creating triangles with C') since such a chord would also exist in C.

(D) So C' must now contain an induced C_{2s+1} with $s \leqslant k-1$; this contradicts the minimality of C. □

Theorem 3. *If G_1 is a BFF graph containing an induced diamond and G_2 a tree, then G_1+G_2 is perfect.*

Proof. Let G' be an induced subgraph of G_1+G_2. If G' contains no induced diamond, then by Lemma 3 G' is a Berge diamond-free graph and it is perfect according to the result of Tucker [13]. So assume G' contains an induced diamond D and let $(w,i),(x,i),(y,i),(z,i)$ be the nodes of D (the edge between (x,i) and (y,i) is missing). It is sufficient to show that there is no chordless odd chain between (x,i) and (y,i) [8]. Suppose there is such a chain C through nodes $(v_0,j_0)=(x,i),(v_1,j_1),\ldots,$ $(v_{2k+1},j_{2k+1})=(y,i)$. If there is an edge $[(v_1,j_1),(z,i)]$, then nodes x,v_1,z,y induce a flag in G_1 since we must have $j_1 = i$. So (z,i) is linked neither to (v_1,j_1) nor to (v_{2k},j_{2k}).

By Lemma 3, $G_1 + G_2$ has no induced C_{2p+1}, so there must be an index q $(2 \leqslant q \leqslant 2k-2)$ such that $[(z,i),(v_q,j_q)]$ and $[(z,i),(v_{q+1},j_{q+1})]$ are present. Observe that we must have $j_q = j_{q+1} = i$ and v_q, v_{q+1}, z, x induce a flag in G_1. Hence, C cannot exist. □

4. Perfectness of a sum of graphs

Using the results of the previous sections, we can now establish necessary and sufficient conditions for a sum of two graphs to be perfect.

Assume first that one of the graphs say G_1, has at most two nodes. If G_2 has one node, then trivially $G_1 + G_2$ is perfect if and only if G_1 is perfect. Let us now examine the case $G_2 = K_2$; we will show that G_1 must be a parity graph. A parity graph is a graph where for any two nodes x, y all chordless chains between x and y have the same parity. Olaru and Sachs have shown that these graphs are perfect [10].

Lemma 4. *If G_1 is a parity graph and $G_2 = K_2$, then $G_1 + G_2$ has no induced C_{2k+1} $(k \geqslant 2)$.*

Proof. Assume there is in $G_1 + G_2$ an induced C_{2k+1} $(k \geqslant 2)$; the 1-edges form an odd cycle C in G_1 (possibly with chords). Notice first that when we traverse C_{2k+1} each 2-edge is followed by at least two consecutive 1-edges. Consider three consecutive nodes x, y, z on C and assume there is a short chord $[x,z]$ in G_1; this implies that in C_{2k+1} we have a 2-edge $[(y,a),(y,b)]$ where a and b are the nodes of $G_2 = K_2$. Hence y cannot be an endpoint of a short chord (this would create a chord in C_{2k+1}).

So we have shown that in C we cannot have two crossing short chords. This contradicts the fact that G_1 is a parity graph (it is well known that in a parity graph, every odd cycle has at least two crossing chords and hence two crossing short chords). □

Theorem 4. *Let G_1 be a graph and $G_2 = K_2$, then $G_1 + G_2$ is perfect if and only if G_1 is a parity graph.*

Proof. Assume first that G_1 is not a parity graph; hence G_1 must contain either an induced C_{2k+1} $(k \geqslant 2)$ or an induced odd cycle of length $\geqslant 5$ with one chord or an odd cycle of length 5 with two noncrossing chords (this graph is called a *gem*). One can verify that in each such case $G_1 + G_2$ contains an induced C_{2k+1} $(k \geqslant 2)$.

Conversely, it is known that in a minimal imperfect graph which is not a C_{2k+1} $(k \geqslant 2)$ every edge belongs to a triangle [7]. In our case, let G' be an induced subgraph of $G_1 + G_2$; if G' contains no 2-edge, then it is perfect (because it is a parity graph) and if it contains some 2-edge e, then e belongs to no triangle and to no C_{2k+1} $(k \geqslant 2)$ by Lemma 4. □

Remark 5. A consequence of Theorem 4 is the existence of a polynomial algorithm for coloring the maximum number of nodes in a parity graph G_1 with two colors. For three or more colors we cannot deduce such a result as simply because G_1+K_q $(q\geqslant 3)$ may not be perfect.

There is a simple coloring algorithm which could have been used for establishing perfection of G_1+G_2 when G_1 is a parity graph and $G_2=K_2$. It runs as follows on an arbitrary connected induced subgraph G' of G_1+G_2: let G'_a (resp. G'_b) be the subgraph of G' induced by nodes of type (x,a) (resp. (x,b)). First, find a minimum coloring of G'_a and G'_b separately (this can be done in polynomial time since they are parity graphs [4]). Next, we introduce the 2-edges one after the other. Assume when we introduce edge $[(x,a),(x,b)]$ both nodes have some color k; we pick up some color $j\neq k$.

Let C_{kj} (x,b) be the connected component containing (x,b) of the subgraph G' induced by nodes of color j and k; we show that C_{kj} (x,b) cannot contain (x,a). If C_{kj} (x,b) did contain (x,a), there would be an even chain between (x,a) and (x,b) with alternating colors k and j. Let C be such a chain. It forms with $[(x,a),(x,b)]$ an odd cycle of length at least 5 without chords.

Since none of the additional 2-edges which will be introduced later belong to a triangle, even if they are chords of C there will still be an induced chordless odd cycle of G' which is impossible by Lemma 4. Hence, we can make a bichromatic exchange on C_{kj} (x,b). The number of colors used in the final coloring is still

$$\max(\chi(G'_a),\chi(G'_b))=\max(\omega(G'_a),\omega(G'_b))=\omega(G').$$

We finally consider the case where both G_1,G_2 have at least three nodes.

Theorem 5. *Let G_1,G_2 be two connected graphs with at least three nodes each. Then G_1+G_2 is perfect if and only if we have one of the following mutually exclusive cases:*
(a) *both are bipartite;*
(b) *one is a TDF graph and the other is a clique;*
(c) *one is a BFF graph with an induced diamond and the other is a tree.*

Proof. The sufficiency follows from Theorems 1–3. To show necessity, observe first that G_1 and G_2 must be perfect (hence they are Berge graphs). If both G_1 and G_2 are bipartite, we are in case (a). So assume G_1 is not bipartite and hence contains a K_3. As seen in the proof of Theorem 2, C_p+K_3 (with $p\geqslant 4$) and $D+K_3$ (D is a diamond) contain an induced C_{2k+1} $(k\geqslant 2)$. So G_2 must be a TDF graph. If G_1 is a clique we are in case (b). So suppose G_1 is not a clique; if G_1 contains an induced diamond D, then G_2 must be bipartite. Since it was a TDF graph, it must be a tree. G_2 contains an induced P_3 and by Remark 3, G_1 contains no induced flag, and hence it is a BFF graph with a diamond and we are in case (c).

Finally, if G_1 (which is not a clique) does not contain any induced diamond, it must contain an induced flag. So G_2 must be a clique and hence contains an induced K_3. As a consequence, G_1 is a TDF graph and we are in case (b). □

The above result has some interpretation in terms of list coloring: we have seen that finding a q-coloring of $G=(V,E)$ is equivalent to finding an independent set S of $G+K_q$ with $|S|=|V|$: we may now consider the case where each node x has a list $\varphi(x)\subset\{1,2,\ldots,q\}$ of feasible colors.

In $G+K_q$ we simply have to remove nodes (x,i) whenever $i\notin\varphi(x)$.

Such a coloring can be found in polynomial time (due to the perfectness of $G+K_q$) when G is a TDF (i.e., a block graph) (see [6]). When G is a line graph, $G+K_q$ is perfect when G is a forest (see, for instance, [14] where a polynomial algorithm is given for coloring the edges of a forest when each edge e has a set $\varphi(e)$ of feasible colors).

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