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Abstract

We define two classes of lower bounds using either one or two simplices for the minimization of a concave function over a polytope. For each of them, a procedure is developed to compute the best lower bound of the class. These two lower bound procedures are embedded in a normal conical algorithm to solve the concave minimization problem. Computational results are presented.

Keywords: Concave Minimization, Normal Conical Algorithm, Lower Bound, Global Optimization.

Résumé

Nous définissons deux classes de bornes inférieures pour la minimisation d'une fonction concave sur un polytope. Pour chacune d'elle, nous développons un algorithme pour le calcul de la meilleure borne possible. Ces deux procédures de calcul de bornes sont incluses dans un algorithme d'énumération implicite utilisant des subdivisions coniques normales. Des résultats numériques sont présentés.

Mots clés: minimisation concave, subdivisions coniques normales, borne inférieure, optimisation globale.

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1 Introduction

We consider the following concave minimization problem:

$$(CP) \qquad \min\{f(x) \mid x \in P\}$$

where f is a concave function defined on $\mathbb{R}^n$ and $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ is a full dimensional polytope. (This last assumption is not restrictive as it is easy to check whether P has a non-empty interior and to project on a lower-dimensional space if not.)

These last years, concave minimization - and particularly concave minimization over a polytope - has received a lot of attention (see, for example, the book of Horst and Tuy [7] and the surveys of Benson [2], Horst [3] and Pardalos and Rosen [13]).

From a mathematical point of view, concave minimization is a difficult class of problems which nevertheless possesses some nice properties that make it solvable in practice for instances of moderate size. Its main difficulty is the possible existence of local minima, that cause the standard tools developed for convex minimization to be insufficient. This difficulty is also expressed by the fact that concave minimization is NP-hard, even for simple case such as minimizing a concave quadratic function over a hypercube (see Kalantari and Bagchi [10]). Fortunately, concave minimization has several other mathematical properties that can be exploited to design algorithms for solving (CP). One of the most important is that the minimum of a concave function over a polytope is attained at a vertex of the polytope (see, e.g., Rockafellar [14], Horst and Tuy [7], Benson [2] for this and other properties).

Although many methods have been proposed to solve the concave minimization problem, they are essentially based on three approaches: vertex enumeration, successive approximation and successive partitioning (branch-and-bound). This last approach can be further subdivided depending on whether the elements of the partitions are cones, simplices or rectangles.
In this paper, we focus on the conical partitioning approach, which is one of the winners of the computational experiments of Horst and Thoai [5]. This approach originated in 1964 with the paper of Tuy [17] where conical subdivisions are used in conjunction with concavity cuts (now called Tuy cuts). In 1980, Thoai and Tuy [15] proposed the first conical branch-and-bound method. Their lower bounding procedure is strongly related to the Tuy cut: if the cut does not allow the elimination of the cone, the cutting hyperplane is translated (by means of the solution of a linear program) until it supports the portion of the polytope contained in the cone. The lower bound is then defined as the minimum of the function f over the set of extreme points of the simplex defined by the hyperplane and the cone. This bound was later used by Horst, Thoai and Benson [6] for the minimization of a concave function over a convex set.
A second lower bounding method, based on the same principle but which does not require the solution of a linear program, is proposed in Tuy and Thai [21] (see also Tuy, Thieu and Thai [22]). In this method, the hyperplane is a supporting hyperplane to the convex set at

a point of the intersection of the cone with the boundary of the convex set. Unfortunately, if some edge of the cone does not intersect the hyperplane, it is not possible to compute a lower bound by this way.
The reader is referred to Horst and Tuy [7] and Benson [2] for more detail.

The purpose of this paper is to propose two new classes of lower bounds, to give for each of them a procedure for the computation of the best possible bound and finally to embed them in a conical branch-and-bound algorithm for the concave minimization. The first class, called simplicial bound, contains the Thoai-Tuy and the Tuy-Thai lower bounds as a special case. The second, which involves two simplices, is called double-simplicial bound.

The paper is organized as follows. Section 2, which is devoted to the two classes of lower bounds and constitutes therefore the core of this paper, is subdivided in four paragraphs. In Paragraph 2.1, we recall the notion of γ-extension, which is used by the two lower bound procedures. In Paragraph 2.2, we present the simplicial lower bound. Paragraph 2.3 is devoted to the presentation of the double-simplicial bound, assuming that an additional cone satisfying some conditions is available. Finally in Paragraph 2.4, we explain how the simplicial bound procedure can be used to determine such a cone. In Section 3, we embed these lower bounds in a conical branch-and-bound algorithm and prove its convergence. Some computational results are presented in Section 4 and conclusions are drawn in the last section.

2 Simplicial and Double-Simplicial lower bounds

In this section, we present two lower bounds of f over $K \cap P$, where K is a cone originated at an interior point O of P, with exactly n linearly independent edges.
For the first one, called simplicial bound, a hyperplane H is chosen, which defines with K a simplex containing $K \cap P$.
The second one, called double-simplicial bound, needs the introduction of an additional cone K'. A hyperplane H is then chosen, defining with K and K' two simplices, the union of which contains $K \cap P$.
For each of these bounds, we characterize the hyperplane H giving the best possible value (i.e, largest) and present a procedure for its computation.

2.1 Definition and properties

We recall below the definition of a γ-extension, where we use a reference vector instead of a reference feasible point as in Horst and Tuy [7].

For $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ fixed, consider the function $\gamma \mapsto \overline{\lambda}(z,u;\gamma)$ defined on $(-\infty, f(z)]$:

$$\overline{\lambda}(z,u;\gamma) = \max\{\lambda \geq 0 \mid f(z+\lambda u) \geq \gamma\}. \tag{1}$$

The point (possibly at infinity) $z+\overline{\lambda}(z,u;\gamma)u$ is called *γ-extension of z along the direction u.*

Proposition 1 *Let z and u fixed.*

a) If $\overline{\lambda}(z,u;\gamma_0) = +\infty$ for some $\gamma_0 < f(z)$, then $\overline{\lambda}(z,u;\gamma) = +\infty$ for all $\gamma < f(z)$.

b) If $\overline{\lambda}(z,u;\gamma_0)$ is finite for some $\gamma_0 < f(z)$, then the mapping $\gamma \mapsto \overline{\lambda}(z,u;\gamma)$ is concave and strictly decreasing over $(-\infty, f(z))$.

Proof:

It is well known that a concave function over a halfline is continuous and either reaches its minimum at the origin, or is unbounded from below (see, e.g., Tuy [20, p. 134]).

If f reaches its minimum at the origin, there cannot exist $\lambda \geq 0$ such that $f(z+\lambda u) < \gamma$ for $\gamma < f(z)$, thus $\overline{\lambda}(z,u;\gamma) = +\infty$ for all $\gamma < f(z)$.

Assume now that f is unbounded from below. Clearly, for any $\gamma < f(z)$, there exists a $\overline{\lambda}$ such that $f(z+\lambda u) \leq \gamma$ for all $\lambda \geq \overline{\lambda}$. Thus $\overline{\lambda}(z,u;\gamma)$ is finite for each $\gamma < f(z)$. The function $\gamma \mapsto \overline{\lambda}(z,u;\gamma)$ is concave as it is the supremum of a concave function over a convex set (see, e.g., Tuy [20, Proposition 2.6]). Finally the decreasivity is shown by contradiction: assume that there exist $\gamma_1 < \gamma_2 < f(z)$ such that $\overline{\lambda}(z,u;\gamma_1) \leq \overline{\lambda}(z,u;\gamma_2)$. Then $\gamma_1 = f(z+\overline{\lambda}(z,u;\gamma_1)u) \geq \gamma_2$ by concavity of f, which is a contradiction. ∎

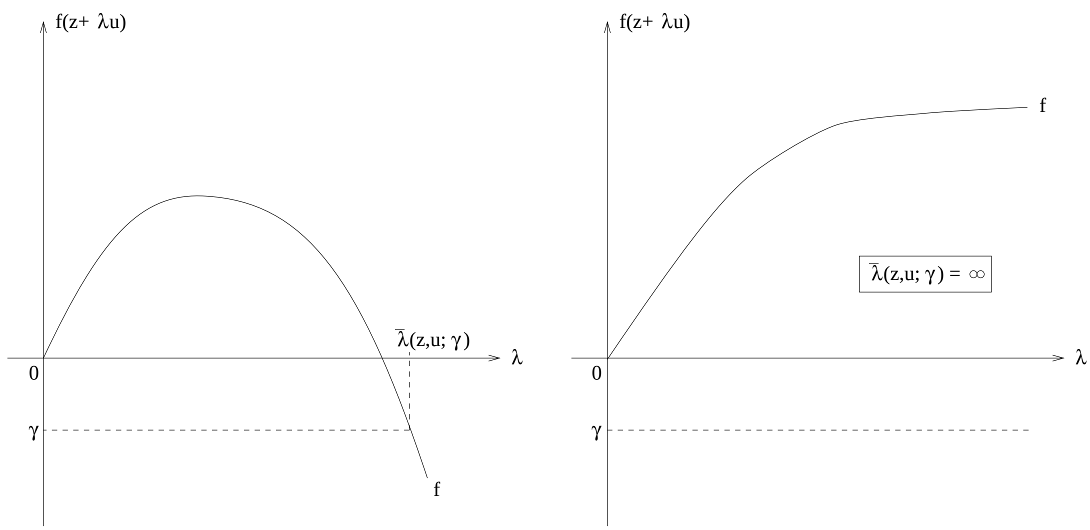

Figure 1: Possible configurations of f over a halfline

Figure 1 illustrates the two cases of Proposition 1.

2.2 Simplicial lower bound

Let $\mathcal{H}$ be the set of hyperplanes H defining with K a generalized simplex (i.e., with possibly some extreme points at infinity) containing $K \cap P$.

Consider the simplicial lower bound function SLB which, to each $H \in \mathcal{H}$, associates the lower bound $SLB(H) = \min\{f(O), f(z^1), f(z^2), \ldots, f(z^n)\}$ where z^i, $i = 1, 2, \ldots, n$ are the intersection points of the edges u^i of K with H.

The first bound of this class was proposed by Thoai and Tuy [15] with H being the hyperplane supporting $K \cap P$ and parallel to the hyperplane intersecting the edges of K at points of value $\overline{f}$, where $\overline{f}$ is the value of the current best feasible point of problem (CP). A second one, proposed by Tuy and Thai [21], takes for H a hyperplane supporting $K \cap P$ at a preselected point of $K \cap \delta P$ (note that this construction does not guarantee that H belongs to $\mathcal{H}$, thus one cannot always compute a lower bound by this last method).

In this section, we are interested in identifying the best possible lower bound $SLB(H)$ with respect to $\mathcal{H}$, i.e., in solving the following optimization problem:

$$(SLBP) \quad \max_{H \in \mathcal{H}} SLB(H).$$

We first prove the existence of a solution.

Proposition 2 *There exists always a finite optimal solution for problem* $(SLBP)$.

Proof:

First note that if an optimal solution exists, it is necessary finite. Indeed, by boundedness of P there exists a hyperplane H^0 which together with K defines a simplex containing $K \cap P$ and that intersects each edge of K at finite distance. By continuity of f (which follows from the concavity), the simplicial lower bound $SLB(H^0)$, which provides a lower bound of the optimal simplicial bound, is finite. An upper bound is given by $f(O)$.

When solving $(SLBP)$, we can always assume without loss of generality that H supports $K \cap P$. Indeed, let H' be a supporting hyperplane of $K \cap P$, parallel to H. The generalized simplex defined with H' is included in that defined with H thus $SLB(H') \geq SLB(H)$.

Any hyperplane $H \in \mathcal{H}$ which supports $K \cap P$ can be written under the form $H = \{x \mid ax = b\}$ with:

$$au^i \geq 0 \quad i = 1, 2, \ldots, n \tag{2}$$

$$\|a\| = 1 \tag{3}$$

$$b = \max\{ax \mid x \in K \cap P\} \tag{4}$$

where u^i is the direction of the i^{th} edge of K, $i = 1, 2, \ldots, n$.

Constraints (2) express the fact that H intersects each direction corresponding to an edge of K at the same side with respect to O (note that we could also consider the other direction of the inequalities), (3) is a normalization constraint and (4) expresses that H supports $K \cap P$. Clearly, (2) and (3) define a compact set, which will be noted by D.

The function $b : a \mapsto \max\{ax \mid x \in K \cap P\}$ is continuous as the supremum of an infinite family of convex functions, and bounded from above over the unit ball. Moreover, $b(a) > 0$ for all a in D. Indeed since O is an interior point of P, there exists $\lambda_0 > 0$ such that $x_0 = \lambda_0(u^1 + u^2 + \cdots + u^n) \in K \cap P$. Now for at least one i, $au^i > 0$ since the $u^i, i = 1, 2, \ldots, n$ are linearly independent and $a \neq 0$ by (3). Hence $b(a) \geq ax^0 > 0$.

On the other hand, the intersection points of the edges of K with the hyperplane H are $z^i = (\frac{b(a)}{au^i})u^i$, $i = 1, 2, \ldots, n$. Thus the objective function of problem $(SLBP)$ can be written:

$$SLB(a) = \min \left\{ f(O), f\left((\frac{b(a)}{au^1})u^1 \right), \ldots, f\left((\frac{b(a)}{au^n})u^n \right) \right\}.$$

Let us set $f(\frac{1}{0}u^i) = \lim_{\lambda \to +\infty} f(\lambda u^i)$ for all $i = 1, 2, \ldots, n$. By continuity of f over $\mathbb{R}^n$ and by continuity of the functions $a \mapsto \frac{b(a)}{au^i}, i = 1, 2, \ldots, n$ over D, it follows that the function $SLB(a)$ is continuous over D.

Thus problem $(SLBP)$ can be reduced to the maximization of a continuous function over a compact set which has always a solution using the Theorem of Weierstrass (see, e.g., Mawhin [12, p. 137]). ∎

We next discuss a sufficient condition for a hyperplane to lead to a best possible lower bound.

Theorem 1 *A sufficient condition for a hyperplane $H \in \mathcal{H}$ to be an optimal solution of problem (SLBP) is*

(SC) |
a) $SLB(H) = f(O)$
or
b) H supports $K \cap P$, is parallel to any edge of K on which f is nondecreasing and the intersection points of the remaining edges of K with the hyperplane H have the same value.

This condition can always be satisfied.

Proof:

Let $\tilde{H} = \{x \in \mathbb{R}^n \mid \tilde{a}x = 1\} \in \mathcal{H}$ be a hyperplane satisfying the condition (SC) and let $\tilde{\gamma} = SLB(\tilde{H})$ be the corresponding simplicial bound.

If $\tilde{\gamma} = f(O)$, $\tilde{H}$ is clearly optimal since O is a feasible point of $K \cap P$.

Assume now that $\tilde{\gamma} < f(O)$ and let J_∞ be the set of indices j such that f is nondecreasing on the edge j: we have $|J_\infty| < n$. Assume that there exists a hyperplane $\hat{H} = \{x \in \mathbb{R}^n \mid \hat{a}x = 1\} \in \mathcal{H}$ which yields a simplicial lower bound $\hat{\gamma} = SLB(\hat{H})$ satisfying $\hat{\gamma} > \tilde{\gamma}$. We show that this leads to a contradiction.

Let $\tilde{z}^j$ (respectively $\hat{z}^j$) be the intersection point of the j^{th} edge of K with $\tilde{H}$ (respectively $\hat{H}$) for all $j \notin J_\infty$ (note that $\hat{z}^j$ and $\tilde{z}^j$ are finite points, otherwise the

corresponding simplicial bound would be $-\infty$). By definition of $\hat{\gamma}$, we have $f(\hat{z}^j) \geq \hat{\gamma}$ for all $j \notin J_\infty$ and $f(O) \geq \hat{\gamma}$. Since $\hat{\gamma} > \tilde{\gamma} = f(\tilde{z}^j)$ and by concavity of f, it follows that $\tilde{z}^j$ is outside the segment $[O, \hat{z}^j]$ thus $\hat{a}\tilde{z}^j > 1$ for all $j \notin J_\infty$.
On the other hand, let $\tilde{\omega}$ be a point of $\tilde{H} \cap (K \cap P)$ (such a point exists since $\tilde{H}$ supports $K \cap P$): $\tilde{\omega} = \sum_{j \in J_\infty} \mu_j u^j + \sum_{j \notin J_\infty} \mu_j \tilde{z}^j$ with $\sum_{j \notin J_\infty} \mu_j = 1$ and $\mu_j \geq 0$ for all j. Thus $\hat{a}\tilde{\omega} = \sum_{j \in J_\infty} \mu_j \hat{a} u^j + \sum_{j \notin J_\infty} \mu_j \hat{a} \tilde{z}^j > 1$. This shows that $\tilde{\omega}$ lies outside the generalized simplex defined by K and $\hat{H}$, which is a contradiction. Thus the condition (SC) is sufficient.

Finally, let us show that the condition (SC) can always be satisfied. Let $H^* \in \mathcal{H}$ be an optimal solution of problem ($SLBP$) and let γ^* be its value. If H^* satisfies the condition (SC), we are done.
Otherwise, let J_∞ be the set of indices j such that f is nondecreasing on the edge j (note that $|J_\infty| < n$; otherwise H^* would satisfy part a) of condition (SC)). For $j \notin J_\infty$, denote by z^{j*} the intersection point of the j^{th} edge of K with H^* (note that z^{j*} is a finite point, otherwise we would have $\gamma^* = -\infty$) and by y^{j*} the γ^*-extension of O along the edge u^j. By concavity of f and since $f(O) > \gamma^*$ and $f(z^{j*}) \geq \gamma^*$ for $j \notin J_\infty$, y^{j*} is located at or after z^{j*} on the edge. Let $H^{*'}$ be the hyperplane parallel to the directions u^j, $j \in J_\infty$ and going through the points y^{j*}, $j \notin J_\infty$: this hyperplane still defines a generalized simplex containing $K \cap P$ and yields also the lower bound γ^*. $H^{*'}$ still supports $K \cap P$ (otherwise we could slide $H^{*'}$ until it does, improving strictly the lower bound which would contradict the optimality of γ^*), thus $H^{*'}$ satisfies the condition (SC). ∎

The next two results show, under some condition, the unicity of the optimal hyperplane.

Proposition 3 *Let $\tilde{H}$ be a hyperplane satisfying condition (SC). If $SLB(\tilde{H}) < f(O)$, then $\tilde{H}$ is the unique hyperplane satisfying (SC).*

Proof:

Since $SLB(\tilde{H}) = \tilde{\gamma}$ is the optimal simplicial bound, any hyperplane satisfying (SC) must satisfy the part b) of this condition. Consider an edge of K along which f is not nondecreasing. Since $\tilde{\gamma} < f(O)$ and since f is concave along this edge, there exists a unique point z^j on this edge such that $f(z^j) = \tilde{\gamma}$ (note that for $\tilde{\gamma} = f(O)$, this is not true: f can be constant over a segment starting at O and then decreases). Since the u^j, $j = 1, 2 \ldots, n$ are linearly independent, there exists an unique hyperplane parallel to the edges of K along which f is nondecreasing, and that intersects the other edges at point of value $\tilde{\gamma}$. ∎

We now show under which conditions the (SC) condition is necessary.

Theorem 2 *Assume that the best simplicial bound is not equal to $f(O)$.*
Then condition (SC) is necessary if and only if there exists a support point of the hyperplane satisfying (SC) in the interior of K.

Proof:

Note that the assumption reduces the condition (SC) to its part b).

Denote by $\tilde{H} = \{x \in \mathbb{R}^n \mid \tilde{a}x = 1\}$ the unique hyperplane satisfying the condition (SC) (such a hyperplane exists by Theorem 1 and the unicity follows from Proposition 3).

Assume first that $\tilde{H}$ supports $K \cap P$ at a point $\tilde{\omega}$ that is in the interior of K. Then

$$\tilde{\omega} = \sum_{j \notin J_\infty} \mu_j \tilde{z}^j + \sum_{j \in J_\infty} \mu_j u^j \quad \text{with} \sum_{j \notin J_\infty} \mu_j = 1 \text{ and } \mu > 0,$$

where $\tilde{z}^j$, $j \notin J_\infty$ are the intersection points of the hyperplane $\tilde{H}$ with the edges u^j of K. Let $\hat{H} = \{x \in \mathbb{R}^n \mid \hat{a}x = 1\}$ be an optimal hyperplane: we next show that $\hat{H} = \tilde{H}$. Since the optimal simplicial bound is attained at the points $\tilde{z}^j$, $j \notin J_\infty$, $\hat{H}$ must intersect the corresponding edges of K at points $\hat{z}^j \in [O\tilde{z}^j]$, thus $\hat{a}\tilde{z}^j \geq 1$ for all $j \notin J_\infty$. Since $\hat{H}$ must intersect each edge of K, we have also $\hat{a}u^j \geq 0$ for all $j \in J_\infty$. On the other hand

$$\sum_{j \notin J_\infty} \mu_j \hat{a}\tilde{z}^j + \sum_{j \in J_\infty} \mu_j \hat{a}u^j = \hat{a}\tilde{\omega} \leq 1$$

since $\tilde{\omega} \in K \cap P$. However, as $\mu_j > 0$ for all $j = 1, 2, \ldots, n$ it follows then that $\hat{a}\tilde{z}^j = 1$ for all $j \notin J_\infty$ and $\hat{a}u^j = 0$ for all $j \in J_\infty$. Thus $\hat{H} = \tilde{H}$, which shows the necessity of the condition (SC).

The fact that the hyperplane is not unique if there exists no support point in the interior of K is illustrated in Figure 2 in the two-dimensional case. The simplex defined by the dotted hyperplane is clearly included in the (possibly generalized) simplex defined by the dashed hyperplane, hence yields a not-smaller lower bound. Since the lower bound given by the dashed hyperplane is optimal by assumption, the two simplices actually give the same lower bound. This construction can be easily generalized to higher dimension, showing the non-unicity of the optimal hyperplane when the support point is not in the interior of the cone. ∎

Let J_∞ be the set of edges of cone K on which f is nondecreasing.

Let H be a valid hyperplane (i.e., belonging to $\mathcal{H}$), parallel to the edges $j \in J_\infty$ of K. Let γ be the corresponding simplicial lower bound.

If H does not satisfy condition (SC), a valid hyperplane H'' yielding a better lower bound can be defined as follows.

For $j \notin J_\infty$, let $y^j = \overline{\lambda}_j(\gamma)u^j$ be the γ-extension of O along the direction u^j. Let H' be the hyperplane parallel to the edges j of K for $j \in J_\infty$ and going through y^j, $j \notin J_\infty$: this hyperplane belongs to $\mathcal{H}$ and yields the same simplicial lower bound γ. If H' supports $K \cap P$, H' is optimal by Theorem 1. Otherwise, let H'' be the hyperplane parallel to H' that supports $K \cap P$: this hyperplane yields a lower bound $\gamma'' = SLB(H'') > \gamma$.

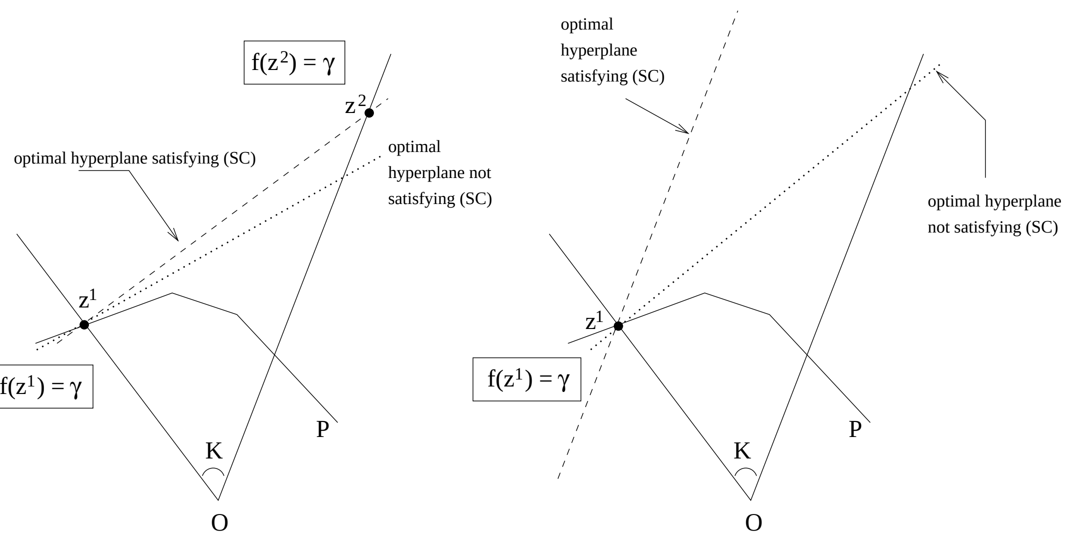

Figure 2: Non necessity of condition (SC)

The hyperplanes going through y^j, $j \notin J_\infty$ and parallel to the edges $j \in J_\infty$ of K are obtained as

$$H_c = \{x \in \mathbb{R}^n \mid x = \sum_{j=1}^{n} \lambda_j u^j;\ \sum_{j \notin J_\infty} \frac{\lambda_j}{\overline{\lambda}_j(\gamma)} = c\}.$$

Note that $H_1 = H'$.
The hyperplane H'' is thus obtained by solving the following linear program

$$LP(\gamma)\ :\ \hat{c}(\gamma) = \max \sum_{j \notin J_\infty} \frac{\lambda_j}{\overline{\lambda}_j(\gamma)}$$

$$\text{s.t.} \begin{cases} x = \sum_{j=1}^{n} \lambda_j u^j \\ Ax \leq b \\ \lambda_j \geq 0 \quad j = 1, 2, \ldots, n. \end{cases}$$

We are now able to describe a procedure which allows the computation of the best simplicial bound.

Procedure BSB

Step 1 (Initialization) : Select $\gamma < f(O)$.
Compute the γ-extensions $y^j = \overline{\lambda}_j(\gamma)u^j$ of O along each edge of K and let $J_\infty = \{j \mid \overline{\lambda}_j(\gamma) = +\infty\}$.

Step 2 (Computing a new hyperplane) : Solve $LP(\gamma)$ to obtain the hyperplane $H = H_{\hat{c}(\gamma)}$. If $\hat{c}(\gamma) = 1$, stop: H is an optimal solution of problem $(SLBP)$.

Step 3 (Computing a new simplicial value) : Let $z^j = \hat{c}(\gamma)y^j$, $j \notin J_\infty$ be the intersection point of the j^{th} edge of K with the hyperplane H.
Compute $\gamma = \min_{j \notin J_\infty} \{f(z^j), f(O)\}$.
If $\gamma = f(O)$, stop: H is an optimal solution of problem (SLBP).

Step 4 (Computing new γ-extensions) : For all $j \notin J_\infty$, compute the γ-extensions $y^j = \overline{\lambda}_j(\gamma)u^j$.
Return to Step 2.

Theorem 3 *The procedure BSB converges to an optimal solution of* $(SLBP)$.

Proof:

Denote by γ_k the value of γ at the beginning of iteration k, by $\hat{\lambda}^k = (\hat{\lambda}_1^k, \hat{\lambda}_2^k, \ldots, \hat{\lambda}_n^k)$ and $\hat{c}_k$ respectively the optimal solution and value of problem $LP(\gamma_k)$ and by y^{jk} and z^{jk} the points y^j and z^j, $j \notin J_\infty$. Let $H^k = H_{\hat{c}_k}$.
If the procedure stops at the end of Step 2, we have $\hat{c}_k = 1$. Thus the hyperplane H^k supports $K \cap P$, is parallel to the edges $j \in J_\infty$ of K and intersects the edges $j \notin J_\infty$ at points y^{jk} of same value γ_k: by Theorem 1, H^k is then optimal.
If the procedure stops at the end of Step 3, H^k is optimal by part a) of the sufficient condition (SC) of Theorem 1.
Assume now that the procedure is infinite. The sequence $\{\gamma_k\}$ is increasing from the second iteration on and bounded from above by $f(O)$ thus converges to a limit γ^*. By continuity of the functions $\overline{\lambda}_j(\gamma)$, the sequences $\{y^{jk}\}_k$ converge to $y^{j*} = \overline{\lambda}_j(\gamma^*)u^j$ and we have $f(y^{j*}) = \gamma^*$ for all $j \notin J_\infty$. Moreover, the mapping $\gamma \mapsto \hat{c}(\gamma)$ is nondecreasing as for any feasible λ and x, the objective function of $LP(\gamma)$ is nondecreasing with respect to γ by Proposition 1. This entails that the sequence $\hat{c}_k = \hat{c}(\gamma_k)$ is nondecreasing. Since in addition $\hat{c}_k$ is bounded from above by 1, it follows that $\hat{c}_k \to \hat{c}^*$. Thus the sequences $\{z^{jk}\}_k$ converge to $z^{j*} = \hat{c}^* y^{j*}$. Let us show that $\hat{c}^* = 1$. Let j_k be an index such that $f(z^{j_k k}) = \gamma_k$: by definition of the points y^{jk} we have $y^{j_k, k+1} = z^{j_k, k}$. By considering a subsequence if necessary, we can assume that $j_{k_s} = i$ for all k_s with $i \notin J_\infty$. Thus $y^{i*} = z^{i*}$. Since z^{i*} is outside the segment $[O, \underline{z}^i]$ where $\underline{z}^i$ is the intersection of the edge i with the boundary δP of P, we have $z^{i*} \neq O$ thus $\hat{c}^* = 1$.
Let $\hat{\omega}^k = \sum_{j=1}^{n} \hat{\lambda}_j^k u^j$. Since $\hat{\omega}^k \in K \cap \delta P$ for all k, there exists a subsequence k_s such that $\hat{\omega}^{k_s} \to \hat{\omega}$. Since $\hat{\omega}^{k_s}$ belongs to the hyperplane $H_{\hat{c}_{k_s}}$ for all k_s, it follows that $\hat{\omega}$ belongs to $H_{\hat{c}^*} = \lim_{s \to \infty} H_{\hat{c}_{k_s}}$.
Thus $H_{\hat{c}^*}$ supports $K \cap P$, is parallel to the edges j of K, $j \in J_\infty$ and intersects the edges $j \notin J_\infty$ at points z^{j*} of same value γ^*. By Theorem 1, $H_{\hat{c}^*}$ is an optimal solution of problem $(SLBP)$. ∎

Observe that in Step 2, the value of $\hat{c}(\gamma)$ allows us to locate the best simplicial bound with respect to γ. Indeed, if $\hat{c}(\gamma) \leq 1$ then γ is a simplicial bound, hence the best simplicial bound is greater than γ (actually all subsequent simplicial bounds given by the procedure are greater than γ). On the other hand, $\hat{c}(\gamma) > 1$ means that the simplex S defined by K and the hyperplane passing through the y^j does not contain $K \cap P$. By definition of the y^j, any (hypothetical) simplicial bound $\gamma' > \gamma$ would be obtained with a simplex $S' \subseteq S$, which clearly does not contain $K \cap P$. Thus if $\hat{c}(\gamma) > 1$, there does not exist simplicial bound with value greater than γ.

Note also that for some functions f, the above procedure gives the best simplicial bound after only one iteration. This is in particular the case if there exists a function h such that $f(\lambda u) = h(\lambda, f(u))$ for all $\lambda \geq 0$ and all $u \in \mathbb{R}^n$.

Of course, procedure BSB can be stopped at any iteration at the end of Step 3, yielding a valid simplicial lower bound $\underline{f}(K)$ of f over $K \cap P$.
In particular, if in Step 1, γ corresponds to the incumbent value $\overline{f}$ of the concave minimization problem (CP) and if only one iteration is performed we obtain the bounding procedure proposed by Thoai and Tuy [15].

We denote by $H(K)$ the hyperplane used to compute the simplicial bound (i.e., such that $\underline{f}(K) = SLB(H(K))$), and by $(\omega(K), \lambda(K))$ the basic optimal solution of the corresponding linear program $LP(\gamma)$.

2.3 Double-simplicial lower bounding procedure

Recall that K is a cone originated at O with exactly n linearly independent edges whose directions are $u^1, u^2, \ldots, u^n$.
Let K' be a cone, originated at z' with exactly n linearly independent edges of directions ${u'}^1, {u'}^2, \ldots, {u'}^n$, which contains $K \cap P$.
Note that since O is an interior point of P, and since the $u^i, i = 1, \ldots, n$ are linearly independent, it follows that $int(K \cap P) \neq \emptyset$. By the inclusion $K \cap P \subseteq K'$, this implies that $K \cap K'$ has a nonempty interior. We assume furthermore that $K \cap K'$ is bounded.

Clearly, since $K \cap P \subseteq K \cap K'$ and by concavity of f, $\min\limits_{x \in vert(K \cap K')} f(x)$ is a lower bound of f over $K \cap P$, where $vert(K \cap K')$ denotes the set of extreme points of $K \cap K'$. Note however that $vert(K \cap K')$ can contain an exponential number of points since, e.g., the hypercubes of $\mathbb{R}^n$ can be viewed as the intersection of two cones.
Therefore we next explore how to outer-approximate $K \cap K'$ by a set containing less extreme points.

Let $\mathcal{H}_{K'}$ be the set of hyperplanes $H = \{x \in \mathbb{R}^n \mid ax = b\}$ defining with K and K' two full dimensional generalized simplices $S(H) = K \cap \{x \in \mathbb{R}^n \mid ax \leq b\}$ and $S'(H) = K' \cap \{x \in \mathbb{R}^n \mid ax \geq b\}$ such that $K \cap K' \subseteq S(H) \cup S'(H)$ (see Figure 3 for an illustration).

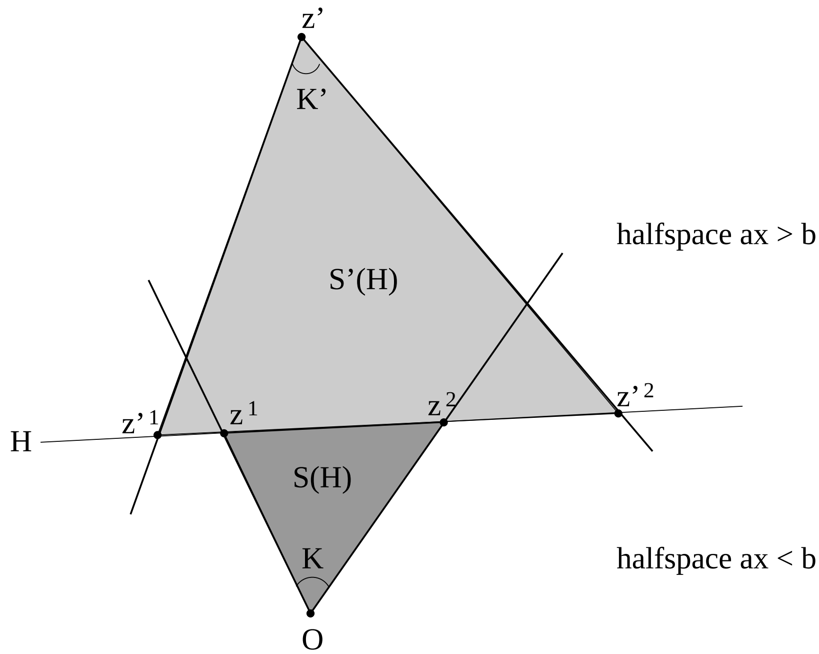

Figure 3: Double-simplicial lower bound

We define the double-simplicial lower bound $DSLB(H)$ as the minimum of f over $S(H) \cup S'(H)$, i.e., since f is concave, over $vert(S(H)) \cup vert(S'(H))$.
In the following, we consider the problem of finding the best possible bound of this class, i.e., of solving the problem

$$(DSLBP) \qquad \max_{H \in \mathcal{H}_{K'}} DSLB(H).$$

The first question of importance is the characterization of $\mathcal{H}_{K'}$.

Proposition 4 *$\mathcal{H}_{K'}$ is the set of hyperplanes $H = \{x \in I\!R^n \,|\, ax = b\}$ with (a, b) satisfying*

$$\begin{cases} au^i \geq 0 & i = 1, 2, \ldots, n & (5) \\ au'^i \leq 0 & i = 1, 2, \ldots, n & (6) \\ az' = 1 & & (7) \\ 0 < b < 1 & & (8). \end{cases}$$

This set is nonempty and bounded.

Proof:

Let $H = \{x \in \mathbb{R}^n \mid ax = b\}$ be a hyperplane of $\mathcal{H}_{K'}$. Since $S(H) = \{x = \sum_{i=1}^n \lambda_i u^i \mid \lambda_i \geq 0, i = 1, 2, \ldots, n; ax \leq b\}$ is defined by $n+1$ inequalities, its vertices can only be

$$\begin{aligned} z^0 &= O \\ z^i &= \left(\frac{b}{au^i}\right) u^i \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{9}$$

In order that these points actually belong to $S(H)$ and that $S(H)$ is full dimensional, we must have $aO < b$, i.e., $b > 0$ and $\frac{b}{au^i} > 0$ for $i = 1, 2, \ldots, n$, i.e., $au^i \geq 0$ for $i = 1, 2, \ldots, n$.

Similarly, the vertices of $S'(H)$ can only be

$$\begin{aligned} {z'}^0 &= z' \\ {z'}^i &= z' + \left(\frac{b - az'}{a{u'}^i}\right) {u'}^i \qquad i = 1, 2, \ldots, n \end{aligned} \tag{10}$$

from which we deduce the conditions $az' > b$ and $a{u'}^i \leq 0$, $i = 1, 2, \ldots, n$.

From $az' > b$ and $b > 0$ we deduce $az' > 0$. By dividing a and b by az', we obtain $H = \{x \in \mathbb{R}^n \mid \tilde{a}x = \tilde{b}\}$ with $(\tilde{a}, \tilde{b})$ satisfying (5)-(8).

Conversely, if (a, b) satisfies (5)-(8), then clearly the hyperplane $H = \{x \in \mathbb{R}^n \mid ax = b\}$ belongs to $\mathcal{H}_{K'}$.

Note that only (8) involves the variable b, and the set described by this double inequality is clearly nonempty and bounded. Therefore to show that $\mathcal{H}_{K'}$ is nonempty and bounded, it is sufficient to show these properties for a satisfying (5)-(7).

Let us show first that this set is nonempty. Since $K \cap K'$ is bounded and $z' \neq O$, the following linear problem

$$\begin{array}{ll} \max & \displaystyle\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \lambda'_i \\ \text{s.t.} & \left\{ \begin{array}{l} \displaystyle\sum_{i=1}^n \lambda_i u^i = z' + \sum_{i=1}^n \lambda'_i {u'}^i \\ \lambda, \lambda' \geq 0 \end{array} \right. \end{array}$$

has a finite positive optimal solution, thus its dual

$$\begin{array}{ll} \min & az' \\ \text{s.t.} & \left\{ \begin{array}{lll} -au^i & \leq -1 & i = 1, 2, \ldots, n \\ a{u'}^i & \leq -1 & i = 1, 2, \ldots, n \end{array} \right. \end{array}$$

is feasible and has an optimal solution $\tilde{a}$ with $\tilde{a}z' > 0$. Hence $\frac{\tilde{a}}{\tilde{a}z'}$ is a feasible solution for (5)-(7).

We now show that the set of a satisfying (5)-(7) is bounded. To do that, it suffices to show that the following linear program

$$\begin{array}{ll} \max & ae \\ \text{s.t.} & \begin{cases} -au^i & \leq 0 \qquad i = 1, 2, \ldots, n \\ a{u'}^i & \leq 0 \qquad i = 1, 2, \ldots, n \\ az' = 1 & \end{cases} \end{array} \tag{11}$$

has a finite optimal value for any vector e satisfying $\|e\| = 1$. Since we already know that this program is feasible, by the duality theorem of linear programming, it suffices to show that the dual is feasible. The dual is:

$$\begin{array}{ll} \min & \eta \\ \text{s.t.} & \begin{cases} -\sum_{i=1}^{n} \lambda_i u^i + \sum_{i=1}^{n} \lambda'_i {u'}^i + \eta z' = e \\ \lambda, \lambda' \geq 0 \end{cases} \end{array} \tag{12}$$

Since the $u^i = 1, \ldots, n$ are linearly independent, there exists $\tilde{\lambda}$ such that

$$-\sum_{i=1}^{n} \tilde{\lambda}_i u^i = e. \tag{13}$$

On the other hand, since the interior of $K \cap K'$ is nonempty, there exists $\dot{\lambda}$ and $\dot{\lambda}'$ such that

$$-\sum_{i=1}^{n} \dot{\lambda}_i u^i + \sum_{i=1}^{n} \dot{\lambda}'_i {u'}^i + z' = 0 \tag{14}$$

with $\dot{\lambda}_i, \dot{\lambda}'_i > 0$ for $i = 1, \ldots, n$. A feasible solution to the dual can then be constructed by adding to (13) the equation (14) premultiplied by a sufficiently large η. Hence problem (12) - and therefore (11) - has a finite optimal value, which shows that the set of a satisfying (5)-(7) is bounded. ∎

We now give a reformulation of problem $(DSLBP)$.

Proposition 5 *For any $\gamma \leq \min\{f(O), f(z')\}$, let $\overline{\lambda}_i(\gamma) = \overline{\lambda}(O, u^i; \gamma)$ and $\overline{\lambda}'_i(\gamma) = \overline{\lambda}(z', {u'}^i; \gamma)$, $i = 1, 2, \ldots, n$. Denote by J_∞ the set of edges of K along which f is nondecreasing, and by J'_∞ the set of edges of K' along which f is nondecreasing. Then problem $(DSLBP)$ is equivalent to*

$$
(DSP) \qquad \max \gamma
$$
$$
s.t. \begin{cases}
\overline{\lambda}_i(\gamma) a u^i & \geq b & i \notin J_\infty & (15) \\
-\overline{\lambda}'_i(\gamma) a u'^i & \geq 1-b & i \notin J'_\infty & (16) \\
a u^i & \geq 0 & i = 1,2,\ldots,n & (17) \\
a u'^i & \leq 0 & i = 1,2,\ldots,n & (18) \\
a z' & = 1 & & (19) \\
0 < b < 1 & & & (20) \\
\gamma \leq \min\{f(O), f(z')\}. & & & (21)
\end{cases}
$$

Proof:

First note that the constraint (21) is innocuous: since O and z' are feasible point of $K \cap K'$, there cannot exist a double-simplicial lower bound with a value γ greater than $f(O)$ or $f(z')$. This constraint is merely here to ensure that the $\overline{\lambda}_i(\gamma)$ and $\overline{\lambda}'_i(\gamma)$ are defined (see Section 2.1).

Also note that the constraints (17)-(20) are those who characterize $\mathcal{H}_{K'}$ (see Proposition 4).

In order to prove Proposition 5, it suffices to show the identity between the set of hyperplanes defined by constraints (15)-(20) and the subset of $\mathcal{H}_{K'}$ of hyperplanes $H = \{x \in \mathrm{IR}^n \mid ax = b\}$ satisfying $DSLB(H) \geq \gamma$.

Assume that (a, b) satisfies constraints (15)-(20). Since (a, b) satisfies (17)-(20), by Proposition 4, $H = \{x \in \mathrm{IR}^n \mid ax = b\}$ is in $\mathcal{H}_{K'}$. Let $y^i = \overline{\lambda}_i(\gamma) u^i, i \notin J_\infty$ and $y'^i = z' + \overline{\lambda}'_i(\gamma) u'^i, i \notin J'_\infty$. Let z^i (respectively z'^i), $i = 1, 2, \ldots, n$ be the (possibly infinite) intersection points of the edges of K (respectively K') with the hyperplane H: the expression of these points is given respectively by (9) and (10). Then for $i \notin J_\infty$, z^i lies between O and y^i by constraint (15), thus by concavity, $f(z^i) \geq \gamma$. For $i \in J_\infty$, f is greater than $f(O)$ on the entire edge, thus again $f(z^i) \geq \gamma$. Similarly, constraints (16) and (19) imply that the z'^i lie between z' and y'^i for $i \notin J'_\infty$, thus $f(z'^i) \geq \gamma$ for all $i = 1, 2, \ldots, n$. It follows that $DSLB(H) \geq \gamma$.

Conversely, assume that $H \in \mathcal{H}_{K'}$ satisfies $DSLB(H) \geq \gamma$. By Proposition 4, there exists (a, b) satisfying (17)-(20) such that $H = \{x \in \mathrm{IR}^n \mid ax = b\}$. Now H intersects the edges of K at point $z^i = \frac{b}{au^i} u^i$ of value greater than or equal to γ. By definition of the $\overline{\lambda}_i(\gamma)$, we have then $\frac{b}{au^i} \leq \overline{\lambda}_i(\gamma)$ for $i \notin J_\infty$. Multiplying both side by au^i (which is positive by constraint (17) and since z^i cannot be infinite for $i \notin J_\infty$), we obtain relation (15). Similarly, constraint (16) is satisfied since the intersection points z'^i of H with the edges of K' must have a value greater than or equal to γ. Thus H satisfies the constraints (15)-(20). ∎

Note that if $b = 0$ in (20), $K \cap P \subseteq K' \cap \{x \in \mathrm{IR}^n \mid ax \geq 0\} = S'(H)$ since $K \cap P \subseteq K'$ and by (17), thus a lower bound of f over $K \cap P$ is the simplicial bound $\min\limits_{x \in S'(H)} f(x)$. By

(16), its value is greater than or equal to γ.
Similarly, if $b = 1$, we have $K \cap P \subseteq S(H)$ and the simplicial bound $\min_{x \in S(H)} f(x)$ is greater than or equal to γ.
Thus, by slightly modifying the definition of the double-simplicial bound to include these two extreme cases, we can assume that the double inequality (20) is a nonstrict one. We denote by $\overline{\mathcal{H}}_{K'}$ the set of hyperplanes $H = \{x \in \mathrm{I\!R}^n \mid ax = b\}$ where (a, b) satisfies constraints (5)-(7) and $0 \leq b \leq 1$.

There are different ways to solve problem (DSP).
First, it may be restated as

$$\max \gamma$$
$$\text{s.t.} \begin{cases} (a,b) \in C & \\ h_i(a,b;\gamma) \leq 0 & \text{for } i \notin J_\infty \\ h'_i(a,b;\gamma) \leq 0 & \text{for } i \notin J'_\infty \end{cases}$$

where $C = \{(a,b) \in \mathrm{I\!R}^{n+1} \,|\, az' = 1;\, au^i \geq\ 0\,(i = 1,2,\ldots,n);\, au'^i \leq\ 0\,(i = 1,2,\ldots,n);\, 0 \leq b \leq 1\}$, $h_i(a,b;\gamma) = b - (au^i)\overline{\lambda}_i(\gamma)$ for $i \notin J_\infty$ and $h'_i(a,b;\gamma) = 1 - b + (au'^i)\overline{\lambda}'_i(\gamma)$ for $i \notin J'_\infty$.
Since C is convex and since the functions h_i and h'_i are linear-convex functions over $C \times \mathrm{I\!R}$ (i.e., $(a,b) \mapsto h(a,b,\gamma)$ is linear for each fixed $\gamma \in \mathrm{I\!R}$ and $\gamma \mapsto h(a,b;\gamma)$ is convex for each fixed $(a,b) \in C$), this problem can be solved by the algorithm of Horst, Muu and Nast [4].

A second approach is as follows. For each γ, define $CDSP(\gamma)$ as the set of vectors (a,b) such that (a,b,γ) is a feasible solution to problem (DSP). By noting that the function $\gamma \mapsto CDSP(\gamma)$ is nonincreasing, i.e., $CDSP(\gamma) \subseteq CDSP(\gamma')$ when $\gamma \geq \gamma'$, problem (DSP) can be solved by a dichotomy search over $[\underline{\gamma}, \overline{\gamma}]$ where $\underline{\gamma}$ is such that $CDSP(\underline{\gamma}) \neq \emptyset$ and $\overline{\gamma}$ is such that $CDSP(\overline{\gamma}) = \emptyset$.

We choose to focus our attention on a third method. Assume that γ is fixed and less than $\min\{f(O), f(z')\}$. If (a,b) satisfies at equality none of the constraints (15) and (16), then $\gamma' = DSLB(a,b)$ improves strictly the current value of γ. Indeed, assume that the value γ' is attained at the intersection point of the i^{th} edge of K with the hyperplane $ax = b$. Then $\gamma' = f((\frac{b}{au^i})u^i)$ and $\frac{b}{au^i} < \overline{\lambda}_i(\gamma)$. Thus $\gamma' > \gamma$ by definition of $\overline{\lambda}_i(\gamma)$. Similarly, if γ' is attained at an intersection point of an edge of K' with the hyperplane $ax = b$, since $\gamma < f(z')$, we have $\gamma' > \gamma$.
If a vector (a,b) satisfying strictly all the constraints (15) and (16) exists, it can be found by solving the following linear program:

$$DSLP(\gamma) \qquad \max \xi$$
$$\text{s.t.} \begin{cases} \overline{\lambda}_i(\gamma)au^i & \geq b+\xi & i \notin J_\infty \\ -\overline{\lambda}'_i(\gamma)au'^i & \geq 1-b+\xi & i \notin J'_\infty \\ au^i & \geq 0 & i \in J_\infty \\ au'^i & \leq 0 & i \in J'_\infty \\ az' & = 1 & \\ 0 \leq b \leq 1. & & \end{cases}$$

This leads to the following procedure.

Procedure BDSB

Step 1 (Initialization) : Initialize γ to a double-simplicial bound (for example, consider a hyperplane $\hat{H}$ belonging to $\mathcal{H}_{K'}$ and take $\gamma = DSLB(\hat{H})$; such a hyperplane can be constructed by finding a feasible point of the system $\{au^i \geq 1\,(i = 1, 2, \ldots, n), au'^i \leq -1\,(i = 1, 2, \ldots, n)\}$ (see the proof of Proposition 4)).
Compute $\overline{\lambda}_j(\gamma)$ and $\overline{\lambda}'_j(\gamma)$ for $j = 1, 2, \ldots, n$. Let $J_\infty = \{j \mid \overline{\lambda}_j(\gamma) = +\infty\}$ and $J'_\infty = \{j \mid \overline{\lambda}'_j(\gamma) = +\infty\}$.

Step 2 (Computing a new hyperplane) : Solve the linear problem $DSLP(\gamma)$. Let $(\hat{a}, \hat{b}, \hat{\xi})$ be an optimal solution. Let $\hat{H} = \{x \in \mathbb{R}^n \mid \hat{a}x = \hat{b}\}$.
If $\hat{\xi} = 0$, stop: $\hat{H}$ is an optimal solution to problem $(DSLBP)$ with value γ.

Step 3 (Computing a new double-simplicial bound) : Let $z^j = \frac{\hat{b}}{\hat{a}u^j}u^j$ for $j \notin J_\infty$ and $z'^j = z' + \left(\frac{\hat{b}-1}{\hat{a}u'^j}\right) u'^j$ for $j \notin J'_\infty$. Compute $\gamma = \min\{f(O), f(z'), \min\limits_{j \notin J_\infty} f(z^j), \min\limits_{j \notin J'_\infty} f(z'^j)\}$.
If $\gamma = \min\{f(O), f(z')\}$, stop: $\hat{H}$ is an optimal solution of problem $(DSLBP)$.

Step 4 (Computing new γ-extensions) : For all $j \notin J_\infty$, compute $\overline{\lambda}_j(\gamma)$; for all $j \notin J'_\infty$, compute $\overline{\lambda}'_j(\gamma)$.
Return to Step 2.

Theorem 4 *Procedure BDSB described above converges to an optimal solution of problem* $(DSLBP)$.

Proof:

Denote by γ_k the value of γ at iteration k, and by $\hat{a}^k, \hat{b}_k, \hat{\xi}_k$ the optimal solution of problem $DSLP(\gamma_k)$.
The sequence γ_k is increasing and bounded from above by $\min\{f(O), f(z')\}$, thus converges to a limit γ^*. If $\gamma^* = \min\{f(O), f(z')\}$ we are done, hence we assume that $\gamma^* < \min\{f(O), f(z')\}$.
For any k, $(\hat{a}^{k+1}, \hat{b}_{k+1}, \hat{\xi}_{k+1})$ is a feasible solution of problem $DSLP(\gamma_k)$. Thus $\hat{\xi}_{k+1} \leq \hat{\xi}_k$. Since $\hat{\xi}_k$ is positive (because γ_k is a double-simplicial lower bound), it follows that the sequence $\hat{\xi}_k$ converges to a limit $\hat{\xi}^*$. Let us show that this limit is 0. If the sequence

is finite, it is easy to see that if $\hat{\xi}^* > 0$, the best double-simplicial lower bound can be strictly improved. Hence $\hat{\xi}^* = 0$. Assume now that the sequence is infinite. Then there exists a subsequence $\{k_h\}$ of $\{k\}$ such that a) for all k_h, $\gamma_{k_h+1} = f(z^{k_h\ell})$ for some $\ell \notin J_\infty$, or b) for all k_h, $\gamma_{k_h+1} = f({z'}^{k_h\ell})$ for some $\ell \notin J'_\infty$. Since the proof is similar for the two cases, we only give the details for the case b). We have then

$$\overline{\lambda}'_\ell(\gamma_{k_h+1}) = \frac{\hat{b}^{k_h} - 1}{\hat{a}^{k_h}{u'}^\ell}. \tag{22}$$

But $(\hat{a}^{k_{h+1}}, \hat{b}_{k_{h+1}}, \hat{\xi}_{k_{h+1}})$ is a feasible solution of problem $DSLP(\gamma_{k_h+1})$, hence

$$-\overline{\lambda}'_\ell(\gamma_{k_h+1})\hat{a}^{k_{h+1}}{u'}^\ell \geq 1 - \hat{b}_{k_{h+1}} + \hat{\xi}_{k_{h+1}}. \tag{23}$$

Observe that $(\hat{a}^{k_h}, \hat{b}^{k_h})$ is bounded by Proposition 4. Hence considering a subsequence if necessary, we may assume that $\hat{a}^{k_h} \to \hat{a}^*$ and $\hat{b}^{k_h} \to \hat{b}^*$. We distinguish between two cases depending on whether $\hat{a}^*{u'}^\ell = 0$ or not. In the first case, since $\overline{\lambda}'_\ell(\gamma^*) < \infty$, we have $\hat{b}^* = 1$ by (22). Using (23), we obtain $0 \geq 1 - \hat{b}^* + \hat{\xi}^*$ which shows that $\hat{\xi}^* = 0$. Consider now the second case in which $\hat{a}^*{u'}^\ell \neq 0$. Using (22) and (23), and simplifying by $\hat{a}^*{u'}^\ell$, we obtain $-(\hat{b}^* - 1) \geq 1 - \hat{b}^* + \hat{\xi}^*$, which again shows that $\xi^* = 0$.

Now assume that there exists a double-simplicial lower bound $\tilde{\gamma} > \gamma^*$. Let $\tilde{H} = \{x \in \mathbb{R}^n \mid \tilde{a}x = \tilde{b}\}$ be the corresponding hyperplane. Then $\tilde{a}u^i > \frac{\tilde{b}}{\overline{\lambda}_i(\gamma^*)}$, $i \notin J_\infty$ and $-a{u'}^i > \frac{1-\tilde{b}}{\overline{\lambda}'_i(\gamma^*)}$, $i \notin J'_\infty$.

Let $\tilde{\xi} = \min\left\{\min_{i\notin J_\infty}\{\overline{\lambda}_i(\gamma^*)\tilde{a}u^i - \tilde{b}\}, \min_{i\notin J'_\infty}\{-\overline{\lambda}'_i(\gamma^*)\tilde{a}{u'}^i - 1 + \tilde{b}\}\right\}$. Then $(\tilde{a}, \tilde{b}, \tilde{\xi})$ is a feasible solution of problem $DSLP(\gamma^*)$ with $\tilde{\xi} > 0$, which is a contradiction. Thus γ^* is the best double-simplicial lower bound. ∎

In the next section, we explain how the simplicial lower bound presented in Paragraph 2.2 can be used to choose a valid cone K'.

2.4 Combination of the simplicial and double-simplicial lower bound

Let $H(K) = \{x \in \mathbb{R}^n \mid ax = 1\}$ be a hyperplane corresponding to a simplicial lower bound, as defined in Section 2.2. Let $\omega(K)$ be an extreme point of $K \cap P$ at which $H(K)$ supports $K \cap P$.

Let K' be a cone originated at $z' = \omega(K)$, contained in the halfspace $\{x \in \mathbb{R}^n \mid ax \leq 1\}$ and containing $K \cap P$. Then $H(K)$ clearly belongs to $\overline{\mathcal{H}}_{K'}$.

If $\omega(K)$ is a non-degenerate vertex of $K \cap P$, we can simply take for K' the cone generated by the n edges of $K \cap P$ emanating from $\omega(K)$.

If $\omega(K)$ is degenerate, we can consider the cone induced by the basic variables in the solution of the linear program from which $\omega(K)$ was obtained.

More precisely, recall that $\omega(K) = \sum_{j=1}^{n} \lambda_j^* u^j$ where λ^* is a basic optimal solution of

$$LP(\tilde{\gamma}) \qquad \max \qquad \theta^T \lambda$$
$$\text{s.t.} \qquad \begin{cases} AU\lambda \le b \\ \lambda \ge 0 \end{cases}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ with $\theta_j = \begin{cases} 0 & \text{if } j \in J_\infty \\ \frac{1}{\overline{\lambda}_j(\tilde{\gamma})} & \text{if } j \notin J_\infty \end{cases}$ for some value $\tilde{\gamma}$ and U is the $n \times n$ matrix of columns $u^j,\ j = 1, 2, \ldots, n$.

Let us introduce the slack variables s and let $y = (\lambda, s)$. Let B be the submatrix of (AU, I) consisting of the columns corresponding to the basic variables (denoted y_B). Let N be the submatrix of (AU, I) corresponding to the nonbasic variables y_N. Let (θ_B, θ_N) be the corresponding partitioning of vector $(\theta, 0)$. Then

$$\begin{aligned} By_B + Ny_N &= b \\ y_B, y_N &\ge 0, \end{aligned}$$

thus $\begin{pmatrix} y_B \\ y_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} + \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} y_N$. As $y = (\lambda, s)$, by setting μ to y_N, we get

$$\lambda = \lambda^* + V\mu, \quad \mu \ge 0,$$

where λ^* is some subvector of $\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ and V some $n \times n$ submatrix of $\begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}$.

It follows that $K \cap P = \{x \in \mathbb{R}^n \mid x = U\lambda; AU\lambda \le b; \lambda \ge 0\}$ is included in the set $K' = \{x \in \mathbb{R}^n \mid x = U\lambda^* + UV\mu \; ; \; \mu \ge 0\}$. Note that $U\lambda^* = \omega(K)$ and that the columns of UV can be identified to the vectors $u'^j, j = 1, 2, \ldots, n$. Let $a = \frac{\theta^T U^{-1}}{\theta^T \lambda^*}$ and $b = 1$. By the optimality condition, we have $\theta_N^T - \theta_B^T B^{-1} N \le 0$, i.e., $\theta^T V \le 0$, thus $aUV = \frac{\theta^T U^{-1}(UV)}{\theta^T \lambda^*} \le 0$, hence conditions (6) of Proposition 4 are fulfilled. Conditions (5) are satisfied by construction of $LP(\tilde{\gamma})$. Finally, $a\omega(K) = \frac{\theta^T U^{-1} U \lambda^*}{\theta^T \lambda^*} = 1$, which shows that condition (7) is also satisfied. Thus, with this choice of K', $H = \{x \in \mathbb{R}^n \,|\, ax = 1\}$ belongs to $\overline{\mathcal{H}}_{K'}$.

Since $H(K)$ belongs to $\overline{\mathcal{H}}_{K'}$, $H(K)$ is a valid hyperplane for the computation of a double-simplicial lower bound and $DSLB(H(K)) = SLB(H(K))$. This is true in particular if $H(K)$ is the hyperplane corresponding to the best simplicial lower bound. Then the best double-simplicial lower bound is at least as good as the best simplicial bound. The following result gives a sufficient condition to obtain a strict improvement.

Proposition 6 *If $H(K) = \{x \in I\!R^n | ax = 1\}$ supports $K \cap P$ at an unique, non-degenerate, extreme point z', then either the minimum of f over $K \cap P$ is attained at z', or the simplicial bound $SLB(H(K))$ can be strictly improved.*

Proof:

Let $\gamma = SLB(H(K))$. If $f(z') = \gamma$, then γ is the minimum of f over $K \cap P$ since z' is a feasible point of $K \cap P$.

Assume now that $f(z') > \gamma$. Then $\overline{\lambda}'_i(\gamma) = \overline{\lambda}(z', u'^i; \gamma)$ is strictly positive for $i = 1, 2, \ldots, n$, and $a(\overline{\lambda}_i(\gamma)u^i) \geq 1$ for all $i = 1, 2, \ldots, n$ by definition of γ. Furthermore, we have $au'^i < 0$, $i = 1, 2, \ldots, n$. Indeed, assume that $au'^i = 0$ for some i. Then since z' is a non-degenerate extreme point of $K \cap P$, there exists $\eta > 0$ such that $z' + \eta u'^i$ is also an extreme point of $K \cap P$. Since $a(z' + \eta u'^i) = az' = 1$, it follows that z' is not the unique support point, in contradiction with the assumption.

Now let $\hat{a} = a$, $\hat{\xi} = \min \left\{ 1, \frac{1}{2} \min_{i \notin J'_\infty} \{-\overline{\lambda}'_i(\gamma)\hat{a}u'^i\} \right\}$ and $\hat{b} = 1 - \hat{\xi}$. By construction, $(\hat{a}, \hat{b}, \hat{\xi})$ is a feasible solution of problem $DSLP(\gamma)$ with value $\hat{\xi} > 0$. Hence the double-simplicial lower bound can be strictly improved. ∎

3 Embedding of the lower bounds in a branch-and-bound algorithm

In a conical branch-and-bound algorithm, the subdivision point is often chosen as the point $\omega(K)$, which is a byproduct of the computation of the simplicial lower bound (see Section 2.2). This raises the question of the theoretical convergence of the conical algorithm when using the new lower bounds considered in this paper. Also, because of the interrelation between lower bound and branching, it seems more reasonable to compare the lower bounds on subproblems produced by the branch-and-bound algorithm rather than on random subproblems (it may happen that the lower bounds are very good on average on random subproblems, but that a branch-and-bound algorithm using these lower bounds perform badly, for example because the algorithm tends to generate subproblems for which the lower bound is not good). For these reasons, we have embedded the lower bounds in a conical branch-and-bound algorithm. The subdivision rule is recalled in Paragraph 3.1; in Paragraph 3.2 we present the branch-and-bound algorithm and finally in Paragraph 3.3 we establish its convergence.

3.1 Subdivisions

We propose to consider normal conical subdivisions. We recall their definition and main properties. The reader is referred to Horst and Tuy [7] and Tuy [20] for more details.

A partition of P into initial cones can be constructed in a standard way, either by choosing O as an interior point of P in which case the partition consists of $n+1$ cones, or by choosing O as a non-degenerate vertex of P in which case the partition consists of one cone defined by O and its adjacent extreme points. Note that in the latter case, we can consider instead of P the polyhedron P' obtained from P by removing the n constraints binding at O: O then satisfies our assumption of being an interior point for P'.

Let K^0 be one of the initial cone, $u^{01}, \dots, u^{0n}$ be a point on each of the edges of K^0 (distinct from O) and let $F_0 = conv\{u^{01}, \dots, u^{0n}\}$. Consider a cone $K \subseteq K^0$ and let $U = K \cap F_0 = [u^1, u^2, \dots, u^n]$ be the section of K by the facet F_0: we say that U is the *base* of K.
Let v be an arbitrary point of U such that

$$v = \sum_{i=1}^{n} \lambda_i u^i, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0 \, (i = 1, 2, \dots, n). \tag{24}$$

Let $I = \{i \mid \lambda_i > 0\}$ and for all $i \in I$ let K_i be the cone of edges $Ou^1, \dots, Ou^{i-1}$, $Ov, Ou^{i+1}, \dots, Ou^n$. It is easy to verify that the cones K_i, $i \in I$ form a partition of the cone K. Denote by $\delta(U)$ the length of a longest edge of U. If $v = \alpha u^p + (1-\alpha) u^q$ where $\|u^p - u^q\| = \delta(U)$ and $0 < \alpha \leq \frac{1}{2}$, the partition is called a *bisection* of ratio α. On the other hand, if v belongs to the halfline $O\omega(K)$ where $\omega(K)$ was defined in Section 2.2, we say that the cone K is ω-subdivided.

A subdivision rule is said *normal* if only bisections or ω-subdivisions are used and if bisections occur infinitely many times in every infinite sequence of nested cones. In this paper, we assume in addition that ω-subdivisions also occur infinitely many times in every infinite sequence of nested cones. This is a natural assumption since the concept of normality was introduced to derive proved convergent algorithms that are a compromise between the algorithms that use only bisections but perform poorly, and the algorithms with a pure ω-subdivisions strategy that perform better in practice but whose convergence has not yet been proved (see Tuy [18])[1]. A typical normal rule is defined as follows: Denote by $\tau(K)$ the generation index of the cone K (i.e., $\tau(K) = 0$ if K is one of the initial cone; $\tau(K) = \tau(K') + 1$ if K is the son of cone K') and select a natural number N. Then if $\tau(K)$ is divisible by N, bisect K; otherwise, ω-subdivide K (see, e.g., Tuy [20, p. 149-150]).

The following Proposition is useful to establish the convergence of the algorithm described in the next section.

Proposition 7 *(Tuy [19, p.20])*
Let $U^k = [u^{k1}, u^{k2}, \dots, u^{kn}]$, $k = 1, 2, \dots$ be a sequence of nested $(n-1)$-simplices such that any U^{k+1} is a son of U^k in a subdivision via some $v^k \in U^k$. If there exists an infinite sequence $\Delta \subset \{1, 2, \dots\}$ such that for every $k \notin \Delta$ the subdivision of U^k is a bisection of ratio $\alpha_k \geq \alpha_0 > 0$ then, whenever the sequence $\{v^k, k \in \Delta\}$ is infinite, at least one of its cluster points is a vertex of $U^* = \bigcap_{k=1}^{\infty} U^k$.

[1] Since this paper has been written, the convergence with a pure ω-subdivision strategy has been proven, independently and by different approaches, by some of the authors [9] and by Locatelli [11].

3.2 Algorithm

We propose the following conical branch-and-bound algorithm for obtaining an ε-optimal solution of the concave programming problem (CP):

Algorithm CBB

Step 1 (initialization) : Select the tolerance $\varepsilon \geq 0$.

Construct an initial conical partition $\mathcal{P}$ of P and define a normal subdivision rule (see Section 3.1).

Initialize the incumbent value $\overline{f}$ and solution $\overline{x}$ with the best intersection point of the edges of $K \in \mathcal{P}$ with the boundary δP of P. For each cone K in $\mathcal{P}$, compute a simplicial lower bound, or a simplicial bound followed by a double-simplicial bound, as explained in Section 2. Denote by $\underline{f}(K)$ this lower bound. Let $\omega(K)$ be the point of $K \cap P$ corresponding to the simplicial bound. If for some $K \in \mathcal{P}$, $f(\omega(K)) < \overline{f}$ then set $\overline{f} \leftarrow f(\omega(K))$ and $\overline{x} \leftarrow \omega(K)$. Set $\mathcal{L}$ to $\mathcal{P}$.

Step 2 (subdivision) : let $\tilde{K} \in \arg\min\{\underline{f}(K) \mid K \in \mathcal{L}\}$. Bisect $\tilde{K}$, or ω-subdivide it via the point $\omega(\tilde{K})$ according to the normal subdivision rule. Let $\mathcal{P}$ be the set of new cones. Set $\mathcal{L} \leftarrow (\mathcal{L} \backslash \{\tilde{K}\}) \cup \mathcal{P}$.

Step 3 (bounding) : for each cone $K \in \mathcal{P}$, compute a simplicial lower bound $\underline{f}(K)$, using the simplicial bound computed for $\tilde{K}$ as initial value for γ in Step 1 of procedure BSB (see Section 2.2). Optionally, improve $\underline{f}(K)$ by computing a double-simplicial lower bound as explained in Sections 2.3 and 2.4. Finally let $\underline{f}(K) \leftarrow \max\{\underline{f}(K), \underline{f}(\tilde{K})\}$.

Let $\omega(K)$ be the point of $K \cap P$ corresponding to the simplicial bound. If for some $K \in \mathcal{P}$, $f(\omega(K)) < \overline{f}$ then set $\overline{f} \leftarrow f(\omega(K))$; $\overline{x} \leftarrow \omega(K)$.

Step 4 (fathoming) : delete every cone $K \in \mathcal{L}$ for which $\underline{f}(K) \geq \overline{f} - \varepsilon$.

If $\mathcal{L} = \emptyset$ then terminate: $\overline{x}$ is an ε-optimal solution of problem (CP); otherwise return to Step 2.

3.3 Convergence

Theorem 5 *The algorithm CBB either terminates in a finite number of iterations with an ε-optimal solution of (CP), or is infinite. In this latter case, which can occur only if $\varepsilon = 0$, any cluster point of the sequence $\overline{x}$ is a global optimal solution of problem (CP).*

Proof:

Let $\mathcal{D}$ be the set of cones deleted at Step 4 since the beginning of the algorithm. At any iteration, $\mathcal{L} \cup \mathcal{D}$ forms a conical partition of P and $f(x) \geq \underline{f}(K) \geq \overline{f} - \varepsilon$ for all $K \in \mathcal{D}$ and all $x \in K \cap P$. In particular, if the algorithm stops at the end of Step 4, $f(x) \geq \overline{f} - \varepsilon = f(\overline{x}) - \varepsilon$ for all $x \in P$, which shows that $\overline{x}$ is an ε-optimal solution of (CP).

Assume now that the algorithm is infinite. Then it generates at least an infinite sequence of nested cone K^k of limit K^*.

Denote by $U^k = [u^{1k}, \ldots, u^{nk}]$ the base of K^k. Let $\overline{f}_k$ be the incumbent value at the iteration of the selection of K^k, $\underline{f}_k = \underline{f}(K^k)$ be the lower bound of f over $K^k \cap P$ and γ_k be the simplicial bound. Let γ'_k be the value of γ at the beginning of the last iteration of procedure BSB (see Section 2.2). Let c_k, y^{ik}, z^{ik} and J^k_∞ be respectively the value $\hat{c}(\gamma)$, the points y^i and z^i and the set J_∞ at the last iteration of procedure BSB. Finally, let v^k and q^k be the points where the ray through $\omega^k = \omega(K^k)$ meets the simplex U^k and the generalized simplex $conv_{i \notin J^k_\infty}\{y^{ik}\} + cone_{i \in J^k_\infty}\{u^{ik}\}$. Clearly, $\omega^k = c_k q^k$ and $z^{ik} = c_k y^{ik}$ for all $i \notin J^k_\infty$.
Clearly, $\overline{f}_k$ is nonincreasing and bounded from below by $\min\limits_{x \in P} f(x)$ thus converges to a limit $\overline{f}^*$.
Since γ_k is used as initial value for γ in Step 1 of procedure BSB and since procedure BSB produces a sequence of increasing simplicial values, it follows that $\gamma'_{k+1} \geq \gamma_k \geq \gamma'_k$. Thus γ'_k and γ_k are two imbricated nondecreasing sequences bounded from above by $\min\limits_{x \in P} f(x)$, thus they converge to a same limit γ^*.
Due to the selection of the cone K^k at Step 2, we have $\gamma_k \leq \underline{f}_k = \min\{\underline{f}(K) \mid K \in \mathcal{L}_k\} \leq \min\{f(x) \mid x \in P\}$ where $\mathcal{L}_k$ is the set of cones remaining when cone K^k is selected. By taking the limit, we obtain $\gamma^* \leq \min\{f(x) \mid x \in P\}$.
On the other hand, since K^k is not eliminated at Step 4, we have $\overline{f}^k - \varepsilon > \underline{f}(K^k) \geq \gamma_k$ for all k, thus $\overline{f}^* - \varepsilon \geq \gamma^*$.
Assume first that $\gamma^* = f(O)$. Since $\overline{f}^* \leq f(O)$, this is only possible if $\varepsilon = 0$ and $\overline{f}^* = f(O)$, in which case O is the global optimal solution of problem (CP).
Assume now that $\gamma^* < f(O)$. Let $\Delta = \{k \mid K^k \text{ is } \omega\text{-subdivided}\}$. By Proposition 7 there exists a subsequence k_s of $\{1, 2, \ldots\} \setminus \Delta$ such that $\{v^{k_s}\}$ tends to a vertex of $U^* = \bigcap\limits_{k=1}^{\infty} U^k$, say $v^{k_s} \to \tilde{v} \in U^*$. By considering a subsequence if necessary, we may assume that $J^{k_s}_\infty$ is constant, say equal to $\tilde{J}_\infty$ (note that necessarily $\tilde{J}_\infty \neq \{1, 2, \ldots, n\}$ since $\gamma^* < f(O)$), that $u^{ik_s} \to \tilde{u}^i$ for all $i = 1, 2, \ldots, n$ (so that $U^* = conv\{\tilde{u}^1, \ldots, \tilde{u}^n\}$), that $c_{k_s} \to \tilde{c}$ and that $\omega^{k_s} \to \tilde{\omega}$.
Under these assumptions, the sequence $\{y^{ik_s}\}$ is bounded for all $i \notin \tilde{J}_\infty$. If not, there would exist a subsequence $\{k'_s\}$ of $\{k_s\}$ such that $\{y^{ik'_s}\}$ is unbounded and $f(y^{ik'_s}) = \gamma_{k'_s}$ for all k'_s. Hence, by denoting $\lim\limits_{k'_s \to \infty} \frac{y^{ik'_s}}{\|y^{ik'_s}\|}$ by $\overline{u}$, we would have $\lim\limits_{\lambda \to \infty} f(\lambda \overline{u}) = \gamma^* < f(O)$. This is in contradiction with the fact that a concave function over a halfline is either unbounded below or reaches its minimum at the origin (see, e.g., Tuy [20, Proposition 5.1, p. 134]). Similarly, for $i \in \tilde{J}_\infty$, $f(\lambda u^{ik_s}) \geq f(O)$ for all $\lambda \geq 0$ which, by passing to the limit, implies that $f(\lambda \tilde{u}^i) \geq f(O)$, i.e., that the γ^*-extension along $\tilde{u}^i$ is infinite.
By taking a subsequence if necessary, we can assume that $y^{ik_s} \to \tilde{y}^i$ for all $i \notin \tilde{J}_\infty$. Since $z^{ik_s} \in [Oy^{ik_s}]$ for all $i \notin \tilde{J}_\infty$, we may also assume that $z^{ik_s} \to \tilde{z}^i$ for all $i \notin \tilde{J}_\infty$.

Since $\min_{j \notin \tilde{J}_\infty} \{f(\tilde{z}^j)\} = \gamma^* = f(\tilde{y}^i)$ with $\gamma^* < f(O)$, $\tilde{z}^i = \tilde{c}\tilde{y}^i$ and $\tilde{z}^i \neq O$ for all $i \notin \tilde{J}_\infty$, it follows that $\tilde{c} = 1$.

Now let us show that $f(\tilde{\omega}) = \gamma^*$. Since a vertex of U^* must be one of the points $\tilde{u}^1, \ldots, \tilde{u}^n$, we have for example $\tilde{v} = \tilde{u}^1$. Clearly, $q^{k_s} \to \tilde{q} \in conv_{i \notin \tilde{J}_\infty}\{\tilde{y}^i\} + cone_{i \in \tilde{J}_\infty}\{\tilde{u}^i\}$ and $\tilde{q} = \alpha\tilde{v}$ for some $\alpha > 0$. Therefore $\alpha\tilde{u}^1 = \tilde{q} = \sum_{i \notin \tilde{J}_\infty} \mu_i \overline{\lambda}_i(\gamma^*)\tilde{u}^i + \sum_{i \in \tilde{J}_\infty} \mu_i \tilde{u}^i$ with $\sum_{i \notin \tilde{J}_\infty} \mu_i = 1$. Since the limit of a sequence of finite (respectively infinite) γ-extensions is a finite (respectively infinite) γ-extension as shown above, the set $I = \{i \mid \tilde{u}^i = \tilde{u}^1\}$ cannot contain both elements in $\tilde{J}_\infty$ and elements not in $\tilde{J}_\infty$. Since $\sum_{i \notin \tilde{J}_\infty} \mu_i = 1$, I contains only indices not in $\tilde{J}_\infty$, hence

$$\begin{aligned} &\overline{\lambda}_1(\gamma^*) \sum_{i \in I} \mu_i = \alpha \\ &\mu_i = 0 \qquad \forall i \notin I. \end{aligned}$$

Thus $\sum_{i \in I} \mu_i = \sum_{i \notin \tilde{J}_\infty} \mu_i = 1$. Therefore $\alpha = \overline{\lambda}_1(\gamma^*)$, i.e., $\tilde{q} = \overline{\lambda}_1(\gamma^*)\tilde{v} = \overline{\lambda}_1(\gamma^*)\tilde{u}^1 = \tilde{y}^1$ and $f(\tilde{q}) = f(\tilde{y}^1) = \gamma^*$. Since $\tilde{\omega} = \tilde{c}\tilde{q}$ with $\tilde{c} = 1$, it follows that $f(\tilde{\omega}) = \gamma^*$.

Now $\overline{f}_k = f(\overline{x}^k) \leq f(\omega^k)$ for all $k = 1, 2, \ldots$. Letting k go to ∞, we obtain $\overline{f}^* = f(\overline{x}^*) \leq \gamma^* \leq \min\{f(x) \mid x \in P\}$. Since $\overline{f}^* - \varepsilon \geq \gamma^*$, this is possible only if $\varepsilon = 0$, in which case $\overline{x}^*$ is a global optimal solution of problem (CP). ∎

The conclusion of Theorem 5 still holds if we use in Step 3 the incumbent value $\overline{f}$ (instead of the simplicial bound computed for $\tilde{K}$) as the initial value for γ in procedure BSB, and if only one iteration of this procedure is performed. Since the proof of this result is very similar to that of Theorem 5, we only give the differences. Let p^k be the $\overline{f}_{k+1}$-extension along the ray going through ω^k. Since $\overline{f}_{k+1} \leq \min\{\overline{f}_k, f(\omega^k)\}$ and since $c_k \geq 1$ (otherwise K^k is eliminated which contradicts the infinity of the sequence), we have $\omega^k \in [p^k q^k]$. We then show that for some subsequence, $p^k \to \tilde{p}$ and $q^k \to \tilde{q}$ with $f(\tilde{p}) = \overline{f}^* = f(\tilde{q})$. This implies $\tilde{p} = \tilde{q}$ and $\tilde{c} = 1$. Consequently, the limit $\tilde{\gamma}$ of the simplicial bound γ_k is equal to $\overline{f}^*$.

4 Computational results

We present below some computational experiments to illustrate the efficiency of the lower bounds presented in this paper. We consider test problems of the following form

$$\begin{array}{ll} \min & f(x) = -\sqrt{\sum_{i=1}^n x_i^2} - \sqrt{\sum_{i=1}^n (x_i - 1)^2} \\ \text{s.t.} & \left\{ \begin{array}{l} Ax \leq b \\ x \geq 0 \end{array} \right. \end{array}$$

where A is a (100% dense) matrix of size 30×7 and b a vector of $\mathbb{R}^{30}$.

Ten instances of the problem were generated randomly as follows. The constraint matrix coefficients are pseudo-randomly generated in the interval $[0, 1]$ for the first row, and in the interval $[-1, 1]$ for the remaining rows. The coefficients of the vector b are equal to the sum of the elements of the corresponding row of A, plus a pseudo-random number in the interval $[0, 2]$. This ensures that the polyhedron $P = \{x \in \mathbb{R}^7 : Ax \leq b, x \geq 0\}$ is nonempty (as it contains the point $(1, \ldots, 1)$) and bounded (due to the definition of the first constraint together with the non-negativity constraints).

The point O, origin of the cones, is chosen as follows. We first minimize an arbitrary objective function (e.g., the sum of the x_i) over the polytope P, obtaining an extreme point O^1. Let p^1 be the gradient of f at this point: we minimize $p^1(x - O^1)$ over the polytope, obtaining an extreme point O^2 satisfying $f(O^2) \leq f(O^1)$ (see, e.g., Tuy [20, p.135]). The point O^2 is taken as the origin O (in our experiments, it never happened that the point was degenerate; if that were the case, we could have taken for O an alternate point, for example an interior point of P). The constraints of P binding at O define a cone that forms our initial partition. Although O is not necessarily a local minimum in the general case, this choice ensures that the algorithm terminates after 1 iteration with the optimal solution in the special case where the objective function is the composition of a one-dimensional function with a linear function. This is in accordance with the remarks of Tuan [16], which observed that problems that reduce to linear programs should not be used to test algorithms for global optimization, and if used, should be solved efficiently.

The tolerance ε is set to 10^{-6}, which is also the precision of the dichotomous procedure used when computing the γ-extensions. The linear problems occurring in the computation of the simplicial and double-simplicial bounds are solved by CPLEX [8]. When several iterations are performed in procedures BSB and BDSB, we use the optimal solution of the linear program solved at the previous iteration as a starting point for the linear problem of the current iteration. Finally, to easily access both the cones of smallest lower bound (Step 2 of algorithm CBB) and of greatest lower bound (Step 4), we store them in a min-max heap (see, e.g., Atkinson *et al.* [1]).

The program has been implemented in C and run on a SUN ULTRA-2/1300 (384 Mram). We first ran several versions of the algorithm CBB, in which the lower bound is a simplicial lower bound, computed by procedure BSB. These versions are denoted CBB(SLBx_y) where x refers to the choice of γ in Step 1 of procedure BSB ($x = A$ means that γ is initialized "from above", i.e., to the value of the best known solution; $x = B$ means that γ is initialized to a lower bound on the best simplicial bound, which is taken equal to the simplicial lower bound of the father cone), and where y stands for the number of iterations in procedure BSB. The best simplicial lower bound corresponds to both $SLBA\_\infty$ and $SLBB\_\infty$, which are approximated in practice respectively by $SLBA\_10$ and $SLBB\_10$. In any cases, we used a ratio of 1 bisection for 100 subdivisions in the normal subdivision rule (see Section 3.1).

Table 1 shows the average computing time (*CPU*) in seconds (average taken over 10 instances of the problem considered), the average number of iterations of algorithm CBB (*iter*), the average number of iterations needed until the optimal solution is found (*iter_opt*), the average of the maximum number of cones simultaneously contained in the min-max heap (*maxc*) and the average number of cones (*cone*).

Although some gain in CPU time and in memory is obtained by performing several iterations of procedure BSB when this procedure is initialized from below, this is not enough to beat the version using the Thoai-Tuy lower bound (1 iteration of BSB; γ initialized to the incumbent). Note that the convergence of the algorithm CBB was shown for versions $CBB(SLBA\_1)$, $CBB(SLBA\_\infty)$ and $CBB(SLBB\_y)$ for $1 \leq y \leq \infty$: this could explain the bad performance of $CBB(SLBA\_2)$.

CBB(.)	CPU	iter	iter_opt	maxc	cones
SLBA_1	142.1	14074	9521	3404	41866
SLBA_2	5320.4	1256399	74041	10867	1346568
SLBA_3	485.2	81894	18785	6161	133960
SLBA_5	254.7	24762	14161	5541	71507
SLBA_10	245.9	24005	13806	5358	69365
SLBB_1	398.9	45297	30230	13073	116770
SLBB_2	298.9	29775	19937	6923	81019
SLBB_3	271.4	26743	17829	5916	74110
SLBB_5	261.0	25424	16799	5527	71194
SLBB_10	231.3	25180	16585	5480	70764

Table 1: Simplicial lower bounds

Based on these results, we reran several versions of algorithm CBB, noted CBB(DSLBx_y), that use a combination of simplicial and double-simplicial lower bounds. The simplicial lower bound used is the one that gave the best result in the first run, that is SLBA_1 if $x = A$ and SLBB_10 if $x = B$. y denotes the number of iterations in the procedure BDSB. The results are given in Table 2.

For both A and B versions, the gain obtained by computing a double-simplicial lower bound is substantial: the CPU time decreases by 40% from CBB(SLBA_1) to CBB(DSLBA_5) and by 44% from CBB(SLBB_10) to CBB(DSLBB_5). The decrease in the iterations number is even greater (respectively 73% and 76%), which suggests that further improvement in the computing time could be obtained by implementing more efficiently the procedure BDSB. A similar decrease can be observed for the memory (indicator *maxc*): 72% and 74% respectively.

When comparing A and B versions, versions B clearly outperform versions A. A small consolation for versions A is that the use of double-simplicial lower (version CBB(DSLBB_5)) allows to beat the versions B that use only simplicial bounds.

CBB(.)	CPU	iter	iter_opt	maxc	cones
DSLBA_1	131.6	8719	5543	2163	26696
DSLBA_2	112.4	6026	3613	1540	19174
DSLBA_3	90.4	4536	3031	1160	14901
DSLBA_5	85.6	3706	2665	928	12312
DSLBA_10	90.8	3452	2451	834	11517
DSLBB_1	173.2	10953	7776	2675	34091
DSLBB_2	146.2	7864	5699	1791	25093
DSLBB_3	141.6	6918	4986	1568	22106
DSLBB_5	128.5	6047	4419	1387	19279
DSLBB_10	133.8	5853	4270	1323	18673

Table 2: Combination of simplicial and double-simplicial lower bounds

5 Conclusions

In this paper, we have investigated further the simplicial lower bound and introduced a new class of lower bound, called double-simplicial lower bound, which can be seen as an extension of the simplicial lower bound. For both simplicial and double-simplicial lower bound, we have characterized the hyperplane yielding the best possible bound and given an iterative algorithm to compute it.

A natural algorithm in which these lower bounds can be used is the conical branch-and-bound algorithm. In such algorithms, cones that cannot be eliminated are subdivided with respect to a point, that is often a byproduct of the computation of lower bound. This raises convergence issues. Therefore we have embedded our lower bounds in a conical branch-and-bound algorithm and proved its convergence for two possible initializations of the lower bound computing procedure. Limited numerical results show that improving the computation of the simplicial lower bound is worth for only one of the strategies, but that the use of the double-simplicial lower bound allows significant reduction in the computing time and in the memory for both strategies.

Further works should be done along several directions. More extensive computational experiments should be performed in order to measure more precisely the gain that can be obtained with the double-simplicial lower bound. The branch-and-bound algorithm should also be optimized further (in particular, we have observed substantial differences in the performance of the algorithm for different choices of the origin of the cones; a solution could be to test several candidates and retain the one that gives the best starting lower bound). Finally, it is well known that a conical branch-and-bound algorithm using simplicial lower bound performs better if there is an interaction between the lower bound computing procedure and the subdivision procedure. Therefore it would be interesting to develop a subdivision procedure based on the computation of the double-simplicial lower bound.

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