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M. Aziz, H. Suprême H. Dagdougui, A. Lesage-Landry

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Optimal energy trading in residential prosumer clusters via graphon mean field games

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Abstract : In this work, we address the optimal energy trading (OET) problem in distribution grids operated by an energy grid operator (EGO) and a large number of prosumer households. We first introduce a cluster-of-clusters (CoC) architecture that partitions the grid into residential prosumer clusters (RPCs), each coordinated by an aggregator responsible for internal energy exchanges and external interactions with neighbouring clusters and the EGO. To optimize energy exchanges within and across clusters, we develop a new decentralized control approach based on major-minor graphon mean field game theory (MM-GMFG). This framework models the OET problem as a decentralized dynamic game, deriving optimal control strategies that minimize each prosumer’s individual energy cost (minor-agents) and optimal strategies for the EGO (major agent). Household (minor-agent) dynamics and cost functions are influenced by: (i) local aggregate effects within their RPC and global interactions across the interconnected CoC network, and (ii) the EGO decisions (major-agent strategies). EGO dynamics and cost function are, in turn, influenced by the aggregate effect of all the minor agents. We derive optimal control strategies interpreted as a Nash equilibrium for the formulated game and establish existence, uniqueness under a full-coupling Lasry–Lions monotonicity condition, and a finite-population ε_N -Nash approximation guarantee under finite-to-continuum cost-consistency conditions. Numerical experiments conducted on a dense energy network with 100 clusters, each containing 200 heterogeneous households, demonstrate the computational feasibility and economic performance of the proposed MM-GMFG approach.

Keywords: Decentralized energy management; graphon mean field game; major-minor agent dynamic games; optimal energy exchange; optimal pricing; energy grid operator; home energy management system

Résumé : Dans ce travail, nous abordons le problème du commerce optimal d’énergie (optimal-energy-trading, OET) dans des réseaux de distribution exploités par un opérateur de réseau énergétique (energy grid operator, EGO) et un grand nombre de ménages prosommateurs. Nous introduisons d’abord une architecture de type agrégation d’agrégations (cluster-of-clusters, CoC) qui partitionne le réseau en grappes résidentielles de prosommateurs (Residential Prosumer Clusters, RPC), chacune coordonnée par un agrégateur responsable des échanges d’énergie internes ainsi que des interactions externes avec les agrégations voisines et l’EGO. Afin d’optimiser les échanges d’énergie à l’intérieur des agrégations et entre ceux-ci, nous développons une nouvelle approche de commande décentralisée fondée sur la théorie des jeux à champ moyen graphon majeur-mineur (major-minor graphon mean field games, MM-GMFG). Ce cadre modélise le problème d’OET comme un jeu dynamique décentralisé et permet de dériver des stratégies de commandes optimales minimisant à la fois le coût énergétique individuel de chaque prosommateur (agents mineurs) et le coût associé aux stratégies optimales de l’EGO (agent majeur). Les dynamiques et fonctions de coût des ménages (agents mineurs) sont influencées par : (i) les effets agrégés locaux au sein de leur RPC ainsi que par les interactions globales à travers le réseau CoC interconnecté, et (ii) les décisions de l’EGO (stratégies de l’agent majeur). En retour, les dynamiques et la fonction de coût de l’EGO sont influencées par l’effet agrégé de l’ensemble des agents mineurs. Nous dérivons des stratégies de commande optimales pouvant être interprétées comme un équilibre de Nash pour le jeu formulé. Nous établissons également des résultats d’existence et d’unicité sous une condition de monotonie de Lasry-Lions en couplage complet, ainsi qu’une garantie d’approximation ε_N -Nash pour une population finie, sous des conditions de cohérence entre les coûts des modèles finis et continus. Enfin, des expériences numériques menées sur un réseau énergétique dense composé de 100 agrégations, chacun regroupant 200 ménages hétérogènes, démontrent la faisabilité computationnelle ainsi que les performances économiques de l’approche MM-GMFG proposée.

Mots clés : Gestion décentralisée de l’énergie; graphon mean field game; optimal energy exchange; mécanismes de tarification optimale; agrégation de prosommateurs résidentiels; système de gestion; énergétique domestique

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1 Introduction

Distribution grids are evolving from a centrally dispatched network, with power flowing only downstream from a substation, to networks populated by residential prosumers equipped with rooftop solar photovoltaic panels (PVs), home batteries, electric vehicles (EVs), and smart appliances, actively seeking to minimize their consumption costs and inducing bidirectional power flows in the process. This shift challenges grid operators to create scalable coordination mechanisms that integrate thousands of small actors while preserving affordability and reliability [23, 30]. Policy frameworks increasingly recognize and enable such end-user participation: the EU Electricity Directive 2019/944 establishes rights for active customers, citizen energy communities, and aggregators; clarifies access to electricity markets, demand response, and data use, while urging distribution system operators (DSOs) to procure flexibility as an alternative to traditional network reinforcement through equipment upgrades [13, 14].

In this paper, we consider residential prosumer clusters (RPCs): geographically or electrically coherent groups of households coordinated by an aggregator that manages internal energy exchanges and interfaces with neighbouring clusters and the energy grid operator (EGO). Fig. 1 depicts the CoC architecture considered in our framework.

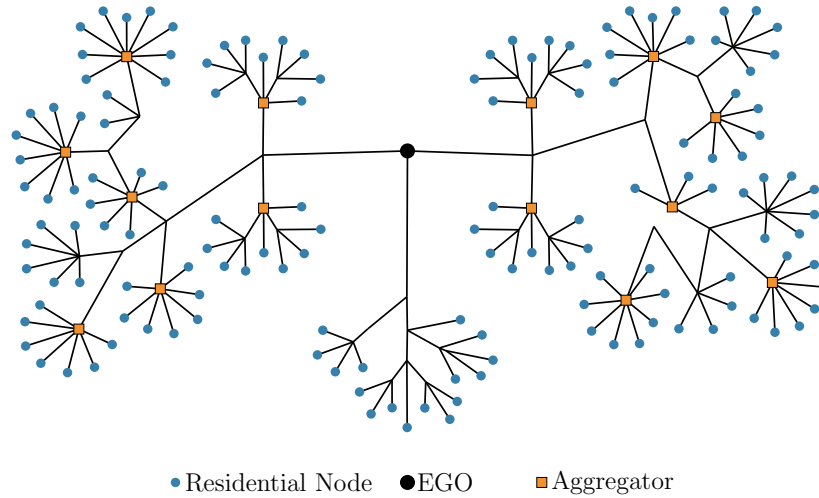


Fig. 1: CoC energy grid: partitioned into residential prosumer clusters (RPCs), each coordinated by an aggregator that trades internally and with neighbouring clusters and the EGO.

This architecture aligns with emerging models for energy communities and aggregation, and with the distribution-level flexibility procurement procedures now being articulated for EGOs in Europe [13, 14]. While related paradigms like transactive energy, virtual power plants (VPPs), and peer-to-peer (P2P) markets, have matured conceptually, their deployment at household scale is often limited by the need for coordination mechanisms that (i) account for network structure, i.e., who interacts with whom, (ii) accommodate multi-scale interactions between a system-level operator and many prosumers, and (iii) remain computationally tractable in dense neighbourhoods [20, 27, 33].

Many existing market/control formulations for prosumer coordination treat interactions as fully aggregative, where each agent responds to an undifferentiated population mean field [12, 26]. This assumption simplifies analysis but neglects the topology of interactions, e.g., feeder neighbourhoods, social or contractual ties. Graphon methods address this by modeling large network limits via measurable kernels that capture dense interaction patterns. Recent advances provide existence, approximation, and intervention results for games and controls on large graphs [18, 31]. Concurrently, major-minor mean field game (MM-MFG) theory extends classical MFGs to include a major agent whose actions influence the entire population, e.g., a system operator, alongside many minor agents with negligible individual impact, yielding equilibrium concepts and ε -Nash approximations suitable for hierarchical

energy systems [9, 21, 28]. Graphon mean field games (GMFGs) generalize MFGs to structured interaction networks and admit ε -Nash equilibrium guarantees linking infinite-population equilibria to large finite networks [7, 8]. Rather than assuming that every agent is affected by the same aggregate mean field, a GMFGs allow different groups of agents to experience different population effects depending on their position in the interaction network.

To fill the literature gap, our approach formulates the optimal energy trading (OET) problem in distribution grids as a major–minor graphon mean field game (MM-GMFG) between an EGO, modelled as the major agent, and a large population of residential households representing minor agents organized into RPCs. Each household’s dynamics and cost couple (i) to local aggregate effects within its cluster and global interactions across clusters through the graphon to capture dense inter-cluster exchange opportunities, and (ii) to the EGO’s pricing strategies that internalize system-level constraints and costs. The EGO’s dynamics and objective, in turn, respond to the aggregate state of all households. This construct directly addresses the three needs above: it retains network structure via the graphon, embeds hierarchy through the major–minor split, and yields scalable equilibrium computation leveraging mean-field limits [8, 28, 31].

Mean field approaches (control and game) have been used to coordinate flexible loads and distributed energy resources (DER) ensembles, e.g., thermostatic loads or EVs, and to design demand response mechanisms that desynchronize or reshape large populations [3]. Recent works also apply MFG ideas to VPP dispatch and ancillary services at scale, underscoring the promise of mean-field approximations for DER coordination [36, 37]. However, most models either ignore detailed network connectivity, i.e., are purely aggregative mean fields, or exclude an explicit system operator as a strategic player. By combining MM-MFG with graphon structure, our framework captures both features and is thus well-suited to RPC–EGO coordination under emerging flexibility markets and distribution system operator (DSO) procurement procedures [13, 14]. This combination is particularly important in distribution systems, where consumers connected to different feeders, transformers, or network areas may have substantially different impacts on congestion, and local flexibility requirements. In summary, our key contributions are as follows:

1. We formalize an RPC architecture in which each cluster is coordinated by an aggregator that clears internal trades and interfaces with neighbouring clusters and the EGO, consistent with the roles granted to aggregators and energy communities in current regulation [14].
2. We formulate the OET problem as an MM-GMFG: households (minors) optimize individual costs subject to local/global aggregate effects induced by a graphon; the EGO (major) optimizes a system-level objective responsive to aggregate household states. We characterize the Nash equilibrium through the fixed-point structure of the coupled household and the best responses of the EGO following MM-MFG [28] and GMFG theory [8]. We establish the existence of an MM-GMFG equilibrium and its uniqueness under a Lasry–Lions monotonicity assumption.
3. We derive scalable, decentralized control laws that constitute an ε_N -Nash equilibrium for the finite-population game under a finite-to-continuum cost-consistency condition, leveraging graphon limits and the MM-MFG decomposition.
4. We demonstrate the economic performance and computational feasibility of the proposed method through numerical experiments on a dense network with 100 clusters, each containing 200 heterogeneous households, and report an additional 200-cluster experiment to examine measured computational scaling.

Our approach advances distributed control for large-scale energy trading by embedding network-aware interactions and operator hierarchy within a single, analyzable game-theoretic construct. It further provides a path to interoperability with transactive energy and VPPs paradigms which are put forward as vehicles for scaling DER participation, while aligning with DSO/EGO procurement processes and consumer participation rights [14, 20, 27].

The remainder of this paper is structured as follows. Section 2 introduces the CoC architecture, the LinDistFlow graphon kernel, and the system dynamics and control variables for a generic household and the EGO, along with the formulation of the decentralized OET problem as an MM-GMFG. Section 3 presents the MM-GMFG approach, detailing the coupled minor-agent and major-agent equations and establishing the existence, uniqueness, and ε_N -Nash property of the fixed-point solution. Section 4 provides a numerical example of the proposed method. Finally, Section 5 sums up the paper and presents potential extensions to the proposed MM-GMFG approach for the OET.

2 Heterogeneous CoC OET via MM-GMFG

Building on the previous OET formulation [2], this work tackles the practical heterogeneous cluster-of-clusters (CoC) case, in which the energy grid operator participates actively and imposes dynamic pricing. In previous work, clusters consisted of uniform prosumers possessing similar load and generation equipment, and the energy grid operator was a passive player with flat pricing; this was established by using uniform technical parameters across all the CoC prosumers' DERs and demand side management (DSM) appliances. In this work, we instead consider a distribution-grid level modelled as a CoC with heterogeneous prosumers and an active EGO. Each household is equipped with a home energy management system (HEMS) set to seek the optimal household controls minimizing the energy bill. Households participate in the local RPC, CoC, and grid-operated markets via their respective aggregators. We consider that the aggregators act as passive players; their role is to mediate the communications and energy trading transactions without any controlling power. Thus, HEMS are solving a fully decentralized optimization problem. Fig. 2 depicts the CoC residential energy network as a weighted, undirected graph, which is modelled as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}, \mathcal{C})$, where $\mathcal{V} \subset \mathbb{N}$ is the set of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ the set of edges, $\mathcal{W} : \mathcal{V} \times \mathcal{V} \rightarrow [0, 1]$ a symmetric weight function representing the influence between nodes.

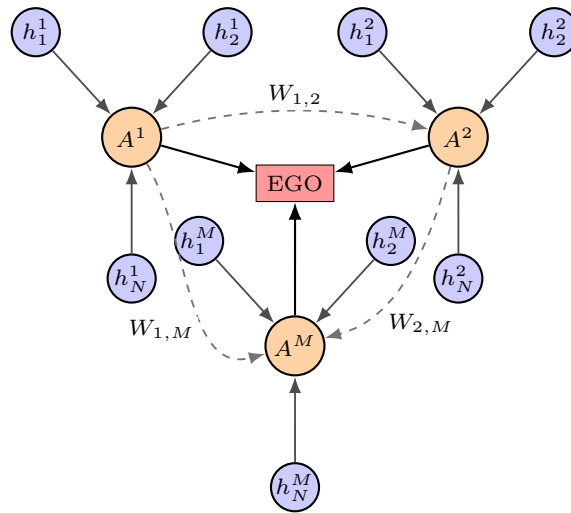


Fig. 2: Cluster-of-Clusters (CoC) Graph Illustration.

For the CoC grid, we adopt the electrical distance as the weight measure, and $\mathcal{C} = \{\mathcal{C}^1, \dots, \mathcal{C}^M\}$, $M \in \mathbb{N}$, is the set of M clusters partitioning the nodes, each cluster $\mathcal{C}^p$ coordinated by an aggregator $\mathcal{A}^p$. As the number of households in the CoC grows large, we pass to the graphon limit of $\mathcal{G}$. Households are then relabelled by a measure-preserving map onto the continuum $x \in [0, 1]$, and the discrete weight function $\mathcal{W}$ is identified with a graphon kernel $\mathcal{W} : [0, 1]^2 \rightarrow [0, 1]$. From this point on, all household-indexed quantities are written as functions of $x \in [0, 1]$, and $\kappa : [0, 1] \rightarrow \{1, \dots, M\}$ denotes the resulting measurable cluster map, with $\kappa(x) = p$ if household x belongs to cluster $\mathcal{C}^p$.

2.1 CoC electrical distance via LinDistFlow

To embed the physical network topology into the graphon interaction kernel, we construct a topology-informed LinDistFlow-based electrical proximity index coupling residential prosumer clusters, enabling the interaction structure to reflect their heterogeneous locations, network dependencies, and contributions to flexibility exchanges [5, 15, 16]. This choice is motivated by the fact that residential distribution feeders are typically radial or weakly meshed, have non-negligible resistance-to-reactance ratios, and are strongly constrained by voltage-magnitude variations. Hence, for prosumer households equipped with PV, batteries, EVs, and flexible loads, voltage sensitivity provides a more distribution-appropriate measure of electrical coupling than a transmission-level line-flow redistribution index.

The construction distinguishes two regimes:

- Intra-cluster: households belonging to the same cluster are treated as electrically close at the market-coupling level; their mutual interaction weight is set to 1.
- Inter-cluster: the coupling strength between two clusters decays with a LinDistFlow-based electrical distance that quantifies how differently injections at the corresponding clusters affect bus voltages.

In the graphon limit, households are indexed by $x \in [0, 1]$ with cluster map $\kappa : [0, 1] \rightarrow \{1, \dots, M\}$. For households in the same cluster, $\mathcal{W}(x, y) = 1$ whenever $\kappa(x) = \kappa(y)$. For $\kappa(x) \neq \kappa(y)$, $\mathcal{W}(x, y)$ is determined by the inter-cluster LinDistFlow electrical distance derived below.

2.1.1 LinDistFlow voltage-sensitivity model

Consider the underlying distribution network with bus set $\mathcal{N}$, where $|\mathcal{N}| = N$, and line set $\mathcal{L}$, where $|\mathcal{L}| = L$. Let $r \in \mathcal{N}$ denote the substation or slack bus. For a radial distribution feeder, the LinDistFlow approximation relates squared voltage magnitudes to active and reactive nodal injections through an affine sensitivity model

$$\mathbf{v}(t) = \mathbf{v}^0(t) - \mathbf{R}p(t) - \mathbf{X}q_r(t), \quad (1)$$

where $\mathbf{v}(t) \in \mathbb{R}^N$ collects squared voltage magnitudes, $\mathbf{v}^0(t)$ is the base-case voltage profile, $p(t), q_r(t) \in \mathbb{R}^N$ are the active and reactive nodal net injections, and $\mathbf{R}, \mathbf{X} \in \mathbb{R}^{N \times N}$ are the active- and reactive-power voltage-sensitivity matrices induced by the feeder topology and line impedances.

For a radial feeder, the entries of $\mathbf{R}$ and $\mathbf{X}$ can be computed from the common electrical path between buses and the substation:

$$\mathbf{R}_{ij} = 2 \sum_{e \in \mathcal{P}_i \cap \mathcal{P}_j} r_e, \quad \mathbf{X}_{ij} = 2 \sum_{e \in \mathcal{P}_i \cap \mathcal{P}_j} x_e,$$

where $\mathcal{P}_i$ is the set of lines on the unique path from the substation to bus i , and r_e and x_e are the resistance and reactance of line e . Thus, $\mathbf{R}_{ij}$ and $\mathbf{X}_{ij}$ quantify how an injection at bus j affects the voltage magnitude at bus i under the LinDistFlow approximation.

2.1.2 Cluster aggregation and LinDistFlow electrical distance

Let $\mathcal{N}_p \subseteq \mathcal{N}$ denote the set of buses associated with households in cluster $\mathcal{C}^p$. Define the cluster aggregation matrix $\mathbf{A}_{\text{cl}} \in \mathbb{R}^{N \times M}$ by

$$\mathbf{A}_{\text{cl}}(i, p) = \begin{cases} 1, & i \in \mathcal{N}_p, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

If desired, $\mathbf{A}_{\text{cl}}$ may be column-normalized or weighted by the rated flexible capacity of households in each cluster. The cluster-level voltage-sensitivity matrices are

$$\mathbf{R}^{\text{agg}} = \mathbf{R} \mathbf{A}_{\text{cl}}, \quad (3)$$

$$\mathbf{X}^{\text{agg}} = \mathbf{X}\mathbf{A}_{\text{cl}}, \quad (4)$$

where the p^{th} column of $\mathbf{R}^{\text{agg}} \in \mathbb{R}^{N \times M}$ provides the feeder-wide voltage-sensitivity profile associated with simultaneous unit active-power injections at the buses belonging to cluster $\mathcal{C}^p$, according to the binary aggregation matrix $\mathbf{A}_{\text{cl}}$ in (2), and the p^{th} column of $\mathbf{X}^{\text{agg}} \in \mathbb{R}^{N \times M}$ gives the corresponding reactive-power voltage-sensitivity profile. If a unit total cluster injection is desired instead, the columns of $\mathbf{A}_{\text{cl}}$ may be normalized accordingly. For clusters $p, q \in \{1, \dots, M\}$, define the LinDistFlow dissimilarity as

$$d_{pq}^{\text{LDF}} := \left(\|\mathbf{R}_{:,p}^{\text{agg}} - \mathbf{R}_{:,q}^{\text{agg}}\|_2^2 + \beta_Q \|\mathbf{X}_{:,p}^{\text{agg}} - \mathbf{X}_{:,q}^{\text{agg}}\|_2^2 \right)^{1/2}, \quad (5)$$

where $\beta_Q \geq 0$ weights the relative importance of reactive-power voltage sensitivity. If reactive-power control is not explicitly modelled or is unavailable, one may set $\beta_Q = 0$ and use only the active-power sensitivity matrix $\mathbf{R}$. Small d_{pq}^{LDF} indicates that injections from clusters p and q produce similar voltage-impact profiles over the feeder, whereas large values indicate that the two clusters are electrically dissimilar from the perspective of distribution-voltage coupling.

2.1.3 Normalization and interpretation

The raw dissimilarity in (5) depends on feeder impedance magnitudes, voltage base values, and the number of monitored buses. We define the normalized LinDistFlow electrical distance $e_{pq}^{\text{LDF}} \in [0, 1]$ by

$$e_{pq}^{\text{LDF}} := \begin{cases} \frac{d_{pq}^{\text{LDF}}}{\max_{p', q' \in \{1, \dots, M\}} d_{p'q'}^{\text{LDF}}}, & \max_{p', q' \in \{1, \dots, M\}} d_{p'q'}^{\text{LDF}} > 0, \\ 0, & \max_{p', q' \in \{1, \dots, M\}} d_{p'q'}^{\text{LDF}} = 0. \end{cases} \quad (6)$$

By construction, we have,

$$0 \leq e_{pq}^{\text{LDF}} \leq 1, \quad e_{pp}^{\text{LDF}} = 0, \quad e_{pq}^{\text{LDF}} = e_{qp}^{\text{LDF}}. \quad (7)$$

This normalization rescales a pairwise cluster-to-cluster voltage-sensitivity distance so that the graphon decay parameter has a consistent interpretation across test feeders. In the MM-GMFG model, clusters with $e_{pq}^{\text{LDF}} \approx 0$ have strong graphon coupling, while clusters with $e_{pq}^{\text{LDF}} \approx 1$ interact weakly through the global mean-effect terms.

2.1.4 Cluster representative and aggregator location

For each cluster $\mathcal{C}^p$, the aggregator is associated with a representative electrical node. For buses $i, m \in \mathcal{N}$, the corresponding bus-level LinDistFlow electrical dissimilarity is defined as

$$d_{im}^{\text{LDF}} := \left(\|\mathbf{R}_{:,i} - \mathbf{R}_{:,m}\|_2^2 + \beta_Q \|\mathbf{X}_{:,i} - \mathbf{X}_{:,m}\|_2^2 \right)^{1/2}.$$

In the graphon construction, this representative can be chosen as the LinDistFlow electrical medoid

$$a_p = \arg \min_{m \in \mathcal{N}_p} \sum_{i \in \mathcal{N}_p} \omega_i d_{im}^{\text{LDF}}, \quad (8)$$

where ω_i may represent household flexible capacity, inverter size, or be set to one. This node is used as the reference for inter-cluster distance. Operationally, if the cluster is represented as an equivalent injection to the upstream grid, the physical injection point is the cluster point of common coupling (PCC) or root bus. Thus, the medoid defines the interaction reference in the graphon, while the PCC defines the network-equivalent injection point.

2.1.5 Graphon Kernel construction

The inter-cluster interaction weight is defined by

$$\mathcal{W}_{pq} := \exp(-\alpha e_{pq}^{\text{LDF}}), \quad \alpha > 0, \quad (9)$$

where α controls how rapidly coupling weakens with LinDistFlow electrical distance. For households indexed by $x, y \in [0, 1]$, the graphon kernel is

$$\mathcal{W}(x, y) = \begin{cases} 1, & \text{if } \kappa(x) = \kappa(y), \\ \exp(-\alpha e_{\kappa(x), \kappa(y)}^{\text{LDF}}), & \text{if } \kappa(x) \neq \kappa(y). \end{cases} \quad (10)$$

Thus, the graphon preserves full intra-cluster coupling while modulating inter-cluster interactions according to distribution-level voltage sensitivity.

2.1.6 LinDistFlow voltage feasibility constraints

Beyond parameterizing the graphon kernel, the same LinDistFlow sensitivities can be used to enforce distribution-voltage feasibility of scheduled cluster exchanges. Let $\Delta p^{\text{cl}}(t)$ and $\Delta q^{\text{cl}}(t)$ denote the vectors of incremental active and reactive cluster injections. The induced voltage profile is approximated by

$$\mathbf{v}(t) = v^0(t) - R^{\text{agg}} \Delta p^{\text{cl}}(t) - X^{\text{agg}} \Delta q^{\text{cl}}(t). \quad (11)$$

Distribution-voltage feasibility is enforced through

$$\underline{v}_i \leq v_i(t) \leq \bar{v}_i, \quad \forall i \in \mathcal{N}, \quad (12)$$

where $\underline{v}_i$ and $\bar{v}_i$ are the admissible lower and upper voltage limits at bus i . Consequently, LinDistFlow plays a dual role in this framework: it defines the topology-aware graphon coupling through (6)–(12), and it provides a tractable linear proxy for voltage feasibility when such constraints are included in the EGO or aggregator optimization layer.

2.2 HEMS minor-agent system variables

Following the problem formulation in [2], the prosumer's HEMS operates the household's load and generation appliances by observing the states of its DER and DSM appliances. The DER state variables for household $h(x)$ are: (i) state of energy (SoE) for the EV battery denoted by $b_e(x, t)$, (ii) SoE for the battery energy storage system (BESS), denoted by $b_s(x, t)$, and (iii) the solar-panel generated energy, denoted by $\theta(t)$. As for the DSM appliances, the state variables are: (i) base load, denoted by $y_b(x, t)$, representing the energy consumed by appliances and devices that run continuously, (ii) controllable load, denoted by $y_c(x, t)$, representing the energy consumed by the must-run controllable appliances (e.g., HVAC, water heater, etc...), and (iii) shiftable load, denoted by $y_s(x, t)$, representing the appliances whose operation can be deferred or rescheduled within a certain time window (e.g. dishwasher, washer/dryer, etc.). Considering the heterogeneous case, the DSM and DER appliances are non-uniform; thus, the parameters governing these appliances are household-dependent. The partial differential equations modelling the dynamics of household $h(x) \in \mathcal{C}^{\kappa(x)}$ HEMS state variables are presented in this subsection.

2.2.1 DSM system variables

Base load, $y_b(x, t)$ of household $h(x)$ at time t is modelled via the following SDE:

$$\partial y_b(x, t) = z(x, t) \partial t + \epsilon_b^x \partial W_b^x, \quad (13)$$

where $z(x, t)$ is the drift rate of the base load process and W_b^x is a Brownian motion representing the stochastic behaviour of base-load appliances.

Controllable load, $y_c(x, t)$ of household $h(x)$ at time t is modelled via the following increment SDE:

$$\partial y_c(x, t) = u_c(x, t)P_c^x \partial t + \epsilon_c^x \partial W_c^x(t), \quad (14)$$

where $W_c^x(t)$ is a Brownian motion covering the stochastic nature of the controllable appliances, e.g., in case of HVAC, it resembles the uncertainty in insulation and temperature sensors, P_c^x is the rated power for switching the mode of operation for the controllable appliance of household $h(x)$ and $u_c(x, t)$ is the decision variable for the controllable load. Here, we need to note that households are equipped with different appliances, such that $P_c^x \in [P_c^{\min}, P_c^{\max}]$ for all $x \in [0, 1]$. The randomness of the controllable load specs is measured by the probability density function $\mu(P_c, x)$.

Shiftable demand, $y_s(x, t)$ of $h(x)$ is modelled via the following SDE:

$$\partial y_s(x, t) = P_s^x u_s(x, t) \partial t + \epsilon_{y_s}^x \partial W_{y_s}^x, \quad (15)$$

where $W_{y_s}^x$ represents the Brownian motion affiliated with the shiftable appliance performance, P_s^x the rated power to operate the shiftable load, and $u_s(x, t)$ is the decision variable for the shiftable load operation such that,

$$u_s(x, t) = \begin{cases} 1 & \text{if } t_s^{x,*} \leq t \leq t_s^{x,*} + t_d^x \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

where $t_s^{x,*}$ is the optimal switching time, and t_d^x is the duration needed for the appliance cycle at the household $h(x)$. Dealing with heterogeneous CoC, t_d and P_s are both agent-dependent such that $t_d^x \in [t_d^{\min}, t_d^{\max}]$, and $P_s^x \in [P_s^{\min}, P_s^{\max}]$ for all $x \in [0, 1]$. The randomness of the controllable load parameter specs is measured by the probability density function $\mu(P_s, t_d, x)$.

Household $h(x)$ load at time t $y(x, t)$ is defined by:

$$y(x, t) = y_b(x, t) + y_c(x, t) + y_s(x, t). \quad (17)$$

Hence, the dynamics of the household load is modelled via the following SDE:

$$\partial y(x, t) = z(x, t) \partial t + u_c(x, t)P_c^x \partial t + u_s(x, t)P_s^x \partial t + \epsilon_y^x \partial W_y^x, \quad (18)$$

where W_y^x is a Brownian motion representing the sum of the independent Brownian motions (W_b^x , W_c^x , $W_{y_s}^x$).

2.2.2 DER system variables

The DER system variables govern the HEMS's EV, BESS, and solar generation subsystems.

The generated energy, $\theta(x, t)$ of household $h(x)$ at time t is governed:

$$\partial \theta(x, t) = P_{PV}^x(t) \partial t + \epsilon_\theta^x \partial W_\theta^x, \quad (19)$$

where $P_{PV}^x(t)$ is the solar panel generated power at time t , W_θ^x , $1 \leq i \leq N$ are N independent Wiener process, i.e., Brownian Motion, and ϵ_θ is a positive constant.

BESS battery, $b_s(x, t)$ of household $h(x)$ at time t is governed by the following SDE:

$$\partial b_s(x, t) = -r_s(x, t) \partial t + \gamma_s(x, t) \partial t + \epsilon_s^x \partial W_s^x, \quad (20)$$

where ϵ_s^x and W_s^x represent the Brownian motion governing the BESS charging and discharging performance, and $r_s(x, t)$ and $\gamma_s(x, t)$ are the BESS discharging and charging rates, respectively.

Electric Vehicle battery, $b_e(x, t)$ when connected to $h(x)$, HEMS is governed by the following SDE:

$$\partial b_e(x, t) = -r_e(x, t)\partial t + \gamma_e(x, t)\partial t + \epsilon_e^x \partial W_e^x, \quad (21)$$

where ϵ_e^x and dW_e^x represent the Brownian motion governing the electric vehicle charging and discharging performance, and $r_e(x, t)$ and $\gamma_e(x, t)$ are the EV battery discharging and charging rates, respectively.

The statistical information on the performance of the CoC electric vehicles on the road is captured by the probability density function $\mu(o-r, d_{\text{trip}}, \eta_d, t_d, t_a, t, x)$. The o-r term represents the count of EVs on the road, d_{trip} represents the distance travelled by EVs, η_d represents the efficiency of the EV driving mode, and t_d, t_a represent the departure and arrival times of EVs, respectively. In this work, we assume that the charge and discharge of EVs only occur at the HEMS level, i.e., when on the road, the EV is assumed to be in motion or parked without being plugged in. This assumption can be relaxed in future extensions to cover the propagation of EVs from the home cluster to a parking-location cluster.

DER control variables for the BESS and EV battery the control variables are the set of decisions made by HEMS for charging and discharging of the BESS and EV battery and are reflected in the dynamics of $r_{s,e}(x, t)$ and $\gamma_{s,e}(x, t)$. For the heterogeneous case, the BESS and EV battery parameters are agent-dependent. We capture the randomness in household parameters using the probability density function $\mu(P_{g2s}, P_{s2g}, P_{g2e}, P_{e2g}, \eta_s, \eta_e, b_{\min}^s, b_{\max}^s, b_{\min}^e, b_{\max}^e, x)$ which is assumed to be known to all participating prosumers. Based on [2], denote by $\delta(x, t) = \theta(x, t) - y(x, t)$, and for simplicity (x, t) will be replaced by $(\cdot)$, when $\delta(\cdot) > 0$:

$$\gamma_s(\cdot) = \begin{cases} \eta_s^x (u_{\gamma_s}(\cdot) d\delta(\cdot) + u_{\gamma_{g,s}}(\cdot) P_{g2s}^x), & \text{if } b_s(\cdot) < b_{\max}^{s,x} \\ 0, & \text{otherwise,} \end{cases} \quad (22)$$

$$r_s(\cdot) = \begin{cases} \frac{1}{\eta_s^x} u_{r_s}(\cdot) P_{s2g}^x, & \text{if } b_s(\cdot) > b_{\min}^{s,x} \\ 0, & \text{otherwise,} \end{cases} \quad (23)$$

where $\eta_s^x \in (0, 1]$ is the charging/discharging efficiency of BESS, P_{g2s}^x and P_{s2g}^x represent the rated power for charging and discharging of the BESS, and $b_{\min}^{s,x}$ and $b_{\max}^{s,x}$ are the $b_s(x, t)$ SoE capacity limits. If the EV is connected to the HEMS, the charging and discharging energy of the EV battery dynamics are as follows:

$$\gamma_e(\cdot) = \begin{cases} \eta_e^x u_{\gamma_e}(\cdot) P_{g2e}^x, & \text{if } b_e(\cdot) < b_{\max}^{e,x} \\ 0, & \text{otherwise,} \end{cases} \quad (24)$$

$$r_e(\cdot) = \begin{cases} \frac{1}{\eta_e^x} u_{r_e}(\cdot) P_{e2g}^x, & \text{if } b_e(\cdot) > b_{\min}^{e,x} \\ 0, & \text{otherwise,} \end{cases} \quad (25)$$

where $\eta_e^x \in (0, 1]$ is the charging/discharging efficiency of EV, $u_{\gamma_e}(\cdot)$ and $u_{r_e}(\cdot)$ represent the charging and discharging controls of the EV battery, respectively, P_{g2e}^x and P_{e2g}^x are the rated power for EV charging and discharging, and $b_{\min}^{e,x}$ and $b_{\max}^{e,x}$ are the $b_e(\cdot)$ SoE capacity limits.

When $\delta(\cdot) \leq 0$, the control actions and the dynamics are described by:

$$\gamma_s(\cdot) = \begin{cases} \eta_s^x u_{\gamma_{g,s}}(\cdot) P_{g2s}^x & \text{if } b_s(\cdot) < b_{\max}^{s,x} \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

$$r_s(\cdot) = \begin{cases} \frac{1}{\eta_s^x} u_{r_s}(\cdot) P_{s2g}^x & \text{if } b_s(\cdot) > b_{\min}^{s,x} \\ 0 & \text{otherwise,} \end{cases} \quad (27)$$

$$\gamma_e(\cdot) = \begin{cases} \eta_e^x u_{\gamma_e}(\cdot) P_{g2e}^x & \text{if } b_e(\cdot) < b_{\max}^{e,x} \\ 0, & \text{otherwise,} \end{cases} \quad (28)$$

$$r_e(\cdot) = \begin{cases} \frac{1}{\eta_e^x} u_{r_e}(\cdot) P_{e2g}^x & \text{if } b_e(\cdot) > b_{\min}^{e,x} \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

2.2.3 Household $h(x) \in C^{\kappa(x)}$ HEMS objective function

The objective of household $h(x)$ HEMS is to minimize its energy bill by computing the optimal controls of the connected appliances. Denote by $\xi(x, t)$ the net energy exchange where $\xi(\cdot) = d(\cdot) - s(\cdot)$, where $d(x, t)$ and $s(x, t)$ are demand and supply. Thus,

$$\xi(\cdot) = y_b(\cdot) + y_s(\cdot) + y_c(\cdot) + \gamma_e(\cdot) + \gamma_s(\cdot) - \theta(\cdot) - r_e(\cdot) - r_s(\cdot), \quad (30)$$

and the dynamic of $\xi(x, t)$ is represented by the following SDE:

$$\begin{aligned} \partial \xi(\cdot) = & z(\cdot) \partial t + u_c(\cdot) P_c^x \partial t + u_s(\cdot) P_s^x \partial t + \eta_e^x u_{\gamma_e}(\cdot) P_{g2e}^x \partial t \\ & + \eta_s^x u_{\gamma_s}(\cdot) P_{g2s}^x \partial t + u_{\gamma_\delta}(\cdot) \partial \delta(\cdot) - P_{PV}^x \partial t - \frac{1}{\eta_e^x} u_{r_e}(\cdot) P_{e2g}^x \partial t \\ & - \frac{1}{\eta_s^x} u_{r_s}(\cdot) P_{s2g}^x \partial t + \epsilon_\xi^x \partial W_\xi^x, \end{aligned} \quad (31)$$

where W_ξ^x is a Brownian motion representing the sum of the independent system-variable Brownian motions.

The cost function $\ell(x, t)$ is defined as:

$$\begin{aligned} \ell(\cdot) = & \xi(\cdot) \left(w_1(\cdot) p_{\kappa(x)}^{\text{RPC}}(t) + w_2(\cdot) p^{\text{CoC}}(t) + w_3(\cdot) p^G(t) \right) \\ & + O(\theta(\cdot), b_e(\cdot), b_s(\cdot)) + (\tau_{\text{loc}} w_1(\cdot) + \tau_{\text{CoC}} w_2(\cdot)) |\xi(\cdot)|, \end{aligned} \quad (32)$$

where:

- $w_1(x, t)$, $w_2(x, t)$, and $w_3(x, t)$, are time-variant weights for local, inter-cluster (CoC), and grid-level energy exchanges, respectively, such that $w_k(\cdot) \in [0, 1]$ for all $k \in \{1, 2, 3\}$ and $t \in [0, T]$ and $\sum_{k=1}^3 w_k(\cdot) = 1$.
- $p_{\kappa(x)}^{\text{RPC}}(t)$ and $p^{\text{CoC}}(t)$ are the local cluster $\mathcal{C}^{\kappa(x)}$ and global CoC market prices, and $p^G(t)$ is the main-grid set energy price. In this framework, the main grid deploys a dynamic price $p^G(t)$ that depends on the CoC participant's aggregate energy demand.
- $O^x(\theta, b_e, b_s)$ captures DERs operating and maintenance costs.
- $0 \leq \tau_{\text{loc}} \leq \tau_{\text{CoC}}$ are layer-dependent grid-use tariffs charged by the EGO.

Moreover, $w_3(x, t)$ ensures that the grid operator balances imports and exports. We can infer from equation (32) that the household's cost function is affected by local and global effects reflected in the terms $p_{\kappa(x)}^{\text{RPC}}(t)$ and $p^{\text{CoC}}(t)$, respectively, and the EGO (major-agent) $p^G(t)$ decision. In addition, the terms $w_k(x, t)$, for $k \in \{1, 2, 3\}$, are the real-time coefficients for the exchange of energy between the household $h(x)$ and its RPC-CoC-grid. These parameters depend on the net energy at the household level, the aggregate energy at the local RPC, and the aggregate energy available from CoC that can be exchanged with the local cluster aggregator. In our framework, the energy grid operator balances the energy exchange, i.e., the prosumer household can buy or sell the remaining energy from and to the energy grid operator.

Let $\mathbf{X}(x, t)$ and $\mathbf{u}(x, t)$ denote the state and control of a generic agent $x \in [0, 1]$ at time t , respectively,

$$\begin{aligned} \mathbf{X}(\cdot) &:= (\theta(\cdot), y_b(\cdot), y_c(\cdot), y_s(\cdot), b_s(\cdot), b_e(\cdot)) \in \mathbb{R}^6, \\ \mathbf{u}(\cdot) &:= (u_c, u_s, u_{\gamma_\delta}, u_{\gamma_s}, u_{r_s}, u_{\gamma_e}, u_{r_e}) \in \mathcal{U}(x) \subset \mathbb{R}^7. \end{aligned}$$

The population density function of $\mathbf{X}$ at the index x is denoted by $\mu(\theta(\cdot), y_b(\cdot), y_c(\cdot), y_s(\cdot), b_s(\cdot), b_e(\cdot))$. Denote by $\Omega_{\text{par}} = (P_c, P_s, P_{g2e}, P_{e2g}, P_{g2s}, P_{s2g}, \eta_s, \eta_e)$ the set of random parameters and by $\mu_\Omega(\Omega, x)$ the probability density function for the parameters at index x . Recall the derivation of w_k , for $k \in \{1, 2, 3\}$ from [2] and let $(S_{\kappa(x)}(t), D_{\kappa(x)}(t))$, and $(S_{\text{CoC}}(t), D_{\text{CoC}}(t))$ be the aggregate (supply and demand) in RPC $\kappa(x)$, and the CoC as seen in the household x , respectively, given by:

$$\begin{aligned} S_{\kappa(x)}(t) &= \int_{\mathcal{I}_{\kappa(x)}} \int_{\Omega} (-\xi(z, t)) \mathbf{1}_{\{\xi < 0\}} \mu_z(t, \mathbf{X}) d\mathbf{X} dz \\ D_{\kappa(x)}(t) &= \int_{\mathcal{I}_{\kappa(x)}} \int_{\Omega} \xi(z, t) \mathbf{1}_{\{\xi > 0\}} \mu_z(t, \mathbf{X}) d\mathbf{X} dz \\ S_{\text{CoC}}(t) &= \int_0^1 \int_{\Omega} \mathcal{W}(x, y) (-\xi(y, t)) \mathbf{1}_{\{\xi < 0\}} \mu_y(t, \mathbf{Y}) d\mathbf{Y} dy, \\ D_{\text{CoC}}(t) &= \int_0^1 \int_{\Omega} \mathcal{W}(x, y) \xi(y, t) \mathbf{1}_{\{\xi > 0\}} \mu_y(t, \mathbf{Y}) d\mathbf{Y} dy, \end{aligned}$$

where $\mathbf{1}_{\{z\}}$ is one if z is true and zero otherwise. The regularization by $\epsilon_w > 0$ ensures that the weights (w_1, w_2, w_3) are continuous functions of the aggregate quantities (S, D) , and hence continuous with respect to the underlying population measure μ . For $\xi(x, t) = 0$, the weights are $w_1(x, t) = w_2(x, t) = 0$ and $w_3(x, t) = 1$. For the case $\xi(x, t) > 0$, the import weights of the household x are:

$$\begin{aligned} w_1(x, t) &= \min\left(1, \frac{S_{\kappa(x)}(t)}{D_{\kappa(x)}(t) + \epsilon_w}\right) \\ w_2(x, t) &= \min\left(1 - w_1(x, t), \frac{S_{\text{CoC}}(t)}{D_{\text{CoC}}(t) + \epsilon_w}\right) \\ w_3(x, t) &= 1 - w_1(x, t) - w_2(x, t), \end{aligned} \tag{33}$$

where $\epsilon_w > 0$ is a small constant introduced to ensure that all ratios are well-defined without affecting the physical interpretation of aggregates. For the case $\xi(x, t) < 0$, the export weights of the household x are:

$$\begin{aligned} w_1(x, t) &= \min\left(1, \frac{D_{\kappa(x)}(t)}{S_{\kappa(x)}(t) + \epsilon_w}\right) \\ w_2(x, t) &= \min\left(1 - w_1(x, t), \frac{D_{\text{CoC}}(t)}{S_{\text{CoC}}(t) + \epsilon_w}\right) \\ w_3(x, t) &= 1 - w_1(x, t) - w_2(x, t). \end{aligned} \tag{34}$$

2.2.4 Local mean effect

Define cluster C^k aggregate by $a^k(t)$ which is estimated via:

$$a^k(t) = \int_{\Omega_{\text{par}}} \int_{I_k} \xi(z, t) \mu(z, t) \mu_\Omega(\Omega_{\text{par}}, z) dz d\Omega_{\text{par}}, \tag{35}$$

where $I_k \subset [0, 1]$ is the set of all $x \in [0, 1]$ such that $\kappa(x) = k$. Thus, the local mean effect is:

$$p_k^{\text{RPC}}(t) = \mathcal{P}_{\text{loc}}(a^k(t)), \tag{36}$$

where $\mathcal{P}_{\text{loc}}$ is the local price function that maps $a^k(t) \mapsto p_k^{\text{RPC}}(t)$.

2.2.5 Global mean effect

Let the CoC-net energy seen at node x be defined as:

$$A(x, t) = \int_{\Omega} \int_0^1 \mathcal{W}(x, y) \xi(t, y) \mu(t, y) \mu_\Omega(\Omega, y) dy d\Omega, \tag{37}$$

and the scalar aggregate as:

$$\begin{aligned} M_g(t) &= \int_0^1 A(x, t) dx \\ &= \int_0^1 \int_{\Omega} \int_0^1 \mathcal{W}(x, y) \xi(t, y) \mu(t, y) \mu_{\Omega}(\Omega, y) dy d\Omega dx. \end{aligned} \quad (38)$$

Then, the CoC mean effect is as follows:

$$p^{\text{CoC}}(t) = \mathcal{P}_{\text{glob}}(M_g(t)), \quad (39)$$

where $\mathcal{P}_{\text{glob}}$ is the global price function that maps $M_g(t) \mapsto p^{\text{CoC}}(t)$.

In this paper, the price mapping functions $\mathcal{P}_{\text{loc}}$ and $\mathcal{P}_{\text{glob}}$ are modelled using two-sided (buy/sell) linear-supply functions [4], such that:

$$\begin{aligned} \mathcal{P}_{\text{loc}}(m_{\text{loc}}^{\kappa(x)}(t)) &= C_{\text{RPC}} + \beta_{\text{loc}}^b \left(\max \left(m_{\text{loc}}^{\kappa(x)}(t), 0 \right) \right) \\ &\quad + \beta_{\text{loc}}^s \left(\min \left(m_{\text{loc}}^{\kappa(x)}(t), 0 \right) \right) \end{aligned} \quad (40)$$

$$\begin{aligned} \mathcal{P}_{\text{CoC}}(M_g(t)) &= C_{\text{CoC}} + \beta_{\text{glob}}^b \left(\max \left(M_g(t), 0 \right) \right) \\ &\quad + \beta_{\text{glob}}^s \left(\min \left(M_g(t), 0 \right) \right), \end{aligned} \quad (41)$$

where C_{RPC} and C_{CoC} are positive constants representing the cost of energy in the RPC and CoC, respectively, and $\beta_{\text{loc}}^{s,b}, \beta_{\text{glob}}^{s,b} \geq 0$ are the buy/sell price-sensitivity constants. Because the slopes on both sides of the kink at zero are nonnegative, $\mathcal{P}_{\text{loc}}$ and $\mathcal{P}_{\text{glob}}$ are monotone non-decreasing, Lipschitz continuous with constants $L_{\text{loc}} = \max(\beta_{\text{loc}}^b, \beta_{\text{loc}}^s)$ and $L_{\text{glob}} = \max(\beta_{\text{glob}}^b, \beta_{\text{glob}}^s)$, and bounded on any bounded domain. This two-sided piecewise-affine form is thus a special case of the general price-map class postulated in Assumption 4 below, so the existence, uniqueness, and ε_N -Nash results of Section 3 apply directly to the price model used in the numerical study of Section 4.

2.3 OET via graphon mean field game theory

2.3.1 Household $h(x)$ minor-agent

Recall the SDEs of the household state variables in (18)–(21). The compact SDE for $\mathbf{X}$ can be written as:

$$d\mathbf{X} = f(\mathbf{X}, \mathbf{u}, m) dt + \Sigma dW_t. \quad (42)$$

where $m = (p_{\kappa(x)}^{\text{RPC}}, p^{\text{CoC}}, p^G)$ collects local/global/major-agent mean field arguments. Recall the HEMS objective function in (32) and define the grid-use tariff by

$$\tau(x, t) = (\tau_{\text{loc}} w_1(x, t) + \tau_{\text{CoC}} w_2(x, t)) |\xi(x, t)|. \quad (43)$$

The continuum representation for the household $h(x)$ instantaneous cost function $\ell(x, t)$ is:

$$\begin{aligned} \ell(x, t) &= \xi(\cdot) \left(w_1(\cdot) p_{\kappa(x)}^{\text{RPC}}(t) + w_2(\cdot) p^{\text{CoC}}(t) + w_3(\cdot) p^G(t) \right) \\ &\quad + O(\theta(\cdot), b_e(\cdot), b_s(\cdot)) + \tau(\cdot), \end{aligned} \quad (44)$$

The finite-horizon cost-to-go problem of household $h(x)$ is

$$\min_{\mathbf{u} \in \mathcal{U}(x)} J_x(\mathbf{u}, p^G, \mu) = \mathbb{E} \left[\int_0^T \ell(x, t) dt + g(\mathbf{X}(x, T)) \right], \quad (45)$$

where $g(\mathbf{X}(x, T))$ is the terminal cost. Let $V(t, x, \mathbf{X})$ denote the household value function. Its Hamilton-Jacobi-Bellman (HJB) equation is

$$-\partial_t V = \min_{u \in \mathcal{U}(x)} \left\{ \ell(x, t) + \nabla_{\mathbf{X}} V^\top f(\mathbf{X}, u, m) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top \nabla_{\mathbf{X}}^2 V) \right\}, \quad (46)$$

with $V(T, x, \mathbf{X}) = g(\mathbf{X})$. During the pointwise Hamiltonian minimization, the HEMS observes the realized state, net energy exchange, allocation weights, and market prices at time t . These quantities determine the transaction cost incurred at the current decision instant. The control selected at time t affects the subsequent evolution of the household states through the controlled drift and, consequently, future costs through the continuation value. Hence, the realized instantaneous cost $\ell(x, t)$ is predetermined from the currently observed quantities and is held fixed with respect to the newly selected control during the pointwise Hamiltonian minimization. Thus,

$$\frac{\partial \ell(x, t)}{\partial u_j(t)} = 0, \quad \forall u_j(t) \in \mathcal{U}_j(x, t). \quad (47)$$

Denote the value-function gradient by

$$\begin{aligned} \nabla_{\mathbf{X}} V^\top &= \left(\frac{\partial V}{\partial \theta}, \frac{\partial V}{\partial y_b}, \frac{\partial V}{\partial y_c}, \frac{\partial V}{\partial y_s}, \frac{\partial V}{\partial b_s}, \frac{\partial V}{\partial b_e} \right) \\ &\equiv (V_\theta, V_{y_b}, V_{y_c}, V_{y_s}, V_{b_s}, V_{b_e}). \end{aligned}$$

Then, the switching scalars of each control in $\nabla_{\mathbf{X}} V^\top$ are:

$$\begin{aligned} \text{Loads:} \quad & S_c = P_c V_{y_c}, \quad S_s = P_s V_{y_s} \\ \text{BESS:} \quad & S_{\gamma\delta, s} = \eta_s \delta^+ V_{b_s}, \quad S_{\gamma, s} = \eta_s P_{g2s} V_{b_s}, \quad S_{r, s} = -\frac{P_{s2g}}{\eta_s} V_{b_s} \\ \text{EV:} \quad & S_{\gamma, e} = (\eta_e P_{g2e}) V_{b_e}, \quad S_{r, e} = -\left(\frac{P_{e2g}}{\eta_e}\right) V_{b_e}. \end{aligned} \quad (48)$$

Because the Hamiltonian operator in (46) is affine in each control component, the pointwise minimizer lies at an endpoint of its interval, which yields the bang-bang rules derived from the switching scalars in (48) and explicitly derived by:

$$u_c^*(t, x, \mathbf{X}) = \begin{cases} -1, & S_c > 0, \\ 1, & S_c < 0, \\ 0, & S_c = 0, \end{cases} \quad (49)$$

$$u_s^*(t, x, \mathbf{X}) = \begin{cases} 1, & S_s < 0, \\ 0, & S_s \geq 0, \end{cases} \quad (50)$$

$$u_{\gamma\delta}^*(t, x, \mathbf{X}) = \begin{cases} 1, & S_{\gamma\delta, s} < 0, \delta(\cdot) > 0, b_s(\cdot) < b_{s, \max}^x, \\ 0, & \text{otherwise,} \end{cases} \quad (51)$$

$$(u_{\gamma, s}^*, u_{r, s}^*) = \begin{cases} (1, 0), & S_{\gamma, s} < \min\{0, S_{r, s}\}, \\ (0, 1), & S_{r, s} < \min\{0, S_{\gamma, s}\}, \\ (0, 0), & \text{otherwise,} \end{cases} \quad (52)$$

$$(u_{\gamma, e}^*, u_{r, e}^*) = \begin{cases} (1, 0), & S_{\gamma, e} < \min\{0, S_{r, e}\}, \\ (0, 1), & S_{r, e} < \min\{0, S_{\gamma, e}\}, \\ (0, 0), & \text{otherwise.} \end{cases} \quad (53)$$

Equations (49)–(53) define the optimal household feedback policy $\mathbf{u}^*(t, x, \mathbf{X})$. Under this feedback policy, the controlled household state process satisfies

$$d\mathbf{X} = f(\mathbf{X}, \mathbf{u}^*(t, x, \mathbf{X}), m) dt + \Sigma dW_t. \quad (54)$$

The corresponding population density evolves forward in time according to the Fokker-Planck-Kolmogorov (FPK) equation

$$\partial_t \mu + \nabla_{\mathbf{X}} \cdot (\mu f(\mathbf{X}, \mathbf{u}^*(t, x, \mathbf{X}), m)) - \frac{1}{2} \nabla_{\mathbf{X}}^2 : (\mu \Sigma \Sigma^\top) = 0, \quad (55)$$

with initial condition $\mu(0, \cdot) = \mu_0(\cdot)$.

2.3.2 EGO as DSO with own generation (major-agent)

In this framework, EGO, the major agent, serves as a DSO equipped with utility-owned DERs. Unlike the minor agents whose individual impact is negligible, the EGO's decisions influence every household in the CoC simultaneously. Define the net aggregate energy exchanged between the CoC and the EGO by $\mathcal{E}_\mu(t)$ calculated as follows:

$$\mathcal{E}_\mu(t) := \int_0^1 w_3(x, t) \xi^\mu(x, t) dx, \quad (56)$$

where $\xi^\mu(x, t)$ is the net household exchange induced by the equilibrium controls u^* and density μ . When $\mathcal{E}_\mu(t) > 0$, the CoC is a net importer from the main grid; when $\mathcal{E}_\mu(t) < 0$, it is a net exporter. Let $\text{Tar}_\mu(t)$ be the aggregate grid-use tariff collected by the EGO, calculated as follows:

$$\text{Tar}_\mu(t) := \int_0^1 (\tau_{\text{oc}} w_1(x, t) + \tau_{\text{CoC}} w_2(x, t)) |\xi^\mu(x, t)| dx. \quad (57)$$

Let $X_0(t) \in \mathbb{R}$ denote the EGO's net energy reserve position, representing the running balance of energy generated minus energy delivered to the CoC. Its dynamics are governed by:

$$dX_0 = (q(t) - \mathcal{E}_\mu(t))dt + \sigma_0 dW_t^0. \quad (58)$$

where $q(t) \geq 0$ is the EGO's controllable generation dispatch rate, i.e., the DSO dispatch, $\sigma_0 > 0$ captures uncertainty in generation output, and $W_0(t)$ is a scalar Wiener process, independent of the household-level noise.

Recall the LinDistFlow voltage-feasibility (11), which expresses the bus-voltage profile $v(t)$ in terms of the generic incremental cluster injection vectors $\Delta p^{\text{cl}}(t)$ and $\Delta q^{\text{cl}}(t)$. Also recall w_1, w_2, w_3 and $\xi(x, t)$ in equations (33)–(34) and (31), respectively. The active-power exchange between cluster $\mathcal{C}^p$ and the EGO is:

$$\Delta p_p^{\text{cl}}(t) := \int_{I_p} w_3(x, t) \xi(x, t) dx, \quad p = 1, \dots, M, \quad (59)$$

with $\Delta q_p^{\text{cl}}(t)$ defined analogously from the reactive-power component of household exchange where reactive control is modelled ($\beta_Q > 0$). By construction, $\sum_{p=1}^M \Delta p_p^{\text{cl}}(t) = \mathcal{E}_\mu(t)$. Substituting (59) into (11) ties $v_i(t)$ directly to the equilibrium household exchange profile and EGO payoff function. In this framework, we assume that the EGO incurs a convex cost $C_{\text{gen}}(q) = c_0 q + c_1 q^2$ with $c_0, c_1 > 0$. Then the instantaneous payoff of EGO's $\pi_0(t)$ is given by:

$$\begin{aligned} \pi_0(t) = & p^G(t) \mathcal{E}_\mu(t) + \text{Tar}_\mu(t) - C_{\text{gen}}(q(t)) - \frac{\beta}{2} (p^G(t) - p^{\text{ref}})^2 \\ & - \sum_{i \in \mathcal{N}} \lambda_i^v \left[\max(0, v_i(t) - \bar{v}_i)^2 + \max(0, \underline{v}_i - v_i(t))^2 \right] \\ & - \lambda \|q(t) - \mathcal{E}_\mu(t)\|^2, \end{aligned} \quad (60)$$

representing a linear function of both DSO and TSO payoffs covering: energy sales/purchase revenue, tariff revenue, generation cost, supply-demand imbalance penalty weighted by coefficient λ , and voltage-violation penalty weighted by λ_i^v . Then EGO expected cumulative payoff-to-go is given as follows:

$$\max_{\substack{p^G \in [\underline{p}, \bar{p}] \\ q \geq 0}} J_{\text{EGO}}(p^G, q; \mu) = \mathbb{E} \left[\int_0^T \pi_0(t) dt + g_0(\mathbf{X}_0(T)) \right], \quad (61)$$

Denote the EGO value function by $V_0(t, X_0)$. Its HJB equation is:

$$-\partial_t V_0(\cdot) = \max_{p^G, q} \left\{ \pi_0(t) + \partial_{\mathbf{X}_0} V_0(\cdot) (q(t) - \mathcal{E}_\mu(t)) + \frac{1}{2} \sigma_0^2 \partial_{\mathbf{X}_0}^2 V_0(\cdot) \right\}. \quad (62)$$

with terminal condition $V_0(T, X_0) = g_0(X_0)$. The optimal generation dispatch $q^*(t)$:

$$q^*(t) = \text{Proj}_{[0, \bar{q}]} \left(\frac{\partial_{X_0} V_0(t, X_0) - c_0 + 2\lambda \mathcal{E}_\mu(t)}{2(c_1 + \lambda)} \right), \quad (63)$$

and the optimal EGO price $p^{G,*}(t)$ is:

$$p^{G,*}(t) = \text{Proj}_{[p, \bar{p}]} \left(p^{\text{ref}} + \frac{\mathcal{E}_\mu(t)}{\beta} \right), \quad (64)$$

where $\text{Proj}_{[p, \bar{p}]}(\cdot)$ denotes projection onto the admissible price interval. The EGO raises the energy price above p^{ref} when the CoC is a large net importer ($\mathcal{E}_\mu \geq 0$) and lowers it when the CoC is a net exporter ($\mathcal{E}_\mu < 0$), with the sensitivity governed by β^{-1} . For Hydro-Québec, p^{ref} is the flat rate 7 ¢/kWh.

In this framework, we seek Nash equilibrium strategies for the major-minor agents by applying MM-GMFG to solve the OET problem in a CoC distribution grid, considering real-time operation and simultaneous decision-making for all participating players. First, consider the following definition:

Definition 1 (Nash Equilibrium). $(p_{\kappa(x)}^{\text{RPC},*}, p^{\text{CoC},*}, p^{G,*}, q^*, \mathbf{u}^*, \mu^*)$ is a Nash equilibrium if:

1. $\mathbf{u}^*$ solves the household HJB (46) given $(p^{G,*}, \mu^*)$;
2. μ^* solves the FPK (55) under $\mathbf{u}^*$;
3. $(p^{G,*}, q^*)$ solves the EGO HJB (62) given μ^* ;
4. $p_{\kappa(x)}^{\text{RPC},*}$ and $p^{\text{CoC},*}$ are computed from μ^* via (36) and (39).

Let $\mathcal{B}_H(p^G, \mu)$ map (p^G, μ) to the household best response density μ' , obtained by solving (46)–(55), and let $\mathcal{B}_E(\mu)$ map μ to the EGO best response $(p^{G,*}, q^*)$ via (63) and (64). A Nash equilibrium is a fixed point of $(\mathcal{B}_E, \mathcal{B}_H)$:

$$(p^{G,*}, q^*, \mu^*, \mathbf{u}^*) = (\mathcal{B}_E(\mu^*), \mathcal{B}_H(p^{G,*}, \mu^*)). \quad (65)$$

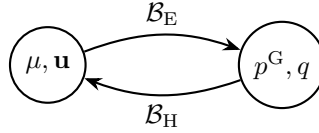


Fig. 3: Fixed-point mapping for MM-GMFG: the EGO best response $\mathcal{B}_E$ maps μ to (p^G, q) , and the household best response $\mathcal{B}_H$ maps (p^G, μ) back to an updated density μ .

Fig. 3 depicts the MM-GMFG loop, and the fixed-point solution for the loop represents the Nash equilibrium strategies for the major and minor agents with the convergent probability measure for the CoC.

3 Existence, uniqueness, and convergence of the MM-GMFG Nash equilibrium

The MM-GMFG formulated in Section 2 couples a continuum of heterogeneous minor agents (households) with a single major agent (EGO) through a graphon-weighted interaction structure and endogenous price formation. The objective of the MM-GMFG approach is to find the joint equilibrium tuple denoted by $\mathbf{e}^*$ and defined as

$$\mathbf{e}^* := (\mu^*, \mathbf{u}^*, p_{\kappa}^{\text{RPC},*}, p^{\text{CoC},*}, p^{G,*}, q^*), \quad (66)$$

where $\mu^* \in C([0, T]; \mathcal{P}_2(\mathbb{R}^d))$ is the population state distribution, $\mathbf{u}^*(x, t)$ is the household optimal control (minor agents), $p_{\kappa}^{\text{RPC},*}(t)$ and $p^{\text{CoC},*}(t)$ are the endogenous local and global prices induced

by μ^* , and $(p^{G,*}, q^*)$ are the EGO's optimal price and generation dispatch (major agent). Prices are functions of μ , controls depend on prices and μ , and μ depends on controls, creating a closed-loop fixed point.

Our analysis proceeds in four stages, drawing on the general theory of mean field games [24], [22], and its graphon extensions [6], [8].

1. **Operator decomposition:** We decompose the equilibrium into four explicit operators: a price operator Π , a minor-agent best-response operator $\mathcal{B}_H$ (HJB-FPK), a major-agent best-response operator $\mathcal{B}_E$ (EGO HJB), and their composition $\mathcal{T}$. The Nash equilibrium is equivalent to a fixed point $\mu^* = \mathcal{T}(\mu^*)$.
2. **Existence:** We verify that $\mathcal{T}$ maps a compact convex set of probability measures into itself and is continuous, then invoke the Schauder fixed-point theorem [32] to guarantee at least one equilibrium exists.
3. **Uniqueness:** We establish uniqueness under a Lasry-Lions monotonicity condition assumed on the complete population-dependent cost coupling [25]. Sufficient economic conditions include monotone price mappings, ordered trading-layer prices, and an allocation mechanism that preserves the required signed monotonicity.
4. **Finite-population approximation:** Following the ε -Nash theory for GMFGs [7] extended to major-minor settings [28], the continuum equilibrium yields an ε_N -Nash equilibrium.

3.1 Preliminaries

3.1.1 Spaces

Let $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_1)$ denote the space of Borel probability measures with finite second moment, metrized by the Wasserstein-1 distance. Define

$$\mathcal{M} := C([0, T]; \mathcal{P}_2(\mathbb{R}^d)), \quad (67)$$

$$\mathcal{P} := \{p^G : [0, T] \rightarrow [\underline{p}, \bar{p}] \text{ measurable}\}, \quad (68)$$

$$\mathcal{Q} := \{q : [0, T] \rightarrow [0, \bar{q}] \text{ measurable}\}. \quad (69)$$

Define the uniform-in-time Wasserstein metric $\mathcal{M}$ by:

$$d_{\mathcal{M}}(\mu, \nu) := \sup_{t \in [0, T]} \mathcal{W}_1(\mu_t, \nu_t). \quad (70)$$

For $R, K > 0$, define

$$\begin{aligned} \mathcal{M}_{R,K} := \Big\{ \mu \in \mathcal{M} : \mu(0) = \mu_0, \sup_{t \in [0, T]} \int_{\mathbb{R}^d} \|X\|^2 d\mu_t(X) \leq R, \\ \mathcal{W}_1(\mu_t, \mu_s) \leq K|t - s|^{1/2}, \forall s, t \in [0, T] \Big\}. \end{aligned} \quad (71)$$

3.1.2 Operators

First, we define the price formation Π . Given $\mu \in \mathcal{M}_{R,K}$, prices are determined by the monotone price maps $\mathcal{P}_{\text{loc}}$ and $\mathcal{P}_{\text{glob}}$ applied to the graphon-weighted aggregates:

$$\begin{aligned} \Pi_{\text{loc}}^k(\mu)(t) &:= \mathcal{P}_{\text{loc}}(m_{\text{loc}}^{\mu,k}(t)) = p_k^{\text{RPC}}(t), \\ \Pi_{\text{glob}}(\mu)(t) &:= \mathcal{P}_{\text{glob}}(M_g^{\mu}(t)) = p^{\text{CoC}}(t). \end{aligned} \quad (72)$$

Next, the minor-agent best response $\mathcal{B}_H$ is as follow. Given $(p^G, \mu) \in \mathcal{P} \times \mathcal{M}$: solve the household HJB backward with $m = (\Pi(\mu), p^G)$, and find $u^{\mu} = \arg \min_{u \in \mathcal{U}(x)} \{\ell + \nabla_X V \cdot f\}$. The corresponding FPK equation is then solved forward, yielding μ' .

$$\mathcal{B}_H(p^G, \mu) := (\mathbf{u}^{\mu}, \mu'). \quad (73)$$

Then, for the major-agent best response $\mathcal{B}_E$, given $\mu \in \mathcal{M}_{R,K}$, compute the aggregates $\mathcal{E}_\mu, \text{Tar}_\mu$ and solve the EGO HJB:

$$\mathcal{B}_E(\mu) := (p^{G,\mu}, q^\mu). \quad (74)$$

Finally, for the composite MM-GMFG operator $\mathcal{T}$, let $(p^{G,\mu}, q^\mu) = \mathcal{B}_E(\mu)$. The composite population-flow operator is then

$$\mathcal{T}(\mu) := \pi_2(\mathcal{B}_H(p^{G,\mu}, \mu)), \quad (75)$$

where $\pi_2(u^\mu, \mu') = \mu'$ extracts the updated population density. Fig. 4 presents the operators loop and the path to the fixed-point solution. Once μ^* is found, the full tuple (66) is recovered from

$$\begin{aligned} \mathbf{u}^* &= \mathbf{u}^{\mu^*}, \quad p_k^{\text{RPC},*} = \Pi_{\text{loc}}^k(\mu^*), \\ p^{\text{CoC},*} &= \Pi_{\text{glob}}(\mu^*), \quad (p^{G,*}, q^*) = \mathcal{B}_E(\mu^*). \end{aligned} \quad (76)$$

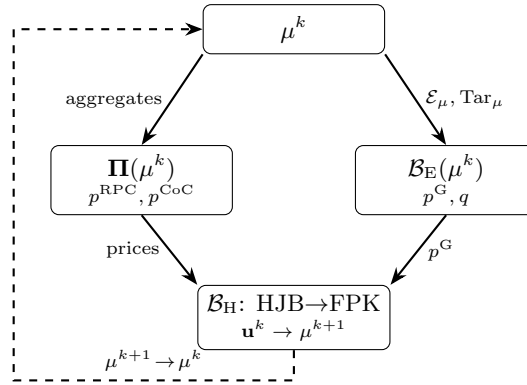


Fig. 4: MM-GMFG operator loop: Fixed point: $\mu^* = \mathcal{T}(\mu^*)$.

3.2 Standing assumptions

The equilibrium analysis below relies on a small set of regularity conditions whose role is to ensure that each component of the MM-GMFG loop, in Fig. 4 is well-posed and responds continuously to changes in the population state. For the remainder of this work, we make the following assumptions.

Assumption 1 (Drift and diffusion). The household drift $f(\mathbf{X}, \mathbf{u}, m)$ is locally Lipschitz in $(\mathbf{X}, \mathbf{u})$ uniformly in m : for every $R > 0$, there exists a constant $L_f(R) > 0$ such that

$$\|f(\mathbf{X}_1, \mathbf{u}_1, m) - f(\mathbf{X}_2, \mathbf{u}_2, m)\| \leq L_f(R)(\|\mathbf{X}_1 - \mathbf{X}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\|), \quad (77)$$

whenever $\|\mathbf{X}_i\| + \|\mathbf{u}_i\| \leq R$ for $i \in \{1, 2\}$. Moreover, f satisfies the linear-growth condition

$$\|f(\mathbf{X}, \mathbf{u}, m)\| \leq C_f(1 + \|\mathbf{X}\| + \|\mathbf{u}\|), \quad (78)$$

for some constant $C_f > 0$, uniformly in m . The diffusion $\Sigma(x)$ is bounded and uniformly elliptic: $\Sigma \Sigma^\top \succeq \underline{\Sigma}^2 I > 0$. The EGO diffusion satisfies $\sigma_0 > 0$.

At the household level, the assumptions guarantee well-defined stochastic state trajectories, admissible optimal controls, and a well-posed HJB-FPK system.

Assumption 2 (Cost structure and admissible controls). The running cost $\ell(x, t)$ is bounded below and jointly continuous in $(t, \mathbf{X}, \mathbf{u}, m)$. For each x , the admissible control set $\mathcal{U}(x)$ is nonempty and compact. The terminal cost $g : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$ is C^2 with bounded second derivatives and convex in $\mathbf{X}$. Moreover, the Hamiltonian in (46) is affine in each component of u (through the controlled drift) and $\mathcal{U}(x)$ is compact; therefore, an optimal control exists and is bang-bang. Each switching function in (48) is assumed to vanish only on a set of Lebesgue measure zero, so the point-wise minimizer is unique almost everywhere in t .

Assumption 3 (Graphon). $\mathcal{W} : [0, 1]^2 \rightarrow [0, 1]$ is symmetric, measurable, and bounded ($\|\mathcal{W}\|_\infty \leq 1$). The parameter density $\mu_\Omega(\Omega_{\text{par}}, x)$ is bounded and compactly supported. The LinDistFlow construction in (10) satisfies this assumption because $e_{pq}^{\text{LDF}} \in [0, 1]$ and $\alpha > 0$, which implies $0 < \mathcal{W}(x, y) \leq 1$ and symmetry follows from (7).

Assumption 4 (Price functions). $\mathcal{P}_{\text{loc}}, \mathcal{P}_{\text{glob}} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ are monotone non-decreasing, Lipschitz with constants $L_{\text{loc}}, L_{\text{glob}} > 0$, and bounded on bounded sets. This includes, as a special case, the single-slope affine parametrization $\mathcal{P}_{\text{loc}}(a) = p_0^{\text{loc}} + \alpha_{\text{loc}} a$ and $\mathcal{P}_{\text{glob}}(M_g) = p_0^{\text{glob}} + \alpha_{\text{glob}} M_g$, with $p_0^{\text{loc}}, p_0^{\text{glob}} \geq 0$ and $\alpha_{\text{loc}}, \alpha_{\text{glob}} \geq 0$, as well as the two-sided (buy/sell) piecewise-affine parametrization of (40)–(41) used in the numerical study, with $L_{\text{loc}} = \max(\beta_{\text{loc}}^b, \beta_{\text{loc}}^s)$ and $L_{\text{glob}} = \max(\beta_{\text{glob}}^b, \beta_{\text{glob}}^s)$.

At the market level, bounded graphon interactions and Lipschitz price mappings ensure that small changes in the population distribution do not produce discontinuous changes in the RPC and CoC prices.

Assumption 5 (EGO). Let $0 < \underline{p} \leq \bar{p} < +\infty$, $0 < \bar{q} < +\infty$. The function C_{gen} is C^2 and strictly convex: $C_{\text{gen}}''(q) \geq 2c_1 > 0$, the social-welfare parameter is $\beta > 0$, and the tariffs as $0 \leq \tau_{\text{loc}} \leq \tau_{\text{CoC}} < +\infty$.

Finally, the EGO assumptions provide a well-defined and unique best response to the aggregate household exchange. Together these conditions allow the household, market, and EGO components to be assembled into a continuous fixed-point operator.

Assumption 6 (Aggregate-map continuity). On $\mathcal{M}_{R,K}$, the aggregate mappings

$$\mu \mapsto m_{\text{loc}}^{\mu,k}, \quad \mu \mapsto M_g^\mu, \quad \mu \mapsto E_\mu, \quad \mu \mapsto \text{Tar}_\mu$$

are continuous with respect to $d_{\mathcal{M}}$, uniformly in time. In particular, if $d_{\mathcal{M}}(\mu^n, \mu) \rightarrow 0$, then

$$\|m_{\text{loc}}^{\mu^n,k} - m_{\text{loc}}^{\mu,k}\|_\infty + \|M_g^{\mu^n} - M_g^\mu\|_\infty \rightarrow 0, \quad (79)$$

$$\|E_{\mu^n} - E_\mu\|_\infty + \|\text{Tar}_{\mu^n} - \text{Tar}_\mu\|_\infty \rightarrow 0. \quad (80)$$

Assumption 6 is consistent with the bounded graphon kernel and parameter distributions in Assumption 3 and with the $\epsilon_w > 0$ regularization in (33) and (34), which prevents singularities in the allocation weight ratios.

3.3 Well-posedness of the component operators

The next lemmas establish the three links required to close the MM-GMFG feedback loop.

Lemma 1 (Price operator). Under Assumptions 3, 4 and 6,

$$\Pi : \mathcal{M}_{R,K} \rightarrow \mathcal{P}_{\text{loc}} \times \mathcal{P}_{\text{glob}} \quad \text{is continuous.}$$

Proof. By Assumption 6, the aggregate maps $\mu \mapsto m_{\text{loc}}^{\mu,k}$ and $\mu \mapsto M_g^\mu$ are continuous under $d_{\mathcal{M}}$. Composition with the Lipschitz price mappings $\mathcal{P}_{\text{loc}}$ and $\mathcal{P}_{\text{glob}}$ from Assumption 4 yields the continuity of Π . $\square$

A population distribution determines the RPC and CoC prices continuously through the price operator, together with the EGO price, these signals determine a household best response and hence an updated population distribution through the HJB-FPK system.

Lemma 2 (Minor-agent operator). Under Assumptions 1–4, for each $(p^G, \mu) \in \mathcal{P} \times \mathcal{M}_{R,K}$:

1. the household HJB admits a unique viscosity solution and a measurable minimizing selector u^μ ;
2. the corresponding FPK equation admits a unique weak solution $\mu' \in \mathcal{M}_{R,K}$;

3. the density component of $\mathcal{B}_H$ is continuous:

$$\|p_n^G - p^G\|_\infty + d_{\mathcal{M}}(\mu_n, \mu) \rightarrow 0 \implies d_{\mathcal{M}}(\mu'_n, \mu') \rightarrow 0. \quad (81)$$

Proof. HJB comparison and stability under perturbations of (p^G, μ) give well-posedness in the viscosity sense. Compactness of the admissible action set yields a measurable minimizing selector, with almost-everywhere uniqueness following from Assumption 2. The Lipschitz and linear-growth properties of the selected drift give FPK well-posedness and continuous dependence of the state law on the inputs. $\square$

Simultaneously, the same population distribution determines the aggregate exchange observed by the EGO and therefore its optimal pricing and dispatch response.

Lemma 3 (Major-agent operator). Under Assumptions 1, 5, and 6, the EGO problem admits a unique best response and $\mathcal{B}_E : \mathcal{M}_{R,K} \rightarrow \mathcal{P} \times \mathcal{Q}$ is continuous. In particular,

$$d_{\mathcal{M}}(\mu_n, \mu) \rightarrow 0 \implies \|p^{G, \mu_n} - p^{G, \mu}\|_\infty + \|q^{\mu_n} - q^\mu\|_\infty \rightarrow 0. \quad (82)$$

Proof. Strict convexity of the generation cost and the quadratic penalties make the EGO Hamiltonian strictly concave in the admissible controls. By Assumption 6, the aggregate exchange E_μ and tariff Tar_μ depend continuously on μ under $d_{\mathcal{M}}$. Together with continuity of the projection formulas (63) and (64), this yields continuity of the unique EGO best response. $\square$

Thus, a small perturbation of the population flow produces only a small perturbation throughout the complete population–market–EGO loop.

3.4 Existence of Nash equilibrium

We now show that a Nash Equilibrium exists for the MM-GMFG loop of Fig. 4.

Theorem 1 (Existence). Under Assumptions 1–6, there exist constants $R, K > 0$ such that $\mathcal{T} : \mathcal{M}_{R,K} \rightarrow \mathcal{M}_{R,K}$ has at least one fixed point μ^* . The tuple (66) recovered via (76) is a Nash equilibrium.

Proof. We verify the Schauder hypotheses [32, 35].

Step 1 (Self-mapping): Let $\mu' = \mathcal{T}(\mu)$. Under Assumption 1, Itô's formula, standard SDE moment estimates, and Grönwall's inequality [29] give

$$\sup_{t \in [0, T]} \mathbb{E} \|\mathbf{X}_t\|^2 \leq C_1 (1 + \mathbb{E} \|\mathbf{X}_0\|^2). \quad (83)$$

Standard SDE increment estimates also give

$$\mathbb{E} \|\mathbf{X}_t - \mathbf{X}_s\| \leq C_2 |t - s|^{1/2}. \quad (84)$$

Using the joint law of $(\mathbf{X}_t, \mathbf{X}_s)$ as a coupling [34],

$$\mathcal{W}_1(\mu'_t, \mu'_s) \leq C_2 |t - s|^{1/2}. \quad (85)$$

Thus, for sufficiently large R and K , $\mathcal{T}(\mathcal{M}_{R,K}) \subseteq \mathcal{M}_{R,K}$.

Step 2 (Compactness and Convexity): The set $\mathcal{M}_{R,K}$ is nonempty and convex. Its uniform second-moment bound gives spatial tightness and uniform integrability of first moments, while its Wasserstein Hölder bound gives temporal equicontinuity. Prokhorov's theorem and the metric Arzelà–Ascoli criterion [1, 34] therefore imply compactness under $d_{\mathcal{M}}$.

Step 3 (Continuity): By Assumption 6 and Lemmas 1–3, the price, household, and major-agent response operators are continuous under $d_{\mathcal{M}}$. Their composition $\mathcal{T}$ is therefore continuous on $\mathcal{M}_{R,K}$.

Step 4 (Schauder): The operator $\mathcal{T}$ is a continuous self-map of a nonempty, compact, and convex set. Schauder’s theorem [32] implies the existence of $\mu^* \in \mathcal{M}_{R,K}$ such that $\mu^* = \mathcal{T}(\mu^*)$.

Step 5 (Verification): By construction, u^* solves the household HJB, μ^* solves the FPK equation, $(p^{G,*}, q^*)$ solves the EGO HJB, and the local and global prices are computed from μ^* . Therefore, the tuple in (66) calculated via (76) is an MM-GMFG Nash equilibrium. $\square$

3.5 Uniqueness via Lasry–Lions monotonicity

We establish uniqueness under a Lasry–Lions monotonicity condition [24], adapted to the graphon and major–minor setting [8, 10]. For each graphon location $x \in [0, 1]$, let $\mu_t^x \in \mathcal{P}_2(\mathbb{R}^d)$ denote the conditional distribution of the household state $\mathbf{X}$ at time t , and write $\mu_t = (\mu_t^x)_{x \in [0,1]}$ for the population flow. Let

$$F(x, \mathbf{X}, \mu) := \xi(x, \mathbf{X}) p_{\text{eff}}^\mu(x, t) + \tau_{\text{eff}}^\mu(x, t) |\xi(x, \mathbf{X})| \quad (86)$$

denote the complete population-dependent component of the household running cost, where

$$\begin{aligned} p_{\text{eff}}^\mu(x, t) &:= w_1^\mu(x, t) p_{\kappa(x)}^{\text{RPC}, \mu}(t) + w_2^\mu(x, t) p^{\text{CoC}, \mu}(t) \\ &\quad + w_3^\mu(x, t) p^G(t), \end{aligned} \quad (87)$$

$$\tau_{\text{eff}}^\mu(x, t) := \tau_{\text{loc}} w_1^\mu(x, t) + \tau_{\text{CoC}} w_2^\mu(x, t). \quad (88)$$

Before presenting the uniqueness result, we make the following additional assumption.

Assumption 7 (Full-coupling monotonicity). For every pair of admissible population flows μ_1 and μ_2 , the complete coupling F satisfies

$$\int_0^T \int_0^1 \int_{\mathbb{R}^d} [F(t, x, \mathbf{X}, \mu_1) - F(t, x, \mathbf{X}, \mu_2)] d(\mu_{1,t}^x - \mu_{2,t}^x)(\mathbf{X}) dx dt \geq 0. \quad (89)$$

Assumption 7 requires the complete market coupling, rather than each price map separately, to preserve the appropriate signed monotonicity. Economically, greater aggregate net demand should not reduce the effective marginal cost faced by an importer or increase the effective compensation received by an exporter. Non-decreasing local, CoC, and EGO price responses support this property, but because the allocation weights also depend on local and global supply and demand, monotonicity is imposed directly on the complete coupling F . Uniqueness is then proved next.

Theorem 2 (Uniqueness). Suppose Assumptions 1–7 hold. Suppose further that the household Hamiltonian admits almost-everywhere unique minimizing controls, the corresponding FPK equation is unique, and the EGO best response is unique. Then the MM-GMFG Nash equilibrium is unique.

Proof. Let $(V_i, \mu_i, \mathbf{u}_i, p_i^G, q_i)$, $i \in \{1, 2\}$, be two equilibrium solutions with the same initial population law. Cross-testing the two household HJB equations against the corresponding FPK solutions and summing the resulting identities gives the standard HJB–FPK duality inequality [10, 24]:

$$0 \geq \int_0^T \int_0^1 \int_{\mathbb{R}^d} [F(t, x, \mathbf{X}, \mu_1) - F(t, x, \mathbf{X}, \mu_2)] d(\mu_{1,t}^x - \mu_{2,t}^x)(\mathbf{X}) dx dt. \quad (90)$$

The remaining Hamiltonian terms in the duality identity are nonnegative by the minimizing properties of u_1 and u_2 . The Assumption 7 gives the reverse inequality in (90); hence, the equality holds throughout. The almost-everywhere uniqueness of the household minimizing selector then yields

$u_1 = u_2$ almost everywhere. Since both population flows have the same initial law, uniqueness of the corresponding FPK equation implies $\mu_1 = \mu_2$. Finally, uniqueness of the EGO best response and the recovery relations in (76) implies

$$p_1^G = p_2^G, \quad q_1 = q_2, \quad p_1^{\text{RPC}} = p_2^{\text{RPC}}, \quad p_1^{\text{CoC}} = p_2^{\text{CoC}}.$$

Therefore, the complete MM-GMFG equilibrium tuple is unique. $\square$

The monotonicity condition follows the Lasry–Lions uniqueness principle for mean field games [24] and its probabilistic formulations [10, 11]. Equation (89) applies this principle to the complete graphon-dependent market coupling, including the response of the major agent. More general displacement monotonicity conditions may provide alternative uniqueness criteria for nonseparable Hamiltonians [17].

3.6 Finite-population approximation

The preceding equilibrium discussion studied the continuum MM-GMFG. The actual CoC, however, contains a large but finite number of households. The final step is therefore to connect the continuum equilibrium to the finite-player CoC. Consider a CoC graph with one major agent and N minor agents with graphon locations $x_1^N, \dots, x_N^N \in [0, 1]$. Let $\alpha^{N,*} = (\alpha_0^{N,*}, \alpha_1^{N,*}, \dots, \alpha_N^{N,*})$ denote the strategy profile vector for all the minor agents and $\alpha_0^{N,*}$ denotes the major agent strategy. Then,

$$\alpha_i^{N,*}(t) = u^*(t, x_i^N, \mathbf{X}_i^N(t)), \quad i = 1, \dots, N, \quad (91)$$

$$\alpha_0^{N,*}(t) = (p^{G,*}(t), q^*(t)). \quad (92)$$

Let $J_i^N(\alpha_i^N, \alpha_{-i}^N)$ denote the cost of minor agent i in the finite-player game, and let $J_0^N(\alpha_0^N, \alpha_{-0}^N)$ denote the cost of the major agent. The continuum costs are denoted by $J_x(\alpha; \mu^*, \alpha_0^*)$ for a minor agent at graphon location x , and by $J_0(\alpha_0; \mu^*)$ for the major agent. For notational consistency with the minor-agent analysis, we work with the negated EGO objective

$$\begin{aligned} J_0(\alpha_0; \mu^*) &:= -J_{\text{EGO}}(\alpha_0; \mu^*) \\ &= -\mathbb{E} \left[\int_0^T \pi_0(t) dt + g_0(X_0(T)) \right], \end{aligned} \quad (93)$$

and define $J_0^N(\alpha_0^N, \alpha_{-0}^N)$ analogously as the negated finite-player EGO payoff. Maximizing J_{EGO} in (61) is therefore equivalent to minimizing J_0 , and both minor and major agents are treated as cost minimizers in the discussion below.

We now make the following assumptions before stating the main result of this section.

Assumption 8 (Finite-to-continuum cost consistency). There exist deterministic sequences

$$\delta_N^{\text{minor}} \rightarrow 0, \quad \delta_N^{\text{major}} \rightarrow 0,$$

such that the following estimates hold. For every minor agent $i = 1, \dots, N$,

$$\left\| J_i^N(\alpha_i^{N,*}, \alpha_{-i}^{N,*}) - J_{x_i^N}(\mathbf{u}^*; \mu^*, \alpha_0^*) \right\| \leq \delta_N^{\text{minor}}, \quad (94)$$

and, uniformly over every admissible unilateral deviation β_i^N ,

$$\left\| J_i^N(\beta_i^N, \alpha_{-i}^{N,*}) - J_{x_i^N}(\beta_i; \mu^*, \alpha_0^*) \right\| \leq \delta_N^{\text{minor}}, \quad (95)$$

where β_i denotes the corresponding admissible control in the continuum representative-agent problem. For the major agent,

$$\left\| J_0^N(\alpha_0^{N,*}, \alpha_{-0}^{N,*}) - J_0(\alpha_0^*; \mu^*) \right\| \leq \delta_N^{\text{major}}, \quad (96)$$

and, uniformly over every admissible major-agent deviation β_0^N ,

$$\left\| J_0^N \left(\beta_0^N, \alpha_{-0}^{N,*} \right) - J_0 \left(\beta_0; \mu^* \right) \right\| \leq \delta_N^{\text{major}}. \quad (97)$$

Assumption 9 (Quantitative finite-population consistency). Let r_N^W , r_N^μ , r_N^A , and r_N^0 denote, respectively, the finite-graph approximation error, the node-conditioned state-law approximation error, the aggregate-market approximation error, and the major-agent state approximation error. Assume that these quantities converge to zero and that there exist constants $C_{\text{minor}}, C_{\text{major}} > 0$, independent of N , such that

$$\delta_N^{\text{minor}} \leq C_{\text{minor}} \left(r_N^W + r_N^\mu + r_N^A + r_N^0 + \frac{1}{N} \right), \quad (98)$$

$$\delta_N^{\text{major}} \leq C_{\text{major}} \left(r_N^W + r_N^\mu + r_N^A + r_N^0 \right). \quad (99)$$

The estimates hold uniformly over the unilateral deviations admitted in Assumption 8.

Theorem 3 (ε_N -Nash approximation). Suppose $e^* = (\mu^*, u^*, p_\kappa^{\text{RPC},*}, p^{\text{CoC},*}, p^{G,*}, q^*)$ is an MM-GMFG equilibrium and Assumption 8 holds. Then the induced finite-player strategy profile $\alpha^{N,*}$ is an ε_N -Nash equilibrium, with

$$\varepsilon_N = 2 \max \left\{ \delta_N^{\text{minor}}, \delta_N^{\text{major}} \right\}. \quad (100)$$

Consequently, $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$.

Proof. We first consider a minor agent $i \in \{1, \dots, N\}$. Let β_i^N be any admissible unilateral deviation while all other agents retain the strategy profile $\alpha_{-i}^{N,*}$. By the equilibrium consistency estimate (94),

$$J_i^N \left(\alpha_i^{N,*}, \alpha_{-i}^{N,*} \right) \leq J_{x_i^N} (u^*; \mu^*, \alpha_0^*) + \delta_N^{\text{minor}}. \quad (101)$$

Because u^* is an optimal feedback for the representative minor agent at graphon location x_i^N under the equilibrium population flow and major-agent strategy, we have

$$J_{x_i^N} (u^*; \mu^*, \alpha_0^*) \leq J_{x_i^N} (\beta_i; \mu^*, \alpha_0^*). \quad (102)$$

Using the deviation consistency estimate (95),

$$J_{x_i^N} (\beta_i; \mu^*, \alpha_0^*) \leq J_i^N \left(\beta_i^N, \alpha_{-i}^{N,*} \right) + \delta_N^{\text{minor}}. \quad (103)$$

Combining (101) and (103) gives

$$J_i^N \left(\alpha_i^{N,*}, \alpha_{-i}^{N,*} \right) \leq J_i^N \left(\beta_i^N, \alpha_{-i}^{N,*} \right) + 2\delta_N^{\text{minor}}. \quad (104)$$

Because β_i^N is arbitrary, no minor agent can reduce its cost by more than $2\delta_N^{\text{minor}}$ through a unilateral deviation. We next consider the major agent. Let β_0^N be any admissible major-agent deviation. By (96),

$$J_0^N \left(\alpha_0^{N,*}, \alpha_{-0}^{N,*} \right) \leq J_0 \left(\alpha_0^*; \mu^* \right) + \delta_N^{\text{major}}. \quad (105)$$

The major-agent equilibrium strategy is optimal against μ^* , leading to

$$J_0 \left(\alpha_0^*; \mu^* \right) \leq J_0 \left(\beta_0; \mu^* \right). \quad (106)$$

Using (97) in (106) results in

$$J_0 \left(\beta_0; \mu^* \right) \leq J_0^N \left(\beta_0^N, \alpha_{-0}^{N,*} \right) + \delta_N^{\text{major}}. \quad (107)$$

Combining (105) and (107) yields

$$J_0^N \left(\alpha_0^{N,*}, \alpha_{-0}^{N,*} \right) \leq J_0^N \left(\beta_0^N, \alpha_{-0}^{N,*} \right) + 2\delta_N^{\text{major}}. \quad (108)$$

Hence, neither a minor agent nor the major agent can improve its cost by more than

$$2 \max \left\{ \delta_N^{\text{minor}}, \delta_N^{\text{major}} \right\}.$$

This proves that $\alpha^{N,*}$ is an ε_N -Nash equilibrium with ε_N given by (100). Because both consistency errors converge to zero, $\varepsilon_N \rightarrow 0$. Using Assumption 9 the bound on ε_N is:

$$\varepsilon_N \leq 2 \max \left\{ C_{\text{minor}} \left(r_N^W + r_N^\mu + r_N^A + r_N^0 + \frac{1}{N} \right), \right. \\ \left. C_{\text{major}} \left(r_N^W + r_N^\mu + r_N^A + r_N^0 \right) \right\}. \quad (109)$$

□

Theorem 1 establishes that the MM-GMFG loop in Fig. 4 admits an equilibrium, and the Theorem 2 provides sufficient conditions for uniqueness of that equilibrium. Finally, Theorem 3 shows that the continuum solution is an ε_N -Nash equilibrium for the finite CoC.

4 Numerical investigation for the OET problem via MM-GMFG

In our setting, each HEMS observes the state variables of its household, i.e., generation, load, and SoE in the batteries. Using the MM-GMFG algorithm, the HEMS calculates the equilibrium prices for energy exchange within its cluster and CoC using the CoC probability density function. Subsequently, the HEMS finds the optimal control strategies. Simultaneously, EGO finds its best response to the CoC probability density function. We now validate our MM-GMFG approach on a large CoC of 20,000 households, composed of 100 clusters, each with 200 prosumer households.

Algorithm 1 summarizes the steps for finding the fixed point solution for the MM-GMFG operator in (75) and subsequently outputs the Nash-Equilibrium tuple of (66) for a generic prosumer household of the CoC grid. Each household applies the Algorithm 1, and once the HEMS finds a converging fixed-point solution (i.e., Algorithm 1 meets the stopping criteria), it determines the optimal controls for its observed state trajectory.

Algorithm 1 MM-GMFG Fixed-Point Iteration.

Require: μ^0 ; tolerance $\varepsilon > 0$; $\mathcal{W}$; & K_{max} .

- 1: $k \leftarrow 0$.
 - 2: **repeat**
 - 3: **Prices:** $p_j^{\text{RPC},k} = \Pi_{\text{loc}}^j(\mu^k)$, $p^{\text{CoC},k} = \Pi_{\text{glob}}(\mu^k)$.
 - 4: **Weights:** w_1^k, w_2^k, w_3^k from μ^k .
 - 5: **EGO:** $(p^{G,k}, q^k) = \mathcal{B}_E(\mu^k)$.
 - 6: **HJB (backward):** solve for $V^k, \mathbf{u}^k$.
 - 7: **FPK (forward):** solve for μ^{k+1} .
 - 8: **Update:** μ^{k+1} .
 - 9: $k \leftarrow k + 1$.
 - 10: **until** $d_{\mathcal{M}}(\mu^k, \mu^{k-1}) < \varepsilon$ or $k > K_{\text{max}}$.
 - 11: **return** $(\mu^k, u^k, p^{\text{RPC},k}, p^{\text{CoC},k}, p^{G,k}, q^k)$.
-

4.1 Numerical simulation case study

The numerical values for the SDE parameters for generic agents, local and global pricing mechanism parameters, grid tariffs, and the rest of the parameters used in the numerical simulations are found in Table 1.

Table 1: Summary of system parameters.

Category	Parameters
Appliances applications	$P_c = 2.5 \text{ kW}$, $P_s = 2.0 \text{ kW}$
BESS battery characteristics	$P_{g2s} = P_{s2g} = 3.0 \text{ kW}$, $\eta_s = 0.95$ $b_{\min}^s = 1.0 \text{ kWh}$, $b_{\max}^s = 12.0 \text{ kWh}$
EV battery applications	$P_{g2e} = P_{e2g} = 7.0 \text{ kW}$, $\eta_e = 0.93$ $b_{\min}^e = 5.0 \text{ kWh}$, $b_{\max}^e = 50.0 \text{ kWh}$
Grid Tariff	$\tau_{\text{loc}} = 0.01 \text{ \$/kWh}$ $\tau_{\text{CoC}} = 0.015 \text{ \$/kWh}$ $C_{\text{RPC}} = 0.02 \text{ \$/kWh}$
Prices	$\beta_{\text{loc}}^s = 0.005 \text{ \$/kWh}^2$ $\beta_{\text{loc}}^b = 0.009 \text{ \$/kWh}^2$ $C_{\text{CoC}} = 0.035 \text{ \$/kWh}$ $\beta_{\text{glob}}^s = 0.004 \text{ \$/kWh}^2$ $\beta_{\text{glob}}^b = 0.01 \text{ \$/kWh}^2$
Spatial scaling factor	$\alpha = 3$
Convergence tolerance	$\epsilon_{\text{con}} = 10^{-4}$
EGO parameters	$C_{\text{gen}} : c_0 = 0.005 \text{ \$/kWh}$, $\beta = 4\text{\$}^{-1}$ $p^{\text{ref}} = 0.07 \text{ \$/kWh}$, $\bar{p} = 0.15 \text{ \$/kWh}$, $p = 0.04 \text{ \$/kWh}$, $c_1 = 0.001 \text{ \$/kWh}^2$ $\bar{\lambda} = 100$, $\lambda_i^v = 100$

We consider non-uniform prosumers and each parameter in Table 1 has its respective probability density function. The parameter density functions are assumed to be common information for all the participating prosumer households in the CoC network. In addition, the initial density function μ^0 for the state variables is calculated from historical data and it is assumed to be common information as well. Thus, all density functions needed for the MM-GMFG are common information and available for a generic agent HEMS to run the algorithmic steps in Algorithm 1. Finally, the case study is set within the Hydro-Québec context, which influences the EGO parametrization and the adopted electricity-price ranges.

The graphon kernel is calculated from the LinDistFlow voltage-sensitivity electrical distance [19], and clusters are defined by geographical proximity subject to minimizing the internal LinDistFlow electrical distance; that is, households are grouped so that their injections have similar voltage-impact profiles, which supports the intra-cluster trading approximation used in the local RPC market.

Fig. 5 presents the graphon kernel of the distributed CoC network based on electric distance; from the figure, we can infer the lossless intra-cluster exchange and larger influence on closer clusters.

Fig. 6 depicts the data-driven initial probability density function for the state variables, and Fig. 7 depicts the fixed-point convergence density function represented in the GMFG-loop in Fig. 4.

The probability density functions show how the best response controls affect the evolution of the initial density function and the solution of the GMFG-FPK equation. Comparing both figures, we can see the aggregate load behavioural changes mainly around the shiftable load start time and a lower trend when the effective price is high. Regarding b_s and b_e , the controlled probability evolution shows the optimal operation trends, where b_s is used before peak solar generation and charging during the peak solar window. Fig. 8 presents the shiftable load switching time t_s probability density function, where we can see the highest probabilities are concentrated around the solar peak time.

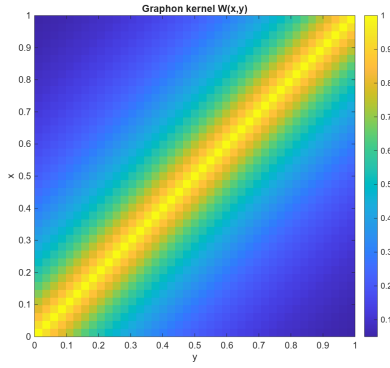


Fig. 5: Graphon kernel $\mathcal{W}(x, y)$ over the CoC network. The block-diagonal structure reflects lossless intra-cluster coupling ($\mathcal{W} = 1$), while off-block entries decay with LinDistFlow-based electrical distance between clusters.

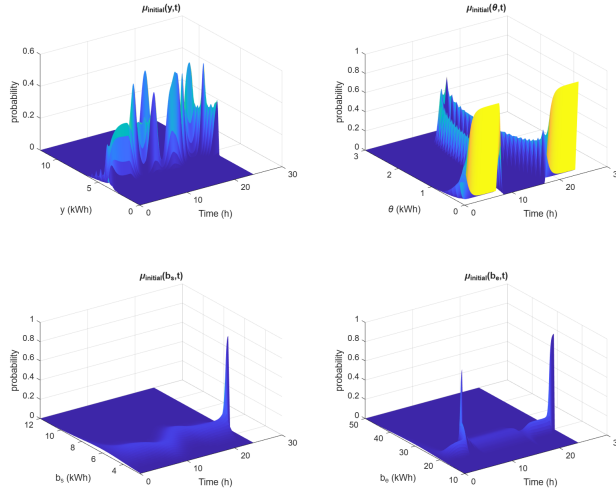


Fig. 6: Initial probability density function $\mu_0(y, b_e, b_s, t)$.

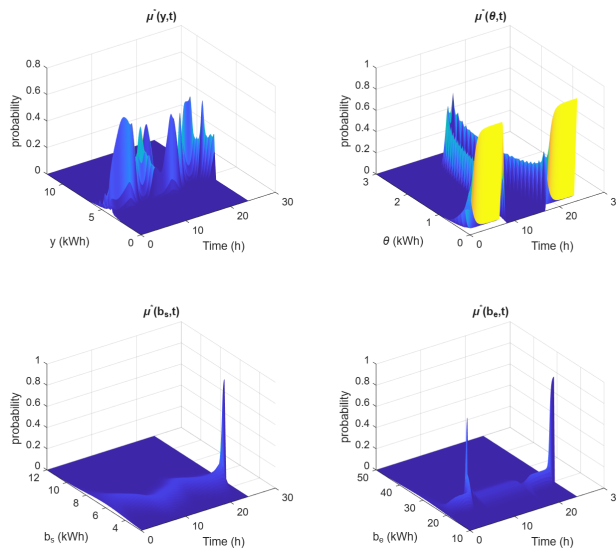


Fig. 7: Convergent probability density function $\mu^*(y, b_e, b_s, t)$.

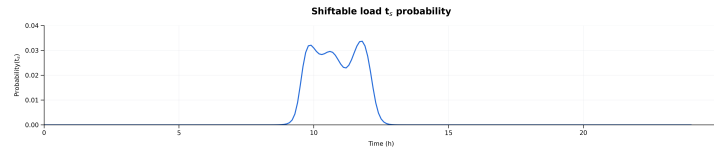


Fig. 8: Shiftable load switching time probability.

Fig. 9 presents the major-agent, EGO, best-response, and state-variables evolution over time. Here, we can see that there is no voltage violation, dispatch energy, and the grid dynamical price. As for the generic minor agent, the best responses and state variables are shown in Fig. 10.

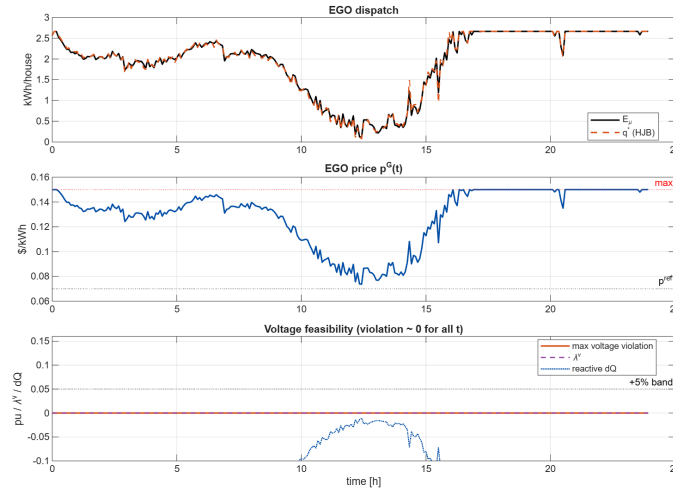


Fig. 9: EGO energy dispatch, $p_g^*(t)$ & Tariff.

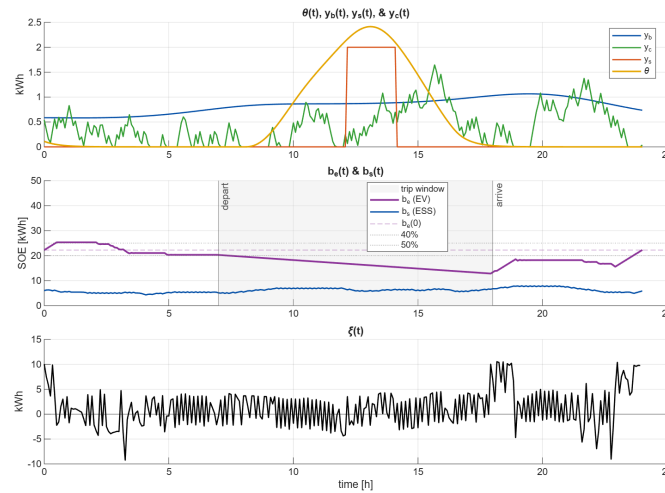


Fig. 10: Generic-agent states and controls.

In our case study, we assumed that the generic agent's EV is disconnected from 8:00 to 18:00, and the EV battery SOC has at least 40% when connected to the HEMS and the BESS SOC has at least 20%. The CoC aggregates representing the w_k for $k \in [1, 3]$, and the equilibrium prices are depicted in Fig. 11.

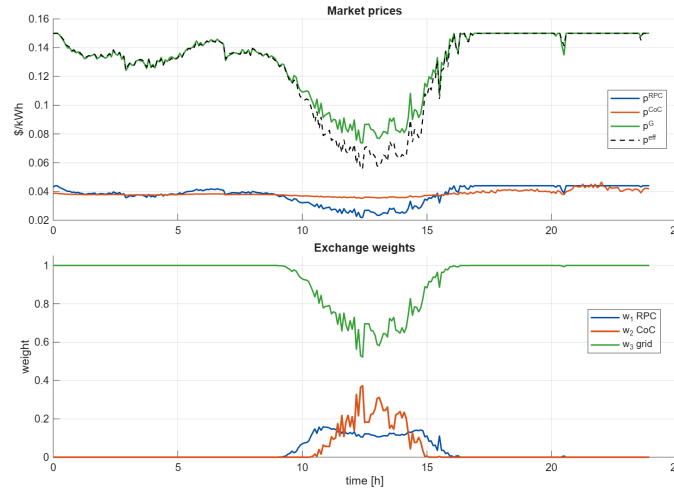


Fig. 11: Markets equilibrium prices and perspective weights.

4.2 Discussion

4.2.1 Economical validity

To validate the economic advantage of the proposed CoC distribution grid, we analyze the energy bill for the generic household operating in three different connection configurations: (i) grid-only participation, i.e., trading only with the grid provider, (ii) RPC-only participation, i.e., trading with the grid operator and its RPC, and (iii) CoC participation, i.e., trading in the local RPC, CoC, and grid operator markets.

Fig. 12 depicts the daily energy cost for the generic household, and from the results we can infer a clear economic benefit associated with the proposed CoC framework. Specifically, the grid-only configuration yields the highest daily energy cost of \$10.48/day, while the RPC-only and CoC configurations reduce the daily cost to \$9.11/day and \$8.22/day, respectively. Compared to conventional grid-only operation, the RPC-only mode achieves a cost reduction of approximately 13.1%, while the proposed CoC mode provides the largest savings, reducing the daily energy bill by about 21.6%. The CoC configuration outperforms the RPC-only mode by offering an additional 9.7% reduction in energy cost. These results illustrate that enabling participation in local energy markets through the CoC structure improves the utilization of DERs' flexibility and provides more cost-effective energy transactions, thus improving the overall economic performance for residential consumers.

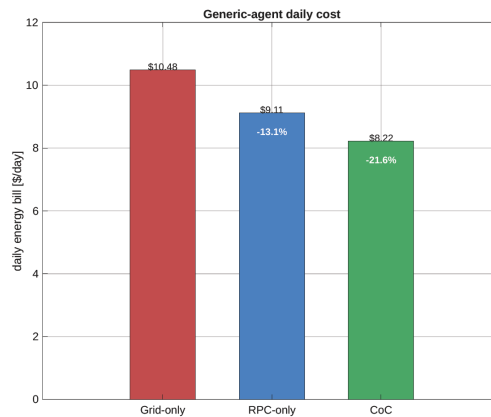


Fig. 12: Daily energy bill comparison for the generic agent.

4.2.2 Numerical performance

The MM-GMFG approach presented in this paper solves the fixed-point solution for the EGO operator and the minor agents best response. Once this fixed-point solution in controls and the probability density function is achieved, CoC's household finds their optimal controls on the trajectory of their states. Thus, the core computational cost relies mainly on the MM-GMFG approach.

Table 2: Measured Wall-Clock Runtime and Step Sizes.

RPC $\times$ House	Run Time sec	$\Delta\theta$ kWh	Δy kWh	Δt min	Δb_e kWh	Δb_s kWh
100 $\times$ 200	305.7	0.2	0.2	5	1	0.5
200 $\times$ 200	502.2	0.2	0.2	5	1	0.5

Table 2 summarizes the step sizes for the states and the running time for the fixed-point solution. All experiments were conducted on a workstation equipped with Intel Core Ultra 7 255H 16 Cores, 32GB of RAM, and running MATLAB R2026a. For the CoC in the 100 $\times$ 200 setting, the stopping condition is met at iteration 15, with each iteration requiring approximately 20.38s of wall-clock time. The quality of the resulting controls is highly correlated with the quality of the forecasted exogenous data, including solar generation and EV mobility patterns. In a real-scenario implementation, with forecast updates, the previously converged density μ^* can be used to warm-start the fixed-point iteration following an update of these forecasts. Thus, the proposed MM-GMFG approach offers a promising pathway toward maintaining computational tractability for online applications with forecast learning, where the rate of forecast updates and decision time size is to be investigated in future work.

To measure the scalability of the MM-GMFG approach, we applied Algorithm 1 to a CoC consisting of 200 clusters with 200 households per cluster. In this 200 $\times$ 200 case, the prescribed numerical tolerance was reached after 13 iterations, with each iteration requiring approximately 38.63s of wall-clock time, for a total runtime of 502.2s. Thus, when the simulated population increased from 20,000 to 40,000 households, the measured total runtime increased by a factor of approximately 1.64, from 305.7s to 502.2s. These results provide an empirical assessment of computational scaling for the two tested configurations.

5 Conclusion

This paper formulates the optimal energy trading (OET) problem for a distribution grid organized into residential prosumer clusters (RPCs) as a major-minor graphon mean field game (MM-GMFG) between an active energy grid operator (EGO, the major agent) and a continuum of heterogeneous prosumer households (the minor agents). Household dynamics and costs are coupled to local intra-RPC aggregates and to global inter-cluster aggregates through a LinDistFlow-derived, electrical-distance-based graphon kernel, while the EGO's own generation dispatch and pricing decisions respond to the aggregate energy and tariff flows induced by the household population. We derive the household best-response (HJB-FPK) system and the EGO best-response (HJB) system, characterize their coupled fixed point as a Nash equilibrium, and establish (i) existence of an equilibrium via a Schauder fixed-point argument, (ii) uniqueness under a Lasry-Lions-type monotonicity assumption, and (iii) an ε_N -Nash guarantee linking the continuum equilibrium to the finite-population game. The proposed MM-GMFG fixed-point algorithm is validated on a dense CoC network, comprising 100 clusters and 20,000 heterogeneous households, numerically converging in 15 iterations (305.7s total). Enabling household participation in local RPC and CoC-wide markets, rather than trading only with the EGO, reduces the generic household's daily energy bill by 13.1% (RPC-only) and 21.6% (full CoC), with the CoC configuration outperforming RPC-only trading by a further 9.7%, confirming that the additional network layers captured by the graphon structure translate into a measurable economic benefit.

Building on the MM-GMFG framework developed here, promising directions for future work include a multi-agent reinforcement learning (MARL) extension in which households may be strategic or passive, and prosumers or pure consumers, with the CoC's local and global aggregates estimated online rather than assumed as common knowledge and to consider low-probability, high-impact scenarios in modelling the graphon kernel.

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