Les Cahiers du GERAD ISSN: 0711–2440

Single-level reformulations for network pricing under Wardrop equilibrium with separable affine costs

M. F. Anjos, J. Neto

G–2026–71

October 2026

La collection *Les Cahiers du GERAD* est constituée des travaux de recherche menés par nos membres. La plupart de ces documents de travail a été soumis à des revues avec comité de révision. Lorsqu'un document est accepté et publié, le pdf original est retiré si c'est nécessaire et un lien vers l'article publié est ajouté.

Citation suggérée : M. F. Anjos, J. Neto (Octobre 2026). Single-level reformulations for network pricing under Wardrop equilibrium with separable affine costs, Rapport technique, Les Cahiers du GERAD G– 2026–71, GERAD, HEC Montréal, Canada.

Avant de citer ce rapport technique, veuillez visiter notre site Web (https://www.gerad.ca/fr/papers/G-2026-71) afin de mettre à jour vos données de référence, s'il a été publié dans une revue scientifique.

The series *Les Cahiers du GERAD* consists of working papers carried out by our members. Most of these pre-prints have been submitted to peer-reviewed journals. When accepted and published, if necessary, the original pdf is removed and a link to the published article is added.

Suggested citation: M. F. Anjos, J. Neto (October 2026). Single-level reformulations for network pricing under Wardrop equilibrium with separable affine costs, Technical report, Les Cahiers du GERAD G–2026–71, GERAD, HEC Montréal, Canada.

Before citing this technical report, please visit our website (https://www.gerad.ca/en/papers/G-2026-71) to update your reference data, if it has been published in a scientific journal.

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2026
– Bibliothèque et Archives Canada, 2026

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Legal deposit – Bibliothèque et Archives nationales du Québec, 2026
– Library and Archives Canada, 2026

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Single-level reformulations for network pricing under Wardrop equilibrium with separable affine costs

Miguel F. Anjos [a, b]

José Neto [c]

[a] *School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh, United Kingdom, EH9 3FD*

[b] *GERAD, Montréal (Qc), Canada, H3T 1J4*

[c] *SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, 91120 Palaiseau, France*

miguel.f.anjos@ed.ac.uk

October 2026
Les Cahiers du GERAD
G–2026–71

Copyright © 2026 Anjos, Neto

Les textes publiés dans la série des rapports de recherche *Les Cahiers du GERAD* n'engagent que la responsabilité de leurs auteurs. Les auteurs conservent leur droit d'auteur et leurs droits moraux sur leurs publications et les utilisateurs s'engagent à reconnaître et respecter les exigences légales associées à ces droits. Ainsi, les utilisateurs:

- Peuvent télécharger et imprimer une copie de toute publication du portail public aux fins d'étude ou de recherche privée;
- Ne peuvent pas distribuer le matériel ou l'utiliser pour une activité à but lucratif ou pour un gain commercial;
- Peuvent distribuer gratuitement l'URL identifiant la publication.

Si vous pensez que ce document enfreint le droit d'auteur, contactez-nous en fournissant des détails. Nous supprimerons immédiatement l'accès au travail et enquêterons sur votre demande.

The authors are exclusively responsible for the content of their research papers published in the series *Les Cahiers du GERAD*. Copyright and moral rights for the publications are retained by the authors and the users must commit themselves to recognize and abide the legal requirements associated with these rights. Thus, users:

- May download and print one copy of any publication from the public portal for the purpose of private study or research;
- May not further distribute the material or use it for any profit-making activity or commercial gain;
- May freely distribute the URL identifying the publication.

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Abstract : We consider the problem of setting tolls on a subset of the arcs of a congested network so as to maximize the revenue of a network authority, anticipating that users route according to a Wardrop equilibrium. This problem is naturally cast as a bilevel program in which the leader sets tolls while the followers respond by choosing an equilibrium routing.

We focus on the important special case of separable affine arc cost functions. Under this assumption, Wardrop equilibria admit Beckmann's convex characterization, from which we derive three new single-level reformulations of the bilevel problem, exploiting both Beckmann's formulation and its dual: an arc-based formulation, which we solve exactly, and two path-based formulations, solved by column-and-constraint generation. Beyond the reformulations, we further derive from Beckmann's potential an upper bound on the optimal revenue that can be used to certify optimality gaps across all formulations.

An extensive computational study on instances derived from the Sioux Falls and Eastern Massachusetts networks compares our formulations with the recent mixed-integer nonlinear formulation of Beck, Labbé and Schmidt [3]. The arc-based formulation solves substantially more instances to proven optimality (within a time limit of 1800 s) than the formulation of [3] and, unlike it, returns a feasible solution on every instance, remaining effective on the largest 74-node network, on which [3] solves none. The magnitude of the upper bounds on the tolls proves to be a major driver of tractability: the genuinely hard core of the benchmark emerges once the regulatory cap on tolls is removed. The two path-based formulations, used as heuristics, return a solution within 1% of the optimum on every instance whose optimal value is known, often in seconds where exact formulations reach the time limit.

Acknowledgements: This research benefited from the support of the FMJH Program Gaspard Monge for optimization and operations research and their interactions with data science.

1 Introduction

Pricing mechanisms play a central role in transportation and telecommunication networks, both as a source of revenue and as a means of influencing users' routing decisions in order to improve the overall performance of the system [1, 18]. In many practical situations, however, the travel cost experienced by users also depends on congestion effects generated by the aggregate traffic, a phenomenon well documented in the road traffic literature [4, 19]. As a consequence, pricing decisions and traffic equilibria become strongly coupled, leading naturally to bilevel optimization models in which a central authority sets tolls anticipating the equilibrium response of the network users [6, 11].

In this work, we consider a toll-setting problem in a congested network. The network is represented by a directed graph $G = (V, A)$, where V denotes the set of nodes and A the set of arcs. For each arc $a \in A$, the unit transportation cost is given by a flow-dependent function $c_a(\cdot)$, and no capacity constraints are imposed on the arcs. All arc cost functions are assumed to be nonnegative, continuous, nondecreasing, and separable, meaning that c_a depends only on the total flow routed through arc a.

A set $\mathcal{K} \subseteq V \times V$ of origin-destination (OD) pairs is given. For each commodity $k \in \mathcal{K}$, the pair (o_k, s_k) is associated with a demand $d_k > 0$ that must be fully routed from o_k to s_k. We denote by $\mathcal{P}_k$ the set of elementary paths from o_k to s_k in G, and assume throughout that $\mathcal{P}_k \neq \emptyset$ for every $k \in \mathcal{K}$.

A leader seeks to determine nonnegative tolls $\tau = (\tau_a)_{a \in A}$ belonging to a prescribed compact and convex set $\mathcal{T} \subseteq \mathbb{R}^A_+$. We further assume that $\mathcal{T}$ is *downward closed*, meaning that if $\tau \in \mathcal{T}$ and $0 \leq \tau' \leq \tau$ componentwise, then $\tau' \in \mathcal{T}$. This assumption is satisfied in particular when $\mathcal{T}$ is defined by independent upper bound constraints on each toll, which is a common setting in the literature. These tolls are added to the flow-dependent arc costs, yielding effective arc costs

$$\widehat{c}_a(f_a, \tau_a) = c_a(f_a) + \tau_a, \qquad a \in A,$$

where f_a denotes the total flow routed through arc a. The followers are nonatomic users who route themselves selfishly according to Wardrop's equilibrium principle [19, 20]: for each commodity, flow is distributed only along minimum-cost paths with respect to the effective costs $\widehat{c}_a$.

The resulting optimization problem can be formulated as the bilevel program

$$\begin{aligned} \max_{\tau \in \mathcal{T},\, f} \quad & F(f, \tau) \\ \text{s.t.} \quad & f \in \mathcal{W}(\tau), \end{aligned} \tag{BW}$$

where $\mathcal{W}(\tau)$ denotes the set of Wardrop equilibrium flows associated with the toll vector τ, and F represents the leader's objective function. Depending on the application, F may model revenue maximization, network efficiency, or other system-level performance criteria [8, 10].

Problem (BW) is inherently difficult to solve, combining a bilevel structure, equilibrium constraints, and a dependence of arc costs on the equilibrium flows. In this work, we focus on the practically and theoretically important special case of separable affine arc cost functions [12, 15]. Under this assumption, Wardrop equilibria admit a characterization through Beckmann's convex potential formulation [4], which provides a powerful and tractable framework for deriving exact single-level reformulations.

The main contribution of this paper is the derivation of new exact single-level reformulations of problem (BW) in the affine separable case, exploiting both Beckmann's formulation and its dual. We introduce three such reformulations — an arc-based formulation, which we solve exactly, and two path-based formulations, which we solve heuristically by column-and-constraint generation — and compare them with the recent formulation proposed by Beck et al. [3]. Beyond the reformulations, we show that the common assumption that every commodity admits a toll-free path induces finite implicit upper bounds on the tolls (Proposition 2), allowing the unbounded case to be treated without arbitrary bounds, and we derive from Beckmann's potential an upper bound on the optimal revenue that certifies optimality gaps uniformly across all formulations. An extensive computational study

on the Sioux Falls and Eastern Massachusetts networks then shows that the arc-based formulation solves substantially more instances to proven optimality and remains effective on the largest, 74-node network, on which the formulation of Beck et al. solves none, while the two path-based heuristics recover the optimum — or a solution within less than 1% of it — wherever it is known.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 recalls Beckmann's potential formulation for Wardrop equilibria and derives the dual formulation used throughout the paper. Single-level reformulations are then introduced and analyzed in Section 4. Bounds and a reduction of the model size are discussed in Section 5, and computational experiments are reported in Section 6. Concluding remarks are given in Section 7.

2 Related works

Network pricing problems have been extensively studied in the literature, especially in the context of transportation and telecommunication networks; see [1, 18] for surveys covering a broad range of models and applications. A large part of the literature considers the case where arc costs are fixed, that is, independent of congestion effects. In this setting, the followers' problem reduces to a collection of independent shortest path computations, one for each origin-destination pair. When the leader seeks to maximize toll revenues, the resulting bilevel problem corresponds to the classical *network pricing problem*, investigated for instance in [6, 7, 11]. Complexity aspects of this problem have received considerable attention: [7] establishes asymmetry results in the complexity of the multi-commodity case, and polynomial special cases have been identified in [17].

The framework considered in the present paper differs fundamentally from these models because arc costs depend on the equilibrium flow. Consequently, the lower-level problem is no longer a collection of independent shortest path problems, but rather the computation of a Wardrop equilibrium, which couples all commodities through shared congestion effects.

Another classical line of research concerns congestion pricing and marginal-cost pricing. Under suitable assumptions on the arc cost functions, it is well known that marginal-cost tolls can induce a Wardrop equilibrium that minimizes the total travel cost,

$$\sum_{a \in A} f_a \, c_a(f_a),$$

thereby achieving a socially optimal outcome. Such results are classical in transportation science and algorithmic game theory [4, 14]. However, these approaches focus on achieving socially optimal equilibria through analytically derived tolls, whereas problem (BW) considers a general bilevel optimization framework in which the leader may pursue arbitrary objectives, including revenue maximization or system efficiency, subject to a feasible toll set $\mathcal{T}$. The structure of $\mathcal{T}$ itself has been the subject of dedicated studies: Harks et al. [10] investigate the impact of toll caps in privatized road networks, showing that restricting the toll space can significantly affect equilibrium outcomes and network performance.

Several works have investigated pricing models involving equilibrium constraints under assumptions that differ from those considered here. In [5], the authors consider fair resource allocation problems in capacitated networks, followed by a revenue optimization phase based on Lagrange multipliers associated with capacity constraints. Başar and Srikant [2] study a Stackelberg game under Nash equilibrium constraints, where followers correspond to finitely many users choosing among predetermined routes, and where congestion and pricing jointly influence user utilities. Dempe and Zemkoho [8] investigate theoretical properties and optimality conditions for bilevel road pricing problems with separable arc costs, providing a rigorous analytical foundation for this class of models.

To the best of our knowledge, the work of Beck et al. [3] is currently the only contribution proposing both single-level reformulations and computational experiments for a bilevel toll-setting problem with Wardrop equilibrium constraints in a setting directly comparable to ours. Notably, Beck et al. also

consider a robust variant of the problem under budgeted uncertainty on the travel costs, whereas the present paper focuses on the deterministic case. We build upon the affine separable structure of the arc cost functions to derive alternative exact single-level reformulations based on Beckmann's potential formulation [4] and duality arguments, introduce bounds, a model size reduction and provide a computational study of the resulting formulations.

3 Wardrop equilibria and Beckmann reformulation

In this section, we recall classical results on Wardrop equilibria that will be used throughout the paper. We first present Beckmann's convex formulation [4] for the computation of Wardrop equilibria under separable arc cost functions. We then specialize to the case of affine arc costs and derive the corresponding dual formulation, which plays a central role in the single-level reformulations introduced in Section 4.

3.1 Beckmann's formulation and KKT conditions

In the context of congested traffic networks, a flow is said to be a *Wardrop equilibrium* if, for each commodity $k \in \mathcal{K}$, flow is routed only along paths of minimum effective cost from o_k to s_k [19, 20]. In other words, no user can unilaterally reduce their travel cost by switching to a different route.

Given a flow $f \in \mathbb{R}^A_+$ and a toll vector $\tau \in \mathbb{R}^A_+$, the effective cost of path $p \in \mathcal{P}_k$ for commodity $k \in \mathcal{K}$ is defined as

$$\widehat{c}_{p,k}(f,\tau) = \sum_{a \in p} \widehat{c}_a(f_a, \tau_a),$$

and we denote by $\pi_k(f,\tau)$ the minimum effective cost of an (o_k, s_k)-path. For each commodity $k \in \mathcal{K}$ and arc $a \in A$, let $f_{p,k}$ denote the flow routed through path $p \in \mathcal{P}_k$, and $f_{a,k}$ the corresponding arc flow for commodity k. The total arc flow is then

$$f_a = \sum_{k \in \mathcal{K}} f_{a,k}, \qquad a \in A.$$

Throughout this section and the remainder of the paper, the letter f may refer either to the path-flow vector $(f_{p,k})_{k \in \mathcal{K},\, p \in \mathcal{P}_k}$, to the arc-flow vector $(f_a)_{a \in A}$, or to both simultaneously, with the intended meaning clear from context.

Using the notation above, a flow f is a Wardrop equilibrium with respect to a fixed toll vector $\tau \in \mathbb{R}^A_+$ if and only if it satisfies the following system:

$$\sum_{p \in \mathcal{P}_k} f_{p,k} = d_k \qquad k \in \mathcal{K} \tag{1a}$$

$$\widehat{c}_{p,k}(f,\tau) - \pi_k \geq 0 \qquad p \in \mathcal{P}_k,\ k \in \mathcal{K} \tag{1b}$$

$$f_{p,k}\left(\widehat{c}_{p,k}(f,\tau) - \pi_k\right) = 0 \qquad p \in \mathcal{P}_k,\ k \in \mathcal{K} \tag{1c}$$

$$f_{p,k} \geq 0 \qquad p \in \mathcal{P}_k,\ k \in \mathcal{K} \tag{1d}$$

$$\pi_k \geq 0 \qquad k \in \mathcal{K} \tag{1e}$$

Constraints (1a) ensure that all demands are fully satisfied. Constraints (1b) impose that, for each commodity $k \in \mathcal{K}$, every path in $\mathcal{P}_k$ has length at least π_k. Complementarity constraints (1c) ensure that only shortest paths may carry a strictly positive amount of flow. Constraints (1d)–(1e) enforce nonnegativity of path flows and shortest-path costs, respectively.

System (1) corresponds exactly to the KKT optimality conditions of Beckmann's formulation [4], given by the following convex program:

$$\min_f \ B(f,\tau) = \sum_{a \in A} \int_0^{f_a} \widehat{c}_a(x, \tau_a)\, dx \tag{2a}$$

$$\text{s.t.} \sum_{p \in \mathcal{P}_k} f_{p,k} = d_k \qquad k \in \mathcal{K} \tag{2b}$$

$$\sum_{k \in \mathcal{K}} \sum_{\substack{p \in \mathcal{P}_k : \\ a \in p}} f_{p,k} = f_a \qquad a \in A \tag{2c}$$

$$f_{p,k} \geq 0 \qquad k \in \mathcal{K}, p \in \mathcal{P}_k \tag{2d}$$

$$f_a \geq 0 \qquad a \in A \tag{2e}$$

Under the assumptions above, a Wardrop equilibrium always exists [4, 15]. Moreover, if the arc cost functions are strictly increasing, the equilibrium arc flow vector is unique [4]. When arc costs are only nondecreasing, multiple Wardrop equilibria may exist and may yield different values for the leader's objective in (BW). In this work, we adopt the standard *optimistic* assumption, meaning that among all lower-level optimal solutions, one maximizing the leader's objective is selected.

3.2 Affine arc cost functions and dual formulation

In this subsection, we specialize to the case of affine arc cost functions $c_a \colon \mathbb{R}_+ \to \mathbb{R}$ of the form

$$c_a(f_a) = \alpha_a \cdot f_a + \beta_a, \qquad (\alpha_a, \beta_a) \in \mathbb{R}^2_+,$$

which are nonnegative and nondecreasing by assumption. Under this structure, Beckmann's formulation (2) takes the following explicit form for a fixed toll vector $\tau \in \mathbb{R}^A_+$.

$$\min_{f \geq 0} \; B(f, \tau) = \sum_{a \in A} \left(\frac{\alpha_a}{2} \cdot f_a^2 + (\beta_a + \tau_a) \cdot f_a \right) \tag{3a}$$

$$\text{s.t.} \sum_{p \in \mathcal{P}_k} f_{p,k} = d_k, \qquad k \in \mathcal{K} \tag{3b}$$

$$\sum_{k \in \mathcal{K}} \sum_{\substack{p \in \mathcal{P}_k : \\ a \in p}} f_{p,k} - f_a \leq 0, \qquad a \in A \tag{3c}$$

Compared with (2), the flow conservation equality (2c) has been relaxed to inequality (3c). This relaxation is valid because, under nonnegative and nondecreasing arc costs, there always exists an optimal solution of (3) satisfying the equality at optimality: increasing f_a beyond the actual routed flow may only increase the objective.

We now derive the dual of (3) by forming the Lagrangian with multipliers $\pi_k \in \mathbb{R}$ for the demand constraints and $\lambda_a \geq 0$ for the inequalities (3c). The Lagrangian is given by

$$\mathcal{L}(f, \pi, \lambda) = B(f, \tau) + \sum_{k \in \mathcal{K}} \pi_k \cdot \left(\sum_{p \in \mathcal{P}_k} f_{p,k} - d_k \right) + \sum_{a \in A} \lambda_a \cdot \left(\sum_{k \in \mathcal{K}} \sum_{\substack{p \in \mathcal{P}_k : \\ a \in p}} f_{p,k} - f_a \right)$$

with $f \geq 0$, $\lambda \geq 0$.

The Lagrangian decomposes separably over arcs and commodities as follows:

$$\mathcal{L}(f, \pi, \lambda) = \sum_{a \in A} \mathcal{L}_{1,a}(f_a, \lambda_a) + \sum_{k \in \mathcal{K}} \mathcal{L}_{2,k}(f_{\cdot,k}, \pi_k, \lambda) - \sum_{k \in \mathcal{K}} \pi_k \cdot d_k$$

where $f_{\cdot,k} = (f_{p,k})_{p \in \mathcal{P}_k}$ denotes the path-flow vector for commodity k, and

$$\mathcal{L}_{1,a}(f_a, \lambda_a) = \frac{\alpha_a}{2} \cdot f_a^2 + (\beta_a + \tau_a - \lambda_a) \cdot f_a$$

$$\mathcal{L}_{2,k}\left(f_{\cdot,k},\pi_k,\lambda\right)=\sum_{p\in\mathcal{P}_k}\left(\pi_k+\left(\sum_{a\in p}\lambda_a\right)\right)\cdot f_{p,k}.$$

The dual function of the Lagrangian is defined by $g\left(\pi,\lambda\right)=\min_{f\geq 0}\mathcal{L}\left(f,\pi,\lambda\right)$. Let $A_0=\{a\in A\colon \alpha_a=0\}$ denote the set of arcs with constant cost function, and let $A_1=A\setminus A_0$. This partition arises naturally when minimizing the Lagrangian, as the two cases yield different expressions. Indeed, minimizing the Lagrangian term by term over $f\geq 0$, we obtain:

$$\min_{f_a\geq 0}\mathcal{L}_{1,a}\left(f_a,\lambda_a\right)=\begin{cases}0 \text{ if } \beta_a+\tau_a-\lambda_a\geq 0\\ -\infty \text{ otherwise}\end{cases}\qquad \text{if } a\in A_0$$

$$\min_{f_a\geq 0}\mathcal{L}_{1,a}\left(f_a,\lambda_a\right)=\begin{cases}0 \text{ if } \beta_a+\tau_a-\lambda_a\geq 0\\ -\frac{(\beta_a+\tau_a-\lambda_a)^2}{2\cdot\alpha_a} \text{ otherwise}\end{cases}\qquad \text{if } a\in A_1$$

$$\min_{f_{\cdot,k}\geq 0}\mathcal{L}_{2,k}\left(f_{\cdot,k},\pi_k,\lambda\right)=\begin{cases}0 \text{ if } -\pi_k\leq\sum_{a\in p}\lambda_a,\forall p\in\mathcal{P}_k\\ -\infty \text{ otherwise.}\end{cases}$$

We thus deduce, after the change of variables $\pi_k\leftarrow-\pi_k$, the dual problem

$$\max_{\lambda\geq 0,\pi}\sum_{k\in\mathcal{K}}\pi_k\cdot d_k-\sum_{a\in A_1}\frac{1}{2\cdot\alpha_a}\left[\min\left(0,\ \beta_a+\tau_a-\lambda_a\right)\right]^2 \tag{4a}$$

$$\text{s.t. } \pi_k\leq\sum_{a\in p}\lambda_a, \qquad k\in\mathcal{K},p\in\mathcal{P}_k, \tag{4b}$$

$$\beta_a+\tau_a-\lambda_a\geq 0 \qquad a\in A_0 \tag{4c}$$

Remark 1. Without loss of generality, one may restrict attention to optimal solutions (λ^*,π^*) of (4) satisfying $\beta_a+\tau_a-\lambda_a^*\leq 0$ for all $a\in A_1$. Indeed, if this condition is violated for some arc $\tilde{a}\in A_1$, then increasing $\lambda_{\tilde{a}}^*$ up to $\beta_{\tilde{a}}+\tau_{\tilde{a}}$ preserves feasibility and leaves the objective value unchanged, since $[\min(0,\beta_{\tilde{a}}+\tau_{\tilde{a}}-\lambda_{\tilde{a}})]^2=0$ throughout this range; repeating this for every such arc yields an optimal solution with the required property.

In light of Remark 1, the objective of (4) simplifies, yielding the equivalent dual formulation:

$$\max_{\lambda\geq 0,\pi}\sum_{k\in\mathcal{K}}\pi_k\cdot d_k-\sum_{a\in A_1}\frac{1}{2\cdot\alpha_a}\left(\beta_a+\tau_a-\lambda_a\right)^2 \tag{5a}$$

$$\text{s.t. } \pi_k\leq\sum_{a\in p}\lambda_a, \qquad k\in\mathcal{K},p\in\mathcal{P}_k \tag{5b}$$

$$\beta_a+\tau_a-\lambda_a\geq 0 \qquad a\in A_0. \tag{5c}$$

The strong duality between (3) and (5) stated in the next proposition will play a central role in the single-level reformulations derived in Section 4.

Proposition 1. *Strong duality holds between* (3) *and* (5).

Proof. By assumption, $\mathcal{P}_k\neq\emptyset$ for all $k\in\mathcal{K}$, so the feasible set of (3) is nonempty. Although the feasible set is unbounded due to the relaxation of (2c) to an inequality, an optimal solution exists that satisfies the latter with equality, using the fact that the objective $B(f,\tau)$ is nondecreasing in f_a. Therefore, we may restrict the minimization to the set

$$\mathcal{F}=\left\{f\geq 0\colon \text{(3b)},\ \text{(3c)},\ f_a\leq\sum_{k\in\mathcal{K}}d_k,\ a\in A\right\},$$

which is nonempty, compact, and polyhedral. Since $B(f,\tau)$ is continuous, it attains its minimum over $\mathcal{F}$, and an optimal primal solution $f^\star$ exists. We consider such an optimal solution satisfying (3c) with equality.

Define the candidate dual solution by

$$\lambda_a^\star = \widehat{c}_a(f_a^\star, \tau_a) = \alpha_a f_a^\star + \beta_a + \tau_a, \quad a \in A, \tag{6a}$$

$$\pi_k^\star = \min_{p \in \mathcal{P}_k} \left\{ \sum_{a \in p} \widehat{c}_a(f_a^\star, \tau_a) \right\}, \quad k \in \mathcal{K}. \tag{6b}$$

We show that $(\lambda^\star, \pi^\star)$ is feasible for (5) and that the primal and dual objective values coincide, thus establishing strong duality.

Dual feasibility. Since arc costs are nonnegative, $\lambda_a^\star = \widehat{c}_a(f_a^\star, \tau_a) \geq 0$ for all $a \in A$. For each $k \in \mathcal{K}$ and $p \in \mathcal{P}_k$,

$$\sum_{a \in p} \lambda_a^\star = \sum_{a \in p} \widehat{c}_a(f_a^\star, \tau_a) = \widehat{c}_{p,k}(f^\star, \tau) \geq \pi_k^\star,$$

so constraint (5b) is satisfied. For arcs $a \in A_0$, we have $\alpha_a = 0$, hence $\lambda_a^\star = \beta_a + \tau_a$, so the constraint $\beta_a + \tau_a - \lambda_a^\star \geq 0$ holds with equality. Thus $(\lambda^\star, \pi^\star)$ is feasible for (5).

Equality of objective values. For $a \in A_1$, since $\lambda_a^\star = \alpha_a f_a^\star + \beta_a + \tau_a$, we have $(\beta_a + \tau_a) f_a^\star = \lambda_a^\star f_a^\star - \alpha_a (f_a^\star)^2$, while for $a \in A_0$, $\alpha_a = 0$ gives $(\beta_a + \tau_a) f_a^\star = \lambda_a^\star f_a^\star$. Therefore the primal objective expands as

$$\begin{aligned} B(f^\star, \tau) &= \sum_{a \in A} \left(\frac{\alpha_a}{2} (f_a^\star)^2 + (\beta_a + \tau_a) f_a^\star \right) \\ &= \sum_{a \in A} \lambda_a^\star f_a^\star - \sum_{a \in A_1} \frac{\alpha_a}{2} (f_a^\star)^2 \\ &= \sum_{a \in A} \lambda_a^\star f_a^\star - \sum_{a \in A_1} \frac{(\lambda_a^\star - \beta_a - \tau_a)^2}{2\alpha_a}. \end{aligned}$$

It remains to show that $\sum_{a \in A} \lambda_a^\star f_a^\star = \sum_{k \in \mathcal{K}} \pi_k^\star d_k$. Using the Wardrop complementarity conditions (1c), which state that $f_{p,k}^\star > 0$ only if $\widehat{c}_{p,k}(f^\star, \tau) = \pi_k^\star$, together with the demand constraints (1a), we obtain

$$\begin{aligned} \sum_{k \in \mathcal{K}} \pi_k^\star d_k &= \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}_k} \pi_k^\star f_{p,k}^\star = \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}_k} \widehat{c}_{p,k}(f^\star, \tau) \cdot f_{p,k}^\star \\ &= \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}_k} \sum_{a \in p} \widehat{c}_a(f_a^\star, \tau_a) \cdot f_{p,k}^\star = \sum_{a \in A} \lambda_a^\star \sum_{k \in \mathcal{K}} \sum_{\substack{p \in \mathcal{P}_k : \\ a \in p}} f_{p,k}^\star = \sum_{a \in A} \lambda_a^\star f_a^\star, \end{aligned}$$

where the last equality uses the fact that $f_a^\star = \sum_{k \in \mathcal{K}} \sum_{\substack{p \in \mathcal{P}_k : \\ a \in p}} f_{p,k}^\star$. Hence $B(f^\star, \tau)$ equals the objective value of (5) at $(\lambda^\star, \pi^\star)$, which, combined with weak duality, proves the claim. □

The following corollary is an immediate consequence of Proposition 1 and its proof.

Corollary 1. *Given an optimal solution $f^\star$ of the primal problem (3), the pair $(\lambda^\star, \pi^\star)$ defined by (6) is an optimal solution of the dual problem (5). Moreover, the dual optimal solution admits the following interpretation: $\lambda_a^\star$ equals the effective arc cost at equilibrium, and $\pi_k^\star$ equals the shortest-path cost for commodity k under the effective arc costs induced by $f^\star$.*

For affine nondecreasing arc cost functions, a Wardrop equilibrium can be computed by the Frank–Wolfe algorithm [15] or by convex quadratic programming. We adopt the latter and solve Beckmann's problem (3) directly. In our implementation, the exponentially many path-flow variables $(f_{p,k})$ of (3) are replaced by the equivalent arc-commodity variables $(f_{a,k})$ together with the associated flow conservation constraints, which yields a formulation of polynomial size.

The primal and dual formulations derived in this section provide the foundation for the single-level reformulations of problem (BW) developed in the next section.

4 Single-level reformulations of problem (BW)

In this section, we derive several single-level reformulations of problem (BW) in the affine separable case, focusing on the profit-maximization objective $F(f,\tau) = \sum_{a\in A} f_a\tau_a$, which is the objective considered in [3] and adopted throughout the remainder of this paper. We first recall the mixed-integer nonlinear programming formulation recently proposed by Beck et al. [3] (Section 4.1), which serves as a benchmark in our computational experiments. We then introduce three new single-level reformulations. The first two exploit the primal-dual structure of Wardrop equilibria established in Section 3: enforcing primal feasibility, dual feasibility, and strong duality simultaneously eliminates the lower-level equilibrium problem and yields an equivalent single-level program. An arc-flow formulation and a path-flow formulation are derived in this way, differing in the choice of flow variables and the structure of the resulting constraints. The third reformulation takes a different route, directly encoding the Wardrop equilibrium conditions through the complementarity system (1), leading to a formulation that can be handled through SOS1 constraints in modern mixed-integer optimization solvers.

4.1 MINLP reformulation proposed by Beck et al. [3]

Beck et al. [3] propose the following mixed-integer nonlinear single-level reformulation of the toll-setting problem. The formulation introduces, for each commodity $k \in \mathcal{K}$ and arc $a \in A$, a binary variable $z_{a,k}$ indicating whether arc a may belong to a shortest path used by commodity k, and a variable $t_{i,k}$ representing the shortest-path distance from node i to the destination s_k for commodity k.

$$\max_{\tau,f,t,z} \sum_{a\in A} f_a \cdot \tau_a \tag{7a}$$

$$\text{s.t. } Nf_{\cdot,k} = b_k, \qquad k \in \mathcal{K} \tag{7b}$$

$$\sum_{k\in\mathcal{K}} f_{a,k} = f_a, \qquad a \in A \tag{7c}$$

$$t_{i,k} - t_{j,k} \le \alpha_a \cdot f_a + \beta_a + \tau_a \qquad a=(i,j)\in A,\ k\in\mathcal{K} \tag{7d}$$

$$t_{i,k} - t_{j,k} \ge \alpha_a \cdot f_a + \beta_a + \tau_a - M_{a,k}\,(1 - z_{a,k}) \qquad a=(i,j)\in A,\ k\in\mathcal{K} \tag{7e}$$

$$t_{s_k,k} = 0 \qquad k \in \mathcal{K} \tag{7f}$$

$$f_{a,k} \le d_k \cdot z_{a,k} \qquad a \in A,\ k \in \mathcal{K} \tag{7g}$$

$$\tau \in \mathcal{T},\ f \ge 0,\ t \ge 0 \tag{7h}$$

$$z_{a,k} \in \{0,1\} \qquad a \in A, k \in \mathcal{K} \tag{7i}$$

where $N \in \mathbb{R}^{V\times A}$ is the node-arc incidence matrix of G, and $b_k \in \mathbb{R}^V$ is the demand vector for commodity k, with $(b_k)_{o_k} = d_k$, $(b_k)_{s_k} = -d_k$, and all other entries equal to zero. The constants $M_{a,k}$ are big-M parameters, for which explicit sufficiently large values are provided in [3]. Constraints (7d)–(7i) ensure that positive flow for commodity k can only be routed along shortest (o_k, s_k)-paths.

Variable bounds and big-M constants

Due to the nonnegativity of the costs on the arcs, problem (7) admits an optimal solution routing no flow on circuits, which allows several variables to be bounded or fixed without loss of optimality. We record here the bounds used in the sequel and valid inequalities additionally proposed by Beck et al. [3].

Flow entering an origin or leaving a destination may be set to zero:

$$f_{a,k} = 0, \qquad a \in \delta^-(o_k) \cup \delta^+(s_k),\ k \in \mathcal{K}, \tag{8}$$

where $\delta^-(v)$ (resp. $\delta^+(v)$) denotes the set of arcs having node v as terminal (resp. initial) endpoint. Combining (8) with the demand bound $f_{a,k} \le d_k$ implied by (7g) (since $z_{a,k} \le 1$), the equilibrium arc flow on every arc $a \in A$ satisfies

$$0 \le f_a \le f_a^{\mathtt{up}} := \sum_{\substack{k\in\mathcal{K}:\\ a\notin\delta^-(o_k)\cup\delta^+(s_k)}} d_k. \tag{9}$$

The quantity $f_a^{\mathtt{up}}$ serves throughout as the reference arc-flow upper bound, both in (7) and in the other formulations of Section 4. These bounds also yield a finite upper bound on the shortest-path variables: telescoping (7d) along any elementary (i, s_k)-path and using $t_{s_k,k} = 0$ together with $f_a \le f_a^{\mathtt{up}}$ and $\tau_a \le \tau_a^{\mathtt{up}} := \max_{\tau \in \mathcal{T}} \tau_a$, one obtains, for every commodity $k \in \mathcal{K}$ and node $i \in V$,

$$0 \le t_{i,k} \le \min_{p \in \mathcal{P}_i^k} \sum_{a \in p} \left(\alpha_a f_a^{\mathtt{up}} + \beta_a + \tau_a^{\mathtt{up}}\right), \tag{10}$$

where $\mathcal{P}_i^k$ denotes the set of elementary (i, s_k)-paths in G. These bounds are used to set the big-M constants $M_{a,k}$ in (7), similar to [3].

Valid inequalities for the routing binary variables

Beck et al. [3] suggest to add two types of inequalities to (7), to improve computational performance. The first is satisfied by every optimal solution, the second holds without loss of optimality. We report them hereafter. For each pair of reverse arcs $(i, j), (j, i) \in A$, every optimal solution satisfies

$$z_{(i,j),k} + z_{(j,i),k} \le 1, \qquad k \in \mathcal{K}. \tag{11}$$

Moreover, there exists an optimal solution satisfying

$$\tau_a \ge \tau_a^{\mathtt{up}} \Big(1 - \sum_{k \in \mathcal{K}} z_{a,k}\Big), \qquad a \in A, \tag{12}$$

i.e., an arc that no commodity routes along a shortest path (so $z_{a,k} = 0$ for all k) is assigned the maximal toll $\tau_a^{\mathtt{up}}$.

4.2 Strong duality based formulations

The strong duality result established in Proposition 1 provides a natural route to eliminate the lower-level equilibrium problem: the lower-level constraint $f \in \mathcal{W}(\tau)$ is equivalently reformulated by jointly imposing primal feasibility of (3), dual feasibility of (5), and strong duality, yielding an equivalent single-level reformulation. We present two variants of this approach, differing in their choice of flow variables: the first relies on arc-flow variables and flow-conservation constraints, while the second additionally introduces path-flow variables, leading to a formulation that is structurally closer to the original Wardrop conditions.

4.2.1 Arc-based formulation

Applying Proposition 1 to problem (BW) and expressing flow conservation through arc-flow variables yields the following single-level reformulation.

$$\max_{\tau, f, \pi, \lambda} \sum_{a \in A} f_a \cdot \tau_a \tag{13a}$$

$$\text{s.t. } N f^k = b^k, \qquad k \in \mathcal{K} \tag{13b}$$

$$\sum_{k \in \mathcal{K}} f_{a,k} = f_a, \qquad a \in A \tag{13c}$$

$$\sum_{a \in A} \Big(\frac{\alpha_a}{2} \cdot f_a^2 + (\beta_a + \tau_a) \cdot f_a\Big) \le \sum_{k \in \mathcal{K}} \pi_k \cdot d_k - \sum_{a \in A_1} \frac{1}{2 \cdot \alpha_a} (\beta_a + \tau_a - \lambda_a)^2 \tag{13d}$$

$$\pi_k \le \sum_{a \in p} \lambda_a, \qquad k \in \mathcal{K}, p \in \mathcal{P}_k, \tag{13e}$$

$$\beta_a + \tau_a - \lambda_a \geq 0 \qquad a \in A_0 \tag{13f}$$
$$0 \leq f_{a,k} \leq d_k \qquad k \in \mathcal{K}, a \in A \tag{13g}$$
$$\tau \in \mathcal{T}, \lambda \geq 0 \tag{13h}$$

Corollary 1 shows that, at optimality, the dual variables λ_a coincide with the corresponding equilibrium arc costs $\alpha_a f_a + \beta_a + \tau_a$. These variables can therefore be eliminated from formulation (13) through the substitutions

$$\lambda_a = \alpha_a f_a + \beta_a + \tau_a, \qquad a \in A,$$

leading to the more compact formulation (14).

$$\max_{\tau,f,\pi,t} \; t \tag{14a}$$
$$\text{s.t. } Nf^k = b^k, \qquad k \in \mathcal{K} \tag{14b}$$
$$\sum_{k\in\mathcal{K}} f_{a,k} = f_a, \qquad a \in A \tag{14c}$$
$$t = \sum_{a\in A} f_a \cdot \tau_a \tag{14d}$$
$$t + \sum_{a\in A} \left(\alpha_a \cdot f_a^2 + \beta_a \cdot f_a\right) \leq \sum_{k\in\mathcal{K}} \pi_k \cdot d_k \tag{14e}$$
$$\pi_k \leq \sum_{a\in p} (\alpha_a \cdot f_a + \beta_a + \tau_a), \qquad k \in \mathcal{K}, p \in \mathcal{P}_k, \tag{14f}$$
$$0 \leq f_{a,k} \leq d_k \qquad k \in \mathcal{K}, a \in A \tag{14g}$$
$$\tau \in \mathcal{T}, t \geq 0 \tag{14h}$$

A notable structural feature of formulation (14) is that all nonconvexity is concentrated in the single bilinear constraint (14d). The remaining constraints are either linear or convex quadratic (constraint (14e)).

4.2.2 Path-based formulation

An alternative formulation is obtained by replacing the arc-flow conservation constraints by a path-flow representation. Although the number of paths may be exponentially large, this formulation can serve as the basis for a column-and-constraint generation approach: one starts with a restricted set of paths and iteratively adds new paths together with their associated constraints, with the aim of identifying high-quality solutions while keeping the size of the formulation manageable. The performance of such an approach will be discussed in Section 6.

$$\max_{\tau,f,\pi,t} \; t \tag{15a}$$
$$\text{s.t. } \sum_{p\in\mathcal{P}_k} f_{p,k} = d_k, \qquad k \in \mathcal{K} \tag{15b}$$
$$\sum_{k\in\mathcal{K}} \sum_{\substack{p\in\mathcal{P}_k:\\ a\in p}} f_{p,k} - f_a = 0, \qquad a \in A \tag{15c}$$
$$t = \sum_{a\in A} f_a \cdot \tau_a \tag{15d}$$
$$t + \sum_{a\in A} \left(\alpha_a \cdot f_a^2 + \beta_a \cdot f_a\right) \leq \sum_{k\in\mathcal{K}} \pi_k \cdot d_k \tag{15e}$$
$$\pi_k \leq \sum_{a\in p} (\alpha_a \cdot f_a + \beta_a + \tau_a), \qquad k \in \mathcal{K}, p \in \mathcal{P}_k, \tag{15f}$$
$$\tau \in \mathcal{T}, f \geq 0, t \geq 0 \tag{15g}$$

4.3 Complementarity-based formulation

The previous formulations exploit Beckmann's formulation and strong duality to eliminate the lower-level problem. An alternative approach consists in directly enforcing the Wardrop equilibrium conditions through the complementarity system (1). Recall that a Special Ordered Set of type 1 (SOS1) constraint imposed on a pair of variables (x, y) enforces that at most one of the two variables is nonzero, which is precisely the structure of the complementarity constraints (16e). SOS1 constraints are supported by several modern mixed-integer optimization solvers.

$$\max_{\tau,f,\pi} \sum_{a\in A} f_a \cdot \tau_a \tag{16a}$$

$$\text{s.t.} \sum_{p\in\mathcal{P}_k} f_{p,k} = d_k \qquad k \in \mathcal{K} \tag{16b}$$

$$\sum_{k\in\mathcal{K}} \sum_{\substack{p\in\mathcal{P}_k:\\ a\in p}} f_{p,k} - f_a = 0, \qquad a \in A \tag{16c}$$

$$\widehat{c}_{p,k}(f,\tau) - \pi_k \geq 0 \qquad p \in \mathcal{P}_k,\ k \in \mathcal{K} \tag{16d}$$

$$f_{p,k}\left(\widehat{c}_{p,k}(f,\tau) - \pi_k\right) = 0 \qquad p \in \mathcal{P}_k,\ k \in \mathcal{K} \tag{16e}$$

$$\pi_k \geq 0 \qquad k \in \mathcal{K} \tag{16f}$$

$$\tau \in \mathcal{T}, f \geq 0 \tag{16g}$$

5 Bounds and size reduction

The formulations introduced in Section 4 can be tightened with bounds that exploit structural properties of the toll-setting problem. When the data permit, they can also be reduced in size. This section establishes three distinct families of bounds, and one model size reduction.

The first family of bounds (Section 5.1) is concerned with bounding the toll set $\mathcal{T}$. In Beck et al. [3], explicit upper bounds on toll values are assumed, which may be interpreted as regulatory restrictions but are otherwise arbitrary. We show that, under the common assumption that every commodity admits a toll-free alternative path (e.g., [6, 7, 11]) — interpretable as a route offered by a competitor of the leader — finite upper bounds on toll values can be derived endogenously, without any arbitrary choice. Moreover, as our computational experiments will demonstrate, the magnitude of these bounds can have a significant impact on computational performance.

The second (Section 5.2) concerns default bounds on the variables appearing in the formulations of Section 4. These bounds are either valid inequalities, satisfied by every feasible solution, or constraints that can be added without loss of optimality, in the sense that at least one optimal solution satisfies them. In both cases they may improve the quality of continuous relaxations and thereby accelerate the solution process. They are added to the new formulations (i.e. the ones of Sections 4.2 and 4.3) and kept active throughout our computational experiments.

The third family (Section 5.3) is of a different nature: an upper bound on the *optimal value* $z^\star$ of (BW), derived from Beckmann's formulation. It can be used to certify optimality gaps but unlike the previous two it is not added to any formulation in our computational experiments.

Lastly, we show (in Section 5.4) how to reduce the model size for formulations involving per-commodity arc-flow variables ($f_{a,k}$). When the arc costs are independent of the commodity, commodities sharing a common destination can be merged without affecting the set of optimal solutions. This grouping reduces the number of per-commodity arc-flow variables and of the associated coupling constraints.

5.1 Implicit bounds induced by toll-free alternatives

We now show that the toll-free path assumption introduced above can be used to derive upper bounds on toll values. Let $A^{\texttt{toll}} \subseteq A$ denote the set of toll arcs, i.e., arcs $a \in A$ for which $\tau_a^{\texttt{up}} = \max_{\tau \in \mathcal{T}} \tau_a > 0$, and let $A \setminus A^{\texttt{toll}}$ denote the set of toll-free arcs. For each commodity $k \in \mathcal{K}$, let $\mathcal{P}_k^0 \subseteq \mathcal{P}_k$ denote the set of (o_k, s_k)-paths containing no toll arc, and assume that $\mathcal{P}_k^0 \neq \emptyset$ for all $k \in \mathcal{K}$.

For each commodity k, define

$$\pi_k^{\infty} = \min_{p \in \mathcal{P}_k^0} \sum_{a \in p} (\alpha_a f_a^{\texttt{up}} + \beta_a),$$

where $f_a^{\texttt{up}}$ is the arc-flow upper bound defined in (9).

Finally, let

$$\tau^{\infty} = \max_{k \in \mathcal{K}} \pi_k^{\infty}.$$

Proposition 2. *Consider problem (BW) with the profit-maximization objective $F(f, \tau) = \sum_{a \in A} f_a \tau_a$. Under the assumption that $\mathcal{P}_k^0 \neq \emptyset$ for every commodity $k \in \mathcal{K}$, problem (BW) admits an optimal solution (τ, f) whose toll vector satisfies*

$$\tau_a \leq \tau_a^{\infty} := \max(0, \tau^{\infty} - \beta_a), \qquad a \in A^{\texttt{toll}}.$$

In particular, the toll set $\mathcal{T}$ can be assumed to be bounded without loss of optimality.

Proof. There exists an optimal solution $(\tau^\star, f^\star)$ of (BW) satisfying

$$f_a^\star \leq f_a^{\texttt{up}}, \qquad a \in A.$$

Indeed, if $f^\star$ carries positive flow along a cycle of zero effective cost, then, since $\alpha_a, \beta_a, f_a^\star, \tau_a^\star \geq 0$ for every arc a on the cycle, the effective cost $\alpha_a f_a^\star + \beta_a + \tau_a^\star = 0$ forces $\tau_a^\star = 0$ on every such arc. Removing the flow along the cycle therefore preserves both the Wardrop equilibrium conditions (1) and the leader's objective value. Repeating this operation for every such cycle yields an optimal solution whose per-commodity flows decompose into elementary (o_k, s_k)-path flows only, and thus satisfy the zero-flow conditions (8). Summing over commodities, each routing exactly d_k units, gives $f_a^\star \leq f_a^{\texttt{up}}$ as defined in (9). We henceforth fix such an optimal solution $(\tau^\star, f^\star)$.

Hence, for every non-toll arc a,

$$c_a(f_a^\star) = \alpha_a f_a^\star + \beta_a \leq \alpha_a f_a^{\texttt{up}} + \beta_a.$$

It follows that, for every commodity $k \in \mathcal{K}$, the cheapest toll-free (o_k, s_k)-path has effective cost at most π_k^{∞} under $\tau^\star$, and hence the shortest-path cost satisfies $\pi_k^\star \leq \pi_k^{\infty} \leq \tau^{\infty}$.

Now suppose that some toll arc $a \in A^{\texttt{toll}}$ satisfies $\tau_a^\star > \max(0, \tau^{\infty} - \beta_a)$. Since arc costs are nonnegative, the effective cost of any (o_k, s_k)-path p containing arc a satisfies

$$\widehat{c}_{p,k}(f^\star, \tau^\star) \geq \beta_a + \tau_a^\star > \max(\beta_a, \tau^{\infty}) \geq \tau^{\infty} \geq \pi_k^\star$$

for every commodity k. Hence every path containing arc a is strictly more expensive than the shortest-path cost, so the Wardrop conditions imply $f_a^\star = 0$.

Define τ^{new} by setting $\tau_a^{\text{new}} = \max(0, \tau^{\infty} - \beta_a)$ and $\tau_{a'}^{\text{new}} = \tau_{a'}^\star$ for all $a' \neq a$. Since $0 \leq \tau_a^{\text{new}} \leq \tau_a^\star$ and $\mathcal{T}$ is downward closed, $\tau^{\text{new}} \in \mathcal{T}$. We show that $(\tau^{\text{new}}, f^\star)$ is feasible for (BW) and achieves the same objective value as $(\tau^\star, f^\star)$.

Since only τ_a was decreased, the cost of every path not containing a is unchanged, while the cost of every path containing a can only decrease. In particular, every path $p \in \mathcal{P}_k$ with $f_{p,k}^\star > 0$ avoids

arc a (since $f_a^\star = 0$) and thus retains its cost $\pi_k^\star$. Moreover, every path containing a has cost at least $\beta_a + \tau_a^{\text{new}} = \max(\beta_a, \tau^\infty) \geq \pi_k^\star$ under τ^{new}. Hence $\pi_k^\star$ remains the shortest-path cost under τ^{new}, and $f^\star$ still satisfies the Wardrop conditions (1), so that $f^\star \in \mathcal{W}(\tau^{\text{new}})$.

Since $f_a^\star = 0$, the leader's objective satisfies $F(f^\star, \tau^{\text{new}}) = \sum_{a' \in A} f_{a'}^\star \tau_{a'}^{\text{new}} = \sum_{a' \in A} f_{a'}^\star \tau_{a'}^\star = F(f^\star, \tau^\star)$. Therefore $(\tau^{\text{new}}, f^\star)$ is a feasible solution of (BW) achieving the optimal value, and is thus itself optimal.

Repeating this operation for every toll arc violating the bound yields an optimal solution satisfying

$$\tau_a \leq \max\left(0,\, \tau^\infty - \beta_a\right), \qquad a \in A^{\texttt{toll}}.$$

This proves the result. □

Remark 2. The argument above does not use the optimality of $(\tau^\star, f^\star)$: applied to any feasible solution (τ, f) of (BW), it yields a feasible solution with the same objective value whose tolls satisfy $\tau_a \leq \tau_a^\infty$ for all $a \in A^{\texttt{toll}}$. Hence, under the assumption that $\mathcal{P}_k^0 \neq \emptyset$ for every $k \in \mathcal{K}$, the compactness assumption on $\mathcal{T}$ made in the introduction can be relaxed: when $\mathcal{T}$ is closed, convex and downward closed but not necessarily bounded, the supremum of (BW) over $\mathcal{T}$ equals its supremum over the compact set $\mathcal{T} \cap \{\tau \colon \tau_a \leq \tau_a^\infty,\ a \in A^{\texttt{toll}}\}$.

5.2 Default variable bounds

Throughout this subsection and Section 5.3, we assume that the toll set $\mathcal{T}$ is bounded, so that $\tau_a^{\texttt{up}} := \max_{\tau \in \mathcal{T}} \tau_a$ is finite for every $a \in A$. This entails no loss of optimality: by Proposition 2, an unbounded toll set can be replaced, without affecting the optimal value, by its intersection with the bounds $\tau_a \leq \max(0, \tau^\infty - \beta_a)$. We also use the arc-flow upper bound $f_a^{\texttt{up}}$ defined in (9), which holds without loss of optimality (see the proof of Proposition 2).

All formulations of Section 4 that involve commodity-specific arc-flow variables are equipped, by default, with the native bounds of the formulation (7) of [3]. First, the zero-flow conditions (8) are imposed for every commodity. Second, for every arc $a \in A$ and commodity $k \in \mathcal{K}$,

$$0 \leq f_{a,k} \leq d_k. \tag{17}$$

Together with (8), these bounds yield the arc-flow bound $0 \leq f_a \leq f_a^{\texttt{up}}$ of (9).

Remark 3. Tighter upper bounds on the arc flows could in principle be computed by maximizing f_a over the feasible set of (7) for each arc $a \in A$. Since the objective $\sum_{a \in A} f_a \tau_a$ is then replaced by the linear objective f_a, each such subproblem is a mixed-integer *linear* program. Preliminary experiments, however, showed that the computational cost of solving these subproblems outweighs the benefit of the resulting tighter bounds, especially on the larger instances. We therefore do not pursue this approach in our computational study.

The shortest-path variables π_k are bounded by

$$0 \leq \pi_k \leq \min_{p \in \mathcal{P}_k} \sum_{a \in p} \left(\alpha_a f_a^{\texttt{up}} + \beta_a + \tau_a^{\texttt{up}}\right), \qquad k \in \mathcal{K}, \tag{18}$$

the right-hand side upper-bounding the effective cost of the shortest (o_k, s_k)-path obtained by replacing each arc flow and toll by $f_a^{\texttt{up}}$ and $\tau_a^{\texttt{up}}$, respectively. Inequality (18) plays, for the shortest-path value π_k, the role that the node-potential bounds (10) on the variables $t_{i,k}$ play in (BLS). Like the flow bounds above, these bounds are imposed in every formulation carrying the corresponding variables for all our computational experiments.

5.3 A Beckmann-based upper bound on the optimal value

The convex structure of Beckmann's formulation yields an upper bound on the optimal value $z^\star$ of (BW). Unlike the bounds of Section 5.2, this bound is not added to the formulations in our computational experiments; it is used only to certify optimality gaps.

Let $\Phi(\tau)$ denote the optimal objective value of problem (3) for a fixed toll vector τ. Since the objective $B(f,\tau)$ is nondecreasing in τ for every fixed $f \geq 0$, the value function Φ is nondecreasing as well:

$$\tau^1 \leq \tau^2 \quad \Longrightarrow \quad \Phi(\tau^1) \leq \Phi(\tau^2).$$

In any feasible solution (τ, f) of (BW), the flow f is a Wardrop equilibrium and therefore minimizes $B(\cdot,\tau)$, so that $B(f,\tau) = \Phi(\tau)$. Combined with the monotonicity of Φ and the upper bound $\tau \leq \tau^{\mathtt{up}} := (\tau_a^{\mathtt{up}})_{a\in A}$, this yields the valid inequality

$$B(f,\tau) \;=\; \Phi(\tau) \;\leq\; \Phi(\tau^{\mathtt{up}}). \tag{19}$$

For formulation (14), inequality (19) can be rewritten as

$$t + \sum_{a\in A}\left(\frac{\alpha_a}{2} f_a^2 + \beta_a f_a\right) \leq \Phi(\tau^{\mathtt{up}}). \tag{20}$$

Moreover, since f is feasible for problem (3) with zero tolls, we have

$$\Phi(0) \leq B(f,0) = \sum_{a\in A}\left(\frac{\alpha_a}{2} f_a^2 + \beta_a f_a\right),$$

and we deduce the following upper bound on the leader's optimal objective value $z^\star$:

$$z^\star \;\leq\; \overline{z}_{\mathtt{B}} \;:=\; \Phi(\tau^{\mathtt{up}}) - \Phi(0). \tag{21}$$

Remark 4. The bound (21) refines the elementary product bound induced by (9) alone. Indeed, since $\tau_a \leq \tau_a^{\mathtt{up}}$ and $f_a \leq f_a^{\mathtt{up}}$ at every feasible point of (BW), the leader's revenue satisfies

$$\sum_{a\in A} f_a \tau_a \;\leq\; \sum_{a\in A} f_a^{\mathtt{up}}\, \tau_a^{\mathtt{up}}.$$

The Beckmann-based bound (21) is always at least as tight as this, and is typically much tighter. To see this, let f^0 be a Wardrop equilibrium at zero tolls. Being a pointwise minimum over f of the maps $\tau \mapsto B(f,\tau)$, each affine in τ, the value function Φ is concave, and any such f^0 is a supergradient of Φ at $\tau = 0$ (since $\nabla_\tau B(f,\tau) = f$). The resulting first-order inequality $\Phi(\tau^{\mathtt{up}}) \leq \Phi(0) + \sum_{a\in A} f_a^0\, \tau_a^{\mathtt{up}}$ gives

$$\Phi(\tau^{\mathtt{up}}) - \Phi(0) \;\leq\; \sum_{a\in A} f_a^0\, \tau_a^{\mathtt{up}} \;\leq\; \sum_{a\in A} f_a^{\mathtt{up}}\, \tau_a^{\mathtt{up}},$$

the last inequality following from $f_a^0 \leq f_a^{\mathtt{up}}$. The product bound treats flow and toll as independently maximizable on each arc and thus ignores the equilibrium response: it grows linearly in the magnitude of the toll bounds, whereas the value of (21) remains bounded, since raising the tolls only diverts flow onto the toll-free paths guaranteed by $\mathcal{P}_k^0 \neq \emptyset$.

5.4 Grouping commodities by destination

The arc-based formulations of Section 4 carry one flow variable $f_{a,k}$ per arc a and commodity k. When the arc costs do not depend on the commodity—as is the case in [3] and our computational experiments—commodities sharing a common destination can be merged without affecting the set of optimal

solutions, which reduces the number of flow variables and of the associated coupling constraints. We therefore group commodities by destination: a grouped commodity k has a single destination s_k and a set of origins O_k, its demand vector b^k having entries $(b^k)_v = d_{k,v}$ for $v \in V \setminus \{s_k\}$ and $(b^k)_{s_k} = -d_k$, writing $d_{k,v}$ for the demand of commodity k issued from node v (with $d_{k,v} = 0$ when $v \notin O_k$ and $d_{k,v} > 0$ otherwise), and $d_k = \sum_{o \in O_k} d_{k,o}$ the total demand.

The default variable bounds of Section 5.2 can be adjusted to this grouping setting. The destination-side condition of (8) (no flow leaves s_k) remains valid for every grouped commodity. The origin-side condition (no flow enters the origin) can no longer be imposed verbatim once a commodity has several origins, since flow may enter one origin to serve a demand of the group issued from another node. It is replaced by the following bound: every arc $a = (u, v)$ satisfies

$$f_{a,k} \;\le\; \begin{cases} 0, & u = s_k, \\ d_k - d_{k,v}, & u \neq s_k, \end{cases} \qquad k \in \mathcal{K}. \tag{22}$$

Indeed, by the argument in the proof of Proposition 2, problem (BW) admits an optimal solution whose flow is a Wardrop equilibrium whose per-commodity flows decompose into elementary (o_k, s_k)-path flows. For a single-origin commodity, (22) recovers the origin-side condition of (8) and the bound $f_{a,k} \le d_k$; for a multi-origin commodity it tightens the latter to $d_k - d_{k,v}$ whenever v is one of its origins. Summing (22) over commodities gives the arc-flow bound (23), the multi-origin counterpart of (9); the derived bounds (10) and (18) are then evaluated from this expression of $f_a^{\mathtt{up}}$.

$$f_a \;\le\; f_a^{\mathtt{up}} \;=\; \sum_{k \in \mathcal{K}\,:\; s_k \neq u} \big(d_k - d_{k,v}\big), \qquad a = (u, v). \tag{23}$$

6 Computational experiments

6.1 Experimental setup

All experiments were carried out on a machine equipped with a 13th Gen Intel Core™ i7-1355U processor (1.70 GHz) and 32 GB of RAM, running a 64-bit operating system. The formulations were solved with Gurobi 13.0.2 [9], exploiting its ability to handle convex quadratic constraints as well as nonconvex bilinear terms; the latter is enabled by setting the parameter `NonConvex` to 2. Gurobi solves the resulting nonconvex quadratically constrained problems using a spatial branch-and-bound framework combined with outer approximation techniques. All other parameters are left at their default values. Because the i7-1355U has a hybrid architecture (two performance and eight efficiency cores), we fix the number of threads to 4 (`Threads` = 4) and pin the solver to the four logical processors of the two performance cores, so that every run executes on identical cores. Wardrop equilibria are computed by a deterministic single-threaded barrier method. They arise in the computation of the bounds and, for the constraint-generation formulations, at each iteration, where the equilibrium under the current toll vector is recomputed to turn the relaxation flow into a bilevel-feasible incumbent. All these computations are included in the reported solution times; each is solved in at most a few seconds, even on the largest (Eastern Massachusetts) instances. To further improve the reproducibility of the reported solve times, Turbo Boost was disabled, fixing the performance cores at their nominal base frequency of 1.70 GHz. The relative optimality gap tolerance is left at its default value of 0.01%, so that a run is considered solved to optimality once this tolerance is reached. As a consequence, objective values reported for different formulations on the same instance may differ slightly. Unless stated otherwise, a time limit of 1800 seconds is imposed on each run.

Instances. Our benchmark is built from three road networks of increasing size, all drawn from the `TransportationNetworks` data repository [16]. Throughout Section 6 we essentially work with a subnetwork of the Sioux Falls network [12], composed of 12 nodes and 36 arcs and already considered in [3]; we turn to the two larger networks — the *complete* Sioux Falls network (24 nodes, 76 arcs) and

the Eastern Massachusetts network (74 nodes, 258 arcs) — only to assess how the formulations scale with the topology. For each network we vary the number of toll arcs in $|A^{\mathtt{toll}}| \in \{2, 4, 6, 8, 10\}$ and the number of commodities along a nested demand ladder, where the level $|\mathcal{K}| = k$ uses the first k demands in a fixed per-network ordering, so that smaller demand levels are contained in larger ones. Further details are given in Appendices A, E and F.

Grouping commodities by destination. Following Section 5.4, commodities are grouped by destination in all the considered formulations; this is possible here because the arc costs of our instances do not depend on the commodity. Since origins and destinations play symmetric roles once arcs and OD pairs are reversed, we assume without loss of generality that the number of distinct destinations does not exceed the number of distinct origins, reversing arcs and OD pairs whenever this is not the case. The reduction is applied uniformly to all formulations, ensuring a fair comparison; in particular, the formulation (7) of Beck et al. is used in its grouped-by-destination form. This form is equivalent to the original one in terms of optimal solutions, while offering a computational advantage over the ungrouped version that grows with instance size, so that our comparison is not biased against [3].

Formulations under comparison. We compare four formulations, which fall into two groups according to the guarantees they provide:

- Two *exact* formulations, which solve problem (BW) to global optimality: the formulation of Beck et al. (7) and the arc-based strong-duality formulation (14).
- Two *path-based* formulations, namely the path-flow formulation (15) and the complementarity-based formulation (16). Both involve a number of path variables and associated constraints that grows exponentially with the network size, and are therefore solved by column-and-constraint generation. Starting from a restricted set of represented paths, one iteratively solves the current restriction and searches for violated path constraints; whenever such a constraint is found, both the constraint and the path variable associated with the corresponding path are added to the formulation. In our implementation, the set of paths represented in the constraints coincides with the set of paths represented by the variables. The procedure terminates when no violated path inequality remains. Since the restricted models are nonconvex, this termination certifies feasibility but not optimality: it does not rule out the existence of a path whose addition, as a variable, would improve the objective. The returned solution is therefore not guaranteed to be optimal for (BW), and these two formulations are used as heuristics (see Section 6.5).

The bounds from Sections 5.1 and 5.2 are always included in the formulations whenever applicable (i.e. if the bounded variables appear in the formulation) for our experiments.

Cost structure for toll arcs. Unless stated otherwise, the cost structure on toll arcs follows [3] (see also Appendix A), and the toll upper bound on each toll arc $a \in A^{\mathtt{toll}}$ is set to

$$\tau_a^{\mathtt{up}} = \min\left(c_a^{\mathtt{fft}}, \tau_a^{\infty}\right),$$

the smaller of the free-flow arc cost $c_a^{\mathtt{fft}}$ used as toll bound in [3] and the implicit bound τ_a^{∞} of Proposition 2. (Here $c_a^{\mathtt{fft}}$ is the original free-flow cost serving as the regulatory factor, whereas the intercept β_a entering τ_a^{∞} is that of the toll arc's cost function, equal to $c_a^{\mathtt{fft}}/2$ unless stated otherwise.) By Proposition 2, taking this minimum is optimality-preserving. The same bound is imposed on every formulation. We refer to this as the *reference configuration*. Other choices for the definition of bounds and costs on toll arcs will be investigated in Sections 6.3 and 6.4, respectively.

6.2 Comparison of the exact formulations

We first compare the two exact formulations, namely the formulation (BLS) (7) of Beck et al. and our arc-based strong-duality formulation (SD-A) (14), on the reference configuration.

Solving (SD-A) by constraint generation

We solve the arc-based formulation (SD-A) (14) by generating the path inequalities (14f) dynamically, since their number grows exponentially with the network size. We start from a restriction of (14) in which only a subset of the inequalities (14f) is present, namely those associated, for each commodity, with a shortest path computed with respect to the effective arc costs $\alpha_a f_a + \beta_a + \tau_a$ of the Wardrop equilibrium at the two toll vectors $\tau_a = 0$ $(a \in A)$ and $\tau_a = \tau_a^{\mathtt{up}}$ $(a \in A)$. This restriction is then solved to optimality, yielding a solution $(\tau^\star, f^\star, \pi^\star, t^\star)$; we write $\lambda_a^\star = \alpha_a f_a^\star + \beta_a + \tau_a^\star$ for the associated effective arc costs, $a \in A$.

Since the inequalities (14f) read $\pi_k \le \sum_{a \in p} \lambda_a$, the most violated one for commodity k is found by a single shortest-path computation with respect to the costs $\lambda_a^\star$. Two cases arise.

Case 1: no inequality is violated. Then $(\tau^\star, f^\star, \pi^\star, t^\star)$ satisfies all inequalities (14f) and is thus feasible for (14); since the restriction is a relaxation of (14), it is optimal for it, so that $f^\star$ is a Wardrop equilibrium for $\tau^\star$ and $(\tau^\star, f^\star)$ is optimal for (BW). The procedure stops.

Case 2: at least one inequality is violated for some commodity k. For each such k we recover a minimizing path and add the corresponding inequality (14f) to the restriction, which is solved anew, and the procedure iterates.

The results are reported in Table 1, with the following information. The fields $|\mathcal{K}|$, $(\#o, \#d)$, and $|A^{\mathtt{toll}}|$ describe the instance:

- $|\mathcal{K}|$: number of OD pairs;
- $(\#o, \#d)$: number of distinct origins, followed by the number of distinct destinations, in the instance as given, i.e. before the possible reversal of arcs and OD pairs described in Section 6.1;
- $|A^{\mathtt{toll}}|$: the number of toll arcs, i.e., arcs $a \in A$ with $\tau_a^{\mathtt{up}} > 0$.

The remaining fields report, for each formulation:

- objective: a valid lower bound $\underline{z}$ on the optimal value $z^\star$ of (BW). When the instance is solved to optimality, $\underline{z} = z^\star$. When the time limit is reached, $\underline{z}$ is a feasible objective value for (BW), obtained as follows. For (BLS), which is solved as a single model, the best feasible solution returned by the solver is already feasible for (BW), and its objective is reported. For (SD-A), which is solved by constraint generation, the flow of the current restriction need not be a Wardrop equilibrium; we therefore compute a Wardrop equilibrium for the tolls of the best feasible solution of the restriction (by convex quadratic programming) and report the corresponding leader's revenue, which is feasible for (BW). A dash (–) indicates that no feasible solution was found.
- time: the total solution time in seconds; "TL" indicates that the time limit was reached.
- nodes: the number of branch-and-bound nodes explored, as reported by the solver. For (SD-A), this is the cumulative number over all constraint-generation iterations.
- $\mathrm{gap}_{\mathtt{own}}$: the relative optimality gap certified by the formulation *alone*, defined as $(\overline{z}_{\mathtt{dual}} - \underline{z})/\underline{z}$, where $\overline{z}_{\mathtt{dual}}$ is the dual bound returned by the solver at termination for that formulation. For (BLS), solved as a single model, $\overline{z}_{\mathtt{dual}}$ is the dual bound of the full model. For (SD-A), it is the dual bound of the last restriction solved; since that restriction is obtained from (14) by omitting a subset of the path inequalities (14f), it is a relaxation of (14), and its dual bound is therefore a valid upper bound on $z^\star$. Both quantities are thus valid upper bounds on $z^\star$, and $\mathrm{gap}_{\mathtt{own}}$ is a valid certificate for the corresponding formulation taken in isolation.
- gap: a certified relative optimality gap, defined as

$$gap = \frac{\overline{z} - \underline{z}}{\underline{z}}, \qquad \overline{z} = \min\left(\overline{z}_{\mathtt{dual}}^{\mathtt{BLS}},\ \overline{z}_{\mathtt{dual}}^{\mathtt{SD\text{-}A}},\ \overline{z}_{\mathtt{B}}\right),$$

where $\overline{z}^{\texttt{BLS}}_{\texttt{dual}}$ and $\overline{z}^{\texttt{SD-A}}_{\texttt{dual}}$ are the dual bounds returned by the solver at termination for the exact formulations (BLS) and (SD-A), and $\overline{z}_{\texttt{B}}$ is the Beckmann-based upper bound (21). Each is a valid upper bound on $z^\star$ for the given instance, hence so is their minimum.

For (SD-A), we additionally report:

- it: the number of constraint-generation iterations;
- $\#p_i$, $\#p_g$: the number of path inequalities (14f) present after initialization, and the number generated subsequently by separation, respectively.

Main observations. The results highlight a marked difference in scalability between the two formulations. Within the time limit, (BLS) solves 33 of the 55 instances and (SD-A) solves 52. The formulation (BLS) solves every instance with up to 7 commodities and still solves four of the five instances for $|\mathcal{K}| \in \{8, 10, 15\}$, but only one of five for $|\mathcal{K}| = 20$ and none for $|\mathcal{K}| \in \{25, 30, 132\}$; on eight instances — $|\mathcal{K}| = 30$ with 2, 4 and 8 toll arcs, and all five instances with $|\mathcal{K}| = 132$ — it returns no feasible solution at all within the time budget. In contrast, (SD-A) returns a feasible solution on every instance and solves all five instances with $|\mathcal{K}| = 132$, in between 5.4 and 36.0 seconds. On the 32 instances solved by both formulations, (SD-A) is faster on 20, with a median solution time of 7.4 s against 18.4 s for (BLS) and a median speed-up of 1.5×. This advantage is not uniform: on the instance $|\mathcal{K}| = 8$ with 6 toll arcs, (SD-A) is markedly slower, taking 1268 s against 874 s for (BLS), and this single instance accounts for most of its cumulative time (1817 s, against 3790 s for (BLS)).

The two gap columns reveal a second, more qualitative difference. On every instance solved to optimality, each formulation certifies its own solution: $\text{gap}_{\texttt{own}}$ is then at most 0.01%, i.e. the solver tolerance. On the time-limited runs, however, the two formulations behave in opposite ways. For (BLS), $\text{gap}_{\texttt{own}}$ ranges from 0.051% to 111.4%, with a median of 21.3%, whereas the pooled gap never exceeds 0.881% and is at most 0.01% on 12 of the 14 such runs: (BLS) frequently reaches an optimal, or near-optimal, solution, but its own dual bound remains too loose to certify it. Its difficulty on these instances is therefore one of certification rather than of primal search. For (SD-A), on the contrary, $\text{gap}_{\texttt{own}}$ coincides with the pooled gap on two of its three time-limited runs — $|\mathcal{K}| = 8$ and $|\mathcal{K}| = 30$, both with 10 toll arcs, at 0.985% and 0.281% — and pooling improves it only on the third, $|\mathcal{K}| = 7$ with 10 toll arcs, from 0.023% to 0.010%. The strong-duality formulation is thus essentially self-certifying: pooling the dual bounds improves its gaps only marginally, so that the gaps reported for (SD-A) are not an artefact of borrowing bounds from (BLS).

The constraint-generation procedure terminates in a handful of iterations. The shortest-path initialization provides a single path per commodity for each of the two reference toll vectors $\tau = 0$ and $\tau = \tau^{\texttt{up}}$; being deliberately sparse, it leaves at least one violated inequality on every instance of the benchmark, so that separation is triggered throughout. The exact separation nonetheless terminates the procedure quickly: at most 11 path inequalities are generated on any instance, in at most 5 iterations. As each separation round amounts to one reverse-Dijkstra computation per distinct destination, the cost of (SD-A) is dominated by the solution of the successive restrictions — each a nonconvex quadratically constrained program — rather than by the separation itself.

We emphasize that (SD-A) does not dominate (BLS) uniformly. On $|\mathcal{K}| = 7$ with 10 toll arcs, (BLS) is solved within the time limit (in 1739.1 s) while (SD-A) reaches it; on $|\mathcal{K}| = 8$ with 6 toll arcs, both formulations solve the instance to optimality, but (BLS) is the faster of the two (874.3 s against 1268.4 s); and on $|\mathcal{K}| = 8$ with 10 toll arcs, where both reach the time limit, (BLS) returns the better feasible solution (38 825.91 against 38 785.82, an advantage of about 0.1%). Conversely, (SD-A) is the stronger method on the $|\mathcal{K}| = 30$ instances: with 8 toll arcs it solves the instance to optimality (in 439 s) whereas (BLS) finds no feasible solution, and with 10 toll arcs, where both time out, it returns the better feasible solution (107 221.60 against 107 075.08). Moreover, the three instances on which (SD-A) reaches the time limit (highlighted in bold in Table 1) are not the largest ones: (SD-A) solves $|\mathcal{K}| = 132$ with 10 toll arcs in 27.7 s, exploring only 225 branch-and-bound nodes, whereas $|\mathcal{K}| = 8$ with

10 toll arcs requires more than 3.7 million nodes without closing the gap. The relative performance of the two formulations, and the difficulty of an instance for (SD-A), therefore depend on the structure of the instance rather than on its size alone.

Table 1: Comparison of the exact formulations (BLS) and (SD-A) (time limit TL = 1800 s). The "objective" field reports a valid lower bound $\underline{z}$ on $z^\star$; "gap$_{\text{own}}$" the gap certified by the formulation alone, and "gap" the certified gap $(\overline{z}-\underline{z})/\underline{z}$ obtained with the pooled instance-level bound $\overline{z}$; see the field descriptions above for details. A dash (–) indicates that no feasible solution was found.

$\lvert\mathcal{K}\rvert$	$(\#o,\#d)$	$\lvert A^{\texttt{toll}}\rvert$	(BLS)					(SD-A)							
			objective	time	nodes	gap$_{\text{own}}$ (%)	gap (%)	objective	time	$\#p_i$	$\#p_g$	nodes	it	gap$_{\text{own}}$ (%)	gap (%)
4	(4,3)	2	11 787.08	2.7	5 007	0.010	0.000	11 787.08	3.6	4	2	3	3	0.000	0.000
		4	16 287.08	2.7	3 944	0.008	0.000	16 287.08	4.7	4	2	3	3	0.000	0.000
		6	18 987.08	1.8	982	0.000	0.000	18 987.08	4.7	4	2	3	3	0.010	0.000
		8	23 870.26	6.8	1 780	0.004	0.004	23 870.26	6.7	5	1	2 921	2	0.009	0.004
		10	26 212.70	24.1	34 751	0.007	0.003	26 212.69	3.7	5	4	961	3	0.003	0.003
5	(5,4)	2	11 787.08	1.2	1 083	0.000	0.000	11 787.08	5.6	5	2	3	3	0.000	0.000
		4	16 287.08	1.2	15	0.000	0.000	16 287.08	5.1	5	2	3	3	0.000	0.000
		6	18 987.08	9.7	14 056	0.000	0.000	18 987.08	4.7	5	2	3	3	0.008	0.000
		8	23 870.26	8.6	2 177	0.001	0.001	23 870.26	8.1	6	1	4 069	2	0.009	0.001
		10	26 603.80	18.4	22 183	0.007	0.000	26 603.80	5.2	7	4	1 314	3	0.000	0.000
6	(6,5)	2	11 743.58	3.5	1 988	0.003	0.003	11 743.58	5.8	6	5	4	4	0.009	0.003
		4	19 199.33	5.9	4 099	0.005	0.005	19 199.33	20.3	6	6	8 151	5	0.008	0.005
		6	22 799.33	6.4	1 509	0.007	0.005	22 799.33	22.9	6	6	2 045	5	0.005	0.005
		8	30 343.02	59.8	53 522	0.006	0.006	30 343.01	26.3	8	4	24 142	3	0.010	0.006
		10	39 619.54	118.3	143 176	0.009	0.009	39 619.54	178.1	8	4	137 951	3	0.010	0.009
7	(7,6)	2	8 593.37	4.4	1 744	0.004	0.004	8 593.37	6.3	8	3	252	3	0.009	0.004
		4	19 477.39	15.2	12 037	0.004	0.004	19 477.39	17.5	8	5	790	5	0.005	0.004
		6	22 177.40	26.5	10 505	0.010	0.007	22 177.39	17.8	8	5	616	5	0.007	0.007
		8	28 931.57	332.9	385 594	0.010	0.010	28 931.56	26.9	9	4	10 191	4	0.010	0.010
		10	**31 005.14**	**1739.1**	**1 391 779**	**0.010**	**0.010**	**31 005.08**	**TL**	**9**	**3**	**820 390**	**3**	**0.023**	**0.010**
8	(8,7)	2	8 612.07	7.6	1 903	0.008	0.005	8 612.07	5.0	9	4	3	3	0.005	0.005
		4	18 540.90	27.7	14 589	0.007	0.007	18 540.89	8.2	11	2	1 255	2	0.009	0.007
		6	26 508.83	874.3	656 144	0.009	0.009	26 508.83	1268.4	9	5	2 714 674	3	0.010	0.009
		8	29 508.83	359.3	292 940	0.010	0.010	29 508.83	41.3	9	5	22 025	3	0.010	0.010
		10	**38 825.91**	**TL**	**579 276**	**6.695**	**0.881**	**38 785.82**	**TL**	**10**	**3**	**3 723 867**	**3**	**0.985**	**0.985**
10	(10,9)	2	11 055.09	11.2	3 929	0.008	0.003	11 055.09	8.7	12	3	39	3	0.003	0.003
		4	17 863.25	136.1	214 345	0.010	0.006	17 863.25	10.6	12	3	2 487	3	0.006	0.006
		6	30 245.72	18.3	1 813	0.004	0.004	30 245.72	12.1	13	2	2 073	2	0.009	0.004
		8	37 612.33	TL	784 915	0.535	0.009	37 612.33	37.1	13	5	2 088	3	0.009	0.009
		10	40 612.33	366.7	159 702	0.010	0.005	40 612.33	47.9	13	5	2 789	3	0.005	0.005
15	(12,10)	2	14 000.00	72.7	29 500	0.009	0.008	14 000.00	4.9	15	5	3	3	0.008	0.008
		4	28 362.90	88.5	12 455	0.010	0.007	28 362.90	6.2	15	5	3	3	0.007	0.007
		6	39 606.76	139.0	20 978	0.000	0.000	39 606.76	6.2	17	4	3	3	0.010	0.000
		8	49 379.93	847.3	414 651	0.010	0.007	49 379.91	16.6	17	4	968	3	0.007	0.007
		10	61 791.00	TL	474 673	0.445	0.003	61 791.00	20.7	18	6	268	4	0.003	0.003
20	(12,10)	2	29 695.88	TL	614 968	0.051	0.003	29 695.88	7.7	20	6	3	3	0.003	0.003
		4	45 359.25	TL	239 884	1.035	0.009	45 359.25	6.5	20	6	3	3	0.009	0.009
		6	54 452.42	191.1	21 242	0.005	0.001	54 452.41	6.7	20	7	3	3	0.001	0.001
		8	65 252.42	TL	168 891	20.494	0.000	65 252.42	21.1	20	7	17	3	0.000	0.000
		10	78 057.43	TL	115 784	44.854	0.001	78 057.41	11.4	22	6	11	3	0.001	0.001
25	(12,11)	2	27 009.92	TL	138 625	12.603	0.009	27 010.26	10.4	27	5	8	3	0.008	0.008
		4	54 577.76	TL	345 202	22.027	0.010	54 577.74	9.7	26	7	3	3	0.010	0.010
		6	66 477.74	TL	161 183	29.256	0.010	66 477.74	13.0	26	8	3	3	0.010	0.010
		8	83 277.74	TL	184 512	36.067	0.009	83 277.74	10.3	26	9	9	3	0.009	0.009
		10	98 674.12	TL	113 948	47.880	0.009	98 674.11	26.5	27	4	2 534	2	0.009	0.009
30	(12,12)	2	–	TL	112 931	–	–	26 666.18	11.0	33	11	4	4	0.010	0.010
		4	–	TL	106 703	–	–	39 102.74	47.8	36	5	19 428	2	0.008	0.008
		6	56 712.73	TL	133 978	111.374	0.010	56 712.73	80.9	37	5	30 523	2	0.010	0.010
		8	–	TL	107 598	–	–	85 373.21	439.0	37	6	76 447	2	0.010	0.010
		10	**107 075.08**	**TL**	**124 470**	**87.170**	**0.418**	**107 221.60**	**TL**	**36**	**6**	**2 292 726**	**2**	**0.281**	**0.281**
132	(12,12)	2	–	TL	40 796	–	–	97 020.71	6.7	138	4	2	2	0.009	0.009
		4	–	TL	41 086	–	–	134 696.98	5.4	138	4	2	2	0.006	0.006
		6	–	TL	48 997	–	–	206 690.32	19.8	138	6	55	3	0.000	0.000
		8	–	TL	42 353	–	–	279 290.32	36.0	137	6	149	3	0.002	0.002
		10	–	TL	51 818	–	–	335 336.97	27.7	141	2	225	2	0.000	0.000

6.3 Impact of the toll bounds

In the reference configuration of Section 6.1, the toll upper bound is $\tau_a^{\texttt{up}} = \min(c_a^{\texttt{fft}}, \tau_a^{\infty})$, the smaller of the regulatory cap $c_a^{\texttt{fft}}$ used in [3] and the implicit bound τ_a^{∞} of Proposition 2. We now assess how

the magnitude of this bound affects tractability, by comparing it with a tighter and a looser choice:

$$\tau_a^{\mathtt{up}} \in \left\{ \tfrac{1}{2}\min\left(c_a^{\mathtt{fft}}, \tau_a^{\infty}\right), \quad \min\left(c_a^{\mathtt{fft}}, \tau_a^{\infty}\right), \quad \tau_a^{\infty} \right\}, \qquad a \in A^{\mathtt{toll}}. \tag{24}$$

The intermediate setting is the reference configuration, used in Section 6.2 and in the remainder of the paper; it keeps the toll set $\mathcal{T}$ bounded. The loosest one retains only the endogenous bound τ_a^{∞}. The tightest one halves the reference bound.

Taking the minimum with τ_a^{∞} is never weaker than the regulatory cap alone, and is strictly tighter whenever $\tau_a^{\infty} < c_a^{\mathtt{fft}}$; on the instances considered here, however, this never occurs, so that the reference bound coincides with the cap $c_a^{\mathtt{fft}}$ of [3] on all toll arcs. The three settings of (24) may therefore be read, on this benchmark, as half the regulatory cap, the regulatory cap, and no regulatory cap.

On the 23 instances solved to optimality under all three settings, the optimal value $z^\star$ increases by a median factor of 2.34 — and by up to 6 — from the tightest to the loosest bound.

Tractability. Table 2 shows a clear, monotone effect: the looser the toll bound, the harder the instances, for both formulations. As the bound widens from $\frac{1}{2}\min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$ to τ_a^{∞}, the number of instances solved by (SD-A) falls from all 55 to 23, and by (BLS) from 41 to 8, while the shifted geometric mean of the solution time grows by more than an order of magnitude. This is consistent with the spatial branch-and-bound applied to the bilinear terms $f_a\tau_a$: a wider toll range enlarges the box over which the McCormick relaxation [13] is built, so that more branching is required to close the gap. Across all three settings, (SD-A) dominates (BLS), solving from 1.3 times as many instances under the tightest bound to close to three times as many under the loosest, and remaining between 3 and 12 times faster in shifted geometric mean; (SD-A) always returns a feasible solution, whereas (BLS) fails to do so on up to 14 instances under the loosest bound. Under the tightest bound, (SD-A) solves every instance, all but two of them with at most 100 branch-and-bound nodes (see Table B1).

Table 2: Effect of the toll upper bound (24) on the tractability of the exact formulations (time limit TL = 1800 s, 55 instances). "solved" counts instances solved to proven optimality; "no sol." counts instances for which no feasible solution was found; "time" is the shifted geometric mean of the solution time (shift 10 s, unsolved runs counted at the time limit). Per-instance results are reported in Tables 1, B1 and B2.

$\tau_a^{\mathtt{up}}$	(BLS)			(SD-A)		
	solved	no sol.	time (s)	solved	no sol.	time (s)
$\frac{1}{2}\min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$	41/55	7	60.5	55/55	0	5.1
$\min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$	33/55	8	206.1	52/55	0	24.5
τ_a^{∞}	8/55	14	1149.2	23/55	0	403.9

Gaps at the time limit. The residual gaps confirm the reading of Section 6.2 and extend it across regimes. On the instances that one formulation solves but the other does not, the failing formulation stays within a few hundredths of a percent of optimality — at most 0.054% for (BLS), with a mean of 0.007% over 31 such instances, and at most 0.010% for (SD-A) over 2 — so that these failures are again of certification rather than of primal search. On the instances left unsolved by *both* formulations, the picture depends on the regime. Under the tightest bound there are none; under the reference bound there are two — $|\mathcal{K}| = 8$ and $|\mathcal{K}| = 30$, both with 10 toll arcs — with pooled gaps of at most 0.985%. Under τ_a^{∞}, by contrast, 31 instances are left unsolved by both; on the 21 of them on which (BLS) returns a feasible solution, the pooled gaps average about 20% for (BLS) and 38% for (SD-A), and exceed 100% on the hardest instances. The genuinely hard core of the problem thus emerges only once the regulatory cap is removed; there, on these 21 instances, the feasible solutions returned by (BLS) are markedly better than those of (SD-A) (a mean certified gap of 20.4% against 37.6%), even though (SD-A) solves far more instances overall and is the only one to return a feasible solution on the remaining 10.

6.4 Impact of the cost structure on toll arcs

In the reference configuration of Section 6.1, the cost function of a toll arc $a \in A^{\texttt{toll}}$ follows [3]: both coefficients read from the instance file are halved, so that its slope is $\alpha_a/2$ and its intercept is $\beta_a = c_a^{\texttt{fft}}/2$. We now assess the effect of this choice by retaining instead the original coefficients, i.e. slope α_a and intercept $\beta_a = c_a^{\texttt{fft}}$. All other settings, including the toll upper bound $\tau_a^{\texttt{up}} = \min(c_a^{\texttt{fft}}, \tau_a^{\infty})$, are those of the reference configuration; only the cost of the toll arcs changes.

Retaining the original cost coefficients lowers the optimal revenue — the median ratio is 0.74 over the 44 instances solved under both settings, with extreme values 0.56 and 0.96 — while making the instances harder for both formulations. The revenue drop is expected: since every commodity retains a toll-free alternative, increasing the un-tolled cost of the toll arcs reduces the margin available to the leader. Note that the toll upper bound itself is unchanged here: on every toll arc of the benchmark $\tau_a^{\infty} > c_a^{\texttt{fft}}$, so that $\tau_a^{\texttt{up}} = c_a^{\texttt{fft}}$ under both cost structures.

Tractability. Table 3 shows that retaining the original coefficients penalizes both formulations: the number of instances solved to optimality drops from 33 to 22 for (BLS) and from 52 to 45 for (SD-A), while the shifted geometric mean of the solution time roughly doubles for both. The ranking of the two formulations is unaffected, and the advantage of (SD-A) actually grows: on the instances solved by both, its median speed-up over (BLS) increases from 1.5× to 2.4×, and it remains the only formulation that returns a feasible solution on every instance, whereas (BLS) fails to do so on 10 of them. The degradation is not uniform in the number of commodities: (SD-A) still solves all five instances with $|\mathcal{K}| = 132$ and all five with $|\mathcal{K}| = 15$, but only two of five for $|\mathcal{K}| = 10$ — another indication that the difficulty of an instance is not governed by its size alone. Finally, the constraint-generation procedure remains equally effective under the original coefficients. Separation is triggered on essentially all instances in both cost structures, but converges in at most five iterations, generating at most 11 path inequalities per instance under the reference coefficients and at most 16 under the original ones. The initial seed is slightly larger under the original coefficients — on average 0.85 more inequalities per instance, and more on 39 of the 55 — and the total number of inequalities generated over the benchmark rises from 248 to 301, consistent with a steeper congestion term spreading the equilibrium flow over a larger number of paths.

Table 3: Effect of the cost structure on toll arcs on the tractability of the exact formulations (time limit TL = 1800 s, 55 instances). Columns are as in Table 2: "solved" counts instances solved to proven optimality, "no sol." those for which no feasible solution was found, and "time" is the shifted geometric mean of the solution time (shift 10 s, unsolved runs counted at the time limit). Per-instance results are reported in Table 1 and Appendix C.

(slope, intercept) on $A^{\texttt{toll}}$	(BLS)			(SD-A)		
	solved	no sol.	time (s)	solved	no sol.	time (s)
$(\alpha_a/2,\ c_a^{\texttt{fft}}/2)$ (reference)	33/55	8	206.1	52/55	0	24.5
$(\alpha_a,\ c_a^{\texttt{fft}})$	22/55	10	471.7	45/55	0	56.0

Gaps at the time limit. The picture of Sections 6.2 and 6.3 carries over. On the instances that (SD-A) solves but (BLS) does not, (BLS) is again close to optimality — on average 0.008% and at most 0.033% over the 14 such instances — so that these failures remain ones of certification rather than of primal search; symmetrically, every instance solved by (BLS) is also solved by (SD-A). On the ten instances left unsolved by *both* formulations, the residual pooled gaps remain small: on the nine where (BLS) also returns a feasible solution, they average 0.840% for (BLS) against 1.046% for (SD-A), so that (BLS) returns slightly better feasible solutions on this hard core.

6.5 Path-based formulations as heuristics

We now evaluate the two path-based formulations, (SD-P) (15) and (CC) (16), solved by column-and-constraint generation as described in Section 6.1. As explained there, the termination of this procedure

does not certify global optimality for (BW); these formulations are therefore used as heuristics, and we assess them in terms of the quality of the solutions they produce and of the associated computational effort.

Evaluation by bounding. For each instance, the heuristic returns a valid lower bound $\underline{z}$ on the optimal value $z^\star$ of (BW), obtained as follows. If the procedure terminates within the time limit (no violated constraint is found), the flow of the current restriction is a Wardrop equilibrium for the corresponding tolls, so the restriction's optimal value is feasible for (BW) and is reported as $\underline{z}$. If the time limit is reached, we take the tolls of a feasible solution of the current restriction, compute a Wardrop equilibrium for these tolls (by solving (3)), and report the resulting leader's revenue, which is feasible for (BW).

The certified relative gap

$$\frac{\overline{z} - \underline{z}}{\underline{z}}$$

is then an upper bound on the true gap between the heuristic solution and the optimum.

Results. The results are reported in Table D1, for both (SD-P) and (CC).

Solution quality. Both heuristics prove remarkably accurate. On the 53 instances for which one of the exact formulations of Section 6.2 proved optimality, (SD-P) and (CC) both return a solution whose value coincides with $z^\star$, up to the solver tolerance, on *all* 53 of them. On the two remaining instances — $|\mathcal{K}| = 8$ with 10 toll arcs and $|\mathcal{K}| = 30$ with 10 toll arcs — for which no exact formulation closed the gap within the time limit, both heuristics match the best feasible solution found by any method. The certified gaps of Table D1 are correspondingly small: at most 0.01% on 52 of the 55 instances for (SD-P) and on 53 for (CC), with a median of 0.005%. The larger values, up to 0.881%, occur on the two instances that no exact formulation closed — where the heuristic solution is the best one known and the residual gap may reflect the looseness of $\overline{z}$ — and, for (SD-P), on $|\mathcal{K}| = 6$ with 10 toll arcs (0.011%), where the returned value lies within the solver tolerance of the optimum.

Comparison of the two heuristics. The two formulations return essentially the same objective values — the largest relative difference between their values over the whole benchmark is $2.2 \cdot 10^{-5}$ — but they differ sharply in computational effort. (CC) is faster on 54 of the 55 instances, by a median factor of 13.6 and by up to nearly three orders of magnitude; over the whole benchmark it requires 20.5 seconds and 25 069 branch-and-bound nodes in total, against 1026 seconds and 2.3 million nodes for (SD-P). Both terminate within the time limit on every instance. The contrast is sharpest on the instance with $|\mathcal{K}| = 8$ and 10 toll arcs, where (SD-P) needs 475 seconds and more than 1.5 million nodes while (CC) terminates in 0.8 seconds after 1 176 nodes, both returning the same objective value. Since the two formulations share the same column-and-constraint generation scheme, the same initialization and, as observed above, the same objective values, this gap in performance is attributable to the way the equilibrium conditions are expressed: the complementarity conditions handled through SOS1 constraints in (CC) are processed far more efficiently than the strong-duality inequality of (SD-P), whose quadratic term couples all arc flows.

Comparison with the exact formulations. The heuristics are most useful precisely where the exact formulations struggle. On the three instances on which (SD-A) reaches the time limit, (CC) returns in at most 2.9 seconds a solution at least as good as the incumbent (SD-A) reaches after 1800 seconds; on $|\mathcal{K}| = 30$ with 8 toll arcs, where (BLS) finds no feasible solution at all, it returns the best solution known in under two seconds. The constraint-generation procedure behaves as it does for (SD-A): the sparse shortest-path initialization leaves at least one violated inequality on every instance, so that separation is triggered throughout, but it converges in at most five iterations, generating at most 13 path inequalities on any instance.

Sensitivity to the initial path set. Initializing with the zero-toll equilibrium alone preserves the accuracy of both formulations: each still returns an optimal solution on every instance whose optimal

value is known. The burden merely shifts to the separation, which generates about 60% more path inequalities over the benchmark (402 against 250). Initializing with the equilibrium at $\tau = \tau^{\mathrm{up}}$ alone, by contrast, degrades the results markedly, and does so slightly more for (CC) than for (SD-P): (SD-P) returns an optimal solution on 46 of the 53 instances, with a worst-case shortfall of 10.2%, and (CC) on 42, with a worst case of 11.2%.

The explanation is structural. At $\tau = \tau^{\mathrm{up}}$ the tolled arcs are expensive, so the corresponding equilibrium largely avoids them, and the resulting path set offers little capacity to route flow through the toll arcs — precisely the flow on which revenue is raised. Although the separation is still triggered on every instance, it is driven by the relaxation flow and does not, on its own, recover the missing revenue-carrying routes: the column-and-constraint generation then converges to an impoverished restricted model, leaving the returned solution suboptimal on 7 of the 53 instances for (SD-P) and on 11 for (CC). Consistently with this, the two formulations, which return essentially identical solutions under the reference initialization, no longer do so under this one.

These experiments show that the quality of the solutions returned by the path-based formulations is not an automatic consequence of the column-and-constraint generation scheme: it relies on the initial path set containing the routes that users take when tolls do not deter them. They also indicate that the two equilibria used for initialization play complementary roles, the zero-toll one being essential to solution quality and the one at $\tau = \tau^{\mathrm{up}}$ mainly reducing the number of path inequalities that must be generated.

Table 4: Sensitivity of the path-based formulations to the initial path set, in the reference configuration (time limit TL = 1800 s, 55 instances). "optimal" counts the instances, among the 53 whose optimal value is known from an exact formulation, on which the returned solution is optimal; "$\sum \#p_g$" is the total number of path inequalities generated, and "separated" the number of instances on which at least one was generated. Per-instance results are reported in Tables D2 and D3 (Appendix D).

	initial path set	optimal	$\sum \#p_g$	separated	time (s)	worst shortfall
(SD-P)	$\tau = 0$ and $\tau = \tau^{\mathrm{up}}$	**53/53**	250	55/55	1026.1	0.002%
	$\tau = 0$ only	**53/53**	402	55/55	2722.2	0.002%
	$\tau = \tau^{\mathrm{up}}$ only	46/53	326	55/55	516.7	10.230%
(CC)	$\tau = 0$ and $\tau = \tau^{\mathrm{up}}$	**53/53**	247	55/55	20.5	0.001%
	$\tau = 0$ only	**53/53**	402	55/55	26.2	0.001%
	$\tau = \tau^{\mathrm{up}}$ only	42/53	317	55/55	16.0	11.246%

6.6 The full Sioux Falls and Eastern Massachusetts networks

The results obtained with the reference configuration on the two larger networks are summarized in Table 5, together with those on the subnetwork, with the per-instance results in Appendices E and F. The exact formulations (upper part) and the heuristics (lower part) are reported with the same fields, so that the two parts can be read side by side — with the caveat that termination within the time limit ("term.") certifies optimality only for the exact formulations.

As the network grows from 12 to 24 to 74 nodes, (BLS) degrades rapidly: it solves 33 of the 55 subnetwork instances, 7 of the 55 full-Sioux-Falls instances, and *none* of the 60 Eastern Massachusetts instances, on 45 of which it returns no feasible solution at all. (SD-A), by contrast, remains effective and always feasible — solving 52, 35 and 26 instances respectively. The "opt" column shows that part of the difficulty of (BLS) is one of certification: it returns an optimal solution on 45, 14 and 11 of the instances whose optimum is known, while proving optimality on only 33, 7 and 0 of them. This is not a matter of instance size alone: on all three networks, some small-demand instances remain unsolved while some large ones are solved. It is a matter of formulation. The strong-duality reformulation remains usable on networks on which the big-M linearization of (BLS) no longer even locates a feasible point.

Table 5: Summary of the results on the three networks (reference configuration, time limit TL = 1800 s), for the exact formulations (top) and the path-based heuristics (bottom), reported with the same fields. "Subnetwork", "Full SF" and "East. Mass." denote the 12-node Sioux Falls subnetwork, the complete Sioux Falls network and the Eastern Massachusetts network, respectively. "# inst." is the number of instances and "# known" the number of them whose optimal value is known, i.e. solved to proven optimality by an exact formulation. For each method, "term." counts the runs that terminate within the time limit — for an exact formulation this certifies optimality, whereas for a heuristic it only means that no violated path inequality remains; "opt" counts the instances, among the "# known" ones, on which the returned solution is optimal up to the solver tolerance (0.01%), whether or not the run terminates; "best" counts the instances on which the returned solution matches, within the same tolerance, the best solution returned by any of the four methods; "no feas." counts the instances on which no feasible solution is returned; "sgm" is the shifted geometric mean of the solution time (in seconds, 10 s shift, time-limited runs counted at the time limit).

Exact formulations

Network	# inst.	# known	(BLS)					(SD-A)				
			term.	opt	best	no feas.	sgm	term.	opt	best	no feas.	sgm
Subnetwork	55	53	33	45	46	8	206.1	52	53	53	0	24.5
Full SF	55	36	7	14	19	31	1305.6	35	35	39	0	232.7
East. Mass.	60	26	0	11	12	45	1800.0	26	26	29	0	392.5

Path-based heuristics

Network	# inst.	# known	(SD-P)					(CC)				
			term.	opt	best	no feas.	sgm	term.	opt	best	no feas.	sgm
Subnetwork	55	53	55	53	55	0	6.7	55	53	55	0	0.4
Full SF	55	36	45	32	43	0	48.4	53	34	52	0	11.5
East. Mass.	60	26	40	26	42	0	112.0	59	26	59	0	8.4

The two path-based heuristics extend this reach further. Wherever an exact formulation certifies an optimum, both recover it: on all 53 such instances on the subnetwork, on 32 (SD-P) and 34 (CC) of the 36 on the full Sioux Falls, and on all 26 on Eastern Massachusetts, with worst-case shortfalls below 0.2%; and where the exact formulations run out of time or feasibility, the heuristics still return a solution on every instance. Between the two, (CC) is consistently the faster — with a shifted geometric mean of 0.4, 11.5 and 8.4 seconds, against 6.7, 48.4 and 112.0 for (SD-P) — and, on the larger networks, the more robust: it terminates within the time limit on 53 of the 55 full-Sioux-Falls instances and on 59 of the 60 Eastern Massachusetts ones, against 45 and 40 for (SD-P). It is also the method that most often returns the best solution known: on 52 and 59 instances of the two larger networks, against 39 and 29 for (SD-A) and 43 and 42 for (SD-P). In particular, on the instances on which (SD-A) itself reaches the time limit, (CC) returns a solution at least as good as the (SD-A) incumbent on all 3 such instances of the subnetwork, on 19 of the 20 of the full Sioux Falls network, and on all 34 of the Eastern Massachusetts network.

7 Conclusion

In this paper, we studied the toll-setting problem (BW) under Wardrop equilibrium constraints, in the case of separable affine arc cost functions and with the objective of maximizing the leader's revenue. Starting from Beckmann's convex formulation of Wardrop equilibria and its dual, we derived three new single-level reformulations of the problem: an arc-based formulation (SD-A), which we solve exactly, and two path-based formulations, (SD-P) and (CC), solved by column-and-constraint generation. We showed that the common assumption that every commodity admits a toll-free path induces finite implicit bounds on the tolls (Proposition 2), which makes it possible to treat the case of unbounded tolls without resorting to arbitrary bounds, and we derived from Beckmann's potential an upper bound on the optimal revenue that applies uniformly to all the formulations considered and is used to certify optimality gaps.

On the computational side, we conducted an extensive study on instances derived from the Sioux Falls and Eastern Massachusetts networks, comparing our formulations with the recent formulation of Beck et al. [3]. The arc-based formulation (SD-A) proved markedly more robust: it solved to proven optimality substantially more instances than (BLS) and, unlike (BLS), returned a feasible solution on every instance of every network. The difference widened with network size — on the largest network, (BLS) solved none of the instances and returned no feasible solution on most of them, whereas (SD-A) still solved a sizeable fraction — and it is one of formulation rather than of instance size, since on all three networks some small-demand instances resisted while some larger ones were solved. Pooling the dual bounds of the formulations, rather than relying on each solver's own bound, also allowed us to localize one difficulty: on many instances left unsolved by (BLS) with an incumbent, the latter turned out to be optimal, or nearly so, the obstacle being the certification of optimality rather than the search for a good solution.

Used as heuristics, the two path-based formulations proved remarkably accurate: they returned an optimal solution on every instance whose optimal value is known on the subnetwork and on the Eastern Massachusetts network, and on all but a few on the full Sioux Falls, always within a fraction of a percent of the optimum; and they returned a feasible solution on every instance on which the exact formulations found none, often in a few seconds where those formulations reached the time limit. The complementarity-based formulation (CC) proved consistently and substantially cheaper than (SD-P), and, on the larger networks, the more robust of the two; the two return essentially identical solutions on the subnetwork, but no longer coincide as the network grows. This accuracy rests on the initial set of paths: an ablation of the initialization shows that the shortest paths at the zero-toll equilibrium are important to it.

Finally, and beyond the specific figures, our experiments highlight that the magnitude of the toll bounds is a significant factor affecting computational performance.

Several directions for future research emerge from this work. First, the path-based formulations (SD-P) and (CC) are used here as heuristics: termination of the column-and-constraint generation guarantees feasibility, but not optimality. Given their accuracy in our experiments, devising a way to turn them into exact methods is a natural next step. Second, the paths and feasible solutions generated by the path-based heuristics could be used to initialize or warm-start the exact formulation (SD-A); our observation that the bottleneck often lies in certification rather than in primal search suggests, however, that the benefit of a better incumbent alone may be limited, and that such an initialization would have to be combined with techniques improving the dual bound. Third, a more systematic study of the influence of the toll-arc cost structure, only briefly touched upon here, would be of interest.

Finally, another promising direction for future research is to develop tailored solution approaches that exploit the specific structure of the reformulations, such as the fact that the nonconvexity of (SD-A) is concentrated in a single bilinear constraint.

A Instances based on the Sioux Falls network

A.1 Description of the network and demands

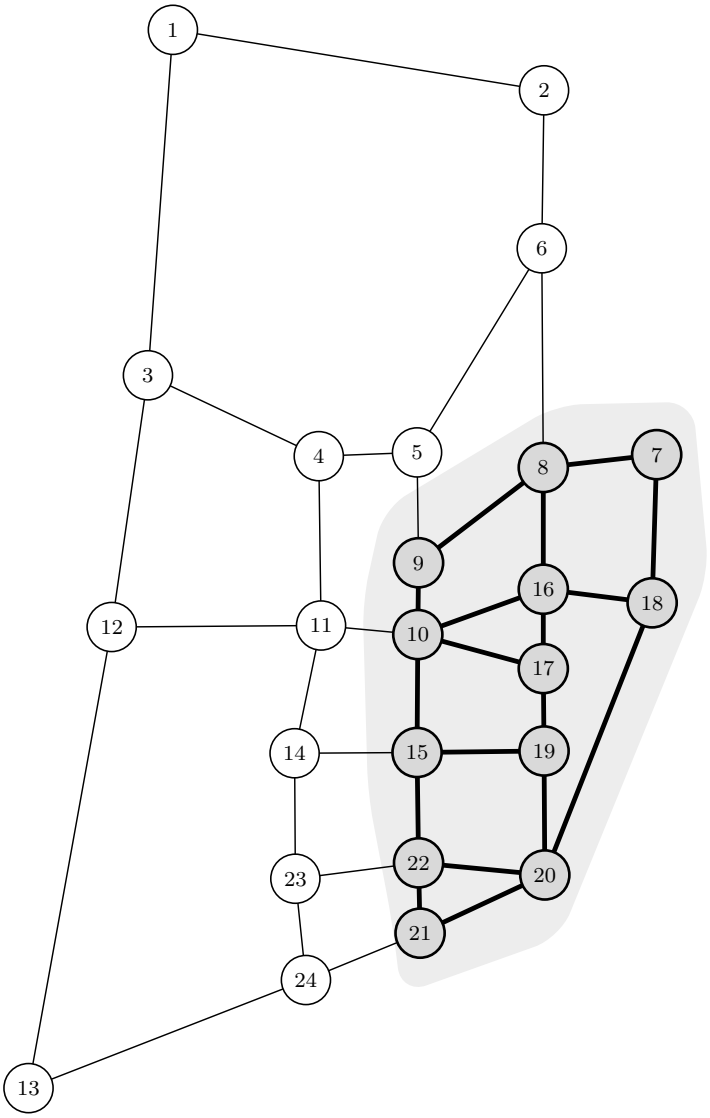
In the computations reported in Sections 6.2–6.5, we use a subnetwork of the Sioux Falls network [12, 16], already considered in [3]. It is induced by the 12 nodes $\{7, 8, 9, 10, 15, 16, 17, 18, 19, 20, 21, 22\}$ of the original network and has 36 arcs; see Figure A1.

For the arc cost functions, we use the ones proposed in [3]:

$$c_a(f_a) = c_a^{\mathtt{fft}} \left(1 + 1.5 \cdot \frac{f_a}{u_a}\right), \qquad a \in A, \tag{A1}$$

where $c_a^{\mathtt{fft}}$ is the so-called *free flow time* given in the description of the network and u_a is the capacity associated with arc $a \in A$; in the notation of Section 3, $\alpha_a = 1.5\, c_a^{\mathtt{fft}}/u_a$ and $\beta_a = c_a^{\mathtt{fft}}$. Following [3],

Figure A1: The Sioux Falls network (24 nodes, 76 arcs), drawn from the node coordinates and links of the benchmark data [16], with the original node identifiers. The shaded subnetwork induced by the 12 nodes $\{7, 8, 9, 10, 15, 16, 17, 18, 19, 20, 21, 22\}$ (36 arcs), used in the main experiments, is drawn with thicker edges. Each drawn edge represents a pair of opposite arcs.



once the toll arcs have been selected (Appendix A.2), both coefficients of the cost function of each toll arc are halved in the reference configuration, which yields the slope $\alpha_a/2$ and the intercept $c_a^{\mathtt{fft}}/2$ for $a \in A^{\mathtt{toll}}$; the alternative in which the original coefficients are retained is studied in Section 6.4. The same cost functions, built from the free-flow times and capacities of the network data [16], are used for the complete Sioux Falls and Eastern Massachusetts networks (Appendices E and F).

For the demands, our *complete* set $\mathcal{K}$ refers to the restriction, to the subgraph under consideration, of the demands of the original instance having both endpoints in this subgraph; there are 132 such demands. When considering a smaller number k of commodities, with $k \in \{4, 5, 6, 7, 8, 10, 15, 20, 25, 30\}$, we mean the first k demands as ordered in Table A1. The case $k = 132$ corresponds to the complete set $\mathcal{K}$.

Table A1: OD pairs considered in our instances based on a subnetwork of the Sioux Falls network, with the demand values and identifiers. The ordering of the demands (column #) determines, for each $k \leq 30$, the first k commodities used in the corresponding instances.

#	(o_k, s_k)	demand	#	(o_k, s_k)	demand	#	(o_k, s_k)	demand
1	(9,21)	300.0	11	(10,21)	1200.0	21	(19,21)	400.0
2	(8,20)	900.0	12	(18,22)	300.0	22	(22,16)	1200.0
3	(16,21)	600.0	13	(9,17)	900.0	23	(21,7)	200.0
4	(17,22)	1700.0	14	(8,19)	700.0	24	(20,9)	600.0
5	(21,18)	100.0	15	(7,22)	500.0	25	(9,22)	700.0
6	(20,10)	2500.0	16	(10,7)	1900.0	26	(8,21)	400.0
7	(15,7)	500.0	17	(16,20)	1600.0	27	(7,19)	400.0
8	(19,9)	400.0	18	(18,9)	200.0	28	(10,19)	1800.0
9	(22,8)	500.0	19	(17,18)	600.0	29	(16,9)	1400.0
10	(7,17)	1000.0	20	(15,20)	1100.0	30	(18,15)	200.0

A.2 Setting the toll arcs

The following procedure is used for the three networks considered. Let t denote the desired number of toll arcs; in our computations we consider $t \in \{2, 4, 6, 8, 10\}$. First, we compute a Wardrop equilibrium assuming zero tolls on all arcs, and we let $f_a^{\mathtt{WE}}$ denote the equilibrium flow on arc $a \in A$. Then, for each commodity $k \in \mathcal{K}$, we compute a shortest path with respect to the arc costs $c_a(f_a^{\mathtt{WE}})$, and we order the arcs by decreasing number of shortest paths in which they appear, ties being broken arbitrarily. We then process the arcs in this order. For each candidate arc, we set it as a toll arc provided that a toll-free path still exists for every commodity afterwards; otherwise, the arc is left toll-free and the next candidate is processed similarly. The procedure stops once t toll arcs have been set, or once all arcs have been processed. Being greedy, it offers no guarantee of reaching t toll arcs while preserving a toll-free path for every commodity, even when such a set of toll arcs exists; on all the instances considered in our experiments, however, the desired number t was reached. Finally, since the candidate arcs are processed in a fixed order that does not depend on t, and since the decision taken for a candidate arc depends only on the arcs already selected, the procedure yields a nested family of toll-arc sets: for $t_1 < t_2$, the set of toll arcs of the instance with t_1 toll arcs is contained in that of the instance with t_2 toll arcs.

B Detailed results for the impact of the toll bounds

Tables B1 and B2 give the per-instance results underlying the summary of Table 2 (Section 6.3), under the tightest bound $\frac{1}{2}\min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$ and the loosest bound τ_a^{∞} respectively; the reference bound $\min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$ is reported in Table 1.

C Detailed results for the impact of the cost structure on toll arcs

Table C1 gives the per-instance results for the original toll-arc cost coefficients $(\alpha_a, c_a^{\mathtt{fft}})$, underlying the summary of Table 3 (Section 6.4); the results with the reference coefficients $(\alpha_a/2, c_a^{\mathtt{fft}}/2)$ are reported in Table 1.

D Detailed results for the heuristics

This appendix reports the full per-instance results for the two path-based heuristics of Section 6.5: Table D1 for the reference configuration, and Tables D2 and D3 for the initialization ablation summarized in Table 4.

E Experiments on the complete Sioux Falls network

The experiments of Sections 6.2–6.5 are conducted on the twelve-node, thirty-six-arc subnetwork of the Sioux Falls network described in Appendix A. To assess how the formulations scale with the network, we compare them here, in the reference configuration of Section 6.1 only, on the *complete* Sioux Falls network, which has 24 nodes and 76 arcs (see Figure A1).

The instances are generated as on the subnetwork — the same Wardrop-equilibrium-based selection of the toll arcs (Appendix A.2), the same reference cost structure, and the same toll upper bound $\tau_a^{\mathtt{up}} = \min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$ — and follow the same 55-instance design: eleven demand levels, ranging from $|\mathcal{K}| = 4$ up to the full origin–destination demand $|\mathcal{K}| = 528$, each combined with $|A^{\mathtt{toll}}| \in \{2, 4, 6, 8, 10\}$ toll arcs. For the smaller demand levels ($k \leq 30$), the first k commodities are taken in the order given in Table E1. Table E2 reports the per-instance results of the exact formulations, with the same fields and the same pooled instance-level bound $\bar{z}$ as Table 1.

Scalability. The larger network sharpens the contrast already visible on the subnetwork. Within the time limit, (BLS) solves only 7 of the 55 instances — all with $|\mathcal{K}| \leq 6$ — and, more tellingly, returns no feasible solution at all on 31 of them: on every instance with $|\mathcal{K}| \geq 10$, and on one further instance with $|\mathcal{K}| = 7$. Its shifted geometric mean solution time, 1306 seconds, is close to the time limit itself. In contrast, (SD-A) solves 35 of the 55 instances, returns a feasible solution on every instance, and is more than five times faster in shifted geometric mean (233 seconds). On the six instances solved by both formulations, (SD-A) is faster on five, with a median speed-up of 21×. A single instance — $|\mathcal{K}| = 4$ with 6 toll arcs — is solved by (BLS) but not by (SD-A), confirming that neither formulation dominates the other; but the overall gap in robustness is wide, (SD-A) solving five times as many instances.

Difficulty and network size. As on the subnetwork, the difficulty of an instance is not governed by the number of commodities alone. (SD-A) solves all five full-demand instances ($|\mathcal{K}| = 528$), in between 149 and 1355 seconds, from an initial path set of about 540 paths; yet it reaches the time limit on three of the five instances for each of $|\mathcal{K}| \in \{20, 25, 30\}$, and even on two of the five four-commodity instances ($|\mathcal{K}| = 4$ with 6 and 10 toll arcs).

Gaps at the time limit. The certified gaps show that, here too, the failures of both formulations are primarily of certification rather than of primal search. On the ten instances that (SD-A) solves and on which (BLS) returns a feasible solution without proving its optimality, that solution is within 0.79% of the optimum, and 0.093% on average. On the nineteen instances that neither formulation solves, the pooled certified gap of (SD-A) remains moderate — a median of 2.2% and a maximum of 13.2% — so that even where optimality is out of reach, the strong-duality formulation delivers solutions of controlled quality. Overall, the complete network confirms the conclusion drawn on the subnetwork: (SD-A) is the more robust and more reliable of the two exact formulations, and the margin in its favour widens as the network grows.

Path-based heuristics. We also evaluate the two path-based heuristics on the complete network; Table E3 reports the per-instance results, with the certified gap computed against the same pooled bound $\bar{z}$ as in Table E2. Where the optimum is known — the 36 instances solved by an exact formulation — both heuristics remain accurate: (SD-P) attains it on 32 of them and (CC) on 34, and the largest shortfall of a returned solution is 0.175%. Their value is clearest, however, where the exact formulations fail. On the 31 instances on which (BLS) returns no feasible solution, both heuristics return one; and on the 20 instances on which (SD-A) reaches the time limit, (CC) returns, in at most 548 seconds, a solution strictly better than the (SD-A) incumbent on 18 of them, by up to 10%. Across the benchmark the certified gap is at most 0.01% on 32 of the 55 instances for (SD-P) and 33 for (CC); the larger gaps fall on the intermediate demand levels, which neither the exact formulations nor the heuristics close.

The two heuristics no longer coincide. On the subnetwork (Section 6.5), (SD-P) and (CC) returned essentially identical solutions and (CC) dominated in speed almost uniformly. On the complete network the two heuristics part ways. Their values now differ by up to 8.4% (on $|\mathcal{K}| = 30$ with 8 toll arcs), and, where they differ, (CC) is generally the stronger: it is faster than (SD-P) on 46 of the 55 instances, by a median factor of 10; it reaches the time limit on only two instances, against ten for (SD-P); and on the hard intermediate-demand instances ($|\mathcal{K}| \in \{20, 25, 30\}$) it returns the better feasible solution — for instance a certified gap of 10.7% against 14.7% on $|\mathcal{K}| = 30$ with 10 toll arcs, on which (SD-P) reaches the time limit. The one axis on which (SD-P) retains an edge is full demand: (CC)'s two time-limited instances are both at $|\mathcal{K}| = 528$, where (SD-P)'s generation terminates on all five. Even there, however, the advantage is one of termination rather than of quality — on $|\mathcal{K}| = 528$ with 4 toll arcs (CC) matches the exact optimum while (SD-P), though it terminates, misses it by 0.18%. The SOS1 branching that makes (CC) so effective thus scales less readily to the 528 commodities of the full-demand case, but without ever degrading the solutions it returns. Overall, (CC) remains the method of choice across the benchmark, with (SD-P) a useful complement only at full demand, where it is the more likely of the two to terminate within the time budget.

F Experiments on the Eastern Massachusetts network

The Eastern Massachusetts network, with 74 nodes and 258 arcs, is the largest that we consider. Instances are generated as on the Sioux Falls networks — the same Wardrop-equilibrium-based selection of the toll arcs (Appendix A.2), the same reference cost structure and toll upper bound $\tau_a^{\mathtt{up}} = \min(c_a^{\mathtt{fft}}, \tau_a^{\infty})$ — over twelve demand levels ranging from $|\mathcal{K}| = 4$ to $|\mathcal{K}| = 100$, each combined with $|A^{\mathtt{toll}}| \in \{2, 4, 6, 8, 10\}$ toll arcs, for 60 instances in all; the first 30 demands of the nested demand ladder are listed in Table F1. Tables F2 and F3 report the per-instance results for the exact formulations and for the heuristics, with the same pooled instance-level bound $\overline{z}$ as before.

Scalability. The larger network drives the gap between the two exact formulations to its extreme. Within the time limit, (BLS) solves *none* of the 60 instances and returns a feasible solution on only 15 of them; on the remaining 45 it finds no feasible toll vector at all within the budget. (SD-A), by contrast, solves 26 of the 60 instances and returns a feasible solution on every one, with a shifted geometric mean solution time of 392 seconds, against the full time limit for (BLS). As on the smaller networks, the instances (SD-A) solves are not simply the smallest: it solves all ten instances with $|\mathcal{K}| \in \{4, 5\}$, but only one or two per demand level thereafter, the difficulty again turning on the interaction between commodities and tolled arcs rather than on size alone.

Gaps at the time limit. The certified gaps confirm that the failures are ones of certification rather than of primal search. On the eleven instances that (SD-A) solves and on which (BLS) returns a feasible solution, that solution is optimal up to numerical precision, even though (BLS) never certifies it. On the 34 instances that neither formulation solves, the pooled certified gap of (SD-A) stays moderate on most — a median of 2.1% — though it reaches 42% on the hardest, reflecting how loose the available upper bound becomes once the network is large and the demand intermediate.

Path-based heuristics. The path-based heuristics extend their usefulness to this network. On the 26 instances whose optimum is known, both (SD-P) and (CC) attain it, with a worst-case shortfall of 0.001%; and on the 45 instances on which (BLS) returns nothing, both return a feasible solution. The two heuristics differ mainly in robustness and cost: (CC) is faster than (SD-P) on 58 of the 60 instances, with a median speed-up of 13 over the whole benchmark, and reaches the time limit on a single instance, against twenty for (SD-P). On the 34 instances on which (SD-A) itself reaches the time limit, (CC) returns a solution at least as good as the (SD-A) incumbent on *all* of them, and (SD-P) on 22; where the exact formulations run out of time, the complementarity-based heuristic is thus the most reliable source of a good feasible toll vector on this network.

Table B1: Comparison of (BLS) and (SD-A) under the tightest toll upper bound $\tau_a^{\mathrm{up}} = \frac{1}{2}\min\left(c_a^{\mathtt{fft}}, \tau_a^{\infty}\right)$ of (24) (time limit TL $= 1800$ s). Columns are as in Table 1. A dash (–) indicates that no feasible solution was found.

$\lvert\mathcal{K}\rvert$	$(\#o, \#d)$	$\lvert A^{\mathtt{toll}}\rvert$	(BLS)					(SD-A)							
			objective	time	nodes	$\mathrm{gap}_{\mathtt{own}}$ (%)	gap (%)	objective	time	$\#p_i$	$\#p_g$	nodes	it	$\mathrm{gap}_{\mathtt{own}}$ (%)	gap (%)
4	(4,3)	2	5 980.51	0.4	25	0.000	0.000	5 980.51	1.6	4	1	2	2	0.009	0.000
		4	8 230.51	0.6	215	0.000	0.000	8 230.51	1.7	4	1	2	2	0.009	0.000
		6	9 580.51	1.0	814	0.000	0.000	9 580.51	1.8	4	1	2	2	0.007	0.000
		8	13 006.05	0.9	7	0.000	0.000	13 006.05	1.8	4	1	2	2	0.002	0.000
		10	14 850.00	1.0	242	0.000	0.000	14 850.00	0.1	4	0	1	1	0.005	0.000
5	(5,4)	2	5 980.51	0.4	1	0.000	0.000	5 980.51	2.0	5	1	2	2	0.008	0.000
		4	8 230.51	0.5	8	0.000	0.000	8 230.51	1.4	5	1	2	2	0.009	0.000
		6	9 580.51	1.0	1	0.000	0.000	9 580.51	2.1	5	1	2	2	0.007	0.000
		8	13 006.05	4.4	6 952	0.009	0.002	13 006.05	2.0	5	1	2	2	0.002	0.002
		10	15 150.00	1.7	919	0.000	0.000	15 150.00	1.0	5	0	1	1	0.009	0.000
6	(6,5)	2	5 994.89	1.2	562	0.000	0.000	5 994.89	2.8	6	3	3	3	0.008	0.000
		4	11 282.72	5.7	4 967	0.000	0.000	11 282.72	4.3	6	3	3	3	0.009	0.000
		6	13 082.72	7.6	7 129	0.000	0.000	13 082.72	4.0	6	3	3	3	0.009	0.000
		8	19 780.51	1.5	87	0.001	0.001	19 780.51	3.2	7	1	2	2	0.010	0.001
		10	26 201.54	3.0	1 341	0.000	0.000	26 201.54	1.5	7	0	1	1	0.007	0.000
7	(7,6)	2	4 650.00	5.4	3 235	0.000	0.000	4 650.00	3.6	7	3	3	3	0.001	0.000
		4	12 016.58	8.6	7 235	0.000	0.000	12 016.58	5.9	7	3	3	3	0.009	0.000
		6	13 366.58	5.2	1 834	0.005	0.005	13 366.58	7.2	7	3	3	3	0.009	0.005
		8	19 875.28	7.4	5 977	0.000	0.000	19 873.82	5.6	7	1	2	2	0.007	0.007
		10	20 875.28	9.4	5 213	0.007	0.007	20 875.28	3.5	7	1	3	2	0.007	0.007
8	(8,7)	2	4 650.00	7.2	2 331	0.000	0.000	4 650.00	3.8	8	3	3	3	0.002	0.000
		4	11 457.33	9.3	4 646	0.009	0.008	11 457.33	3.6	9	1	2	2	0.008	0.008
		6	16 221.00	5.8	1 963	0.000	0.000	16 221.00	3.1	8	2	2	2	0.007	0.000
		8	17 721.00	6.0	1 697	0.006	0.006	17 721.00	3.7	8	2	2	2	0.006	0.006
		10	24 106.60	11.2	4 122	0.000	0.000	24 106.60	5.5	8	1	9	2	0.008	0.000
10	(10,9)	2	7 315.88	9.6	1 615	0.000	0.000	7 315.87	4.1	11	2	3	3	0.009	0.000
		4	11 315.88	10.0	1 447	0.002	0.002	11 315.87	6.9	11	2	3	3	0.007	0.002
		6	19 393.13	7.4	1 397	0.000	0.000	19 393.13	9.7	11	2	3	3	0.009	0.000
		8	23 208.88	8.2	1 437	0.006	0.003	23 208.88	5.5	11	1	2	2	0.003	0.003
		10	24 708.88	10.9	1 386	0.001	0.001	24 708.88	6.5	11	1	7	2	0.006	0.001
15	(12,10)	2	7 000.00	20.0	1 262	0.000	0.000	7 000.00	4.1	15	5	3	3	0.009	0.000
		4	14 400.00	40.0	10 427	0.002	0.000	14 400.00	6.3	15	4	3	3	0.009	0.000
		6	21 671.55	22.7	1 368	0.009	0.005	21 671.55	5.1	16	1	2	2	0.005	0.005
		8	27 171.55	26.0	1 358	0.001	0.001	27 171.55	5.8	16	1	2	2	0.009	0.001
		10	32 815.20	136.9	21 344	0.000	0.000	32 815.20	5.9	17	1	7	2	0.002	0.000
20	(12,10)	2	15 600.00	60.8	10 771	0.000	0.000	15 600.00	6.5	20	4	3	3	0.006	0.000
		4	22 800.00	131.8	20 747	0.010	0.004	22 799.99	5.7	20	4	3	3	0.004	0.004
		6	27 500.01	163.5	21 212	0.004	0.004	27 500.00	6.8	20	4	3	3	0.006	0.004
		8	32 900.00	223.4	31 647	0.010	0.004	32 900.00	6.5	20	4	3	3	0.004	0.004
		10	40 459.59	TL	242 818	17.450	0.009	40 459.59	5.5	22	2	2	2	0.009	0.009
25	(12,11)	2	14 250.00	167.4	20 899	0.000	0.000	14 250.00	4.2	26	2	2	2	0.000	0.000
		4	28 549.69	TL	668 243	4.886	0.009	28 549.69	5.2	26	3	2	2	0.009	0.009
		6	34 499.69	52.9	2 413	0.006	0.006	34 499.68	4.9	26	3	2	2	0.009	0.006
		8	42 899.69	TL	210 119	16.578	0.009	42 899.69	6.1	26	3	2	2	0.009	0.009
		10	54 407.86	TL	275 610	8.081	0.000	54 407.86	14.5	26	3	2	2	0.000	0.000
30	(12,12)	2	–	TL	114 863	–	–	14 679.70	8.0	31	10	3	3	0.000	0.000
		4	–	TL	111 642	–	–	24 991.81	17.9	31	10	231	3	0.000	0.000
		6	33 411.50	TL	135 349	43.103	0.000	33 411.50	17.2	34	7	17	3	0.000	0.000
		8	49 312.24	TL	139 449	40.700	0.003	49 312.24	13.8	34	7	1 140	3	0.003	0.003
		10	68 283.57	TL	165 142	18.510	0.003	68 283.57	5.8	31	3	2	2	0.003	0.003
132	(12,12)	2	–	TL	48 311	–	–	52 898.13	5.5	138	4	2	2	0.009	0.009
		4	–	TL	45 825	–	–	74 595.31	5.2	134	8	2	2	0.009	0.009
		6	–	TL	51 579	–	–	110 595.31	4.7	134	8	2	2	0.010	0.010
		8	–	TL	45 186	–	–	146 895.31	10.1	133	9	2	2	0.001	0.001
		10	–	TL	49 562	–	–	175 026.07	11.8	137	5	2	2	0.008	0.008

Table B2: Comparison of (BLS) and (SD-A) under the loosest toll upper bound $\tau_a^{\mathrm{up}} = \tau_a^{\infty}$ of (24) (time limit TL $= 1800$ s). Columns are as in Table 1. A dash (–) indicates that no feasible solution was found.

$\lvert\mathcal{K}\rvert$	$(\#o, \#d)$	$\lvert A^{\mathtt{toll}}\rvert$	(BLS)					(SD-A)							
			objective	time	nodes	$\mathrm{gap}_{\mathtt{own}}$ (%)	gap (%)	objective	time	$\#p_i$	$\#p_g$	nodes	it	$\mathrm{gap}_{\mathtt{own}}$ (%)	gap (%)
4	(4,3)	2	16 266.08	5.3	3 673	0.000	0.000	16 266.08	2.9	8	0	1 395	1	0.007	0.000
		4	19 207.03	207.6	524 310	0.010	0.009	19 207.03	7.0	8	0	8 667	1	0.009	0.009
		6	24 621.96	TL	3 584 499	0.012	0.010	24 621.95	0.3	8	2	2	2	0.010	0.010
		8	27 388.84	TL	1 689 498	92.341	0.145	15 574.24	TL	8	0	177 095	1	76.115	76.115
		10	30 193.95	TL	1 462 295	149.338	0.000	30 193.94	0.6	8	10	5	5	0.000	0.000
5	(5,4)	2	16 266.08	6.9	6 278	0.000	0.000	16 266.08	4.4	9	0	2 096	1	0.003	0.000
		4	19 207.03	50.1	84 738	0.007	0.007	19 207.03	6.1	9	0	6 351	1	0.010	0.007
		6	24 621.96	TL	3 213 388	0.036	0.000	24 621.95	0.3	9	2	2	2	0.000	0.000
		8	27 388.85	TL	1 054 364	99.651	0.160	15 574.24	TL	9	0	181 684	1	76.142	76.142
		10	30 585.05	TL	887 490	169.830	0.000	30 585.05	0.6	10	9	4	4	0.000	0.000
6	(6,5)	2	16 035.71	28.2	35 245	0.001	0.001	16 035.71	11.9	10	1	3 882	2	0.005	0.001
		4	23 369.55	475.8	525 186	0.010	0.010	23 369.48	TL	12	1	467 867	2	0.020	0.010
		6	29 506.91	TL	2 418 455	0.304	0.009	29 506.90	8.4	12	1	1 531	2	0.009	0.009
		8	35 745.20	TL	720 625	118.200	0.010	35 745.19	36.1	12	5	25 574	2	0.010	0.010
		10	43 860.17	TL	483 858	440.529	3.400	32 261.96	TL	12	5	301 060	2	40.572	40.572
7	(7,6)	2	10 977.31	418.6	1 045 321	0.010	0.009	10 977.31	16.4	11	2	11 095	2	0.009	0.009
		4	24 977.49	TL	840 402	8.325	0.006	24 977.49	700.1	14	3	272 899	4	0.006	0.006
		6	30 657.79	TL	450 381	104.985	1.303	23 467.76	TL	14	0	224 419	1	32.340	32.340
		8	38 101.82	TL	393 418	341.561	2.129	26 079.09	TL	14	0	220 864	1	49.211	49.211
		10	39 176.17	TL	333 357	675.696	1.861	32 894.70	TL	14	6	250 042	3	21.312	21.312
8	(8,7)	2	10 863.47	TL	3 036 057	0.013	0.006	10 863.44	59.2	12	4	33 171	3	0.006	0.006
		4	21 588.57	TL	430 937	44.821	0.010	21 588.57	1659.2	15	1	151 807	2	0.010	0.010
		6	34 739.09	TL	257 845	250.466	0.054	34 754.58	1109.2	16	4	277 563	3	0.010	0.010
		8	38 584.87	TL	197 140	585.678	5.230	33 315.28	TL	16	2	452 464	2	21.875	21.875
		10	45 913.64	TL	226 358	1076.156	7.894	30 484.38	TL	16	2	164 287	2	62.503	62.503
10	(10,9)	2	13 891.58	88.0	160 200	0.010	0.009	13 891.51	65.3	16	4	39 561	3	0.009	0.009
		4	23 126.45	TL	328 511	79.613	0.173	23 112.92	TL	18	3	283 005	2	0.232	0.232
		6	34 898.31	TL	251 003	334.018	7.357	33 869.16	TL	19	0	101 760	1	10.619	10.619
		8	41 119.09	TL	176 266	1117.696	18.635	41 471.60	TL	20	3	214 955	2	17.627	17.627
		10	47 326.90	TL	207 058	1299.108	14.663	32 801.81	TL	20	0	81 384	1	65.438	65.438
15	(12,10)	2	23 192.78	TL	715 873	5.151	0.008	23 192.77	20.6	21	5	4 204	3	0.008	0.008
		4	87 051.45	TL	133 023	193.187	0.000	87 051.44	567.8	26	4	66 235	2	0.000	0.000
		6	111 287.54	TL	143 717	448.926	12.212	112 266.45	TL	29	3	202 200	2	11.234	11.234
		8	147 793.20	TL	197 768	591.225	6.473	125 154.28	TL	29	0	86 703	1	25.733	25.733
		10	164 377.85	TL	113 495	1007.194	9.292	134 768.09	TL	30	9	88 641	2	33.304	33.304
20	(12,10)	2	–	TL	158 918	–	–	35 956.20	21.4	28	5	5 008	3	0.003	0.003
		4	43 085.09	TL	148 677	1139.367	78.612	74 902.02	TL	32	0	85 269	1	2.741	2.741
		6	45 408.54	TL	103 281	2930.608	115.754	91 299.90	TL	34	0	106 968	1	7.307	7.307
		8	100 334.04	TL	96 306	2571.770	18.245	105 305.21	TL	36	0	70 316	1	12.663	12.663
		10	101 787.40	TL	83 423	4728.296	50.131	101 713.41	TL	38	0	138 684	1	50.240	50.240
25	(12,11)	2	–	TL	120 121	–	–	55 042.72	260.6	35	2	178 785	2	0.010	0.010
		4	–	TL	98 450	–	–	92 812.63	TL	43	0	242 238	1	0.181	0.181
		6	–	TL	87 379	–	–	116 111.61	TL	44	0	101 123	1	8.930	8.930
		8	–	TL	65 762	–	–	134 804.36	TL	46	0	264 217	1	15.552	15.552
		10	–	TL	77 738	–	–	145 375.13	TL	48	0	118 474	1	34.039	34.039
30	(12,12)	2	–	TL	84 763	–	–	41 512.14	61.2	43	13	22 399	4	0.007	0.007
		4	–	TL	57 403	–	–	50 661.67	TL	47	0	196 770	1	23.922	23.922
		6	–	TL	67 650	–	–	184 050.69	TL	54	0	347 606	1	11.407	11.407
		8	237 407.03	TL	53 460	2475.392	28.233	233 314.05	TL	57	0	320 610	1	30.483	30.483
		10	288 699.19	TL	78 123	4485.315	47.186	175 152.70	TL	58	0	327 153	1	142.603	142.603
132	(12,12)	2	–	TL	44 033	–	–	279 723.36	207.7	178	14	9 327	2	0.000	0.000
		4	–	TL	36 154	–	–	432 303.88	TL	186	0	340 605	1	10.141	10.141
		6	–	TL	36 594	–	–	633 058.67	TL	206	0	272 405	1	27.975	27.975
		8	–	TL	41 203	–	–	1 192 205.05	TL	217	0	277 444	1	79.274	79.274
		10	–	TL	45 657	–	–	1 860 962.73	TL	227	0	285 897	1	45.614	45.614

Table C1: Comparison of (BLS) and (SD-A) when the original cost coefficients of the toll arcs are retained, i.e. without halving α_a and $c_a^{\texttt{fft}}$ (time limit TL $= 1800$ s). The toll upper bound is the reference one, $\tau_a^{\texttt{up}} = \min(c_a^{\texttt{fft}}, \tau_a^{\infty})$. Columns are as in Table 1. A dash (–) indicates that no feasible solution was found.

$\lvert\mathcal{K}\rvert$	$(\#o, \#d)$	$\lvert A^{\texttt{toll}}\rvert$	(BLS)					(SD-A)							
			objective	time	nodes	$\text{gap}_{\texttt{own}}$ (%)	gap (%)	objective	time	$\#p_i$	$\#p_g$	nodes	it	$\text{gap}_{\texttt{own}}$ (%)	gap (%)
4	(4,3)	2	8 954.63	1.1	377	0.000	0.000	8 954.63	2.2	5	1	2	2	0.000	0.000
		4	10 677.56	33.7	88 288	0.010	0.006	10 677.56	9.6	6	1	2 477	2	0.006	0.006
		6	13 377.56	34.9	55 637	0.009	0.009	13 377.56	12.5	6	2	1 823	2	0.010	0.009
		8	14 122.90	1009.2	2 364 954	0.010	0.008	14 122.89	7.4	7	0	4 315	1	0.008	0.008
		10	15 259.98	TL	2 713 898	0.238	0.000	15 259.98	0.3	6	3	2	2	0.000	0.000
5	(5,4)	2	8 954.63	2.1	2 250	0.000	0.000	8 954.63	1.4	6	1	2	2	0.001	0.000
		4	10 677.56	14.4	25 062	0.008	0.007	10 677.56	8.1	7	1	1 891	2	0.007	0.007
		6	13 377.56	14.3	22 351	0.009	0.004	13 377.56	11.5	7	2	1 580	2	0.004	0.004
		8	14 122.90	TL	3 682 431	0.013	0.010	14 122.89	14.5	8	0	8 339	1	0.010	0.010
		10	15 342.20	TL	2 124 802	0.360	0.000	15 342.19	2.9	9	1	2	2	0.000	0.000
6	(6,5)	2	8 824.98	3.9	1 744	0.000	0.000	8 824.98	2.7	7	2	2	2	0.000	0.000
		4	13 475.12	21.5	32 018	0.010	0.003	13 475.12	10.6	8	4	916	3	0.003	0.003
		6	14 387.88	88.5	149 888	0.010	0.007	14 387.88	12.7	9	3	817	3	0.007	0.007
		8	16 920.57	113.4	132 312	0.010	0.005	16 920.56	13.8	9	5	2 351	4	0.005	0.005
		10	19 784.83	TL	1 296 282	0.438	0.066	19 784.80	TL	8	6	417 902	3	0.066	0.066
7	(7,6)	2	6 605.61	8.9	14 012	0.001	0.001	6 605.61	10.4	8	3	3 112	2	0.009	0.001
		4	13 792.25	30.5	23 302	0.004	0.000	13 792.25	29.2	9	3	4 779	3	0.000	0.000
		6	16 492.25	52.0	45 407	0.009	0.008	16 492.25	25.9	9	4	6 783	3	0.008	0.008
		8	18 451.77	224.5	141 886	0.008	0.007	18 451.77	16.6	9	6	5 900	5	0.007	0.007
		10	18 786.83	TL	893 595	1.812	1.113	18 533.94	TL	10	4	383 356	3	2.493	2.493
8	(8,7)	2	6 553.21	25.5	24 294	0.006	0.001	6 553.21	14.8	9	5	3 502	3	0.001	0.001
		4	10 715.73	235.7	166 023	0.009	0.009	10 715.73	178.0	11	2	148 773	2	0.010	0.009
		6	16 570.17	TL	912 082	0.754	0.002	16 570.17	23.7	11	4	3 865	3	0.002	0.002
		8	19 570.17	TL	835 973	7.763	0.002	19 570.17	9.0	11	4	2 676	3	0.002	0.002
		10	24 828.86	TL	625 603	34.511	0.021	24 828.80	TL	12	3	801 136	3	0.021	0.021
10	(10,9)	2	7 356.59	15.3	10 441	0.000	0.000	7 356.59	5.2	12	4	3	3	0.009	0.000
		4	13 051.88	67.2	33 358	0.008	0.002	13 051.88	17.3	12	5	2 191	3	0.002	0.002
		6	17 585.21	TL	598 405	11.199	0.423	17 561.89	TL	14	4	212 167	2	0.556	0.556
		8	20 953.78	TL	476 752	26.516	3.059	20 894.69	TL	15	3	259 937	2	3.350	3.350
		10	23 953.78	TL	548 774	9.481	2.597	23 894.69	TL	15	3	290 736	2	2.851	2.851
15	(12,10)	2	12 512.77	136.0	62 437	0.001	0.000	12 512.77	9.5	15	7	9	3	0.000	0.000
		4	26 595.75	260.4	83 474	0.006	0.006	26 595.75	4.1	16	3	2	2	0.006	0.006
		6	34 071.18	1758.0	575 647	0.010	0.008	34 071.18	20.9	18	5	1 196	3	0.008	0.008
		8	43 994.02	TL	321 998	6.404	0.005	43 994.02	15.3	18	7	144	3	0.005	0.005
		10	54 397.08	TL	246 295	17.549	0.006	54 397.08	12.3	20	6	770	3	0.006	0.006
20	(12,10)	2	18 238.34	TL	741 509	2.095	0.010	18 238.34	513.9	22	6	680 089	3	0.010	0.010
		4	34 194.12	TL	203 762	26.730	0.022	34 194.12	TL	22	7	1 854 752	4	0.022	0.022
		6	40 197.45	TL	185 476	36.122	0.010	40 197.44	50.4	22	6	1 675	3	0.010	0.010
		8	50 997.45	TL	209 590	63.332	0.009	50 997.45	49.4	22	6	1 748	3	0.009	0.009
		10	59 088.53	TL	179 680	75.196	0.244	59 211.15	TL	24	6	331 036	3	0.037	0.037
25	(12,11)	2	24 613.92	TL	143 329	10.970	0.010	24 613.92	4.9	27	4	2	2	0.010	0.010
		4	40 223.94	TL	259 501	84.732	0.016	40 223.94	TL	26	12	2 799 759	3	0.016	0.016
		6	47 699.32	TL	163 439	67.300	0.010	47 699.43	314.6	30	7	280 774	3	0.010	0.010
		8	64 484.54	TL	153 498	63.699	0.033	64 499.43	75.4	29	9	52 810	2	0.010	0.010
		10	73 663.86	TL	137 479	89.264	0.010	73 663.86	89.2	28	11	34 200	3	0.010	0.010
30	(12,12)	2	–	TL	85 993	–	–	23 377.16	7.0	30	11	3	3	0.007	0.007
		4	–	TL	104 473	–	–	28 937.38	86.6	34	10	44 059	3	0.009	0.009
		6	–	TL	88 880	–	–	46 596.39	59.0	33	11	6 361	3	0.010	0.010
		8	–	TL	82 077	–	–	68 708.84	TL	38	7	237 423	2	0.057	0.057
		10	–	TL	81 841	–	–	81 950.63	152.0	38	7	27 902	3	0.007	0.007
132	(12,12)	2	–	TL	42 061	–	–	84 338.86	7.5	132	14	3	3	0.000	0.000
		4	–	TL	42 050	–	–	117 720.93	12.1	136	12	13	3	0.000	0.000
		6	–	TL	42 917	–	–	197 414.38	33.6	140	16	16	4	0.000	0.000
		8	–	TL	44 923	–	–	260 490.63	32.3	140	12	332	3	0.000	0.000
		10	–	TL	51 018	–	–	316 419.91	25.2	136	10	115	2	0.001	0.001

Table D1: Path-based heuristics (SD-P) and (CC) (time limit TL $= 1800$ s). The "obj" field reports the valid lower bound $\underline{z}$ on $z^\star$ produced by the heuristic, and "gap" the certified gap $(\overline{z}-\underline{z})/\underline{z}$ obtained with the pooled instance-level bound $\overline{z}$ of Section 6.2; see Section 6.5 for details. The remaining fields are as described for Table 1.

$\lvert\mathcal{K}\rvert$	$(\#o,\#d)$	$\lvert A^{\texttt{toll}}\rvert$	(SD-P)							(CC)						
			obj	time	$\#p_i$	$\#p_g$	nodes	it	gap	obj	time	$\#p_i$	$\#p_g$	nodes	it	gap
4	(4,3)	2	11 787.07	1.6	4	2	2	3	0.000	11 787.08	0.1	4	2	2	3	0.000
		4	16 287.08	1.5	4	2	2	3	0.000	16 287.08	0.1	4	2	2	3	0.000
		6	18 987.08	1.3	4	2	2	3	0.000	18 987.08	0.1	4	2	2	3	0.000
		8	23 870.26	1.0	5	1	2	2	0.004	23 870.26	0.1	5	1	6	2	0.004
		10	26 212.69	2.0	5	4	4	4	0.003	26 212.69	0.2	5	3	37	3	0.003
5	(5,4)	2	11 787.07	1.5	5	2	2	3	0.000	11 787.08	0.1	5	2	2	3	0.000
		4	16 287.08	1.5	5	2	2	3	0.000	16 287.08	0.1	5	2	2	3	0.000
		6	18 987.08	1.4	5	2	2	3	0.000	18 987.08	0.1	5	2	2	3	0.000
		8	23 870.26	1.0	6	1	2	2	0.001	23 870.26	0.1	6	1	6	2	0.001
		10	26 603.80	3.0	7	5	4	4	0.000	26 603.80	0.2	7	3	37	3	0.000
6	(6,5)	2	11 743.58	2.8	6	5	3	4	0.003	11 743.58	0.2	6	5	12	4	0.003
		4	19 199.33	9.8	6	6	8 806	5	0.005	19 199.33	0.3	6	6	128	5	0.005
		6	22 799.33	9.8	6	6	8 737	5	0.005	22 799.33	0.3	6	6	128	5	0.005
		8	30 343.01	21.4	8	4	69 395	3	0.006	30 343.01	0.5	8	4	1 271	3	0.006
		10	39 618.66	2.9	8	4	3	3	0.011	39 619.54	0.6	8	4	2 168	3	0.009
7	(7,6)	2	8 593.37	2.0	8	3	37	3	0.004	8 593.37	0.1	8	3	32	3	0.004
		4	19 477.39	3.7	8	5	5	5	0.004	19 477.39	0.4	8	5	141	5	0.004
		6	22 177.39	3.6	8	5	5	5	0.007	22 177.39	0.4	8	5	141	5	0.007
		8	28 931.56	7.3	9	3	7 738	3	0.010	28 931.56	0.5	9	3	568	3	0.010
		10	31 005.14	1.8	9	2	2	2	0.010	31 005.14	0.3	9	2	150	2	0.010
8	(8,7)	2	8 612.07	1.9	9	3	3	3	0.005	8 612.07	0.1	9	3	7	3	0.005
		4	18 540.89	2.0	11	2	2	2	0.007	18 540.89	0.2	11	2	19	2	0.007
		6	26 508.83	21.5	9	6	41 095	4	0.009	26 508.83	0.4	9	6	488	4	0.009
		8	29 508.83	17.1	9	6	28 783	4	0.010	29 508.71	0.4	9	6	710	4	0.010
		10	38 825.90	475.4	10	5	1 523 376	4	0.881	38 825.90	0.8	10	5	1 176	4	0.881
10	(10,9)	2	11 055.08	5.0	12	3	618	3	0.003	11 055.09	0.2	12	3	29	3	0.003
		4	17 863.25	7.4	12	3	4 845	3	0.006	17 863.25	0.3	12	3	309	3	0.006
		6	30 245.72	2.8	13	2	2	2	0.004	30 245.72	0.3	13	2	266	2	0.004
		8	37 612.33	87.1	13	4	98 017	3	0.009	37 612.33	0.6	13	4	305	3	0.009
		10	40 612.33	79.5	13	4	88 389	3	0.005	40 612.33	0.5	13	4	281	3	0.005
15	(12,10)	2	14 000.00	0.1	15	5	0	3	0.008	14 000.00	0.1	15	5	1	3	0.008
		4	28 362.90	1.2	15	5	2	3	0.007	28 362.90	0.1	15	5	2	3	0.007
		6	39 606.76	1.9	17	4	3	3	0.000	39 606.76	0.2	17	4	3	3	0.000
		8	49 379.91	5.7	17	4	4 012	3	0.007	49 379.91	0.3	17	4	236	3	0.007
		10	61 791.00	5.9	18	6	897	4	0.003	61 791.00	0.5	18	6	246	4	0.003
20	(12,10)	2	29 695.88	1.8	20	6	2	3	0.003	29 695.88	0.1	20	6	2	3	0.003
		4	45 359.25	1.5	20	6	2	3	0.009	45 359.25	0.1	20	6	2	3	0.009
		6	54 452.42	1.6	20	7	2	3	0.001	54 452.42	0.1	20	7	2	3	0.001
		8	65 252.42	1.5	20	7	2	3	0.000	65 252.42	0.1	20	7	2	3	0.000
		10	78 057.41	3.0	22	6	3	3	0.001	78 057.41	0.2	22	6	3	3	0.001
25	(12,11)	2	27 009.85	2.3	27	5	3	3	0.009	27 009.92	0.1	27	5	3	3	0.009
		4	54 577.74	2.3	26	7	3	3	0.010	54 577.74	0.2	26	7	3	3	0.010
		6	66 477.74	2.1	26	7	3	3	0.010	66 477.74	0.2	26	7	3	3	0.010
		8	83 277.74	1.9	26	8	3	3	0.009	83 277.74	0.2	26	8	3	3	0.009
		10	98 674.11	3.1	27	4	1 066	2	0.009	98 674.11	0.2	27	4	63	2	0.009
30	(12,12)	2	26 666.18	3.5	33	13	4	4	0.010	26 666.18	0.6	33	13	112	4	0.010
		4	39 102.74	17.9	36	6	24 316	2	0.008	39 102.74	0.7	36	6	703	2	0.008
		6	56 712.73	16.7	37	5	18 675	2	0.010	56 712.73	1.0	37	5	1 388	2	0.010
		8	85 373.21	20.1	37	6	6 635	2	0.010	85 373.22	1.4	37	6	4 010	2	0.010
		10	107 438.16	141.2	36	7	346 748	2	0.079	107 438.16	2.9	36	7	9 705	2	0.079
132	(12,12)	2	97 020.71	1.8	138	4	2	2	0.009	97 020.71	0.4	138	4	34	2	0.009
		4	134 696.98	1.8	138	4	2	2	0.006	134 696.98	0.4	138	4	22	2	0.006
		6	206 690.32	2.4	138	6	3	3	0.000	206 690.32	0.7	138	6	43	3	0.000
		8	279 290.32	2.7	137	6	3	3	0.002	279 290.32	0.7	137	6	22	3	0.002
		10	335 336.96	1.7	141	2	2	2	0.000	335 336.96	0.6	141	2	29	2	0.000

Table D2: Path-based formulations (SD-P) and (CC) initialized with the paths of the zero-toll equilibrium only, i.e. $\tau_a = 0$ for all $a \in A$ (reference configuration, time limit TL $= 1800$ s). Columns are as in Table D1.

$\lvert\mathcal{K}\rvert$	$(\#o, \#d)$	$\lvert A^{\texttt{toll}}\rvert$	(SD-P)							(CC)						
			obj	time	$\#p_i$	$\#p_g$	nodes	it	gap	obj	time	$\#p_i$	$\#p_g$	nodes	it	gap
4	(4,3)	2	11 787.07	1.5	4	2	2	3	0.000	11 787.08	0.1	4	2	2	3	0.000
		4	16 287.08	1.4	4	2	2	3	0.000	16 287.08	0.1	4	2	2	3	0.000
		6	18 987.08	1.3	4	2	2	3	0.000	18 987.08	0.1	4	2	2	3	0.000
		8	23 870.26	0.7	4	2	1	2	0.004	23 870.26	0.1	4	2	5	2	0.004
		10	26 212.69	2.2	4	4	3	4	0.003	26 212.69	0.2	4	4	35	4	0.003
5	(5,4)	2	11 787.07	1.6	5	2	2	3	0.000	11 787.08	0.1	5	2	2	3	0.000
		4	16 287.08	1.5	5	2	2	3	0.000	16 287.08	0.1	5	2	2	3	0.000
		6	18 987.08	1.3	5	2	2	3	0.000	18 987.08	0.1	5	2	2	3	0.000
		8	23 870.26	0.8	5	2	1	2	0.001	23 870.26	0.1	5	2	5	2	0.001
		10	26 603.80	2.3	5	6	3	4	0.000	26 603.80	0.2	5	5	35	4	0.000
6	(6,5)	2	11 743.58	3.0	6	5	3	4	0.003	11 743.58	0.2	6	5	12	4	0.003
		4	19 199.33	9.5	6	6	8 806	5	0.005	19 199.33	0.3	6	6	128	5	0.005
		6	22 799.33	10.0	6	6	8 737	5	0.005	22 799.33	0.3	6	6	128	5	0.005
		8	30 343.01	264.2	6	6	547 696	4	0.006	30 343.01	0.6	6	6	2 057	4	0.006
		10	39 618.55	8.7	6	6	11 123	4	0.011	39 619.54	0.6	6	6	1 818	4	0.009
7	(7,6)	2	8 593.37	1.4	7	4	2	3	0.004	8 593.37	0.1	7	4	31	3	0.004
		4	19 477.39	4.2	7	6	506	5	0.004	19 477.39	0.4	7	6	167	5	0.004
		6	22 177.39	4.3	7	6	506	5	0.007	22 177.39	0.4	7	6	167	5	0.007
		8	28 931.56	11.0	7	5	9 370	4	0.010	28 931.56	0.5	7	5	1 043	4	0.010
		10	31 005.14	12.1	7	5	14 388	3	0.010	31 005.14	0.4	7	5	630	3	0.010
8	(8,7)	2	8 612.07	1.7	8	5	2	3	0.005	8 612.07	0.2	8	5	24	3	0.005
		4	18 540.89	6.9	8	8	5 176	4	0.007	18 540.89	0.4	8	8	157	4	0.007
		6	26 508.83	20.0	8	7	38 389	4	0.009	26 508.83	0.5	8	7	1 087	4	0.009
		8	29 508.83	17.3	8	7	33 749	4	0.010	29 508.83	0.4	8	7	573	4	0.010
		10	38 825.90	354.4	8	7	1 354 519	4	0.881	38 825.90	0.8	8	7	1 099	4	0.881
10	(10,9)	2	11 055.09	2.6	10	5	42	3	0.003	11 055.09	0.2	10	5	31	3	0.003
		4	17 863.25	6.1	10	5	4 950	3	0.006	17 863.25	0.2	10	5	297	3	0.006
		6	30 245.72	4.4	10	5	875	3	0.004	30 245.72	0.3	10	5	266	3	0.004
		8	37 612.33	11.8	10	8	22 131	4	0.009	37 612.33	0.6	10	8	418	4	0.009
		10	40 612.33	12.0	10	8	18 555	4	0.005	40 612.33	0.6	10	8	418	4	0.005
15	(12,10)	2	14 000.00	0.1	15	5	0	3	0.008	14 000.00	0.1	15	5	1	3	0.008
		4	28 362.90	1.2	15	5	2	3	0.007	28 362.90	0.1	15	5	2	3	0.007
		6	39 606.76	1.4	15	6	2	3	0.000	39 606.76	0.1	15	6	2	3	0.000
		8	49 379.91	4.6	15	6	3 984	3	0.007	49 379.91	0.2	15	6	231	3	0.007
		10	61 791.00	6.1	15	11	1 578	4	0.003	61 791.00	0.5	15	11	95	4	0.003
20	(12,10)	2	29 695.88	1.8	20	6	2	3	0.003	29 695.88	0.1	20	6	2	3	0.003
		4	45 359.25	1.6	20	6	2	3	0.009	45 359.25	0.1	20	6	2	3	0.009
		6	54 452.42	1.6	20	7	2	3	0.001	54 452.42	0.1	20	7	2	3	0.001
		8	65 252.42	1.5	20	7	2	3	0.000	65 252.42	0.1	20	7	2	3	0.000
		10	78 057.41	1.3	20	8	2	3	0.001	78 057.41	0.2	20	8	2	3	0.001
25	(12,11)	2	27 012.54	3.1	25	7	2	3	0.000	27 009.92	0.1	25	7	2	3	0.009
		4	54 577.74	1.8	25	8	2	3	0.010	54 577.74	0.2	25	8	2	3	0.010
		6	66 477.74	1.7	25	9	2	3	0.010	66 477.74	0.2	25	9	2	3	0.010
		8	83 277.74	1.6	25	10	2	3	0.009	83 277.74	0.2	25	10	3	3	0.009
		10	98 674.11	7.6	25	10	1 485	3	0.009	98 674.11	0.5	25	10	240	3	0.009
30	(12,12)	2	26 666.18	1.8	30	13	2	3	0.010	26 666.18	0.3	30	13	75	3	0.010
		4	39 102.75	16.2	30	16	9 638	4	0.008	39 102.74	1.2	30	16	1 565	4	0.008
		6	56 712.73	36.5	30	16	48 655	4	0.010	56 712.73	2.0	30	16	3 993	4	0.010
		8	85 373.22	41.2	30	17	17 472	5	0.010	85 373.21	2.6	30	17	6 025	5	0.010
		10	106 524.68	TL	30	12	494 654	3	0.937	107 438.16	5.1	30	13	17 560	4	0.079
132	(12,12)	2	97 020.71	1.9	132	14	2	3	0.009	97 020.71	0.6	132	14	51	3	0.009
		4	134 696.98	2.1	132	14	2	3	0.006	134 696.98	0.6	132	14	77	3	0.006
		6	206 690.32	2.1	132	14	2	3	0.000	206 690.32	0.8	132	14	111	3	0.000
		8	279 290.32	2.2	132	15	2	3	0.002	279 290.32	0.7	132	15	64	3	0.002
		10	335 336.96	1.2	132	14	1	2	0.000	335 336.96	0.5	132	14	21	2	0.000

Table D3: Path-based formulations (SD-P) and (CC) initialized with the paths of the equilibrium at the maximal tolls only ($\tau_a = \tau_a^{\text{up}}$) (reference configuration, time limit TL $= 1800$ s). Columns are as in Table D1.

$\lvert\mathcal{K}\rvert$	$(\#o, \#d)$	$\lvert A^{\texttt{toll}}\rvert$	(SD-P)							(CC)						
			obj	time	$\#p_i$	$\#p_g$	nodes	it	gap	obj	time	$\#p_i$	$\#p_g$	nodes	it	gap
4	(4,3)	2	11 787.07	1.6	4	2	2	3	0.000	11 787.08	0.1	4	2	2	3	0.000
		4	16 287.08	1.4	4	2	2	3	0.000	16 287.08	0.1	4	2	2	3	0.000
		6	18 987.08	1.2	4	2	2	3	0.000	18 987.08	0.1	4	2	2	3	0.000
		8	23 870.26	0.8	4	2	1	2	0.004	21 879.96	0.0	4	1	1	2	9.100
		10	26 212.69	1.2	4	5	3	4	0.003	23 611.99	0.1	4	2	1	2	11.018
5	(5,4)	2	11 787.07	1.6	5	2	2	3	0.000	11 787.08	0.1	5	2	2	3	0.000
		4	16 287.08	1.5	5	2	2	3	0.000	16 287.08	0.1	5	2	2	3	0.000
		6	18 987.08	1.3	5	2	2	3	0.000	18 987.08	0.1	5	2	2	3	0.000
		8	23 870.26	0.8	5	2	1	2	0.001	21 879.96	0.0	5	1	1	2	9.098
		10	26 603.79	1.3	5	5	3	4	0.000	23 611.99	0.1	5	2	1	2	12.671
6	(6,5)	2	11 743.58	2.9	6	5	3	4	0.003	11 743.58	0.2	6	5	12	4	0.003
		4	19 199.33	10.0	6	6	8 806	5	0.005	19 199.33	0.3	6	6	128	5	0.005
		6	22 799.33	9.9	6	6	8 737	5	0.005	22 799.33	0.3	6	6	128	5	0.005
		8	30 343.01	192.3	6	6	844 782	4	0.006	30 343.01	0.6	6	6	1 387	4	0.006
		10	39 619.53	8.6	6	5	30 291	4	0.009	39 619.54	0.4	6	5	316	5	0.009
7	(7,6)	2	8 593.37	1.5	7	4	2	3	0.004	8 593.37	0.2	7	4	31	3	0.004
		4	17 955.10	3.2	7	5	4	5	8.483	17 955.10	0.2	7	5	14	5	8.483
		6	20 655.10	3.6	7	5	4	5	7.378	20 655.10	0.2	7	5	10	5	7.378
		8	27 546.70	9.8	7	4	9 949	4	5.038	27 546.70	0.2	7	4	317	4	5.038
		10	29 626.13	2.3	7	3	2	3	4.665	29 626.13	0.2	7	3	106	3	4.665
8	(8,7)	2	8 612.07	1.7	8	5	2	3	0.005	8 612.07	0.2	8	5	24	3	0.005
		4	16 644.19	0.7	8	4	1	2	11.404	16 644.19	0.1	8	4	1	2	11.404
		6	24 245.72	14.2	8	6	19 300	4	9.344	24 245.72	0.3	8	6	664	4	9.344
		8	27 245.72	13.8	8	6	18 508	4	8.317	27 245.72	0.3	8	6	664	4	8.317
		10	37 427.79	3.7	8	6	594	4	4.649	37 427.79	0.2	8	5	97	4	4.649
10	(10,9)	2	11 055.09	1.5	10	4	2	3	0.003	11 055.09	0.1	10	4	20	3	0.003
		4	17 863.25	6.6	10	4	5 363	3	0.006	17 863.25	0.3	10	4	2 639	3	0.006
		6	30 245.72	2.2	10	6	2	3	0.004	30 245.72	0.2	10	6	86	3	0.004
		8	37 612.33	21.7	10	8	53 931	5	0.009	37 612.33	0.9	10	8	615	5	0.009
		10	40 612.33	18.4	10	8	42 533	5	0.005	40 612.33	0.7	10	8	576	5	0.005
15	(12,10)	2	14 000.00	0.1	15	5	0	3	0.008	14 000.00	0.1	15	5	1	3	0.008
		4	28 362.90	1.2	15	5	2	3	0.007	28 362.90	0.1	15	5	2	3	0.007
		6	39 606.76	1.6	15	5	2	3	0.000	39 606.76	0.1	15	5	2	3	0.000
		8	49 379.91	4.1	15	5	3 243	3	0.007	49 379.91	0.2	15	5	81	3	0.007
		10	61 791.00	4.8	15	7	601	4	0.003	61 791.00	0.3	15	7	88	4	0.003
20	(12,10)	2	29 695.88	1.8	20	6	2	3	0.003	29 695.88	0.1	20	6	2	3	0.003
		4	45 359.25	1.6	20	6	2	3	0.009	45 359.25	0.1	20	6	2	3	0.009
		6	54 452.42	1.6	20	7	2	3	0.001	54 452.42	0.1	20	7	2	3	0.001
		8	65 252.42	1.5	20	7	2	3	0.000	65 252.42	0.1	20	7	2	3	0.000
		10	78 057.41	1.8	20	6	2	3	0.001	78 057.41	0.1	20	6	2	3	0.001
25	(12,11)	2	27 009.84	1.5	25	7	2	3	0.009	27 009.92	0.1	25	7	2	3	0.009
		4	54 577.74	1.4	25	7	2	3	0.010	54 577.74	0.1	25	7	2	3	0.010
		6	66 477.74	1.5	25	7	2	3	0.010	66 477.74	0.1	25	7	2	3	0.010
		8	83 277.74	1.5	25	8	2	3	0.009	83 277.74	0.1	25	8	2	3	0.009
		10	98 674.11	1.8	25	6	781	2	0.009	98 674.11	0.2	25	6	103	2	0.009
30	(12,12)	2	26 666.18	3.4	30	13	7	4	0.010	26 666.18	0.5	30	13	50	4	0.010
		4	39 102.74	25.2	30	9	34 886	4	0.008	39 102.74	0.7	30	9	431	4	0.008
		6	56 712.73	44.3	30	10	80 172	4	0.010	56 712.73	0.8	30	10	430	4	0.010
		8	85 373.21	23.3	30	11	22 418	3	0.010	85 373.21	0.6	30	11	669	3	0.010
		10	107 438.16	44.7	30	11	126 906	3	0.079	107 438.16	2.0	30	11	6 596	3	0.079
132	(12,12)	2	97 020.71	0.7	132	8	1	2	0.009	97 020.71	0.4	132	8	3	2	0.009
		4	134 696.98	0.8	132	8	1	2	0.006	134 696.98	0.3	132	8	1	2	0.006
		6	206 690.32	1.7	132	10	2	3	0.000	206 690.32	0.6	132	10	23	3	0.000
		8	279 290.32	1.7	132	10	2	3	0.002	279 290.32	0.6	132	10	23	3	0.002
		10	335 336.96	1.8	132	10	2	3	0.000	335 336.96	0.6	132	10	22	3	0.000

Table E1: OD pairs considered in our instances based on the complete Sioux Falls network, with the demand values and identifiers. The ordering of the demands (column #) determines, for each $k \leq 30$, the first k commodities used in the corresponding instances.

#	(o_k, s_k)	demand	#	(o_k, s_k)	demand	#	(o_k, s_k)	demand
1	(1,10)	1300.0	11	(12,7)	700.0	21	(9,20)	600.0
2	(17,22)	1700.0	12	(13,2)	300.0	22	(23,17)	600.0
3	(20,10)	2500.0	13	(14,8)	400.0	23	(10,13)	1900.0
4	(11,22)	1100.0	14	(18,23)	100.0	24	(15,1)	500.0
5	(2,11)	200.0	15	(21,10)	1200.0	25	(22,8)	500.0
6	(16,4)	800.0	16	(5,22)	200.0	26	(2,16)	400.0
7	(8,1)	800.0	17	(24,16)	300.0	27	(4,23)	500.0
8	(3,16)	200.0	18	(7,22)	500.0	28	(19,7)	400.0
9	(6,3)	300.0	19	(6,20)	300.0	29	(10,2)	600.0
10	(1,20)	300.0	20	(8,13)	600.0	30	(13,9)	600.0

Table E2: Comparison of (BLS) and (SD-A) on the complete Sioux Falls network (24 nodes, 76 arcs; time limit TL = 1800 s). The first 30 origin–destination pairs are listed in Table E1 and the toll arcs are selected as described in Appendix A.2. Columns are as in Table 1. A dash (–) indicates that no feasible solution was found.

$\lvert\mathcal{K}\rvert$	$(\#o, \#d)$	$\lvert A^{\texttt{toll}}\rvert$	(BLS)					(SD-A)							
			objective	time	nodes	$\text{gap}_{\texttt{own}}$ (%)	gap (%)	objective	time	$\#p_i$	$\#p_g$	nodes	it	$\text{gap}_{\texttt{own}}$ (%)	gap (%)
4	(4,2)	2	7 829.62	47.3	21 315	0.009	0.008	7 829.62	4.8	5	3	3	3	0.008	0.008
		4	12 030.89	58.1	29 131	0.010	0.010	12 030.89	221.3	6	3	256 227	3	0.010	0.010
		6	17 428.60	92.8	48 161	0.010	0.010	17 425.21	TL	6	3	1 649 936	2	0.080	0.029
		8	26 046.03	109.7	52 749	0.010	0.000	26 046.03	2.5	6	4	3	3	0.000	0.000
		10	36 277.93	TL	1 829 351	0.958	0.958	35 420.36	TL	6	2	1 350 462	2	3.775	3.402
5	(5,3)	2	7 829.62	116.2	21 370	0.003	0.003	7 829.62	6.0	8	2	3	3	0.005	0.003
		4	10 600.37	235.8	131 248	0.010	0.001	10 600.37	10.8	8	3	1 325	3	0.001	0.001
		6	13 630.89	TL	905 302	18.834	0.056	13 628.84	TL	7	2	1 149 625	3	0.071	0.071
		8	16 362.67	TL	656 199	7.469	0.208	16 348.47	TL	7	2	1 016 000	2	0.295	0.295
		10	21 124.65	TL	775 204	17.873	0.000	21 124.64	0.7	7	4	3	3	0.000	0.000
6	(6,4)	2	5 149.26	1013.6	398 784	0.008	0.007	5 149.21	6.6	6	7	3	3	0.009	0.009
		4	12 978.89	TL	182 912	20.713	0.008	12 978.89	42.0	7	5	7 931	3	0.008	0.008
		6	15 746.97	TL	211 442	37.932	0.024	15 749.63	36.3	7	7	1 572	3	0.007	0.007
		8	19 254.81	TL	197 267	39.006	0.037	19 254.63	TL	8	7	234 888	3	0.038	0.038
		10	24 519.08	TL	146 078	57.526	0.229	23 425.50	TL	8	4	920 978	2	4.908	4.908
7	(7,5)	2	4 697.87	TL	81 099	68.155	0.063	4 700.39	14.5	7	8	69	4	0.010	0.010
		4	12 520.61	TL	94 352	60.037	0.009	12 520.61	11.3	7	6	3	3	0.009	0.009
		6	–	TL	101 951	–	–	19 950.23	25.0	9	5	1 477	3	0.005	0.005
		8	22 720.98	TL	83 668	88.231	0.009	22 720.98	32.9	10	6	2 008	3	0.009	0.009
		10	28 607.54	TL	83 448	119.063	0.169	26 967.56	TL	10	6	438 682	3	6.261	6.261
8	(8,6)	2	4 304.97	TL	72 030	187.962	0.009	4 304.97	22.3	9	4	2 875	3	0.009	0.009
		4	8 935.64	TL	58 232	124.635	0.789	9 005.36	156.3	10	6	34 039	4	0.008	0.008
		6	16 825.58	TL	78 669	80.544	0.010	16 825.58	69.7	9	6	9 955	3	0.010	0.010
		8	23 437.24	TL	59 213	111.189	0.009	23 437.24	58.7	10	7	8 857	3	0.009	0.009
		10	26 497.87	TL	64 901	150.867	0.378	24 058.95	TL	10	5	472 869	2	10.554	10.554
10	(9,8)	2	–	TL	20 101	–	–	6 090.25	14.6	11	7	4	4	0.001	0.001
		4	–	TL	24 739	–	–	11 266.46	37.9	12	8	1 069	4	0.001	0.001
		6	–	TL	20 994	–	–	12 956.59	251.3	13	10	101 148	4	0.010	0.010
		8	–	TL	23 375	–	–	16 360.56	162.6	15	3	51 143	3	0.010	0.010
		10	–	TL	22 710	–	–	19 100.33	TL	14	4	605 719	2	1.284	1.284
15	(14,12)	2	–	TL	5 029	–	–	5 969.11	55.7	16	8	1 181	3	0.008	0.008
		4	–	TL	5 553	–	–	9 724.27	89.9	18	7	11 057	3	0.006	0.006
		6	–	TL	8 298	–	–	16 765.66	274.1	20	6	145 799	3	0.008	0.008
		8	–	TL	11 292	–	–	20 197.03	TL	20	4	395 161	2	1.268	1.268
		10	–	TL	10 895	–	–	24 844.92	TL	18	7	383 173	3	0.031	0.031
20	(17,13)	2	–	TL	4 972	–	–	7 738.08	74.4	22	10	2 731	4	0.008	0.008
		4	–	TL	5 152	–	–	13 774.78	TL	24	10	317 773	3	0.722	0.722
		6	–	TL	6 026	–	–	17 846.48	TL	24	5	53 778	2	3.818	3.818
		8	–	TL	5 174	–	–	21 972.38	TL	27	4	104 407	2	1.072	1.072
		10	–	TL	5 103	–	–	30 952.97	618.5	25	8	116 695	3	0.010	0.010
25	(22,14)	2	–	TL	5 244	–	–	9 466.27	55.6	26	12	608	3	0.009	0.009
		4	–	TL	4 650	–	–	16 237.35	464.2	28	11	54 036	3	0.010	0.010
		6	–	TL	3 276	–	–	24 335.52	TL	32	6	77 358	2	0.984	0.984
		8	–	TL	4 402	–	–	28 328.24	TL	33	6	434 804	2	3.232	3.232
		10	–	TL	4 960	–	–	33 366.85	TL	35	5	132 914	2	3.508	3.508
30	(24,15)	2	–	TL	2 643	–	–	7 861.46	29.0	33	8	374	2	0.000	0.000
		4	–	TL	3 910	–	–	16 830.74	77.1	35	9	5 485	2	0.006	0.006
		6	–	TL	5 105	–	–	25 325.45	TL	33	9	101 653	2	2.208	2.208
		8	–	TL	4 177	–	–	29 621.06	TL	39	8	67 672	2	10.849	10.849
		10	–	TL	3 566	–	–	36 010.16	TL	41	0	120 618	1	13.177	13.177
528	(24,24)	2	–	TL	585	–	–	69 977.71	908.4	533	97	1 417	3	0.001	0.001
		4	–	TL	66	–	–	178 591.69	209.6	537	101	2 828	3	0.000	0.000
		6	–	TL	2 279	–	–	331 152.45	148.9	539	98	652	3	0.002	0.002
		8	–	TL	1 870	–	–	395 240.08	1355.0	536	102	145 094	4	0.006	0.006
		10	–	TL	1 171	–	–	460 184.66	300.5	540	103	1 676	4	0.005	0.005

Table E3: Path-based heuristics (SD-P) and (CC) on the complete Sioux Falls network (24 nodes, 76 arcs; time limit TL = 1800 s). The "gap" column is the certified gap against the pooled bound $\overline{z}$ of Table E2; all other columns are as in Table D1.

$\lvert\mathcal{K}\rvert$	$(\#o,\#d)$	$\lvert A^{\texttt{toll}}\rvert$	(SD-P)							(CC)						
			obj	time	$\#p_i$	$\#p_g$	nodes	it	gap	obj	time	$\#p_i$	$\#p_g$	nodes	it	gap
4	(4,2)	2	7 829.62	1.2	5	3	3	3	0.008	7 829.62	0.1	5	3	3	3	0.008
		4	12 030.89	1.7	6	2	3	3	0.010	12 030.89	0.8	6	2	3 669	3	0.010
		6	17 428.60	4.2	6	3	857	3	0.010	17 428.60	1.7	6	4	15 635	3	0.010
		8	26 046.03	1.1	6	3	3	3	0.000	26 046.03	4.5	6	4	43 357	3	0.000
		10	36 277.75	1.9	6	5	4	4	0.959	36 277.93	6.7	6	5	65 463	4	0.958
5	(5,3)	2	7 829.58	2.2	8	2	3	3	0.004	7 829.62	0.2	8	2	15	3	0.003
		4	10 600.37	3.5	8	3	1 505	3	0.001	10 600.37	0.4	8	3	1 751	3	0.001
		6	13 630.89	2.0	7	2	3	3	0.056	13 630.89	0.8	7	2	3 663	3	0.056
		8	16 362.66	1.9	7	3	3	3	0.208	16 362.66	1.0	7	3	5 263	3	0.208
		10	21 124.64	2.4	7	4	3	3	0.000	21 124.64	3.2	7	4	20 810	3	0.000
6	(6,4)	2	5 149.26	1.3	6	7	2	3	0.007	5 149.26	0.2	6	7	22	3	0.007
		4	12 978.89	3.2	7	5	1 162	3	0.008	12 978.89	0.2	7	5	241	3	0.008
		6	15 749.63	7.6	7	7	3 659	3	0.007	15 749.63	0.8	7	7	3 350	3	0.007
		8	19 122.04	2.7	8	6	4	4	0.731	19 122.04	3.8	8	5	28 100	3	0.731
		10	24 519.08	9.6	8	6	850	3	0.229	24 519.08	3.7	8	6	18 046	3	0.229
7	(7,5)	2	4 700.39	2.2	7	8	3	4	0.010	4 700.39	0.3	7	8	1 057	4	0.010
		4	12 520.61	1.6	7	6	2	3	0.009	12 520.61	0.1	7	6	34	3	0.009
		6	19 950.23	3.0	9	5	46	3	0.005	19 950.23	0.4	9	5	389	3	0.005
		8	22 720.98	16.2	10	5	4 193	3	0.009	22 720.98	0.6	10	5	1 048	3	0.009
		10	28 623.49	TL	10	7	1 005 375	4	0.113	28 623.68	3.2	10	7	17 121	4	0.113
8	(8,6)	2	4 302.51	4.3	9	3	4 497	3	0.066	4 304.97	1.0	9	4	8 463	3	0.009
		4	9 002.91	39.1	10	5	152 421	4	0.036	9 002.91	3.7	10	5	22 591	4	0.036
		6	16 823.12	9.6	9	4	8 011	3	0.024	16 823.12	1.8	9	5	8 311	3	0.024
		8	23 437.24	16.3	10	5	8 222	3	0.009	23 437.24	1.2	10	6	6 338	3	0.009
		10	26 506.14	52.1	10	7	104 885	3	0.347	26 506.14	41.5	10	6	431 407	3	0.347
10	(9,8)	2	6 090.25	3.1	11	7	4	4	0.001	6 090.25	0.3	11	7	37	4	0.001
		4	11 266.46	6.8	12	8	1 196	4	0.001	11 266.46	0.8	12	8	2 205	4	0.001
		6	12 956.59	79.5	13	8	163 838	4	0.010	12 956.46	2.9	13	6	13 087	4	0.011
		8	16 360.56	21.4	15	5	14 031	3	0.010	16 360.56	1.8	15	5	7 620	3	0.010
		10	19 245.79	319.5	14	8	307 963	3	0.519	19 245.79	10.9	14	8	58 025	3	0.519
15	(14,12)	2	5 969.11	9.6	16	8	5 797	3	0.008	5 969.11	0.4	16	8	826	3	0.008
		4	9 724.27	9.5	18	11	2 854	3	0.006	9 724.27	0.7	18	11	291	3	0.006
		6	16 765.66	31.8	20	6	9 042	3	0.008	16 765.66	4.7	20	6	19 634	3	0.008
		8	20 281.36	40.3	20	6	20 201	3	0.847	20 281.36	2.7	20	6	9 383	3	0.847
		10	24 844.10	15.6	18	7	4 801	3	0.035	24 844.10	1.8	18	7	4 757	3	0.035
20	(17,13)	2	7 738.08	163.9	22	10	30 305	4	0.008	7 738.08	0.9	22	10	2 922	4	0.008
		4	13 826.14	TL	24	11	2 832 008	3	0.348	13 826.22	2.4	24	11	10 076	3	0.347
		6	17 839.72	TL	24	5	1 872 774	2	3.857	17 849.09	2.1	24	7	8 831	3	3.803
		8	21 973.41	TL	27	4	498 026	2	1.067	22 004.11	2.0	27	6	3 759	4	0.926
		10	30 952.97	47.3	25	8	7 759	3	0.010	30 952.97	1.4	25	8	1 631	3	0.010
25	(22,14)	2	9 466.27	3.3	26	15	2	3	0.009	9 466.27	0.5	26	15	102	3	0.009
		4	16 237.35	179.3	28	13	164 122	3	0.010	16 237.35	1.3	28	13	3 165	3	0.010
		6	24 335.57	TL	32	6	242 083	2	0.984	24 335.60	5.3	32	7	22 530	3	0.984
		8	27 925.61	TL	33	5	2 990 020	2	4.720	28 329.31	26.5	33	7	174 817	3	3.228
		10	33 370.48	TL	35	6	188 195	2	3.497	33 384.44	43.2	35	8	151 251	3	3.453
30	(24,15)	2	7 861.46	20.3	33	12	5 907	3	0.000	7 861.46	0.6	33	12	163	3	0.000
		4	16 830.74	650.8	35	13	732 516	3	0.006	16 830.74	2.0	35	13	4 137	3	0.006
		6	25 279.95	TL	33	12	1 976 806	3	2.392	25 736.28	3.2	33	12	9 916	3	0.576
		8	27 918.94	TL	39	0	316 931	1	17.607	30 486.34	84.0	39	9	312 775	4	7.703
		10	35 523.31	TL	41	0	315 996	1	14.728	36 831.20	547.7	41	6	1 995 165	3	10.654
528	(24,24)	2	69 977.71	13.1	533	98	3	3	0.001	69 977.71	1587.6	533	98	1 672 419	3	0.001
		4	178 278.68	1112.4	537	101	7 756	3	0.176	178 591.69	TL	537	96	1 857 097	2	0.000
		6	331 152.45	13.4	539	97	3	3	0.002	331 152.46	986.3	539	97	1 186 715	3	0.002
		8	395 239.80	13.4	536	101	3	3	0.006	395 239.80	319.4	536	101	446 803	3	0.006
		10	460 184.65	14.2	540	98	3	3	0.005	460 184.65	TL	540	98	3 277 078	3	0.005

Table F1: The first 30 origin–destination demands used to build the Eastern Massachusetts instances, in the order in which they enter the nested demand ladder ($|\mathcal{K}| = k$ uses demands 1 to k).

#	(o_k, s_k)	demand	#	(o_k, s_k)	demand	#	(o_k, s_k)	demand
1	(36,12)	14.33	11	(58,38)	17.46	21	(66,25)	10.98
2	(50,57)	121.44	12	(54,1)	0.53	22	(39,35)	6.96
3	(1,54)	1.88	13	(12,55)	146.29	23	(69,54)	624.56
4	(1,36)	11.72	14	(62,13)	5.23	24	(54,69)	562.29
5	(64,1)	0.55	15	(2,14)	104.28	25	(12,56)	146.29
6	(57,50)	157.19	16	(55,12)	136.83	26	(56,12)	136.83
7	(38,57)	5.22	17	(36,2)	13.10	27	(1,59)	2.02
8	(13,64)	2.01	18	(1,62)	1.94	28	(1,69)	1.97
9	(51,10)	11.35	19	(53,2)	4.66	29	(1,67)	1.95
10	(10,51)	8.15	20	(25,66)	42.32	30	(1,61)	1.88

Table F2: Comparison of the exact formulations (BLS) and (SD-A) on the Eastern Massachusetts network (74 nodes, 258 arcs; time limit TL $= 1800$ s). The "objective" field reports a valid lower bound $\underline{z}$ on $z^\star$; "gap$_{\text{own}}$" the gap certified by the formulation alone, and "gap" the certified gap $(\overline{z}-\underline{z})/\underline{z}$ obtained with the pooled instance-level bound $\overline{z}$. A dash (–) indicates that no feasible solution was found.

$\lvert\mathcal{K}\rvert$	$(\#o,\#d)$	$\lvert A^{\texttt{toll}}\rvert$	(BLS)					(SD-A)							
			objective	time	nodes	gap$_{\text{own}}$ (%)	gap (%)	objective	time	$\#p_i$	$\#p_g$	nodes	it	gap$_{\text{own}}$ (%)	gap (%)
4	(3,4)	2	3.22	TL	503 972	0.941	0.000	3.22	1.5	4	1	2	2	0.010	0.000
		4	4.48	TL	426 107	3.505	0.006	4.48	4.6	6	1	1 976	2	0.010	0.010
		6	5.65	TL	91 006	14.927	0.010	5.65	32.8	6	3	1 123	2	0.010	0.010
		8	6.74	TL	104 643	31.428	0.005	6.74	9.9	6	3	875	2	0.004	0.004
		10	8.65	TL	55 380	113.798	0.006	8.65	1.2	6	1	2	2	0.007	0.007
5	(4,5)	2	1.99	TL	257 500	1.636	0.000	1.99	2.2	7	1	2	2	0.000	0.000
		4	4.36	TL	150 580	5.151	0.001	4.36	4.8	7	1	149	2	0.001	0.001
		6	4.67	TL	194 846	5.445	0.008	4.67	4.3	7	1	580	2	0.009	0.009
		8	5.36	TL	108 087	13.582	0.009	5.36	14.7	6	4	3	3	0.009	0.009
		10	5.88	TL	97 517	48.963	0.011	5.88	107.2	8	3	50 364	2	0.010	0.010
6	(5,6)	2	20.22	TL	36 403	32.883	0.000	20.22	51.2	7	3	2 816	3	0.001	0.001
		4	37.28	TL	23 776	62.598	0.720	37.43	TL	9	2	191 967	3	0.310	0.310
		6	39.42	TL	23 155	70.048	0.914	39.42	TL	10	1	270 205	2	0.925	0.925
		8	–	TL	21 200	–	–	41.71	TL	10	0	200 610	1	1.888	1.888
		10	39.42	TL	21 649	157.619	8.840	42.02	TL	10	0	216 374	1	2.087	2.087
7	(6,6)	2	–	TL	17 019	–	–	20.22	97.8	8	3	2 681	3	0.002	0.002
		4	–	TL	15 543	–	–	37.40	TL	10	1	91 564	2	0.601	0.601
		6	–	TL	17 124	–	–	39.42	TL	11	1	206 695	2	1.021	1.021
		8	–	TL	18 548	–	–	41.71	TL	11	0	179 151	1	1.875	1.875
		10	40.52	TL	12 730	196.391	5.835	42.02	TL	11	0	187 747	1	2.053	2.053
8	(7,7)	2	–	TL	19 730	–	–	1.70	2.8	11	4	2	2	0.000	0.000
		4	–	TL	15 808	–	–	7.07	11.7	12	4	834	2	0.002	0.002
		6	–	TL	10 911	–	–	31.04	TL	12	7	141 244	5	0.110	0.110
		8	–	TL	5 668	–	–	45.31	TL	14	0	99 156	1	6.285	6.285
		10	–	TL	6 396	–	–	46.62	TL	14	0	161 757	1	7.276	7.276
10	(9,9)	2	–	TL	8 088	–	–	1.69	4.0	13	3	2	2	0.006	0.006
		4	–	TL	5 504	–	–	4.23	17.1	15	5	573	2	0.009	0.009
		6	–	TL	5 157	–	–	4.66	TL	16	0	91 706	1	42.413	42.413
		8	–	TL	5 907	–	–	10.04	TL	17	0	96 568	1	18.016	18.016
		10	–	TL	4 306	–	–	38.90	TL	18	0	102 937	1	3.913	3.913
15	(14,13)	2	–	TL	2 491	–	–	12.80	41.3	18	3	507	2	0.000	0.000
		4	–	TL	2 236	–	–	15.55	170.0	21	4	10 251	2	0.000	0.000
		6	–	TL	2 097	–	–	14.95	TL	22	0	88 782	1	20.962	20.962
		8	–	TL	2 066	–	–	26.76	TL	23	0	75 690	1	13.547	13.547
		10	–	TL	2 190	–	–	36.89	TL	26	0	74 000	1	10.258	10.258
20	(17,16)	2	–	TL	1 476	–	–	16.88	37.3	21	3	2	2	0.000	0.000
		4	–	TL	1 631	–	–	19.74	419.9	24	3	29 614	2	0.000	0.000
		6	–	TL	1 775	–	–	21.36	TL	28	0	125 621	1	1.378	1.378
		8	–	TL	2 310	–	–	33.79	TL	29	0	151 097	1	1.795	1.795
		10	–	TL	2 299	–	–	41.85	TL	30	0	70 200	1	18.194	18.194
25	(20,20)	2	–	TL	666	–	–	20.76	295.6	26	12	380	4	0.000	0.000
		4	–	TL	542	–	–	35.71	TL	30	10	63 960	4	0.142	0.142
		6	–	TL	916	–	–	59.77	TL	32	10	95 536	3	1.073	1.073
		8	–	TL	1 244	–	–	63.15	TL	34	8	57 372	3	2.160	2.160
		10	–	TL	449	–	–	143.90	TL	36	7	94 146	3	0.110	0.110
30	(21,23)	2	–	TL	32	–	–	27.81	525.2	33	11	9 527	4	0.003	0.003
		4	–	TL	208	–	–	53.89	214.0	37	9	13 323	3	0.010	0.010
		6	–	TL	118	–	–	69.10	TL	41	14	99 806	3	0.391	0.391
		8	–	TL	18	–	–	92.38	TL	42	18	101 691	3	0.812	0.812
		10	–	TL	18	–	–	117.63	TL	47	9	84 365	3	2.711	2.711
50	(26,29)	2	–	TL	16	–	–	30.07	202.9	53	18	773	4	0.001	0.001
		4	–	TL	1	–	–	65.14	816.1	59	12	29 279	3	0.010	0.010
		6	–	TL	1	–	–	90.30	TL	61	17	61 099	4	0.890	0.890
		8	–	TL	31	–	–	102.68	TL	65	7	57 566	2	10.302	10.302
		10	–	TL	1	–	–	118.13	TL	72	9	59 018	2	29.360	29.360
100	(32,35)	2	–	TL	1	–	–	26.09	323.9	109	28	617	3	0.010	0.010
		4	–	TL	1	–	–	50.01	TL	110	31	30 190	3	1.068	1.068
		6	–	TL	1	–	–	82.83	TL	129	30	22 010	3	23.212	23.212
		8	–	TL	1	–	–	99.94	TL	140	30	54 794	3	20.063	20.063
		10	–	TL	1	–	–	127.68	TL	141	28	34 094	3	15.651	15.651

Table F3: Path-based heuristics (SD-P) and (CC) on the Eastern Massachusetts network (74 nodes, 258 arcs; time limit TL = 1800 s). The "gap" column is the certified gap against the pooled bound $\bar{z}$ of Table F2; all other columns are as in Table D1. A dash (–) indicates that no feasible solution was found.

$\|\mathcal{K}\|$	$(\#o, \#d)$	$\|A^{\texttt{toll}}\|$	(SD-P)							(CC)						
			obj	time	$\#p_i$	$\#p_g$	nodes	it	gap	obj	time	$\#p_i$	$\#p_g$	nodes	it	gap
4	(3,4)	2	3.22	0.1	4	1	0	2	0.000	3.22	0.1	4	1	0	2	0.000
		4	4.48	0.9	6	1	1 003	2	0.010	4.48	0.2	6	1	16	2	0.010
		6	5.65	1.9	6	3	1 793	2	0.010	5.65	0.2	6	3	8	2	0.010
		8	6.74	18.9	6	2	132 703	2	0.005	6.74	0.4	6	2	197	2	0.005
		10	8.65	0.2	6	1	2	2	0.007	8.65	0.2	6	1	6	2	0.007
5	(4,5)	2	1.99	0.3	7	1	6	2	0.000	1.99	0.1	7	1	2	2	0.000
		4	4.36	0.2	7	1	4	2	0.001	4.36	0.2	7	1	10	2	0.001
		6	4.67	0.5	7	1	197	2	0.009	4.67	0.2	7	1	16	2	0.009
		8	5.36	0.3	6	3	2	2	0.009	5.36	0.2	6	3	2	2	0.009
		10	5.88	3.0	8	3	1 377	2	0.011	5.88	0.3	8	3	14	2	0.011
6	(5,6)	2	20.22	1.3	7	3	1 105	3	0.000	20.22	0.2	7	3	3	3	0.000
		4	37.44	122.0	9	2	331 417	2	0.303	37.44	0.5	9	2	941	2	0.303
		6	39.15	TL	10	0	231 886	1	1.617	39.42	8.0	10	2	83 787	2	0.915
		8	41.41	TL	10	0	644 448	1	2.631	41.78	8.1	10	2	61 653	2	1.712
		10	41.76	TL	10	0	373 967	1	2.728	42.10	20.5	10	2	233 296	2	1.912
7	(6,6)	2	20.22	2.4	8	3	542	3	0.002	20.22	0.2	8	3	3	3	0.002
		4	37.45	65.1	10	2	226 514	2	0.479	37.45	0.5	10	2	1 166	2	0.479
		6	39.15	TL	11	0	502 845	1	1.716	39.42	3.1	11	2	20 179	2	1.013
		8	41.41	TL	11	0	494 961	1	2.623	41.79	11.2	11	2	110 170	2	1.699
		10	41.76	TL	11	0	416 280	1	2.701	42.10	33.8	11	2	385 733	2	1.878
8	(7,7)	2	1.70	0.3	11	3	2	2	0.000	1.70	0.3	11	3	6	2	0.000
		4	7.07	1.7	12	4	328	3	0.002	7.07	0.4	12	1	226	2	0.002
		6	31.04	1.9	12	7	18	4	0.087	31.04	0.6	12	7	142	3	0.087
		8	46.87	TL	14	4	3 514 436	3	2.761	46.87	32.5	14	7	224 218	4	2.743
		10	49.06	708.2	14	2	245 767	2	1.953	49.06	6.0	14	2	37 730	2	1.954
10	(9,9)	2	1.69	0.4	13	3	2	2	0.006	1.69	0.3	13	3	8	2	0.006
		4	4.23	1.0	15	5	2	2	0.009	4.23	0.4	15	5	16	2	0.009
		6	6.20	1.1	16	4	2	2	7.118	6.20	0.4	16	4	56	2	7.118
		8	11.57	2.7	17	2	142	2	2.430	11.57	0.8	17	2	413	2	2.430
		10	39.03	TL	18	3	777 378	3	3.578	39.27	2.1	18	6	1 834	4	2.944
15	(14,13)	2	12.80	1.0	18	3	13	2	0.000	12.80	0.4	18	3	2	2	0.000
		4	15.55	14.3	21	4	4 427	2	0.000	15.55	0.7	21	4	149	2	0.001
		6	15.26	TL	22	0	761 392	1	18.467	16.91	0.7	22	4	142	2	6.909
		8	26.95	TL	23	0	2 403 284	1	12.735	28.98	1.4	23	4	4 524	2	4.824
		10	36.83	TL	26	0	682 674	1	10.447	38.72	2.4	26	4	4 956	3	5.044
20	(17,16)	2	16.88	1.2	21	3	2	2	0.000	16.88	0.5	21	3	2	2	0.000
		4	19.74	8.5	24	3	5 067	2	0.000	19.74	0.6	24	3	2	2	0.001
		6	21.36	1047.3	28	1	1 493 571	2	1.361	21.36	0.5	28	0	141	1	1.363
		8	33.79	7.3	29	0	2 391	1	1.783	33.79	0.6	29	0	195	1	1.783
		10	43.63	19.4	30	2	12 217	2	13.374	43.63	1.1	30	2	424	2	13.374
25	(20,20)	2	20.76	4.5	26	14	4	4	0.000	20.76	1.0	26	14	4	4	0.000
		4	35.72	156.3	30	13	61 957	4	0.126	35.72	3.0	30	13	12 129	4	0.126
		6	60.16	305.1	32	13	396 359	4	0.423	60.16	8.8	32	13	41 254	4	0.423
		8	63.01	TL	34	7	937 934	2	2.380	63.54	17.0	34	10	97 620	3	1.539
		10	143.73	TL	36	0	596 355	1	0.227	143.91	19.0	36	9	95 444	3	0.107
30	(21,23)	2	27.81	30.1	33	13	8 431	4	0.000	27.81	1.6	33	13	313	4	0.003
		4	53.89	18.7	37	11	13 996	3	0.010	53.89	5.2	37	11	21 247	3	0.009
		6	69.10	TL	41	12	1 946 090	4	0.396	69.11	11.1	41	12	38 240	4	0.386
		8	89.83	TL	42	10	1 093 149	2	3.672	92.39	43.4	42	16	210 412	4	0.802
		10	116.30	TL	47	0	620 116	1	3.887	119.61	330.6	47	12	1 581 413	3	1.010
50	(26,29)	2	30.07	31.3	53	18	6 409	4	0.000	30.07	2.5	53	18	1 655	4	0.000
		4	65.14	57.0	59	15	67 294	3	0.010	65.14	12.6	59	15	113 913	3	0.010
		6	90.38	726.3	61	16	245 515	4	0.805	90.38	7.3	61	16	17 324	4	0.805
		8	104.31	TL	65	0	367 880	1	8.583	106.12	427.9	65	13	2 203 071	3	6.730
		10	132.82	TL	72	0	429 314	1	15.049	132.83	TL	72	10	6 654 012	2	15.041
100	(32,35)	2	26.10	24.5	109	45	1 086	4	0.000	26.10	6.4	109	45	430	4	0.000
		4	50.15	697.7	110	48	857 470	4	0.795	50.15	246.1	110	47	766 063	4	0.794
		6	101.35	626.1	129	43	464 324	4	0.701	101.35	22.5	129	43	27 933	4	0.701
		8	117.82	TL	140	23	482 746	2	1.842	118.93	32.7	140	43	36 824	4	0.890
		10	145.23	TL	141	23	428 673	2	1.675	146.48	41.1	141	42	56 525	4	0.806

References

[1] E. Altman, T. Boulogne, R. El-Azouzi, T. Jiménez, and L. Wynter. A survey on networking games in telecommunications. Computers & Operations Research, 33(2):286–311, 2006.

[2] T. Başar and R. Srikant. A stackelberg network game with a large number of followers. Journal of Optimization Theory and Applications, 115(3):479–490, 2002.

[3] Y. Beck, M. Labbé, and M. Schmidt. Toll setting with robust wardrop equilibrium conditions under budgeted uncertainty. Preprint. Available at https://optimization-online.org/?p=26949, 2026.

[4] M.J. Beckmann, C.B. McGuire, and C.B. Winsten. Studies in the Economics of Transportation. Yale University Press, 1956.

[5] M. Bouhtou, M. Diallo, and L. Wynter. Fair network resource allocation and link pricing: A numerical study. Technical Report RR–4680, INRIA, 2002.

[6] L. Brotcorne, M. Labbé, P. Marcotte, and G. Savard. Bilevel model for toll optimization on a multicommodity transportation network. Transportation Science, 35(4):345–358, 2001.

[7] Q.M. Bui, M. Carvalho, and J. Neto. Asymmetry in the complexity of the multi-commodity network pricing problem. Mathematical Programming, 208:425–461, 2024.

[8] S. Dempe and A.B. Zemkoho. Bilevel road pricing: theoretical analysis and optimality conditions. Annals of Operations Research, 196:223–240, 2012.

[9] Gurobi Optimization, LLC. Gurobi Optimizer Reference Manual. https://www.gurobi.com, 2026.

[10] T. Harks, M. Schröder, and D. Vermeulen. Toll caps in privatized road networks. European Journal of Operational Research, 276(3):947–956, 2019.

[11] M. Labbé, P. Marcotte, and G. Savard. A bilevel model of taxation and its application to optimal highway pricing. Management Science, 44(12):1608–1622, 1998.

[12] L.J. LeBlanc, E.K. Morlok, and W.P. Pierskalla. An efficient approach to solving the road network equilibrium traffic assignment problem. Transportation Research, 9:309–318, 1975.

[13] G.P. McCormick. Computability of global solutions to factorable nonconvex programs: Part i – convex underestimating problems. Mathematical Programming, 10(1):147–175, 1976.

[14] N. Nisan, T. Roughgarden, É. Tardos, and V. V. Vazirani, editors. Algorithmic Game Theory. Cambridge University Press, 2007.

[15] M. Patriksson. The Traffic Assignment Problem - Models and Methods. VSP Publishers, Utrecht, Netherlands, 1994.

[16] Transportation Networks for Research Core Team. Transportation networks for research. https://github.com/bstabler/TransportationNetworks.

[17] C. P. M. van Hoesel, A. F. van der Kraaij, C. Mannino, G. Oriolo, and M. Bouhtou. Polynomial cases of the tarification problem. Technical Report RM/03/053, METEOR, Maastricht University, 2003.

[18] C.P.M. van Hoesel. An overview of stackelberg pricing in networks. European Journal of Operational Research, 189(3):1393–1402, 2008.

[19] J. G. Wardrop. Some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers, 1(3):325–362, 1952.

[20] J. G. Wardrop and J. I. Whitehead. Correspondence. some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers, 1(5):767–768, 1952.