Chapter 1

THE MAXIMUM INDEPENDENT SET PROBLEM AND AUGMENTING GRAPHS

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Abstract In the present paper we review the method of augmenting graphs, which is a general approach to solve the maximum independent set problem. Our objective is the employment of this approach to develop polynomial-time algorithms for the problem on special classes of graphs. We report principal results in this area and propose several new contributions to the topic.

1. Introduction

The maximum independent set problem is one of the central problems of combinatorial optimization, and the method of augmenting graphs is one of the general approaches to solve the problem. It is in the heart of the famous solution of the maximum matching problem, which is equivalent to finding maximum independent sets in line graphs. Recently, the approach has been successfully applied to develop polynomial-time algorithms to solve the maximum independent set problem in many other special classes of graphs. The present paper summarizes classical results and recent advances on this topic, and proposes some new contributions to it.

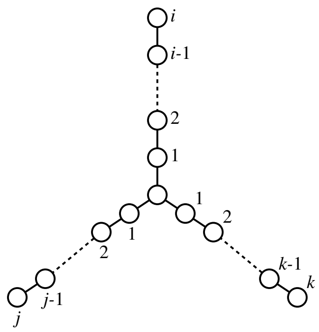

Figure 1.1. The graph $S_{i,j,k}$.

The organization of the paper is as follows. In the rest of this section we introduce basic notations. Section 2 presents general information on the maximum independent set problem, describes its relationship with other problems of combinatorial optimization, shows some applications, etc. In Section 3 we outline the idea of augmenting graphs and prove several auxiliary results related to this notion. Section 4 is devoted to the characterization of augmenting graphs in some special classes, and Section 5 describes algorithms to identify augmenting graphs of various types.

All graphs in this paper are undirected, without loops and multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively, and by $\overline{G}$ the complement of G. Given a vertex x in G, we let $N(x) := \{y \in V(G) \mid xy \in E(G)\}$ denote the neighborhood of x, and $deg(x) := |N(x)|$ the degree of x. The degree of G is $\Delta(G) := \max_{x \in V(G)} deg(x)$. If W is a subset of $V(G)$, we denote by $N_W(x) := N(x) \cap W$ the neighborhood of x in the subset W, and by $N(W) := \bigcup_{x \in W} N_{V(G)-W}(x)$ the neighborhood of W. Also, $N_U(W) := N(W) \cap U$ is the neighborhood of W in a subset $U \subseteq V(G)$. As usual, P_n is the chordless path (chain), C_n is the chordless cycle and K_n is the complete graph on n vertices. By $K_{n,m}$ we denote the complete bipartite graph with parts of size n and m, and by $S_{i,j,k}$ the graph represented in Figure 1.1. In particular, $S_{1,1,1} = K_{1,3}$ is a claw, $S_{1,1,2}$ is a fork (called also a chair), and $S_{0,j,k} = P_{j+k+1}$. The graph obtained from $S_{1,1,k}$ by adding a new vertex adjacent to the two vertices of degree 1 of distance 1 from the vertex of degree 3 will be called $Banner_k$. This is a generalization of $Banner_1$ known in the literature simply as a *banner*.

2. The Maximum Independent Set Problem

An *independent set* in a graph (called also a *stable set*) is a subset of vertices no two of which are adjacent. There are different problems associated with the notion of independent set, among which the most important one is the MAXIMUM INDEPENDENT SET problem. In the *decision version* of this problem, we are given a graph G and an integer K, and the problem is to determine whether G contains an independent set of cardinality at least K. The *optimization version* deals with finding in G an independent set of maximum cardinality. The number of vertices in a maximum cardinality independent set in G is called the *independence (stability) number* of G and is denoted $\alpha(G)$. One more version of the same problem consists in computing the independence number of G. All three versions of this problem are polynomially equivalent and we shall refer to any of them as the MAXIMUM INDEPENDENT SET (MIS) problem.

The MAXIMUM INDEPENDENT SET problem is NP-hard in general graphs and remains difficult even under substantial restrictions, for instance, for cubic planar graphs (Garey, Johnson and Stockmeyer (1976)). Alekseev (1983) has proved that if a graph H has a connected component which is not of the form $S_{i,j,k}$, then the MIS is NP-hard in the class of H-free graphs. On the other hand, it admits polynomial-time solutions for graphs in special classes such as bipartite or, more generally, perfect graphs (Grötschel, Lovász and Schrijver (1984)).

An independent set S is called *maximal* if no other independent set properly contains S. Much attention has been devoted in the literature to the problem of generating *all* maximal independent sets in a graph (see for example Johnson and Yannakakis (1988); Lawler, Lenstra and Rinnooy Kan (1980); Tsukiyama, Ide, Ariyoshi and Shirakawa (1977)). Again, there are different versions of this problem depending on definitions of notions of "performance" or "complexity" (see for example Johnson and Yannakakis (1988) for definitions): polynomial total time, incremental polynomial time, polynomial delay, specified order, polynomial space.

One more problem associated with the notion of independent set is that of finding in a graph a maximal independent set of minimum cardinality, also known in the literature as the INDEPENDENT DOMINATING SET problem. This problem is more difficult than the MAXIMUM INDEPENDENT SET in the sense that it is NP-hard even for bipartite graphs, where MIS can be solved in polynomial time.

In the present paper, we focus on the MAXIMUM INDEPENDENT SET problem. This is one of the central problems in graph theory that is closely related to many other problems of combinatorial optimization. For instance, if S is an independent set in a graph $G = (V, E)$, then S is a clique in the complement $\overline{G}$ of G and $V - S$ is a vertex cover of G. Therefore, the MAXIMUM INDEPENDENT SET problem in a graph G is equivalent to the MAXIMUM CLIQUE problem in $\overline{G}$, and the MINIMUM VERTEX COVER in G. A *matching* in a graph $G = (V, E)$, i.e. an independent set of edges, corresponds to an independent set of vertices in the line graph of G, denoted $L(G)$ and defined as follows: the vertices of $L(G)$ are the edges of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G are adjacent. Thus, the MAXIMUM MATCHING problem coincides with the MAXIMUM INDEPENDENT SET problem restricted to the class of line graphs. Unlike the general case, the MIS can be solved in polynomial time in the class of line graphs, which is due to the celebrated matching algorithm proposed in (Edmonds (1965)).

The weighted version of the MAXIMUM INDEPENDENT SET problem, also known as the VERTEX PACKING problem, deals with graphs whose vertices are weighted with positive integers, the problem being to find an independent set of maximum total weight. In (Ebenegger, Hammer and de Werra (1984)), this problem has been shown to be equivalent to maximizing a pseudo-Boolean function, i.e. a real-valued function with Boolean variables. Notice that pseudo-Boolean optimization is a general framework for a variety of problems of combinatorial optimization such as MAX-SAT or MAX-CUT (Boros and Hammer (2002)).

There are numerous generalizations and variations around notions of independent sets and matchings (independent sets of edges). Consider, for instance, a subset of vertices inducing a subgraph with vertex degree at most k. For $k = 0$, this coincides with the notion of independent set. For $k = 1$, this notion is usually referred in the literature as a *dissociation set*. As shown in (Yannakakis (1981)), the problem of finding a dissociation set of maximum cardinality is NP-hard in the class of bipartite graphs. Both the MAXIMUM INDEPENDENT SET and the MAXIMUM DISSOCIATION SET problems belong to a more general class of hereditary subset problems (Halldórsson (2000)). Another generalization of the notion of independent set has been recently introduced under the name *k-insulated set* (Jagota, Narasimhan and Soltes (2001)).

Consider now a subset of vertices of a graph G inducing a subgraph H with vertex degree exactly 1. The set of edges of H is called an *induced matching* of G (Cameron (1989)). Similarly to the ordinary MAXIMUM MATCHING problem, the MAXIMUM INDUCED MATCHING can be reduced to the MAXIMUM INDEPENDENT SET problem by associating with G an auxiliary graph, which is the square of $L(G)$, where the square of a graph

$H = (V, E)$ is the graph with vertex set V in which two vertices x and y are adjacent if and only if the distance between x and y in H is at most 2. However, unlike the MAXIMUM MATCHING problem, the MAXIMUM INDUCED MATCHING problem is NP-hard even for bipartite graphs with maximum degree 3 (Lozin (2002a)).

One more variation around matchings is the problem of finding in a graph a maximal matching of minimum cardinality, known also as the MINIMUM INDEPENDENT EDGE DOMINATING SET problem (Yannakakis and Gavril (1980)). This problem reduces to MIS by associating with the input graph G the total graph of G consisting of a copy of G, a copy of $L(G)$ and the edges connecting a vertex v of G to a vertex e of $L(G)$ if and only if v is incident to e in G. By exploiting this association, Yannakakis and Gavril (1980) proved NP-hardness of the MAXIMUM INDEPENDENT SET problem for total graphs of bipartite graphs. Some more problems related to the notion of matching can be found in (Müller (1990); Plaisted and Zaks (1980); Stockmeyer and Vazirani (1982)).

Among various applications of the MAXIMUM INDEPENDENT SET (MAXIMUM CLIQUE) problem let us distinguish two examples. The origin of the first one is the area of computer vision and pattern recognition, where one of the central problems is the matching of relational structures. In graph theoretical terminology, this is the GRAPH ISOMORPHISM, or more generally, MAXIMUM COMMON SUBGRAPH problem. It reduces to the MAXIMUM CLIQUE problem by associating with a pair of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ a special graph $G = (V, E)$ (known as the *association graph* (Barrow and Burstal (1976); Pelillo, Siddiqi and Zucker (1999))) with vertex set $V = V_1 \times V_2$ so that two vertices $(i, j) \in V$ and $(k, l) \in V$ are adjacent in G if and only if $i \neq k$, $j \neq l$ and $ik \in E_1 \Leftrightarrow jl \in E_2$. Then a maximum common subgraph of the graphs G_1 and G_2 corresponds to a maximum clique in G.

Another example comes from information theory. The graph theoretical model arising here can be roughly described as follows. An information source sends messages in the alphabet $X = \{x_1, x_2, \ldots, x_n\}$. Along the transmission some symbols of X can be changed to others because of random noise. Let G be a graph with $V(G) = X$ and $x_i x_j \in E(G)$ if and only if x_i and x_j can be interchanged during transmission. Then a noise-resistant code should consist of the symbols of X that constitute an independent set in G. Therefore, a largest noise-resistant code corresponds to a largest independent set in G.

For more information about the MAXIMUM CLIQUE (MAXIMUM INDEPENDENT SET) problem, including application, complexity issues, etc., we refer to (Bomze, Budinich, Pardalos and Pelillo (1999)).

In view of the NP-hardness of the MAXIMUM INDEPENDENT SET problem, one can distinguish three main groups of algorithms to solve this problem:

(1) non-polynomial-time algorithms,

(2) polynomial-time algorithms providing approximate solutions,

(3) polynomial-time algorithms that solve the problem exactly for graphs belonging to special classes.

Non-polynomial-time algorithms are generally impractical even for graphs of moderate size. It has been recently shown in (Håstad (1999)) that non-exact polynomial-time algorithms cannot approximate the size of a maximum independent set within a factor of $n^{1-\epsilon}$, which is viewed as a negative result. The objective of the present paper is the algorithms of the third group.

As we mentioned already, the MAXIMUM INDEPENDENT SET problem has a polynomial-time solution in the classes of bipartite graphs and line graphs. In both cases, MIS reduces to the MAXIMUM MATCHING problem. For line graphs, this reduction has been described above. For bipartite graphs, the reduction is based on the fundamental theorem of König stating that the independence number of a bipartite graph G added to the number of edges in a maximum matching of G amounts to the number of vertices of G.

A polynomial-time solution to the MAXIMUM MATCHING problem is based on Berge's idea (Berge (1957)) that a matching in a graph is maximum if and only if there are no augmenting chains with respect to the matching. The first polynomial-time algorithm to find augmenting chains has been proposed by Edmonds in 1965. The idea of augmenting chains is a special case of a general approach to solve the MAXIMUM INDEPENDENT SET problem by means of augmenting graphs. The next section presents this approach in its general form.

3. Method of Augmenting Graphs

Let S be an independent set in a graph G. We shall call the vertices of S *black* and the remaining vertices of the graph *white*. A bipartite graph $H = (W, B, E)$ with the vertex set $W \cup B$ and the edge set E is called *augmenting* for S (and we say that S *admits* the augmenting graph) if

(1) $B \subseteq S$, $W \subseteq V(G) - S$,

(2) $N(W) \cap (S - B) = \emptyset$,

(3) $|W| > |B|$.

Clearly if $H = (W, B, E)$ is an augmenting graph for S, then S is not a maximum independent set in G, since the set $S' = (S - B) \cup W$ is independent and $|S'| > |S|$. We shall say that the set S' is obtained from S by H-augmentation and call the number $|W| - |B| = |S'| - |S|$ the *increment* of H.

Conversely, if S is not a maximum independent set, and S' is an independent set such that $|S'| > |S|$, then the subgraph of G induced by the set $(S - S') \cup (S' - S)$ is augmenting for S. Therefore, we have proved the following key result.

Theorem of augmenting graphs. *An independent set S in a graph G is maximum if and only if there are no augmenting graphs for S.*

This theorem suggests the following general approach to find a maximum independent set in a graph G: begin with any independent set S in G and as long as S admits an augmenting graph H, apply H-augmentation to S. Clearly the problem of finding augmenting graphs is generally NP-hard, as the maximum independent set problem is NP-hard. However, this approach has proven to be a useful tool to develop approximate solutions to the problem (Halldórsson (1995)), to compute bounds on the independence number (Denley (1994)), and to solve the problem in polynomial time for graphs in special classes. For a polynomial-time solution, one has to

(a) find a complete list of augmenting graphs in the class under consideration,

(b) develop polynomial-time algorithms for detecting all augmenting graphs in the class.

Section 4 of the present paper analyzes problem (a) and Section 5 problem (b) for various graph classes. Analysis of problem (a) is based on characterization of bipartite graphs in classes under consideration. Clearly not every bipartite graph can be augmenting. For instance, a bipartite cycle is never augmenting, since the condition (3) fails for it. Moreover, without loss of generality we may exclude from our consideration those augmenting graphs, which are not minimal. An augmenting graph H for a set S is called *minimal* if no proper induced subgraph of H is augmenting for S. Some bipartite graphs that may be augmenting are never minimal augmenting. To give an example, consider a claw $K_{1,3}$. If it is augmenting for some independent set S, then its subgraph obtained by deleting a vertex of degree 1 also is an augmenting graph for S. The following lemma describes several necessary conditions for an augmenting graph to be minimal.

Lemma 1 *If $H = (B, W, E)$ is a minimal augmenting graph for an independent set S, then*

(i) H is connected;

(ii) $|B| = |W| - 1$;

(iii) for every subset $A \subseteq B$, $|A| < |N_W(A)|$.

Proof. Conditions (i) and (ii) are obvious. To show (iii), assume $|A| \geq |N_W(A)|$ for some subset A of B. Then the vertices in $(B - A) \cup (W - N_W(A))$ induce a proper subgraph of H which is augmenting too. ■

Another notion, which is helpful in some cases, is the notion of maximum augmenting graph. An augmenting graph H for an independent set S is called *maximum* if the increment of any other augmenting graph for S does not exceed the increment of H. The importance of this notion is due to the following lemma.

Lemma 2 *Let S be an independent set in a graph G, and H an augmenting graph for S. Then the independent set obtained by H-augmentation is maximum in G if and only if H is a maximum augmenting graph for S.*

To conclude this section, let us mention that an idea similar to augmenting graphs can be applied to the INDEPENDENT DOMINATING SET problem. In this case, given a maximal independent set S in a graph G, we want to find a smaller maximal independent set. So, we define a bipartite graph $H = (B, W, E)$ to be a *decreasing* graph for S if

(1') $B \subseteq S$, $W \subseteq V(G) - S$,

(2') $N(W) \cap (S - B) = \emptyset$,

(3') $|W| < |B|$,

(4') $(S - B) \cup W$ is a maximal independent set in G.

The additional condition (4') makes the problem of finding decreasing graphs harder than that of finding augmenting graphs, though some results exploiting the idea of decreasing graphs are available in the literature (Boliac and Lozin (2003a)).

4. Characterization of Augmenting Graphs

The basis for characterization of augmenting graphs in a certain class is the description of bipartite graphs in that class. For a bipartite graph

$G = (V_1, V_2, E)$, we shall denote by $\widetilde{G}$ the bipartite complement of G, i.e. $\widetilde{G} = (V_1, V_2, (V_1 \times V_2) - E)$. We call a bipartite graph G *prime* if any two distinct vertices of G have different neighborhoods.

Claw-free ($S_{1,1,1}$-free) graphs. In the class of *claw*-free graphs, no bipartite graph has a vertex of degree more than 2, since otherwise a claw arises. Therefore, every connected *claw*-free bipartite graph is either an even cycle or a chain. Cycles of even length and chains of odd length cannot be augmenting graphs, since they have equal number of black and white vertices. Thus, every minimal *claw*-free augmenting graph is a chain of even length.

P_4-free ($S_{0,1,2}$-free) graphs. It is a simple exercise to show that every connected P_4-free bipartite graph is complete bipartite. Therefore, every minimal augmenting graph in this class is of the from $K_{n,n+1}$ for some value of n.

Fork-free ($S_{1,1,2}$-free) graphs. Connected bipartite fork-free graphs G have been characterized in (Alekseev (1999)) as follows: either $\Delta(G) \leq 2$ or $\Delta(\widetilde{G}) \leq 1$. A bipartite graph G with $\Delta(\widetilde{G}) \leq 1$ has been called a *complex.* Thus, every minimal fork-free augmenting graph is either a chain of even length or a complex.

P_5-free ($S_{0,2,2}$-free) graphs. It has been shown independently by many researchers (and can be easily verified) that every connected P_5-free bipartite graph is $2K_2$-free, where a $2K_2$ is the disjoint union of two copies of K_2. The class of $2K_2$-free bipartite graphs was introduced in the literature under various names such as *chain graphs* (Yannakakis (1981)) or *difference graphs* (Hammer, Peled and Sun (1990)). The fundamental property of a chain graph is that the vertices in each part can be ordered under inclusion of their neighborhoods. Unfortunately, this nice property does not help in finding maximum independent sets in P_5-free graphs in polynomial time (the complexity status of the problem in this class is still an open question). So, augmenting graphs in many subclasses of P_5-free bipartite graphs have been characterized, among which we distinguish $(P_5, banner)$-free and $(P_5, K_{3,3} - e)$-free graphs.

It has been shown in (Lozin (2000a)) that in the class of $(P_5, banner)$-free graphs every minimal augmenting graph is complete bipartite. Here we prove a more general proposition, which is based on the following two lemmas.

LEMMA 3 *Let H be a connected bipartite banner-free graph. If H contains a C_4, then it is complete bipartite.*

Proof. Denote by H' a maximal induced complete bipartite subgraph of H containing the C_4. Let x be a vertex outside H' adjacent to a vertex in the subgraph. Then x must be adjacent to all the vertices in the opposite part of the subgraph, since otherwise H contains an induced banner. But then H' is not maximal. This contradiction proves that $H = H'$ is complete bipartite. ■

LEMMA 4 *No minimal $(S_{1,2,2}, C_4)$-free augmenting graph H contains a $K_{1,3}$ as an induced subgraph.*

Proof. Let vertices a, b, c, d induce a $K_{1,3}$ in H with a being the center. Assume first that a is the only neighbor of b and c in the graph H. Then H is not minimal. Indeed, in case that a is a white vertex, this follows from Lemma 1(iii). If a is a black vertex, then $H[a, b, c]$ is a smaller augmenting graph. Now suppose without loss of generality that b has a neighbor $e \neq a$, and c has a neighbor $f \neq a$ in the graph H. Since H is C_4-free, $e \neq f$ and $ec, ed, fb, fd \notin E(H)$. But now the vertices a, b, c, d, e, f induce an $S_{1,2,2}$. ■

Combining these two lemmas with the characterization of *claw*-free minimal augmenting graphs, we obtain the following conclusion.

THEOREM 1 *Every minimal $(S_{1,2,2}, banner)$-free augmenting graph is either complete bipartite or a chain of odd length. In particular, every minimal $(P_5, banner)$-free augmenting graph is complete bipartite.*

The class of $(P_5, K_{3,3} - e)$-free graphs generalizes $(P_5, banner)$-free graphs. The minimal augmenting $(P_5, K_{3,3} - e)$-free graphs have been characterized in (Gerber, Hertz and Schindl (2004)) as follows.

THEOREM 2 *Every minimal augmenting $(P_5, K_{3,3} - e)$-free graph is either complete bipartite or a graph obtained from a complete bipartite graph $K_{n,n}$ by adding a single vertex with exactly one neighbor in the opposite part.*

$S_{1,2,2}$-free graphs. The class of $S_{1,2,2}$-free bipartite graphs has been provided in (Lozin (2000b)) with the following characterization.

THEOREM 3 *Every prime $S_{1,2,2}$-free bipartite graph is either $K_{1,3}$-free or $\widetilde{P}_5$-free.*

The class of $S_{1,2,2}$-free graphs is clearly an extension of P_5-free graphs. Since the complexity of the problem is still open even for P_5-free graphs, it is worth characterizing subclasses of $S_{1,2,2}$-free graphs that do not contain the entire class of P_5-free graphs. One of such characterizations is given in Theorem 1. Now we extend this theorem to $(S_{1,2,2}, A)$-free bipartite graphs, where A is the graph obtained from a P_6 by adding an edge between two vertices of degree 2 at distance 3.

THEOREM 4 *A prime connected $(S_{1,2,2}, A)$-free bipartite graph G is $S_{1,1,2}$-free.*

Proof. By contradiction, assume G contains an $S_{1,1,2}$ with vertices a, b, c, d, e and edges ab, bc, cd, ce. Since G is prime, there must be a vertex f adjacent to e but not to d. Since G is bipartite, f is not adjacent to a and c. But then the vertices a, b, c, d, e, f induce either an $S_{1,2,2}$, if f is not adjacent to b, or an A, otherwise. ■

$S_{2,2,2}$-free graphs. This is a rich class containing all the previously mentioned classes. Moreover, the bipartite graphs in this class include also all bipartite permutation graphs (Spinrad, Brandstädt and Stewart (1987)) and all biconvex graphs (Abbas and Stewart (2000)). So, again we restrict ourselves to a special subclass of $S_{2,2,2}$-free bipartite graphs that does not entirely contain the class of P_5-free graphs. To this end, we recall that a *complex* is a bipartite graph every vertex of which has at most one non-neighbor in the opposite part. A *caterpillar* is a tree that becomes a path by removing the pendant vertices. A *circular caterpillar* G is a graph that becomes a cycle C_k by removing the pendant vertices. We call G a long circular caterpillar if $k > 4$. The following theorem has been proven in (Boliac and Lozin (2001)).

THEOREM 5 *A prime connected $(S_{2,2,2}, A)$-free bipartite graph is either a caterpillar or a long circular caterpillar or a complex.*

P_k-free graphs with $k \geq 6$ and $S_{1,2,j}$-free graphs with $j \geq 3$. It has been shown in (Mosca (1999)) that (P_6, C_4)-free augmenting graphs are simple augmenting trees (i.e. graphs $M_{r,0}$ with $r \geq 0$ in Figure 1.2). This characterization as well as the characterization of $(P_5, banner)$-free augmenting graphs has been extended in (Alekseev and Lozin (2000)) in two different ways.

THEOREM 6 *In the class of $(P_7, banner)$-free graphs every minimal augmenting graph is either complete bipartite or a simple augmenting tree or an augmenting plant (i.e. a graph $L^2_{r,0}$ with $r \geq 2$ in Figure 1.2).*

In the class of $(S_{1,2,3}, banner)$-free graphs every minimal augmenting graph is either a chain or a complete bipartite graph or a simple augmenting tree or an augmenting plant.

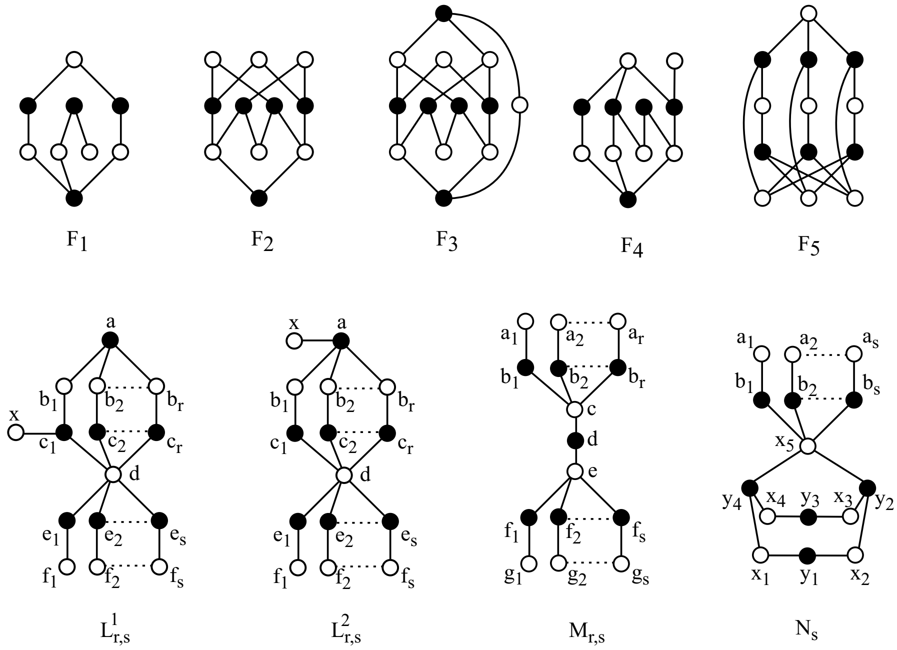

Figure 1.2. minimal augmenting graphs

Finally, in (Gerber, Hertz and Lozin (2004)), these results have been generalized in the following way.

THEOREM 7 *A minimal augmenting* $(P_8, banner)$*-free graph is one of the following graphs (see Figure 1.2 for definitions of the graphs):*
- *a complete bipartite graph* $K_{r,r+1}$ *with* $r \geq 0$,
- *a* $L^1_{r,s}$ *or a* $L^2_{r,s}$ *with* $r \geq 2$ *and* $s \geq 0$,
- *a* $M_{r,s}$ *with* $r \geq 1$ *and* $r \geq s \geq 0$,
- *a* N_s *with* $s \geq 0$.
- *one of the graphs* $F_2, \ldots, F_5$.

A minimal augmenting $(S_{1,2,4}, banner)$*-free graph is one of the following graphs:*
- *a complete bipartite graph* $K_{r,r+1}$ *with* $r \geq 0$,
- *a path* P_k *with* k *odd* ≥ 7,
- *a* $L^2_{r,0}$ *with* $r \geq 2$,
- *a* $M_{r,0}$ *with* $r \geq 1$,
- *one of the graphs* $F_1, \ldots, F_5, L^1_{3,0}, L^1_{2,1}, N_0, N_1$.

Notice that that the set of $(P_8, banner)$-free augmenting graphs can be partitioned into two general groups. The first one contains infinitely many graphs of high vertex degree and "regular" structure, while the second group consists of finitely many graphs of bounded vertex degree. It is not a surprise. With simple arguments it has been shown in (Lozin and Rautenbach (2003)) that for any k and n, there are finitely many connected bipartite $(P_k, K_{1,n})$-free graphs. This observation has been generalized in (Gerber, Hertz and Lozin (2003)) by showing that in the class of $(S_{1,1,j}, K_{1,n})$-free graphs there are finitely many connected bipartite graphs of maximum vertex degree more than 2. Now we extend this result as follows.

THEOREM 8 *For any three integers j, k and n, the class of $(S_{1,2,j}$, $Banner_k$, $K_{1,n})$-free graphs contains finitely many minimal augmenting graphs different from chains.*

Proof. To prove the theorem, consider a minimal augmenting graph H in this class that contains a $K_{1,3}$ with the center a_0. Denote by A_i the subset of vertices of H of distance i from a_0. In particular, $A_0 = \{a_0\}$. Let m be an integer greater than $\max(k+2, j+2)$. Consider a vertex $a_m \in A_m$, and let $P = (a_0, a_1, \ldots, a_m)$ with $a_i \in A_i$ $(i = 1, \ldots, m)$ denote a shortest path connecting a_0 to a_m. Then a_2 has no other neighbor in A_1, except for a_1, since otherwise the vertices of P together with this neighbor would induce a $Banner_{m-2}$.

Since a_0 is the center of a $K_{1,3}$, we may consider two vertices in A_1 different from a_1, say b and c. Assume b has a neighbor d in A_2. Then d is not adjacent to a_3, since otherwise a_3 is the vertex of degree 3 in an induced $S_{1,2,m-3}$ (if $da_1 \notin E(H)$) or in an induced $Banner_{m-3}$ (if $da_1 \in E(H)$). Consequently, d is not adjacent to a_1, since otherwise a_1 is the vertex of degree 3 in an induced $Banner_{m-1}$. But now a_0 is the vertex of degree 3 either in an induced $S_{1,2,m}$ (if $cd \notin E(H)$) or in an induced $Banner_m$ (if $cd \in E(H)$). Therefore, vertices b and c have degree 1 in H, but then H is not minimal. This contradiction shows that $A_i = \emptyset$ for each $i > \max(k+2, j+2)$. Since H is $K_{1,n}$-free, there is a constant bounding the number of vertices in each A_i for $i \leq \max(k+2, j+2)$. Therefore, only finitely many minimal augmenting graphs in the class under consideration contain a $K_{1,3}$. ■

We conclude this section with the characterization of prime $S_{1,2,3}$-free bipartite graphs found in (Lozin (2002b)), which may become a source for many other results on the maximum independent set problem in subclasses of $S_{1,2,3}$-free graphs.

THEOREM 9 *A prime $S_{1,2,3}$-free bipartite graph G is either disconnected or $\widetilde{G}$ is disconnected or G can be partitioned into an independent set and a complete bipartite graph or G is $K_{1,3}$-free or $\widetilde{G}$ is $K_{1,3}$-free.*

5. Finding Augmenting Graphs

5.1 Augmenting Chains

Let S be an independent set in the line graph $L(G)$ of a graph G, and let M be the corresponding matching in G. The problem of finding an aumenting chain for S in $L(G)$ is equivalent to the problem of finding an augmenting chain with respect to M in G, and this problem can be solved by means of Edmonds' polynomial-time algorithm (Edmonds (1965)). In 1980, Minty and Sbihi have independently shown how to extend this result to the class of *claw*-free graphs that strictly contains the class of line graphs. More precisely, both authors have shown that the problem of finding augmenting chains in *claw*-free graphs is polynomially reducible to the problem of finding an augmenting chain with respect to a matching. This result has recently been generalized in two different ways:

- Gerber et al.(2003) have proved that by slightly modifying Minty's algorithm, on can determine augmenting chains in the class of $S_{1,2,3}$-free graphs;
- it is proved in (Hertz, Lozin and Schindl (2003)) that the problem of finding augmenting chains in $(S_{1,2,i}, banner)$-free graphs, for a fixed $i \geq 1$, is polynomially reducible to the problem of finding augmenting chains in *claw*-free graphs.

Both classes of $(S_{1,2,i}, banner)$-free graphs ($i \geq 1$) and $S_{1,2,3}$-free graphs strictly contain the class of *claw*-free graphs. In this section, we first describe Minty's algorithm for finding augmenting chains in *claw*-free graphs. We then describe its extensions to the classes of $S_{1,2,3}$-free and $(S_{1,2,i}, banner)$-free graphs .

5.1.1 Augmenting Chains in *Claw*-free Graphs. Notice first that an augmenting chain for a maximal independent set S necessarily connects two non-adjacent white vertices β and γ, each of which has exactly one black neighbor, respectively, $\overline{\beta}$ and $\overline{\gamma}$. If $\overline{\beta} = \overline{\gamma}$, then the chain $(\beta, \overline{\beta}, \gamma)$ is augmenting for S. We can therefore assume that $\overline{\beta} \neq \overline{\gamma}$. We may also assume that any white vertex different from β and γ is not adjacent to β and γ, and has exactly two black neighbors (the vertices not satisfying the assumption are out of interest, since they cannot occur in any augmenting chain connecting β to γ).

Two white vertices having the same black neighbors are called *similar*. The similarity is an equivalence relation, and an augmenting chain clearly contains at most one vertex in each class of similarity. The similarity classes in the neighborhood of a black vertex b are called the *wings*

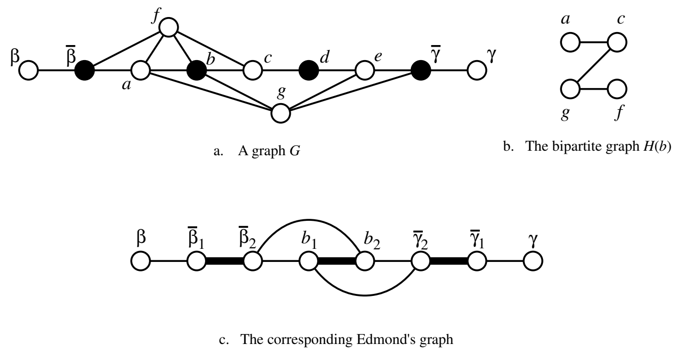

Figure 1.3. Illustration of Minty's algorithm.

of b. Let b be a black vertex different from $\overline{\beta}$ and $\overline{\gamma}$: if b has more than two wings, then b is defined as *regular*, otherwise it is *irregular*. In what follows, R denotes the set of black vertices that are either regular or equal to $\overline{\beta}$ or $\overline{\gamma}$. For illustration, the graph G depicted in Figure 1.3.a has one regular black vertex (vertex b), one irregular black vertex (vertex d), and R is equal to $\{b, \overline{\beta}, \overline{\gamma}\}$.

Definitions An *alternating chain* is a sequence $(x_0, x_1, \ldots, x_k)$ of distinct vertices in which the vertices are alternately white and black. Vertices x_0 and x_k are called the *termini* of the chain. If x_0 and x_k are both black (respectively white) vertices, then the sequence is called a black (respectively white) alternating chain.

Let b_1 and b_2 be two distinct black vertices in R. A black alternating chain with termini b_1 and b_2 is called an IBAP (for *irregular black alternating path*) if it is chordless and if all black vertices of the chain, except b_1 and b_2, are irregular. An IWAP (for *irregular white alternating path*) is a white alternating chain obtained by removing the termini of an IBAP.

For illustration, the graph G depicted in Figure 1.3.a has four IWAPs: (a), (f), (g) and (c, d, e). With this terminology, one can now represent an augmenting chain as a sequence $(I_0 = (\beta), b_0 = \overline{\beta}, I_1, b_1, I_2, \ldots, b_{k-1}, I_{k-1}, b_k = \overline{\gamma}, I_k = (\gamma))$ such that

(a) the b_i $(0 < i < k)$ are distinct black regular vertices,

(b) the I_i $(0 < i < k)$ are pairwise mutually disjoint IWAPs,

(c) each b_i $(0 \le i \le k)$ is adjacent to the final terminus of I_i and to the initial one of I_{i+1},

(d) the white vertices in $I_1 \cup \ldots \cup I_{k-1}$ are pairwise non-adjacent.

Minty has proved that the neighborhood of each black vertex b can be decomposed into at most two subsets $N_1(b)$ and $N_2(b)$, called *node classes*, in such a way that no two vertices in the same node class can occur in the same augmenting chain for S. For vertices $\overline{\beta}$ and $\overline{\gamma}$, such a decomposition is obvious: one of the node classes contains the vertex β (respectively γ) and the other class includes all the remaining vertices in the neighborhood of $\overline{\beta}$ (respectively $\overline{\gamma}$). We assume that $N_1(\overline{\beta}) = \{\beta\}$ and $N_1(\overline{\gamma}) = \{\gamma\}$. For an irregular black vertex b, the decomposition also is trivial: the node classes correspond to the wings of b.

Now let b be a regular black vertex. Two white neighbors of b can occur in the same augmenting chain for S only if they are non-similar and non-adjacent. Define an auxiliary graph $H(b)$ as follows:

- the vertex set of $H(b)$ is $N(b)$
- two vertices u and v of $H(b)$ are linked by an edge if and only if u and v are non-similar non-adjacent vertices in G.

Minty has proved that $H(b)$ is bipartite. The two node classes $N_1(b)$ and $N_2(b)$ of a regular black vertex b therefore correspond to the two parts of the bipartite graph $H(b)$. For illustration, the bipartite graph $H(b)$ associated with b in the graph of Figure 1.3.b defines the partition of $N(b)$ into two node classes $N_1(b) = \{a, g\}$ and $N_2(b) = \{c, f\}$.

We now show how to determine the pairs (u, v) of vertices such that there exists an IWAP with termini u and v. Notice first that u and v must have a black neighbor in R. So let b_0 be a black vertex in R, and let W_1 be one of its wings ($W_1 = N_2(\overline{\beta})$ if $b_0 = \overline{\beta}$, and $W_1 = N_2(\overline{\gamma})$ if $b_0 = \overline{\gamma}$). The following algorithm determines the set P of pairs (u, v) such that u belongs to W_1 and is a terminus of an IWAP:

1. Set $k := 1$;
2. Let b_k denote the second black neighbor of the vertices in W_k; If b_k has two wings then go to Step 3. If b_k is regular and different from b_0 then go to Step 4. Otherwise STOP: P is empty;
3. Let W_{k+1} denote the second wing of b_k. Set $k := k + 1$ and go to Step 2;
4. Construct an auxiliary graph with vertex set $W_1 \cup \ldots \cup W_k$ and link two vertices by an edge if and only if they are non-adjacent in G and belong to two consecutive sets W_i and W_{i+1}. Orient all edges from W_i to W_{i+1};
5. Determine the set P of pairs (u, v) such that $u \in W_1, v \in W_k$ and there exists a path from u to v in the auxiliary graph.

The last important concept proposed by Minty is the *Edmonds' Graph* which is constructed as follows:

- For each black vertex $b \in R$ do the following: create two vertices b_1 and b_2, link them by a black edge, and identify b_1 and b_2 with the two node classes $N_1(b)$ and $N_2(b)$ of b. In particular, $\overline{\beta}_1$ represents $N_1(\overline{\beta}) = \{\beta\}$ and $\overline{\gamma}_1$ represents $N_1(\overline{\gamma}) = \{\gamma\}$;
- Create two vertices β and γ, and link β to $\overline{\beta}_1$ and γ to $\overline{\gamma}_1$ by a white edge.
- Link b_i (i=1 or 2) to b'_j (j=1 or 2) with a white edge if there are two white vertices u and v in G such that $u \in N_i(b)$, $v \in N_j(b')$, and there exists an IWAP with termini u and v. Identify each such white edge with a corresponding IWAP.

The black edges define a matching M in the Edmonds' graph. If M is not maximum, then there exists an augmenting chain of edges $(e_0, \dots, e_{2k})$ such that the even indexed edges are white, the odd-indexed edges are black, e_0 is the edge linking β to $\overline{\beta}_1$, and e_{2k} is the edge linking γ to $\overline{\gamma}_1$. Such an augmenting chain of edges in the Edmonds' graph corresponds to an alternating chain C in G. Indeed, notice first that each white edge e_i with $2 \leq i \leq 2k-2$ corresponds to an IWAP whose termini will be denoted w_{i-1} and w_i. Also, each black edge e_i with $1 \leq i \leq 2k-1$ corresponds to a black vertex b_i. The alternating chain C is obtained as follows:

- replace e_0 by β, e_{2k} by γ, and each white edge $e_i (2 \leq i \leq 2k-2)$ by an IWAP with termini w_{i-1} and w_i;
- replace each black edge $e_i (1 \leq i \leq 2k-1)$ by the vertex b_i.

Minty has proved that C is chordless, and is therefore an augmenting chain for S in G. He has also proved that an augmenting chain for S in G corresponds to an augmenting chain with respect to M in the Edmonds' graph. Hence, determining whether there exists an augmenting chain for S in G, with termini β and γ, is equivalent to determining whether there exists an augmenting chain with respect to M in the Edmonds' graph.

For illustration, the Edmonds' graph associated with the graph in Figure 1.3.a is represented in Figure 1.3.c with bold lines for the black edges and regular lines for the white edges. The four IWAPs (a), (f), (g) and (c, d, e) correspond to the four white edges $\overline{\beta}_2 b_1$, $\overline{\beta}_2 b_2$, $b_1 \overline{\gamma}_2$ and $b_2 \overline{\gamma}_2$, respectively. The Edmonds' graph contains two augmenting chains: $(\beta, \overline{\beta}_1, \overline{\beta}_2, b_1, b_2, \overline{\gamma}_2, \overline{\gamma}_1, \gamma)$ and $(\beta, \overline{\beta}_1, \overline{\beta}_2, b_2, b_1, \overline{\gamma}_2, \overline{\gamma}_1, \gamma)$ which correspond to the augmenting chains $(\beta, \overline{\beta}, a, b, c, d, e, \overline{\gamma}, \gamma)$ and $(\beta, \overline{\beta}, f, b, g, \overline{\gamma}, \gamma)$ for S in G.

In summary, given two non-adjacent white vertices β and γ, each of which has exactly one black neighbor, the following algorithm either builds an augmenting chain for S with termini β and γ, or concludes that no such chain exists.

Minty's algorithm for finding an augmenting chain for S with termini β and γ in a *claw*-free graph

1. Partition the neighborhood of each regular black vertex b into two node classes $N_1(b)$ and $N_2(b)$ by constructing the bipartite graph $H(b)$ in which two white neighbors of b are linked by an edge if and only if they are non-adjacent and non-similar;
2. Determine the set of pairs (u, v) of (not necessarily distinct) white vertices such that there exists an IWAP with termini u and v;
3. Construct the Edmonds' graph and let M denote the set of black edges;
4. If the Edmonds' graph contains an augmenting chain of edges with respect to M, then it corresponds to an augmenting chain for S in G with termini β and γ; otherwise, there are no augmenting chains for S with termini β and γ.

5.1.2 Augmenting Chains in $S_{1,2,3}$-free Graphs. Gerber et al. (2003) have shown that Minty's algorithm can be adapted in order to detect augmenting chains in the class of $S_{1,2,3}$-free graphs that strictly contains the class of *claw*-free graphs. The algorithm described in (Gerber, Hertz and Lozin (2003)) differs from Minty's algorithm in only two points. The first difference occurs in the definition of $H(b)$ where an additional condition is imposed for creating an edge in $H(b)$. More precisely, let us first define *special* pairs of vertices.

Definition A pair (u, v) of vertices is *special* if u and v have a common black regular neighbor b, and if there is a vertex $w \in N(b)$ which is similar neither to u nor to v and such that either both of uw and vw or none of them is an edge in G.

It is shown in (Gerber, Hertz and Lozin (2003)) that if (u, v) is a special pair of non-adjacent non-similar vertices in a $S_{1,2,3}$-free graph, then u and v cannot occur in a same augmenting chain. For a regular black vertex b, the graph $H(b)$ is defined as follows: the vertex set of $H(b)$ is $N(b)$, and two vertices u and v in $H(b)$ are linked by an edge if and

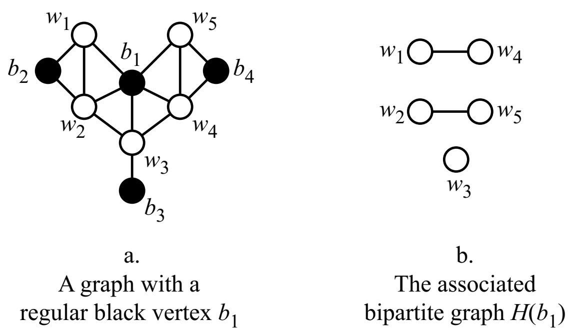

Figure 1.4. Bipartite graph associated with a regular black vertex.

only if (u, v) is a pair of non-special non-similar non-adjacent vertices in G. It is proved in (Gerber, Hertz and Lozin (2003)) that $H(b)$ is bipartite. For illustration, the bipartite graph $H(b_1)$ associated with the regular black vertex b_1 in Figure 1.4.a is represented in Figure 1.4.b.

An isolated vertex in $H(b)$ cannot belong to an augmenting chain. Hence, an IWAP in an augmenting chain necessarily connects two white vertices that are not isolated in the respective bipartite graphs associated with their black neighbors in R. This motivates the following definition.

Definition Let $(b_1, w_1, \ldots, w_{k-1}, b_k)$ be an IBAP. The IWAP obtained by removing b_1 and b_k is *interesting* if w_1 and w_{k-1} are non-isolated vertices in $H(b_1)$ and $H(b_k)$, respectively.

Let W denote the set of white vertices w which have a black neighbor $b \in R$ such that w is an isolated vertex in $H(b)$. The set of pairs (u, v) such that there is an interesting IWAP with termini u and v can be determined in polynomial time by using the algorithm of the previous section that generates all IWAPs, and by removing a pair (u, v) if u or/and v belongs to W.

The Edmonds' graph is then constructed as in Minty's algorithm, except that white edges in the Edmonds' graph correspond to interesting IWAPs.

Now let S be an independent set in a $S_{1,2,3}$-free graph G, let β and γ be two non-adjacent white vertices, each of which has exactly one black neighbor, and let M denote the set of black edges in the corresponding Edmond's graph. It is proved in (Gerber, Hertz and Lozin (2003)) that determining whether there exists an augmenting chain for S with termini β and γ is equivalent to determining whether there exists an augmenting chain with respect to M in the Edmonds' graph. In summary, the algorithm for finding augmenting chains in $S_{1,2,3}$-free graphs works as follows.

Algorithm for finding an augmenting chain for S with termini β and γ in a $S_{1,2,3}$- free graph

1. Partition the neighborhood of each regular black vertex b into two node classes $N_1(b)$ and $N_2(b)$ by constructing the bipartite graph $H(b)$ in which two white neighbors u and v of b are linked by an edge if and only if (u,v) is a pair of non-special non-adjacent non-similar vertices;
2. Determine the set of pairs (u,v) of (not necessarily distinct) white vertices such that there exists an interesting IWAP with termini u and v;
3. Construct the Edmond's graph and let M denote the set of black edges;
4. If the Edmond's graph contains an augmenting chain of edges with respect to M, then it corresponds to an augmenting chain in G with termini β and γ; otherwise, there are no augmenting chains with termini β and γ.

The above algorithm is very similar to Minty's algorithm. It only differs in step 1 where an additional condition is imposed for introducing an edge in $H(b)$, and in step 2 where only interesting IWAPs are considered.

5.1.3 Augmenting Chains in $(S_{1,2,i}, banner)$-free Graphs.

Let $G=(V,E)$ be a $(S_{1,2,i}, banner)$-free graph, where i is any fixed strictly positive integer, and let S be a maximal independent set in G. An augmenting chain for S is of the form $P=(x_0,x_1,x_2,\ldots,x_{k-1},x_k)$ (k is even) where the even-indexed vertices of P are white, and the odd-indexed vertices are black.

Definition Let (x_0,x_k) be a pair of white non-adjacent vertices, each of which has exactly one black neighbor. A pair (L,R) of disjoint chordless alternating chains $L=(x_0,x_1,x_2)$ and $R=(x_{k-m},x_{k-m+1},\ldots,x_{k-1},x_k)$ is said *candidate* for (x_0,x_k) if

- no vertex of L is adjacent to a vertex of R,
- each vertex x_j is white if and only if j is even, and
- $m=2\lfloor\frac{i}{2}\rfloor$.

Augmenting chains with at most $i + 3$ vertices can be detected in polynomial time by inspecting all subsets of black vertices of cardinality at most $\frac{i+4}{2}$. It is proved in (Hertz, Lozin and Schindl (2003)) that larger augmenting chains can be detected by applying the following algorithm for each pair (x_0, x_k) of white non-adjacent vertices, each of which has exactly one black neighbor.

(a) Remove from G all white vertices adjacent to x_0 or x_k as well as all white vertices different from x_0 and x_k which have 0, 1 or more than 3 black neighbors.

(b) Find all candidate pairs (L, R) of alternating chains for (x_0, x_k), and for each such pair, do steps (b.1) through (b.4):

  (b.1) remove all white vertices that have a neighbor in L or in R,

  (b.2) remove the vertices of L and R except for x_2 and x_{k-m},

  (b.3) remove all the vertices that are the center of a *claw* in the remaining graph,

  (b.4) in the resulting *claw*-free graph, determine whether there exists an augmenting chain for S with termini x_2 and x_{k-m}.

Step (b.4) can be performed by using the algorithm described in Section 5.1.1.

5.2 Complete Bipartite Augmenting Graphs

In the present section we describe an approach to finding augmenting graphs every connected component of which is complete bipartite, i.e. P_4-free augmenting graphs. This approach has been applied first to fork-free graphs (Alekseev (1999)) and $(P_5, Banner)$-free graphs (Lozin (2000a)). Then it has been extended to the entire class of *Banner*-free graphs (Alekseev and Lozin (2000)) and to the entire class of P_5-free graphs (Boliac and Lozin (2003b)). We now generalize this approach to the class of $Banner_2$-free graphs that contains all the above mentioned classes.

Throughout the section G stands for a $Banner_2$-free graph and S for a maximal independent set in G. Let us call two white vertices x and y with $N_S(x) = N_S(y)$ *similar*. First, we partition the set of white vertices into similarity classes. Next, each class of similarity C is partitioned into co-components, i.e. subsets each of which forms a connected component in the complement to $G[C]$. Every co-component of a similarity class will be called a *node class*. Two node classes are *non-similar* if their vertices belong to different similarity classes.

Without loss of generality we shall assume that for any node class Q_i the following conditions hold:

$$|N_S(Q_i)| \geq 3, \tag{1}$$

$$\text{each vertex in } Q_i \text{ has a non-neighbor in the same node class.} \tag{2}$$

To meet condition (1), we first find augmenting graphs of the form $K_{1,2}$ or $K_{2,3}$. If S does not admit such augmenting graphs, we may delete node classes non-satisfying (1). Under condition (1), any vertex that has no non-neighbor in its own node class is of no interest to us. So, vertices non-satisfying condition (2) can be deleted.

Assuming (1) and (2), we prove the following lemma.

LEMMA 5 *Let Q_1 and Q_2 be two non-similar node classes. If there is a pair of non-adjacent vertices $x \in Q_1$ and $y \in Q_2$, then one of the following statements holds:*

(a) $\max(|N_S(Q_1) - N_S(Q_2)|, |N_S(Q_2) - N_S(Q_1)|) \leq 1$,

(b) $N_S(Q_1) \subseteq N_S(Q_2)$ *or* $N_S(Q_2) \subseteq N_S(Q_1)$,

(c) $N_S(Q_1) \cap N_S(Q_2) = \emptyset$ *and no vertex in Q_1 is adjacent to a vertex in Q_2.*

Proof. Assume first that the intersection $N_S(Q_1) \cap N_S(Q_2)$ contains a vertex a, and suppose (b) does not hold. Then we denote by b a vertex in $N_S(Q_1) - N_S(Q_2)$ and by c a vertex in $N_S(Q_2) - N_S(Q_1)$. We also assume by contradiction that $N_S(Q_2) - N_S(Q_1)$ contains a vertex $d \neq c$, and finally, according to the assumption (2), we let z be a vertex in Q_2 non-adjacent to y. If x is not adjacent to z, then the vertices a, b, c, x, y, z induce a $Banner_2$ in G. If x is adjacent to z, then a $Banner_2$ is induced by b, c, d, x, y, z.

Now we assume that $N_S(Q_1) \cap N_S(Q_2) = \emptyset$, and we denote by a and b two vertices in $N_S(Q_1)$ and by c and d two vertices in $N_S(Q_2)$. Suppose by contradiction that a vertex z in Q_1 is adjacent to a vertex w in Q_2. If x is not adjacent to w then one can find two vertices v_1 and v_2 on the path connecting z to x in $\overline{G[Q_1]}$ such that w is adjacent to v_1 but not to v_2. But then vertices a, b, c, v_1, v_2, w induce a $Banner_2$ in G, a contradiction. So we can assume that x is adjacent to w. But one can now find two vertices v_1 and v_2 on the path connecting w to y in $\overline{G[Q_2]}$ such that x is adjacent to v_1 but not to v_2. Hence, vertices b, c, d, x, v_1, v_2 induce a $Banner_2$ in G, a contradiction. ■

Let us associate with G and S an auxiliary graph Γ as follows. The vertices of Γ are the node classes of G, and two vertices Q_i and Q_j are defined to be adjacent in Γ if and only if one of the following conditions holds:

- $\max\{|N_S(Q_i) - N_S(Q_j)|, |N_S(Q_j) - N_S(Q_i|\} \leq 1$,
- $N_S(Q_i) \subseteq N_S(Q_j)$ or $N_S(Q_j) \subseteq N_S(Q_i)$,
- each vertex of Q_i is adjacent to each vertex of Q_j in graph G.

In other words, due to Lemma 5, Q_i and Q_j are non-adjacent in Γ if and only if $N_S(Q_i) \cap N_S(Q_j) = \emptyset$ and no vertex in Q_i is adjacent to a vertex in Q_j. To each vertex Q_j of Γ we assign an integer number, the weight of the vertex, equal to $\alpha(G[Q_j]) - |N_S(Q_j)|$.

Consider now an independent set $Q = \{Q_1, \ldots, Q_p\}$ in the graph Γ. Let us associate with each vertex $Q_j \in Q$ a complete bipartite graph $H_j = (B_j, W_j, E_j)$ with $B_j = N_S(Q_j)$ and W_j being an independent set of maximum cardinality in $G[Q_j]$. By definition of the graph Γ, subsets $B_1, \ldots, B_p$ are pairwise disjoint and the union $\cup_{j=1}^{p} W_j$ is an independent set in G. Hence the union of graphs $H_1, \ldots, H_p$, denoted H_Q, is a P_4-free bipartite graph.

The increment of H_Q, equal to $\sum_{j=1}^{p}(|W_j| - |B_j|)$, coincides with the weight of Q, equal to $\sum_{j=1}^{p}(\alpha(G[Q_j]) - |N_S(Q_j)|)$. If the weight of Q is positive, then H_Q is an augmenting graph for S. Moreover, if Q is an independent set of maximum total weight in Γ, then the increment of H_Q is maximum over all P_4-free augmenting graphs for S. Indeed, if H is a P_4-free augmenting graph for S with larger increment, then the node classes corresponding to the components of H form an independent set in Γ the weight of which is obviously at least as large as the increment of H and hence is greater than that of Q, contradicting the assumption. We thus have proved the following lemma

LEMMA 6 *If Q is an independent set of maximum weight in the graph Γ, then the increment of the corresponding graph H_Q is maximum over all possible P_4-free augmenting graphs for S.*

Assume now that S admits no augmenting graphs containing a P_4, and let H be a P_4-free augmenting graph for S with maximum increment. Then, obviously, the independent set obtained from S by H-augmentation is of maximum cardinality. This observation together with Lemma 6 provide a way to reduce the independent set problem in $Banner_2$-free graphs to the following two subproblems:

(P_1) finding augmenting graphs containing a P_4;

(P_2) finding an independent set of maximum weight in the auxiliary graph Γ.

We formally fix the above proposition in the following recursive procedure.

ALPHA(G)

Input: A $Banner_2$-free graph G.
Output: An independent set S of maximum size in G.

1. Find an arbitrary maximal under inclusion independent set S in G. If $S = V(G)$ go to 7.
2. As long as possible apply H-augmentations to S with H containing a P_4.
3. Partition the vertices of $V(G) - S$ into node classes $Q_1, \ldots, Q_k$.
4. For every $j = 1, \ldots, k$, find a maximum independent set $W_j = ALPHA(G[Q_j])$.
5. Construct the auxiliary graph Γ and find an independent set $Q = \{Q_1, \ldots, Q_p\}$ of maximum weight in it.
6. If the weight of Q is positive, augment S by exchanging $N_S(Q_i)$ by W_i for each $i = 1, \ldots, p$.
7. Return S and STOP.

In the rest of this section we show that the problem (P_2), i.e. finding an independent set of maximum weight in the auxiliary graph Γ, has a polynomial-time solution whenever G is a $Banner_2$-free graph.

Let us say that an edge Q_iQ_j in the graph Γ is of type A if $N_S(Q_i) \subseteq N_S(Q_j)$ or $N_S(Q_j) \subseteq N_S(Q_i)$ or $\max(|N_S(Q_i) - N_S(Q_j)|, |N_S(Q_j) - N_S(Q_i|) \leq 1$, and of type B otherwise. Particularly, for every edge Q_iQ_j of type B, we have $N_S(Q_i) - N_S(Q_j) \neq \emptyset$, $N_S(Q_j) - N_S(Q_i) \neq \emptyset$ and each vertex of Q_i is adjacent to each vertex of Q_j in the graph G.

Claim 1 *If vertices Q_1, Q_2, Q_3 induce a P_3 in Γ with edges Q_1Q_2 and Q_2Q_3, then at least one of these edges is of type A.*

Proof. Assume to the contrary that both edges are of type B. Denote by a a vertex of G in $N_S(Q_1) - N_S(Q_2)$ and by b a vertex of G in $N_S(Q_3) - N_S(Q_2)$. Let $q_j \in Q_j$ for $j = 1, 2, 3$. By the assumption (2), q_1 must have a non-neighbor c in Q_1. Since Q_1 is not adjacent to Q_3 in Γ, b has no neighbor in Q_1 and a has no neighbor in Q_3 in G. But now the vertices a, b, c, q_1, q_2, q_3 induce a $Banner_2$ in G. ∎

Claim 2 *If vertices Q_1, Q_2, Q_3, Q_4 induce a P_4 in Γ with edges Q_1Q_2, Q_2Q_3, Q_3Q_4, then the mid-edge Q_2Q_3 is of type B and the other two edges are of type A.*

Proof. Assume by contradiction that the edge Q_1Q_2 is of type B. Then from Claim 1 it follows that Q_2Q_3 is of type A. Let $q_j \in Q_j$ for $j = 1, 2, 3, 4$. Denote by a a vertex of G in $N_S(Q_1) - N_S(Q_2)$ and by b a vertex in Q_1 non-adjacent to q_1. If q_2 is not adjacent to q_3, then we consider a vertex $c \in N_S(Q_2) \cap N_S(Q_3)$ and conclude that a, b, c, q_1, q_2, q_3 induce in G a $Banner_2$. Now let q_2 be adjacent to q_3. If q_3 is adjacent to q_4, then G contains a $Banner_2$ induced by vertices a, b, q_1, q_2, q_3, q_4. If q_3 is not adjacent to q_4, then the edge Q_3Q_4 is of type A and hence there is a vertex c in $N_S(q_3) \cap N_S(q_4)$. But then G contains a $Banner_2$ induced by vertices a, b, q_1, q_2, q_3, c. This contradiction proves that Q_1Q_2 is of type A. Symmetrically, Q_3Q_4 is of type A.

To complete the proof, assume that the mid-edge Q_2Q_3 is of type A too. Remember that $|N_S(Q_i)| \geq 3$ and hence $|N_S(Q_i) \cap N_S(Q_j)| \geq 2$ for any edge Q_iQ_j of type A. Since $N_S(Q_1)$ and $N_S(Q_3)$ are disjoint, we conclude that $N_S(Q_1) \cup N_S(Q_3) \subseteq N_S(Q_2)$. Similarly, $N_S(Q_2) \cup N_S(Q_4) \subseteq N_S(Q_3)$. This is possible only if $N_S(Q_1) = N_S(Q_4) = \emptyset$, which contradicts the maximality of S. ■

Remark. In the concluding part of the proof of Claim 2 we did not use the fact that vertices Q_1 and Q_4 are non-adjacent in Γ, which means that no induced C_4 in Γ has three edges of type A. In conjunction with Claim 1 this implies

CLAIM 3 *In any induced C_4 in the graph Γ, adjacent edges have different types.*

Combining Claims 1, 2 and 3, we obtain

CLAIM 4 *Graph Γ contains no induced $K_{2,3}$, P_5, C_5 and Banner.*

Finally, we prove

CLAIM 5 *Graph Γ contains no induced $\overline{C_k}$ with odd $k > 5$.*

Proof. By contradiction, let $C_k = (x_1, x_2, \ldots, x_k)$ be an induced cycle of odd length $k > 5$ in the complement to Γ. Consider two consecutive vertices of the cycle, say x_1 and x_2. It is not hard to see that pairs $x_{k-2}x_1$ and x_2x_5 form mid-edges of induced P_4's in Γ. Hence by Claim 2 both edges are of type B. Now let us consider the set of edges $F = \{x_1x_5, x_1x_6, \ldots, x_1x_{k-3}, x_1x_{k-2}\}$ in Γ. Any two consecutive edges x_1x_j and x_1x_{j+1} in F belong to a C_4 induced by vertices x_1, x_j, x_2, x_{j+1} in Γ. Hence, by Claim 3, edges of type A in F strictly alternate with edges of type B. Since x_1x_{k-2} is of type B and k is odd, we conclude that x_1x_5 is an edge of type B in Γ. But then vertices x_1, x_5, x_2 induce a P_3 in Γ with both edges of type B, contradicting Claim 1. ■

From Claims 4 and 5 we deduce that Γ is a Berge graph. It is known (Barré and Fouquet (1999); Olariu (1989)) that the Strong Perfect Graph Conjecture is true in (P_5,banner)-free graphs. We hence conclude that

LEMMA 7 *Graph Γ is perfect.*

Lemma 7 together with the result in (Grötschel, Lovász and Schrijver (1984)) show that an independent set of maximum weight in the graph Γ can be found in polynomial time. The weights $\alpha(G[Q_j])$ to the vertices of Γ are computed recursively. Obviously, if every step of a recursive procedure can be implemented in polynomial time, and the number of recursive calls is bounded by a polynomial, then the total time of the procedure is polynomial as well. In algorithm ALPHA the recursion applies to vertex-disjoint subgraphs. Therefore, the number of recursive calls is polynomial. Every step of algorithm ALPHA, other than Step 2, has a polynomial time complexity. Thus, polynomial-time solvability of Step 2 would imply polynomiallity of the entire algorithm. The converse statement is trivial. As a result we obtain

THEOREM 10 *The maximum independent set problem in the class of Banner$_2$-free graphs is polynomially equivalent to the problem of finding augmenting graphs containing a P_4.*

5.3 Other types of Augmenting Graphs

In this section we give additional examples of algorithms that detect augmenting graphs in particular classes of graphs. Throughout this section we assume that G is a $(P_8, banner)$-free graph. As mentioned in Section 4, a minimal augmenting $(P_8, banner)$-free graph is one of the following graphs (see Figure 1.2 for definitions of the graphs):

- a complete bipartite graph $K_{r,r+1}$ with $r \geq 0$,
- a $L^1_{r,s}$ or a $L^2_{r,s}$ with $r \geq 2$ and $s \geq 0$,
- a $M_{r,s}$ with $r \geq 1$ and $r \geq s \geq 0$,
- a N_s with $s \geq 0$.
- one of the graphs $F_2, \ldots, F_5$.

An augmenting F_i $(2 \leq i \leq 5)$ can be found in polynomial time since these graphs have a number of vertices which does not depend on the size of G. Moreover, we know from the preceding section that complete bipartite augmenting graphs can be detected in polynomial time in *banner*-free graphs. In the present section we show how the remaining graphs listed above can be detected in polynomial time, assuming that G is $(P_8, banner)$-free.

We denote by W^i the set of white vertices having exactly i black neighbors. Given a white vertex w, we denote by $B(w) = N(w) \cap S$ the set of black neighbors of w. We first show how to find an augmenting $M_{r,s}$ with $r \geq 1$ and $r \geq s \geq 0$. We assume there is no augmenting $K_{1,2}$ (such augmenting graphs can easily be detected).

Consider three black mutually non-adjacent vertices a_1, c, e such that $a_1 \in W^1$, $|B(c)| \geq |B(e)|$, $B(a_1) \cap B(c) = \{b_1\}$, $B(c) \cap B(e) = \{d\}$ and $B(a_1) \cap B(e) = \emptyset$. Notice that we have chosen, on purpose, the same labeling as in Figure 1.2. The following algorithm determines whether this initial structure can be extended to an augmenting $M_{r,s}$ in G (with $r = |B(c)| - 1$ and $s = |B(e)| - 1$) (Gerber, Hertz and Lozin (2004)).

(a) Determine $A = (B(c) \cup B(e)) - \{b_1, d\}$.

(b) For each vertex $u \in A$, determine the set $N_1(u)$ of white neighbors of u which are in W^1, and which are not adjacent to a_1, c or e.

(c) Let G' be the subgraph of G induced by $\cup_{u \in A} N_1(u)$:

- if $\alpha(G') = |A|$ then $A \cup \{a_1, b_1, c, d, e\}$ together with any maximum independent set in G' induce the desired $M_{r,s}$;
- otherwise, $M_{r,s}$ does not exist in G.

As shown in (Gerber, Hertz and Lozin (2004)), G' is $(banner, P_5, C_5, fork)$-free when G is $(P_8, banner)$-free. Since polynomial algorithms are available for the computation of a maximum independent set in this class of graphs (Alekseev (1999); Lozin (2000a)), the above Step (c) can be performed in polynomial time.

Finding an augmenting N_s with $s \geq 0$ is even simpler. Indeed, consider five white non-adjacent vertices $x_1, \ldots, x_5$ such that $x_i \in W^2$ $(i = 1, \ldots, 4)$, $\cup_{i=1}^{4}(\{x_i\} \cup B(x_i))$ induces a $C_8 = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ in G, and $B(x_5) \cap \{y_1, \ldots, y_4\} = \{y_2, y_4\}$. The following algorithm determines whether this initial structure can be extended to an augmenting N_s in G (with $s = |B(x_5)| - 2$) (Gerber, Hertz and Lozin (2004)).

(a) Determine $A = B(x_5) - \{y_2, y_4\}$.

(b) For each vertex $u \in A$, determine the set $N_1(u)$ of white neighbors of u which are in W^1, and which are not adjacent to $x_1, \ldots, x_4$.

(c) Let G' be the subgraph of G induced by $\cup_{u \in A} N_1(u)$:

- if $\alpha(G') = |A|$ then $A \cup \{x_1, \ldots, x_5, y_1, \ldots, y_4\}$ together with any maximum independent set in G' induce the desired N_s.
- otherwise, N_s does not exist in G.

If G is $(P_8, banner)$-free then G' is the union of disjoint cliques (Gerber, Hertz and Lozin (2004)), which means that a maximum independent set in G' can easily be obtained by choosing a vertex in each connected component of G'.

We finally show how augmenting $L^1_{r,s}$ and $L^2_{r,s}$ with $r \geq 2$ and $s \geq 0$ can be found in polynomial time, assuming there is no augmenting $P_3 = K_{1,2}$, $P_5 = M_{1,0}$ and $P_7 = M_{1,1}$ (these can easily be detected in polynomial time). Consider four white non-adjacent vertices b_1, b_2, d and x such that x belongs to W^1, b_1 and b_2 belong to W^2, $\{b_1, b_2, d\} \cup B(b_1) \cup B(b_2)$ induces a $C_6 = (c_1, b_1, a, b_2, c_2, d)$ in G, and x is adjacent to a or (exclusive) c_1. Notice that we have chosen, on purpose, the same labeling as in Figure 1.2. The following algorithm determines whether this initial structure can be extended to an augmenting $L^1_{r,s}$ or $L^2_{r,s}$ in G (with $r + s = |B(d)|$) (Gerber, Hertz and Lozin (2004)).

(a) Determine $A = B(d) - \{c_1, c_2\}$ as well as the set $\overline{W}$ of white vertices which are not adjacent to x, b_1, b_2 or d.

(b) For each vertex $u \in A$, determine the set $N_1(u)$ of white neighbors of u which are in $W^1 \cap \overline{W}$ as well as the set $N_2(u)$ of white vertices in $W^2 \cap \overline{W}$ which are adjacent to both a and u.

(c) Let G' be the subgraph of G induced by the vertices in $\cup_{u \in A}(N_1(u) \cup N_2(u))$:

- if $\alpha(G') = |A|$ then $A \cup \{a, b_1, b_2, c_1, c_2, d, x\}$ together with any maximum independent set in G' induce the desired $L^1_{r,s}$ (if x is adjacent to c_1) or $L^2_{r,s}$ (if x is adjacent to a).
- otherwise, $L^1_{r,s}$ and $L^2_{r,s}$ do not exist in G.

Once again, it is proved in (Gerber, Hertz and Lozin (2004)) that G' is $(banner, P_5, C_5, fork)$-free when G is $(P_8, banner)$-free, and this implies that the above Step (c) can be performed in polynomial time.

6. Conclusion

In this paper we reviewed the method of augmenting graphs, which is a general approach to solve the maximum independent set problem. As the problem is generally NP-hard, no polynomial-time algorithms are available to implement the approach. However, for graphs in some special classes, this method leads to polynomial-time solutions. In particular, the idea of augmenting graphs has been used to solve the problem for line graphs, which is equivalent to finding maximum matchings in general graphs. The first polynomial-time algorithm for the maximum matching problem has been proposed by Edmonds in 1965. Fifteen years later Minty (1980) and Sbihi (1980) extended independently of each other the solution of Edmonds from line graphs to claw-free graphs. The idea of augmenting graphs did not see any further developments for nearly two decades. Recently Alekseev (1999) and Mosca (1999) revived the interest in this approach, which has led to many new results on the topic. This paper summarizes most of those results and proposes several new contributions. In particular, we show that in the class of $(S_{1,2,j}, Banner_k)$-free graphs for any fixed j and k, there are finitely many minimal augmenting graphs of bounded vertex degree different from augmenting chains. Together with polynomial-time algorithms to find augmenting chains in $(S_{1,2,j}, Banner_1)$-free graphs (Hertz, Lozin and Schindl (2003)) and $S_{1,2,3}$-free graphs (Gerber, Hertz and Lozin (2003)) this immediately implies polynomial-time solutions to the maximum independent set problem in classes of $(S_{1,2,j}, Banner_1, K_{1,n})$-free graphs and $(S_{1,2,3}, Banner_k, K_{1,n})$-free graphs, both generalizing claw-free graphs. The second class extends also $(P_5, K_{1,n})$-free graphs and $(P_2 + P_3, K_{1,n})$-free graphs for which polynomial-time solutions have been proposed by Mosca (1997) and Alekseev (2004), respectively. We believe the idea of augmenting graphs may lead to many further results for the maximum independent set problem and hope the present paper will be of assistance in this respect.

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