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G–2026–61

October 2026

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Citation suggérée : H. Gao, A. Bosso, L. Wang, D. Saussié, B. Yi (October 2026). Data-driven stabilization of continuous-time linear MIMO systems from input–output data, Rapport technique, Les Cahiers du GERAD G– 2026–61, GERAD, HEC Montréal, Canada.

Suggested citation: H. Gao, A. Bosso, L. Wang, D. Saussié, B. Yi (October 2026). Data-driven stabilization of continuous-time linear MIMO systems from input–output data, Technical report, Les Cahiers du GERAD G–2026–61, GERAD, HEC Montréal, Canada.

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The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

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Data-driven stabilization of continuous-time linear MIMO systems from input–output data

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October 2026
Les Cahiers du GERAD
G–2026–61

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Abstract : This paper addresses the problem of data-driven stabilization of continuous-time linear time-invariant systems. The objective is to design a dynamic output-feedback controller directly from a finite input-output trajectory collected during an experiment, without any identification step. The proposed approach applies to general multi-input multi-output systems without a uniform observability index by building on Kreisselmeier's adaptive filter, which acts as an observer of a stabilizable canonical non-minimal realization of the unknown plant. By postprocessing the input-output data through this filter, the stabilization problem is translated into a data-driven linear matrix inequality whose solution yields the feedback gain of a dynamic output-feedback controller. Since the considered non-minimal realization is generally stabilizable rather than controllable, a data-driven state decomposition is introduced to extract the excited coordinates and isolate the stable autonomous modes. The proposed decomposition is then used to implement the controller as a reduced-order system that does not retain the full filter dynamics. Finally, input conditions are provided to guarantee the excitation properties required for data-driven design.

Keywords : Data-driven control; adaptive filters; output feedback; linear matrix inequalities

Acknowledgements: An abridged version of the paper was presented at the 2026 European Control Conference [9]. Corresponding author: Bowen Yi. Tel. +1 (514) 340 4711 ext. 3908.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) under Grants RGPIN-2020-06608 and RGPIN-2024-0478, the Fonds de recherche du Québec (FRQ), the Programme PIED, the European Union's Horizon Europe research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101104404 - IMPACT4Mech, and the National Natural Science Foundation of China under Grant 62573383.

1 Introduction

Obtaining controllers from measured system trajectories has a long history in the control community. Its origins can be traced back to adaptive control [27], where unknown controller parameters are learned online, and it has recently evolved into the broader field of data-driven control [12, 19, 31]. This end-to-end paradigm avoids any intermediate modeling step and may be effective even when the data are not informative enough to perform system identification.

Among different direct data-driven methodologies, an increasingly prominent approach consists of expressing analysis and synthesis conditions as linear matrix inequalities (LMIs) whose coefficients are obtained directly from system trajectories. This leads to a computationally tractable controller synthesis procedure and provides explicit conditions under which a given dataset contains sufficient information for the objective. In the *discrete-time* context, pioneering works on data-driven LMIs are [7] and [33], which mainly focus on the state-feedback stabilization of linear time-invariant (LTI) systems. While the former derives LMIs by developing state-space results inspired by Willems et al.'s fundamental lemma [35], the latter introduces the framework of data informativity to provide necessary and sufficient conditions for data-driven analysis and control. Later, alternative LMIs particularly suited for noisy datasets have been derived from quadratic matrix inequalities [30]. The data-driven LMI paradigm has also been applied to time-varying systems [22], nonlinear systems [7, 30], and output-feedback control, using either non-minimal realizations involving shifted input–output data [1, 7, 16] or the behavioral framework [32].

In contrast, direct data-driven control for *continuous-time* systems remains less developed. This gap is relevant in the control community because most physical systems or processes are naturally modeled in continuous time by ordinary differential equations [20]. Although continuous-time LTI systems admit an exact discrete-time representation under periodic sampling and zero-order-held inputs, such sampling and input conditions are not always guaranteed in an experiment. Moreover, a direct continuous-time treatment provides a useful conceptual and technical foundation for addressing nonlinear systems. Recent progress includes continuous-time versions of the fundamental lemma [18, 28] and the study of the effect of sampling on informativity [8]. Unlike the discrete-time case, continuous-time methods must handle derivative signals, which are usually unavailable in practice. Most state-feedback results derive LMIs assuming that state derivatives are measurable [7, 8, 11]. This is relatively restrictive as derivative signals are typically not measured and their numerical reconstruction can considerably amplify measurement noise [13]. To remove this need, [26] proposes orthogonal polynomial bases, while [23] explores more general sampling functionals.

On the other hand, data-driven control methods that rely purely on input–output data for continuous-time systems remain rare, while most available results assume that the state trajectories are available. The input–output setting is of particular practical interest because measuring every state may require an excessive number of sensors or may even be infeasible when suitable sensors do not exist. Results that address this scenario include [5, 6], which design output-feedback controllers for LTI systems via non-minimal realizations that can be reconstructed by linear filters of the input–output signals. While these results address single-input single-output (SISO) and multi-input multi-output (MIMO) systems with a uniform observability index, respectively, finding an approach for general continuous-time LTI MIMO systems remains an open problem. This paper aims to close this gap with the following contributions.

1. We propose a method to derive an output-feedback stabilizer directly from an input–output trajectory of LTI MIMO systems without restrictions on the observability indices. We obtain this result by showing that Kreisselmeier's adaptive filter [14] acts as an observer of a canonical non-minimal realization of the plant [6].
2. We derive data-driven LMIs for computing a stabilizing feedback gain from filtered input–output data. To handle the fact that the resulting non-minimal realization is generally stabilizable but not completely controllable, we first project out the component represented by a known stable

auxiliary signal and then apply a singular value decomposition (SVD) to the residual. This decomposition separates an excited stabilizable coordinate from stable autonomous modes. We also exploit the known input matrix of the filtered system to relax the standard rank condition.

3. We provide further results on data collection and implementation. In particular, we give input-design conditions that ensure the required excitation and derive a reduced-order dynamic output-feedback controller whose online state contains only the residual coordinate. The complete Kreisselmeier filter is needed only for offline data processing.

Caveat. Compared to the conference version [9], this paper further develops a novel data-driven LMI that exploits the known input matrix of the extended filtered system, derives a projection-based reduced-order online controller, and provides input-design conditions for experimental data collection.

Notation. We use $\mathbb{R}$ and $\mathbb{R}_-$ to denote the sets of real numbers and negative numbers, respectively. 0_j and $0_{j \times k}$ denote the zero column vector of dimension j and the zero matrix of dimension $j \times k$, respectively, and $\mathbf{1}_j$ represents the j -dimensional vector of ones. We use $|\cdot|$ to represent the Euclidean norm of a vector and $\otimes$ for the Kronecker product. Given a matrix $X \in \mathbb{R}^{j \times k}$, the operator $\text{vec}(X)$ denotes the kj -dimensional vector obtained by stacking the columns of X one below another. For matrices $X_1, \dots, X_q$ with appropriate dimensions, $\text{col}(X_1, \dots, X_q) := [X_1^\top \cdots X_q^\top]^\top$. Let $\mathcal{C}^k([t_1, t_2], \mathbb{R}^q)$ denote the set of k -times continuously differentiable functions from $[t_1, t_2]$ to $\mathbb{R}^q$ with $t_1 < t_2$. Given a signal ϕ and ordered sampling instants $\mathbf{t}_N := (t_1, \dots, t_N)$, denote the matrix of samples $[\phi]_{\mathbf{t}_N} := [\phi(t_1) \cdots \phi(t_N)]$. When clear from the context, the arguments of functions and operators are omitted. We conclude the notation by introducing some useful definitions.

Definition 1. [15, 29] A bounded signal $\phi : [t_1, t_2] \rightarrow \mathbb{R}^q$, with $t_1 < t_2$, is said to be *intervally excited* (IE) if

$$\int_{t_1}^{t_2} \phi(s) \phi^\top(s) ds \succ 0. \quad (1)$$

Definition 2. [15] A signal $\phi(\cdot) \in \mathcal{C}^{k-1}([t_1, t_2], \mathbb{R}^q)$, with $t_1 < t_2$ and $k \in \mathbb{N}$, is said to be *rich* of order k if the signal $\text{col}(\phi(\cdot), \dot{\phi}(\cdot), \dots, \phi^{(k-1)}(\cdot))$ is IE.

2 Problem statement

Consider a continuous-time LTI MIMO system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \quad (2)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^p$ is the measured output, while $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ are constant unknown matrices that satisfy the following assumption.

Assumption 1. The pair (A, B) is controllable and the pair (C, A) is observable.

We aim at solving the following input–output data-driven stabilization problem.

Problem 1. Suppose that the input–output dataset

$$\mathcal{E}_D := \{u(t), y(t)\}_{[0, t_D]}, \quad (3)$$

has been acquired from the plant (2) during an experiment of duration $t_D > 0$, under some excitation condition on the input u and from an unknown initial condition $x(0) \in \mathbb{R}^n$. Moreover, suppose that $u(t)$ and $x(t)$ are piecewise continuous and absolutely continuous over $[0, t_D]$, respectively.

Without any prior information on A , B , and C except for the state dimension n , use (3) to design a linear dynamic output-feedback controller of the form

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y, \end{aligned} \quad (4)$$

having state x_c and matrices A_c, B_c, C_c and D_c to be determined, such that the origin $(x, x_c) = 0$ of the interconnection of (2) and (4) is globally exponentially stable (GES).

3 Proposed approach

3.1 Kreisselmeier's adaptive filter

Our approach is inspired by and built on Kreisselmeier's adaptive filter [14], which is given by the following dynamical system:

$$\dot{M} = \Lambda M + [y^\top \otimes I_n \quad u^\top \otimes I_n], \quad (5)$$

where $M \in \mathbb{R}^{n \times \mu}$ is the filter state, with $\mu := (p + m)n$, while $\Lambda \in \mathbb{R}^{n \times n}$ is a Hurwitz matrix having n distinct eigenvalues. In [14], it is shown that the filtered signal $M(t)$ satisfies an algebraic relation with the solution $x(t)$ of system (2), for any initial condition of (2) and (5). Below, we provide a version of this result tailored to our case.

Lemma 1. Consider systems (2), (5) and let Assumption 1 hold. Then, there exist a non-singular matrix $T \in \mathbb{R}^{n \times n}$ and a column vector $\theta \in \mathbb{R}^\mu$ such that, for all initial conditions $x(0) = x_0 \in \mathbb{R}^n$ and $M(0) = M_0 \in \mathbb{R}^{n \times \mu}$,

$$Tx(t) = M(t)\theta + e^{\Lambda t}(Tx_0 - M_0\theta). \quad (6)$$

Proof. Since the pair (C, A) is observable due to Assumption 1, there exists a matrix Γ such that the spectra of $A - \Gamma C$ and Λ coincide. Then, the fact that Λ has distinct eigenvalues implies that $A - \Gamma C$ and Λ are similar, i.e., there exists a non-singular matrix T such that $\Lambda = T(A - \Gamma C)T^{-1}$. Using the change of coordinates $\bar{x} = Tx$, system (2) with initial condition x_0 becomes

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}\bar{x} \quad (7)$$

with $\bar{A} := TAT^{-1}$, $\bar{B} := TB$, $\bar{C} := CT^{-1}$, and initial condition $\bar{x}_0 = Tx_0$. System (7) can be equivalently written as

$$\dot{\bar{x}} = \Lambda\bar{x} + T\Gamma y + TBu. \quad (8)$$

The main results in [14] follow immediately. The solution $\bar{x}(t) = Tx(t)$ of (8) reads as

$$\begin{aligned} Tx(t) &= e^{\Lambda t}\bar{x}_0 + \int_0^t e^{\Lambda(t-s)}[T\Gamma y(s) + TBu(s)]ds \\ &= e^{\Lambda t}\bar{x}_0 + \int_0^t e^{\Lambda(t-s)}[y^\top(s) \otimes I_n \quad u^\top(s) \otimes I_n]ds\theta \\ &= e^{\Lambda t}Tx_0 + (M(t) - e^{\Lambda t}M_0)\theta, \end{aligned}$$

where in the second line we exploited the identities $T\Gamma y = (y^\top \otimes I_n) \text{vec}(T\Gamma)$, $TBu = (u^\top \otimes I_n) \text{vec}(TB)$ and defined

$$\theta := \begin{bmatrix} \text{vec}(T\Gamma) \\ \text{vec}(TB) \end{bmatrix},$$

while in the third line we used the solution $M(t)$ of (5). Rearranging the terms, we obtain (6). $\square$

Remark 1. The filter (5) does not rely on specific values of A, B , or C , but only on their dimensions for the given plant (2). Even in such a case, we have an algebraic relation (6) between the unknown state x and the available filtered variable M , where the vector θ is unknown. Indeed, the use of similar filters to process input-output signals has a longstanding history in adaptive control; see, for example, [24, 27].

3.2 Filter-based output-feedback stabilization

Our approach is motivated by the indirect output-feedback approach [3]. In particular, we first design an observer that achieves an exponentially convergent estimate of the system state, uniformly in the “exogenous” signals (u, y) . Then, we find a “state-feedback” controller for the observer. From Lemma 1, we note that the set

$$\mathcal{A} := \{(x, M) \in \mathbb{R}^n \times \mathbb{R}^{n \times \mu} : Tx = M\theta\} \quad (9)$$

is invariant and attractive. As a consequence, it makes sense to design a feedback law that depends only on M . We aim at achieving the following property:

$$\lim_{t \rightarrow \infty} M(t) = 0 \quad (\text{exp.}),$$

which then implies from (6) that $x(t)$ converges to zero exponentially as $t \rightarrow \infty$. It is important to remember that the system does not start in general on $\mathcal{A}$ because $M(0)$ would require unavailable prior information. Thus, we need to account for the perturbation of the *transverse coordinate*

$$\beta := Tx - M\theta \in \mathbb{R}^n. \quad (10)$$

To facilitate the data-driven controller design, we provide the following corollary of Lemma 1, which highlights the dynamics of the transverse coordinate β .

Corollary 1. *The transverse coordinate $\beta \in \mathbb{R}^n$, defined in (10), satisfies the dynamics*

$$\dot{\beta} = \Lambda\beta, \quad (11)$$

whose origin $\beta = 0$ is GES.

Proof. Rearranging terms in (6), we immediately obtain (11), where Λ is Hurwitz by design. $\square$

Remark 2. Transverse coordinates have been widely explored in various fields, especially in nonlinear control, such as path following of mechanical systems [10], orbital stabilization [36], transverse stability analysis [2], and immersion and invariance control [4].

4 Existence of Kreisselmeier filter-based stabilizing controllers

We now show that the filter (5), once vectorized, defines a canonical non-minimal realization of the plant, and then use this structure to establish the existence of a filter-based output-feedback stabilizing controller. We first recall a definition from [6, Def. 1].

Definition 3. Under Assumption 1, a *canonical non-minimal realization* of system (2) is a system of the form

$$\begin{aligned} \dot{z} &= (F + LH)z + Gu \\ y &= Hz, \end{aligned} \quad (12)$$

where $z \in \mathbb{R}^{n_z}$ and $n_z > n$, such that the following properties hold:

- F is Hurwitz;
- There exists a matrix $\Pi \in \mathbb{R}^{n \times n_z}$ that solves the following equations:

$$\begin{aligned} \Pi(F + LH) &= A\Pi, \quad \Pi G = B \\ H &= C\Pi. \end{aligned} \quad (13)$$

To facilitate the subsequent analysis, we rewrite the Kreisselmeier filter in a vectorized form below.

Lemma 2. Let $\hat{z} := \text{vec}(M)$, where M satisfies (5). Then, it holds that

$$\dot{\hat{z}} = F\hat{z} + Gu + Ly, \quad (14)$$

where

$$\begin{aligned} F &:= I_\mu \otimes \Lambda \\ G &:= \begin{bmatrix} 0_{pn^2 \times m} \\ I_m \otimes \text{vec}(I_n) \end{bmatrix}, \quad L := \begin{bmatrix} I_p \otimes \text{vec}(I_n) \\ 0_{mn^2 \times p} \end{bmatrix}. \end{aligned} \quad (15)$$

The proof is given in Appendix A.

We now show that there exists a matrix H such that the system (12), with F, G and L defined by the vectorized Kreisselmeier filter, provides a canonical non-minimal realization of the plant (2).

Lemma 3. Under the conditions of Lemma 1, system (12), with $z \in \mathbb{R}^{n_z}$, $n_z := n^2(p + m)$, F, G , and L as in (15), and

$$H := C(\theta^\top \otimes T^{-1}), \quad (16)$$

is a canonical non-minimal realization of the plant (2), with (13) solved by the full-row-rank matrix

$$\Pi := \theta^\top \otimes T^{-1}. \quad (17)$$

Moreover, the pair $(F + LH, G)$ is stabilizable.

Proof. By Lemma 1, the set $\mathcal{A}$ defined in (9) is invariant. Hence, for every solution initialized on $\mathcal{A}$,

$$\begin{aligned} x(t) &= T^{-1}M(t)\theta \\ &= (\theta^\top \otimes T^{-1}) \text{vec}(M(t)) = \Pi\hat{z}(t), \quad \forall t \geq 0. \end{aligned} \quad (18)$$

From (16) and (17), it also holds that

$$H = C\Pi. \quad (19)$$

On $\mathcal{A}$, Equations (18) and (19) give $y = Cx = C\Pi\hat{z} = H\hat{z}$. Hence, by (14), the dynamics restricted to $\mathcal{A}$ satisfy

$$\begin{aligned} A\Pi\hat{z} + Bu &= \dot{x} = \Pi\dot{\hat{z}} \\ &= \Pi(F\hat{z} + Gu + Ly) \\ &= \Pi(F + LH)\hat{z} + \Pi Gu. \end{aligned} \quad (20)$$

The identity (20) holds for arbitrary values of $\hat{z}$ and u . To be specific, for any $\hat{z}_0$ and u_0 , one may choose $M(0)$ such that $\text{vec}(M(0)) = \hat{z}_0$, together with $x(0) = \Pi\hat{z}_0$ and $u(0) = u_0$, so that $(x(0), M(0)) \in \mathcal{A}$. Evaluating (20) at $t = 0$ and comparing the coefficients of $\hat{z}_0$ and u_0 gives

$$\Pi(F + LH) = A\Pi, \quad \Pi G = B. \quad (21)$$

Together, (19) and (21) yield (13).

Since (A, B) is controllable, $B \neq 0$. Thus, $TB \neq 0$ and the definition of θ in Lemma 1 gives $\theta \neq 0$. Consequently, $\text{rank } \Pi = \text{rank } (\theta^\top) \text{rank } (T^{-1}) = n$, where we used the rank identity for the Kronecker product. Moreover, $F = I_\mu \otimes \Lambda$ is Hurwitz because Λ is Hurwitz. Thus, all the conditions of Definition 3 hold. Finally, the stabilizability of $(F + LH, G)$ follows from [6, Cor. 1]. $\square$

Remark 3. While vectorizing the matrix variable M reveals that the filter dynamics of $\text{vec}(M)$ resemble those in [6, § VI], the filter (5) does not exploit any structural property of MIMO systems. As a consequence, unlike the design of [6, § VI], our approach does not require knowledge of the observability indices of the pair (C, A) . The trade-off, however, is the introduction of excessive coordinates in the filter. In Section 5.6, we provide a reduced-order Kreisselmeier filter design capable of significantly reducing the filter dimension.

Example 1. Consider a three-dimensional system with two inputs, two outputs ($n = 3$, $p = m = 2$), and

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

We select $\Lambda = \text{diag}(-20, -36, -40)$. Following the proof of Lemma 1, we obtain

$$T = \begin{bmatrix} 6.1083 & -0.2340 & 17.5900 \\ -5.5982 & -2.9025 & -17.1889 \\ -1.4337 & 2.8633 & -0.0868 \end{bmatrix}$$

and the corresponding column vector $\theta = [\theta_1, \theta_2]^\top$ with $\theta_1 = [181.0436, -258.7007, -59.0428, 7.0687, -121.4904, 117.3910]$ and $\theta_2 = [23.6983, -22.7871, -1.5205, 34.9460, -37.2803, 2.6897]$. By lemmata 2 and 3, we obtain $F + LH$ and G . The matrix $F + LH$ has three unstable eigenvalues $\{3.2188, 0.3906 \pm 1.5274i\}$, corresponding to the eigenvalues of A . For all of these eigenvalues, the PBH test yields $\text{rank}[\lambda_i I - (F + LH) \mid G] = 36$, which implies that the pair $(F + LH, G)$ is stabilizable. Note that the observability indices of this system are $\nu_1 = 2$ and $\nu_2 = 1$, and thus the filter in [6] is not applicable.

For arbitrary initial conditions, the definitions of β , $\hat{z}$, and Π give

$$x = \Pi \hat{z} + T^{-1} \beta. \quad (22)$$

Combining Corollary 1, Lemma 2, and $H = C\Pi$, the plant-filter interconnection can be written in the coordinates $(\beta, \hat{z})$ as

$$\begin{aligned} \dot{\beta} &= \Lambda \beta, \\ \dot{\hat{z}} &= (F + LH) \hat{z} + Gu + L(y - H \hat{z}) \\ &= (F + LH) \hat{z} + Gu + LCT^{-1} \beta. \end{aligned} \quad (23)$$

Remark 4. The plant-filter interconnection (23) shows that the vectorized filter (14) can be interpreted as a Luenberger observer for the non-minimal realization (12), with the error dynamics $\frac{d}{dt}(z - \hat{z}) = F(z - \hat{z})$ exponentially stable.

We conclude this section by establishing the existence of a stabilizing controller with the filter state used for feedback.

Proposition 1. Let the conditions of Lemma 1 hold. There exists a matrix $K_z \in \mathbb{R}^{m \times n_z}$ such that the output-feedback controller

$$\begin{aligned} \dot{M} &= \Lambda M + [y^\top \otimes I_n \quad u^\top \otimes I_n], \\ u &= K_z \text{vec}(M) \end{aligned} \quad (24)$$

renders the origin $(x, M) = (0, 0)$ of the closed-loop interconnection with (2) GES.

Proof. For any $K_z \in \mathbb{R}^{m \times n_z}$, applying (24) to (23) gives

$$\begin{bmatrix} \dot{\beta} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ LCT^{-1} & F + LH + GK_z \end{bmatrix} \begin{bmatrix} \beta \\ \hat{z} \end{bmatrix}.$$

By Lemma 3, the pair $(F + LH, G)$ is stabilizable. Hence, K_z can be chosen such that $F + LH + GK_z$ is Hurwitz. Since Λ is Hurwitz, the above block-lower-triangular matrix is then Hurwitz, and $(\beta, \hat{z}) = (0, 0)$ is GES. Finally, the coordinate map from (x, M) to $(\beta, \hat{z})$ is bijective: its inverse is given by

$$x = T^{-1} \beta + \Pi \hat{z}, \quad M = \text{vec}^{-1}(\hat{z}),$$

where T is nonsingular. Therefore, the origin $(x, M) = (0, 0)$ is globally exponentially stable. $\square$

Proposition 1 shows that the remaining design task is to construct, from the input-output data, a gain K_z such that $F + LH + GK_z$ is Hurwitz. Once such a gain is available, (24) is a stabilizing dynamic output-feedback controller. The next section develops a data-driven construction of this gain.

5 Data-driven output-feedback stabilization

The previous section establishes the existence of a stabilizing feedback based on the filter state. In this section, we study how to construct it from the dataset $\mathcal{E}_D$ under suitable excitation conditions.

5.1 Data-compatible realization with available states

Given $\mathcal{E}_D$, we can postprocess the available trajectories over the interval $[0, t_D]$ with the filter (5). Initializing the filter on the invariant set $\mathcal{A}$ would require unavailable plant information. We therefore choose

$$M(0) = 0_{n \times \mu}, \quad (25)$$

so that $\hat{z}(0) = 0_{n_z}$ and $\beta(0) = Tx(0)$. Processing the dataset with (5) gives access to M and $\dot{M}$ (equivalently, $\hat{z}$ and $\dot{\hat{z}}$),¹ whereas the transverse coordinate β in (23) is unavailable because its initial condition is unknown. We select the matrix Λ satisfying the following design condition.² It can be extended to the case of non-diagonal matrices with complex eigenvalues; see Remark 6.

Condition 1. *The Hurwitz matrix Λ in (5) is diagonal with distinct real eigenvalues.*

Under the above condition, define $D := \text{diag}(\beta(0)) \in \mathbb{R}^{n \times n}$. Then, the unavailable β can be represented as

$$\beta(t) = e^{\Lambda t} \beta(0) = D\chi(t), \quad (26)$$

where $\chi(t) \in \mathbb{R}^n$ satisfies the differential equation³

$$\dot{\chi}(t) = \Lambda\chi(t), \quad \chi(0) = \mathbf{1}_n \quad (27)$$

As a consequence, by letting

$$\zeta := \text{col}(\chi, \hat{z}) \in \mathbb{R}^{n_\zeta}, \quad n_\zeta := n + n_z, \quad (28)$$

we obtain the following realization:

$$\begin{aligned} \dot{\zeta} &= \underbrace{\begin{bmatrix} \Lambda & 0 \\ LCT^{-1}D & F + LH \end{bmatrix}}_{A_e} \zeta + \underbrace{\begin{bmatrix} 0 \\ G \end{bmatrix}}_{B_e} u, \\ y &= [CT^{-1}D \quad H] \zeta \end{aligned} \quad (29)$$

Since $(F + LH, G)$ is stabilizable and Λ is Hurwitz, the pair (A_e, B_e) is stabilizable. The initial conditions $\chi(0) = \mathbf{1}_n$ and $\hat{z}(0) = 0_{n_z}$ reproduce the experimental dataset $\mathcal{E}_D$ in (3). Thus, postprocessing $\mathcal{E}_D$ yields the filtered trajectory data

$$\mathcal{E}_F := \{u(t), \zeta(t), \dot{\zeta}(t)\}_{t \in [0, t_D]},$$

where $\dot{\zeta}$ is available from the implemented filter dynamics without numerical differentiation.

Remark 5. System (29) is also a canonical non-minimal realization of (2). To see this, let $\Pi_e := [T^{-1}D \quad \Pi]$ and $H_e := [CT^{-1}D \quad H]$. The definitions of θ , Π , and L give $\Pi L = \Gamma$, and hence $T(A - \Pi LC)T^{-1} = \Lambda$. Since D and Λ are diagonal, it yields $(A - \Pi LC)T^{-1}D = T^{-1}D\Lambda$. Together with (13), this gives $\Pi_e A_e = A\Pi_e$, $\Pi_e B_e = B$, and $H_e = C\Pi_e$. Moreover, Π_e has full row rank and $A_e = \text{diag}(\Lambda, F) + \text{col}(0_{n \times p}, L)H_e$, with $\text{diag}(\Lambda, F)$ Hurwitz.

¹Hereafter, when we say that a derivative is available, we mean that the right-hand side of the corresponding implemented filter is given by available signals and thus can be sampled.

²For ease of presentation, we impose the following condition on the user-selected matrix Λ .

³Since the diagonal entries of Λ are distinct, the components of $\chi(t)$ are distinct exponential functions and hence χ is IE on every interval of positive length.

Remark 6. Condition 1 can be further relaxed to allow the matrix Λ to have complex eigenvalues. Indeed, since Λ has distinct eigenvalues, it is cyclic, that is, its characteristic polynomial coincides with its minimal polynomial. Therefore, there exists a vector g such that (Λ, g) is controllable. It then follows from [6, Lem. 4] that one may always choose $\dot{\chi} = \Lambda\chi$ with $\chi(0) = g$.

The following corollary connects a stabilizing gain for the data-compatible realization (29) to the filter-based controller (24) in Proposition 1.

Corollary 2. Let $K_e = [K_\chi \ K_z]$ be partitioned conformably with (28). If $A_e + B_e K_e$ is Hurwitz, then K_z is a stabilizing gain for $(F + LH, G)$, and controller (24) renders the origin of the closed loop GES.

Now we reformulate Problem 1 as a standard input-state data-driven control problem for the system (29) with $\mathcal{E}_F$ available. As a result, we may apply the methodology in [7] directly to samples of $\mathcal{E}_F$, which requires the existence of ordered sampling instants $\mathbf{t}_N = (t_1, \dots, t_N)$ satisfying $0 \leq t_1 < \dots < t_N \leq t_b$ such that

$$\text{rank } \text{col}([u]_{\mathbf{t}_N}, [\zeta]_{\mathbf{t}_N}) = m + n_\zeta. \quad (30)$$

A sufficient signal-level condition for the existence of such sampling instants is that $\text{col}(u, \zeta)$ be IE [34, Prop. 1]. Compared with this direct application, the extended dynamics (29) has two structural properties:

- P1** Although A_e is unknown, the input matrix B_e is explicitly *known* from the filter construction.
- P2** The pair (A_e, B_e) is stabilizable but need not be controllable, because the extended dynamics may contain uncontrollable modes, which makes condition (30) unlikely to hold.

The next two subsections exploit **P1** and **P2**, respectively.

5.2 Data-driven stabilization with a known input matrix

To address **P1**, the following proposition provides a counterpart of the data-driven stabilization result in [7] for the case with known input matrices.

Proposition 2. Consider the stabilizable linear system $\dot{\mathbf{x}} = \mathfrak{A}\mathbf{x} + \mathfrak{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, the matrix $\mathfrak{A} \in \mathbb{R}^{n \times n}$ is unknown, and $\mathfrak{B} \in \mathbb{R}^{n \times m}$ is known. Given ordered sampling instants $\mathbf{t}_N := (t_1, \dots, t_N)$, define $\mathbf{X} := [\mathbf{x}]_{\mathbf{t}_N}$, $\dot{\mathbf{X}} := [\dot{\mathbf{x}}]_{\mathbf{t}_N}$, and $\mathbf{U} := [\mathbf{u}]_{\mathbf{t}_N}$. Assume that $\text{rank } \mathbf{X} = n$. Then, any matrices $Q \in \mathbb{R}^{N \times n}$ and $Y \in \mathbb{R}^{m \times n}$ satisfying

$$\begin{aligned} \mathbf{X}Q &= Q^\top \mathbf{X}^\top \succ 0, \\ (\dot{\mathbf{X}} - \mathfrak{B}\mathbf{U})Q + \mathfrak{B}Y + ((\dot{\mathbf{X}} - \mathfrak{B}\mathbf{U})Q + \mathfrak{B}Y)^\top &\prec 0, \end{aligned} \quad (31)$$

guarantee that $K = Y(\mathbf{X}Q)^{-1}$ is a stabilizing gain. Conversely, if K is a stabilizing gain for $(\mathfrak{A}, \mathfrak{B})$, then it admits such a representation with Q and Y solutions satisfying (31).

Its proof can be found in Appendix B. The number of decision variables in (31) depends on the number of samples N . The following integral counterpart has a number of decision variables that is independent of N ; its proof is given in Appendix C.

Proposition 3. Consider the system given in Proposition 2 with the data $\{\mathbf{x}(t), \mathbf{u}(t), \dot{\mathbf{x}}(t)\}_{t \in [0, t_b]}$. The LMI

$$\begin{aligned} \mathcal{M}(Y, P) + \int_0^{t_b} \begin{bmatrix} \dot{\mathbf{x}}(s) - \mathfrak{B}\mathbf{u}(s) \\ -\mathbf{x}(s) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}(s) - \mathfrak{B}\mathbf{u}(s) \\ -\mathbf{x}(s) \end{bmatrix}^\top ds \\ \succeq \rho \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \\ P = P^\top \succ 0 \end{aligned} \quad (32)$$

with

$$\mathcal{M}(Y, P) := \begin{bmatrix} -\mathfrak{B}Y - Y^\top \mathfrak{B}^\top & -P \\ -P & 0 \end{bmatrix}, \quad (33)$$

in the variables $P \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{m \times n}$ and $\rho > 0$, is feasible if and only if $\mathbf{x}(\cdot)$ is IE. Moreover, every feasible solution yields a stabilizing gain $K = YP^{-1}$ for the pair $(\mathfrak{A}, \mathfrak{B})$.

Both results reveal the consequence of **P1**: when applied to the system (29) with known B_e , Proposition 2 requires the much weaker condition

$$\text{rank } [\zeta]_{t_N} = n_\zeta, \quad (34)$$

rather than (30). Moreover, the sampled LMI is feasible if and only if (34) holds. Since ζ is absolutely continuous, such sampling instants exist if and only if ζ is IE for continuous signals. Proposition 3 gives the same necessary-and-sufficient condition directly in integral form.

5.3 Data-driven projected-residual decomposition

As specified in **P2**, (A_e, B_e) might be uncontrollable. Consequently, the signal ζ need not be IE. Namely,

$$\mathcal{Z} := \int_0^{t_b} \zeta(s) \zeta(s)^\top ds \quad (35)$$

may fail to be full rank, so the results in subsection 5.2 may not apply. To accommodate this rank loss, we exploit the stabilizability of (A_e, B_e) and require that $\mathcal{Z}$ lose rank only in directions that are unreachable from the input. Namely, we impose the following.

Condition 2. *It holds that*

$$\text{Im } \mathcal{R} \subseteq \text{Im } \mathcal{Z},$$

where $\mathcal{R} := [B_e \ A_e B_e \ \cdots \ A_e^{n_\zeta-1} B_e]$ is the reachability matrix of (A_e, B_e) .

It is clear that Condition 2 is weaker than $\mathcal{Z} \succ 0$ whenever $\text{rank } \mathcal{R} < n_\zeta$. We first exploit the fact that χ is IE—see Footnote 3—and evolves according to the known stable dynamics (27). We remove from $\hat{z}$ the component represented by a linear combination of the rows of χ and apply the SVD to the resulting residual. Following this idea, we propose the data-driven projected-residual decomposition in Algorithm 1, whose properties are summarized by Proposition 4.

Algorithm 1 Projected-residual state decomposition

Input: Filtered signals χ and $\hat{z}$ over $[0, t_b]$.

1: Set $\Sigma \leftarrow \int_0^{t_b} \chi(s) \chi(s)^\top ds$ and

$$P_z \leftarrow \left(\int_0^{t_b} \hat{z}(s) \chi(s)^\top ds \right) \Sigma^{-1}.$$

2: Set $\tilde{z}(t) \leftarrow \hat{z}(t) - P_z \chi(t)$ and $S \leftarrow \int_0^{t_b} \tilde{z}(s) \tilde{z}(s)^\top ds$.

3: Compute $S = \Psi \text{diag}(\Delta, 0) \Psi^\top$ and $r_a \leftarrow \text{rank } S$, where $\Delta \in \mathbb{R}_{>0}^{r_a \times r_a}$.

4: Partition $\Psi \leftarrow [\Psi_a \ \Psi_b]$ conformably with $\text{diag}(\Delta, 0)$.

Output: P_z , Ψ_a , and r_a .

Defining the residual coordinates

$$\hat{z}_a := \Psi_a^\top (\hat{z} - P_z \chi), \quad \hat{z}_b := \Psi_b^\top (\hat{z} - P_z \chi), \quad (36)$$

we have the following proposition.

Proposition 4. Under Conditions 1–2, let P_z , Ψ_a , and r_a be generated by Algorithm 1. The change of coordinates $\text{col}(\hat{z}_a, \hat{z}_b, \chi) = T_{\text{pr}} \zeta$, with the nonsingular matrix

$$T_{\text{pr}} := \begin{bmatrix} -\Psi_a^\top P_z & \Psi_a^\top \\ -\Psi_b^\top P_z & \Psi_b^\top \\ I_n & 0 \end{bmatrix}, \quad (37)$$

transforms (29) into

$$\begin{bmatrix} \dot{\hat{z}}_a \\ \dot{\hat{z}}_b \\ \dot{\chi} \end{bmatrix} = \begin{bmatrix} A_a & A_{ab} & A_{a\chi} \\ 0 & A_b & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \begin{bmatrix} \hat{z}_a \\ \hat{z}_b \\ \chi \end{bmatrix} + \begin{bmatrix} B_a \\ 0 \\ 0 \end{bmatrix} u, \quad (38)$$

where $B_a = \Psi_a^\top G$, A_b is Hurwitz, and (A_a, B_a) is stabilizable.

Proof. Since χ is IE, $\Sigma \succ 0$ and P_z is well defined. Let $\tilde{z} := \hat{z} - P_z \chi$. The definition of P_z gives

$$\int_0^{t_b} \tilde{z}(s) \chi(s)^\top ds = 0. \quad (39)$$

Hence, with

$$T_1 := \begin{bmatrix} I_n & 0 \\ -P_z & I_{n_z} \end{bmatrix}, \quad T_1 \zeta = \text{col}(\chi, \tilde{z}),$$

it follows that $T_1 \mathcal{Z} T_1^\top = \text{diag}(\Sigma, S)$. Applying the orthogonal matrix $\Psi^\top$ to the residual coordinate gives $\text{col}(\chi, \Psi^\top \tilde{z}) = \text{col}(\chi, \hat{z}_a, \hat{z}_b)$, whose data Gramian is $\text{diag}(\Sigma, \Delta, 0)$. Then, permuting the coordinates to $\text{col}(\hat{z}_a, \hat{z}_b, \chi)$ gives the transformation T_{pr} , which is nonsingular as it is the composition of T_1 , $\Psi^\top$, and a coordinate permutation. Moreover,

$$T_{\text{pr}} \mathcal{Z} T_{\text{pr}}^\top = \text{diag}(\Delta, 0, \Sigma), \quad (40)$$

where $\text{rank } \mathcal{Z} = n + r_a$. The transformed dynamics can be written, with conformably partitioned matrices, as

$$\begin{bmatrix} \dot{\hat{z}}_a \\ \dot{\hat{z}}_b \\ \dot{\chi} \end{bmatrix} = \begin{bmatrix} A_a & A_{ab} & A_{a\chi} \\ A_{ba} & A_b & A_{b\chi} \\ 0 & 0 & \Lambda \end{bmatrix} \begin{bmatrix} \hat{z}_a \\ \hat{z}_b \\ \chi \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \\ 0 \end{bmatrix} u \quad (41)$$

with $B_a = \Psi_a^\top G$ and $B_b = \Psi_b^\top G$. Let $\mathcal{R}_{\text{pr}} := T_{\text{pr}} \mathcal{R}$ be the reachability matrix in the new coordinates and let $\mathcal{Z}_{\text{pr}} := T_{\text{pr}} \mathcal{Z} T_{\text{pr}}^\top$. Since T_{pr} is nonsingular, Condition 2 implies

$$\text{Im } \mathcal{R}_{\text{pr}} \subseteq \text{Im } \mathcal{Z}_{\text{pr}}. \quad (42)$$

The block of $\mathcal{Z}_{\text{pr}}$ associated with $\hat{z}_b$ is zero. By the continuity of state trajectories, this implies

$$\begin{aligned} \hat{z}_b(t) &= 0, & t \in [0, t_b], \\ \dot{\hat{z}}_b(t) &= 0 & \text{for almost all } t \in [0, t_b]. \end{aligned} \quad (43)$$

Moreover, as every vector in $\text{Im } \mathcal{Z}_{\text{pr}}$ has a zero $\hat{z}_b$ -component, (42) implies that the $\hat{z}_b$ -block row of $\mathcal{R}_{\text{pr}}$ is zero. In particular, $B_b = 0$. The second block row of the full dynamics (41), together with (43) and $B_b = 0$, gives $[A_{ba} \ A_{b\chi}] \text{col}(\hat{z}_a(t), \chi(t)) = 0$ for almost all $t \in [0, t_b]$. By (40), $\text{col}(\hat{z}_a, \chi)$ is IE. Hence, the only constant matrix that annihilates this signal almost everywhere is the zero matrix, and therefore $A_{ba} = 0$ and $A_{b\chi} = 0$. Substituting these identities and $B_b = 0$ into (41) yields (38).

The pair (A_e, B_e) is stabilizable. Since the $\hat{z}_b$ subsystem is autonomous and unaffected by the input, A_b must be Hurwitz. Finally, because both Λ and A_b are Hurwitz, stabilizability of the complete transformed pair implies that (A_a, B_a) is stabilizable. $\square$

The above result shows that, under Condition 2—which can be ensured by a sufficiently rich input and will be discussed later—Algorithm 1 can decompose the excited subspace and yield the representation (38). In terms of this particular structure, it suffices to stabilize the $\hat{z}_a$ -subspace by finding a gain K_a such that $A_a + B_a K_a$ is Hurwitz. Accordingly, a stabilizing feedback gain for system (29) is given by $K = [-K_a \Psi_a^\top P_z \quad K_a \Psi_a^\top]$.

In Section 5.4, we present a data-driven approach for computing K_a , together with the complete algorithm for solving Problem 1. Before concluding this subsection, we introduce a reduced-order dynamic controller implementation.

Proposition 5. Under Conditions 1–2, let Ψ_a be generated by Algorithm 1 and set $F_a := \Psi_a^\top F \Psi_a$. If $K_a \in \mathbb{R}^{m \times r_a}$ is such that $A_a + B_a K_a$ is Hurwitz, then the dynamic output-feedback controller

$$\begin{aligned}\dot{\xi} &= F_a \xi + \Psi_a^\top G u + \Psi_a^\top L y, \\ u &= K_a \xi,\end{aligned}\tag{44}$$

with $\xi \in \mathbb{R}^{r_a}$ ensures the origin of the plant–controller interconnection (2) and (44) is GES.

Proof. By Proposition 4, the interconnection of (29) and (44) can be represented by the interconnection of (38) and (44). Since the dynamics of χ and $\hat{z}_b$ are autonomous and the matrices Λ and A_b are Hurwitz, the states χ and $\hat{z}_b$ converge exponentially to zero. Therefore, it remains to show that the dynamics of $\hat{z}_a$ are GES under the controller (44).

From Proposition 4, the dynamics of $\hat{z}_a = \Psi_a^\top (\hat{z} - P_z \chi)$ read

$$\dot{\hat{z}}_a = A_a \hat{z}_a + A_{ab} \hat{z}_b + A_{a\chi} \chi + B_a u.\tag{45}$$

Now, we show that ξ , generated from (44), is an exponentially convergent estimate of $\hat{z}_a$. To see this, from the relation $\text{col}(\hat{z}_a, \hat{z}_b, \chi) = T_{\text{pr}} \zeta$ and the second equation in (29), we may rewrite (45) as

$$\begin{aligned}\dot{\hat{z}}_a &= F_a \hat{z}_a + \Psi_a^\top F \Psi_b \hat{z}_b + \Psi_a^\top (F P_z - P_z \Lambda) \chi \\ &\quad + \Psi_a^\top G u + \Psi_a^\top L y.\end{aligned}$$

One has

$$\dot{\overbrace{\xi - \hat{z}_a}^{\cdot}} = F_a (\xi - \hat{z}_a) + \epsilon$$

with the exponentially decaying term $\epsilon := -\Psi_a^\top F \Psi_b \hat{z}_b - \Psi_a^\top (F P_z - P_z \Lambda) \chi$. Under Condition 1, $F = I_\mu \otimes \Lambda \prec 0$ and hence $F_a \prec 0$. Hence, ξ exponentially converges to $\hat{z}_a$ uniformly as $t \rightarrow \infty$. Substituting $u = K_a(\hat{z}_a + \xi - \hat{z}_a)$ into (45) leads to an exponentially stable system with the system matrix $A_a + B_a K_a$, perturbed by some exponentially decaying term, and following the standard perturbation arguments, the interconnection of (29) and (44) is GES. Since (29) realizes the plant (2) through $x = \Pi_e \zeta$, this stability property also holds for the plant–controller interconnection. $\square$

Proposition 5 shows that, for online implementation of the proposed controller, it suffices to run the lower-dimensional filter in (44), rather than the Kreisselmeier filter together with the auxiliary dynamics of χ . This substantially reduces the controller order and improves computational efficiency in online implementation.

Note that the above proposition is a model-based result. The remaining task is therefore to compute the gain matrix K_a directly from the postprocessed data.

5.4 Data-driven controller synthesis

In this section, we present the main result of the paper, i.e., an overall algorithm for input–output data-driven controller design. The proposed algorithm is given in Algorithm 2, and its feasibility and data-driven stabilization guarantees are summarized in Theorem 1.

Algorithm 2 Input/output data-driven stabilization**Input:** Dataset $\mathcal{E}_D$.**Filter setting**1: Choose $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n \times n}$ Hurwitz with distinct real eigenvalues, and construct F , G , and L as in (15).**Data filtering and decomposition**

2: Simulate the filter

$$\dot{M} = \Lambda M + [y^\top \otimes I_n \quad u^\top \otimes I_n], \quad M(0) = 0_{n \times \mu},$$

and set $\hat{z} \leftarrow \text{vec}(M)$.3: Simulate $\dot{\chi} = \Lambda \chi$, $\chi(0) = \mathbf{1}_n$.4: Run Algorithm 1 to obtain P_z , Ψ_a , and r_a .5: Evaluate $\hat{z}_a$ from (36) and compute

$$\hat{z}_a(t) \leftarrow \Psi_a^\top (\hat{z}(t) - P_z \dot{\chi}(t)).$$

Data batches6: Choose $N \geq n + r_a$ sampling instants $t_N = (t_1, \dots, t_N)$ with $0 \leq t_1 < \dots < t_N \leq t_D$ and set $\mathbf{X}_\chi = [\chi]_{t_N}$, $\mathbf{X}_a = [\hat{z}_a]_{t_N}$, $\dot{\mathbf{X}}_a = [\dot{\hat{z}}_a]_{t_N}$, and $\mathbf{U}_D = [u]_{t_N}$.7: Require rank $\begin{bmatrix} \mathbf{X}_\chi \\ \mathbf{X}_a \end{bmatrix} = n + r_a$.8: Set the known input matrix $B_a \leftarrow \Psi_a^\top G$.**Gain computation**9: Find $Q \in \mathbb{R}^{N \times r_a}$ and $Y \in \mathbb{R}^{m \times r_a}$ such that

$$\begin{cases} \mathbf{X}_\chi Q = 0 \\ \mathbf{X}_a Q = Q^\top \mathbf{X}_a^\top \succ 0, \\ \Theta_a + \Theta_a^\top \prec 0 \end{cases} \quad (46)$$

with $\Theta_a := (\dot{\mathbf{X}}_a - B_a \mathbf{U}_D)Q + B_a Y$.

10: Compute

$$K_a \leftarrow Y(\mathbf{X}_a Q)^{-1}. \quad (47)$$

Output: The controller (44), defined by Ψ_a and K_a .

Theorem 1. Consider the plant (2), its input–output dataset $\mathcal{E}_D$, and Algorithm 2 under Assumption 1 and Conditions 1–2. There exist sampling instants $t_1, \dots, t_N$ such that $\text{col}(\mathbf{X}_\chi, \mathbf{X}_a)$ has full row rank. Then, Algorithm 2 is feasible, and every gain K_a computed from (46)–(47) renders $A_a + B_a K_a$ Hurwitz. Consequently, the reduced-order dynamic output-feedback controller (44) solves Problem 1.

Proof. Proposition 4 shows that $\text{col}(\chi, \hat{z}_a)$ is IE. Hence, sampling instants satisfying the rank condition in Algorithm 2 exist. On these samples, (38) gives

$$\dot{\mathbf{X}}_a = A_{a\chi} \mathbf{X}_\chi + A_a \mathbf{X}_a + B_a \mathbf{U}_D.$$

Since (A_a, B_a) is stabilizable, there exist a gain $\bar{K}_a$ and a matrix $P = P^\top \succ 0$ such that

$$(A_a + B_a \bar{K}_a)P + P(A_a + B_a \bar{K}_a)^\top \prec 0.$$

The full row rank of $\text{col}(\mathbf{X}_\chi, \mathbf{X}_a)$ ensures the existence of Q satisfying $\mathbf{X}_\chi Q = 0$ and $\mathbf{X}_a Q = P$. Setting $Y = \bar{K}_a P$ then makes (46) feasible.

Conversely, let (Q, Y) satisfy (46), and define $P := \mathbf{X}_a Q$ and $K_a := Y P^{-1}$. Since $\mathbf{X}_\chi Q = 0$, the sampled dynamics give

$$\Theta_a = A_a P + B_a Y = (A_a + B_a K_a)P.$$

Thus, $\Theta_a + \Theta_a^\top \prec 0$ in (46) is a strict Lyapunov inequality for $A_a + B_a K_a$, which is therefore Hurwitz. Proposition 5 shows that controller (44) renders the origin of the closed loop GES and hence solves Problem 1. $\square$

Although Conditions 1–2 guarantee the existence of instants $t_1, \dots, t_N$ satisfying the rank condition, we still need to find such instants in the above approach. To avoid this additional step and simplify the design, we may instead adopt the integral-type LMI in Proposition 3 for the controller synthesis. Then, we may remove Steps 6–7 in the above algorithm, while the remaining steps are replaced as in Algorithm 3.

Algorithm 3 An alternative to Algorithm 2

Steps 1–5 remain the same as in Algorithm 2.

6: Set the known input matrix $B_a \leftarrow \Psi_a^\top G$.

7: For $t \in [0, t_D]$, set

$$\begin{aligned} s_a(t) &\leftarrow \dot{\hat{z}}_a(t) - B_a u(t), \\ d_a(t) &\leftarrow s_a(t) - \left(\int_0^{t_D} s_a(\tau) \chi(\tau)^\top d\tau \right) \Sigma^{-1} \chi(t). \end{aligned}$$

Gain computation

8: Find $P = P^\top \succ 0$, $Y \in \mathbb{R}^{m \times r_a}$, and $\rho > 0$ such that

$$\begin{aligned} \begin{bmatrix} -B_a Y - Y^\top B_a^\top & -P \\ -P & 0 \end{bmatrix} + \int_0^{t_D} \begin{bmatrix} d_a(t) \\ -\dot{\hat{z}}_a(t) \end{bmatrix} \begin{bmatrix} d_a(t) \\ -\dot{\hat{z}}_a(t) \end{bmatrix}^\top dt \\ \succeq \rho \begin{bmatrix} I_{r_a} & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

9: Compute $K_a \leftarrow Y P^{-1}$.

Output: The controller (44), defined by Ψ_a and K_a .

5.5 Input design for the experimental dataset

The success of the projected-residual decomposition relies on Condition 2, which ensures that every data-null direction belongs to the stable unreachable dynamics. The following proposition provides a sufficient condition on input design such that Condition 2 is ensured.

Proposition 6. Consider the plant (2) and its associated extended dynamics (29). Let n_c be the dimension of the reachable subspace of (A_e, B_e) , and consider a controllability decomposition with controllable state ζ_c and uncontrollable state ζ_u . Suppose that, during the collection of $\mathcal{E}_D$, the input u is rich of order n_c . Then, ζ_c is IE on $[0, t_D]$. If, in addition, the resulting trajectory is nondegenerate in the sense that

$$\ell_1^\top \zeta_c(t) + \ell_2^\top \zeta_u(t) = 0, \quad \forall t \in [0, t_D] \implies \ell_1 = 0, \quad (48)$$

then Condition 2 holds.

Proof. Let T_c be a nonsingular matrix that puts (A_e, B_e) into a controllability decomposition, and define $\bar{\zeta} := T_c \zeta = \text{col}(\zeta_c, \zeta_u)$. Then

$$\bar{A}_e := T_c A_e T_c^{-1} = \begin{bmatrix} A_c & A_x \\ 0 & A_u \end{bmatrix}, \quad \bar{B}_e := T_c B_e = \begin{bmatrix} B_c \\ 0 \end{bmatrix},$$

where (A_c, B_c) is controllable. Accordingly, $\bar{\mathcal{R}} := T_c \mathcal{R} = \text{col}(\mathcal{R}_c, 0)$ and $\text{rank } \mathcal{R}_c = n_c$. Since u is rich of order n_c , Lemma 5 applied to the controllable subsystem implies that ζ_c is IE on $[0, t_D]$.

It remains to prove that Condition 2 holds. Define $\bar{\mathcal{Z}} := T_c \mathcal{Z} T_c^\top = \int_0^{t_D} \bar{\zeta}(s) \bar{\zeta}(s)^\top ds$. For any $v = \text{col}(v_c, v_u) \in \text{Ker } \bar{\mathcal{Z}}$,

$$0 = v^\top \bar{\mathcal{Z}} v = \int_0^{t_D} |v_c^\top \zeta_c(s) + v_u^\top \zeta_u(s)|^2 ds.$$

By continuity and (48), this implies $v_c = 0$, and hence $\bar{\mathcal{R}}^\top v = \mathcal{R}_c^\top v_c = 0$. Therefore, $\text{Ker } \bar{\mathcal{Z}}^\top \subseteq \text{Ker } \bar{\mathcal{R}}^\top$, which is equivalent to Condition 2 because T_c is nonsingular. $\square$

Remark 7. The nondegeneracy condition (48) is mild and generally holds. Indeed, ζ_u evolves according to stable autonomous LTI dynamics and is completely determined by its initial condition. Hence, for almost every input u that is rich of order n_c , there are no constant vectors $\ell_1 \neq 0$ and ℓ_2 such that $\ell_1^\top \zeta_c(t) + \ell_2^\top \zeta_u(t) = 0$ for all $t \in [0, t_D]$.

5.6 Optional reduction of the filtering step

Algorithm 2 uses the complete Kreisselmeier filter only when processing the experimental data offline; the online controller is the r_a -dimensional system (44). Nevertheless, the full filter has dimension

$n^2(m+p)$ and may still make the offline processing costly for large n . The following result exploits its structure to generate the relevant filter components with only $n(m+p)$ states.

Proposition 7. Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ be Hurwitz with distinct eigenvalues. Consider the reduced-order filter

$$\dot{\hat{\xi}}_r = (I_{p+m} \otimes \Lambda) \hat{\xi}_r + \begin{bmatrix} I_p \otimes \mathbf{1}_n \\ 0_{mn \times p} \end{bmatrix} y + \begin{bmatrix} 0_{pn \times m} \\ I_m \otimes \mathbf{1}_n \end{bmatrix} u, \quad (49)$$

where $\hat{\xi}_r = \text{col}(r_1, \dots, r_{p+m})$, $r_i \in \mathbb{R}^n$. Define $\hat{M}_r := [\text{diag}(r_1) \dots \text{diag}(r_{p+m})]$. Then, the set $\{(M, \hat{\xi}_r) : M = \hat{M}_r\}$ is GES with $M(t)$ generated by the Kreisselmeier filter (5), i.e., $\|M(t) - \hat{M}_r(t)\| \leq ce^{-\gamma t} \|M(0) - \hat{M}_r(0)\|$ for some $c, \gamma > 0$.

Proof. Let $q := \text{col}(y, u) \in \mathbb{R}^{p+m}$ and partition the matrix M as $M = [M_1 \dots M_{p+m}]$, $M_i \in \mathbb{R}^{n \times n}$. From the Kreisselmeier filter (5), and since $[y^\top \otimes I_n \quad u^\top \otimes I_n] = [q_1 I_n \dots q_{p+m} I_n]$, each block M_i satisfies

$$\dot{M}_i = \Lambda M_i + q_i I_n, \quad i = 1, \dots, p+m.$$

On the other hand, the reduced-order filter gives $\dot{r}_i = \Lambda r_i + \mathbf{1}_n q_i$. Since $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, the diagonal entries of M_i satisfy

$$\dot{M}_{i,kk} = \lambda_k M_{i,kk} + q_i, \quad k = 1, \dots, n,$$

whereas the off-diagonal entries satisfy

$$\dot{M}_{i,k\ell} = \lambda_k M_{i,k\ell}, \quad k \neq \ell.$$

Defining the error $E = M - \hat{M}_r =: [E_1 \dots E_{p+m}]$, it is straightforward to obtain $\dot{E}_i = \Lambda E_i$, whose origin is clearly GES. This completes the proof. $\square$

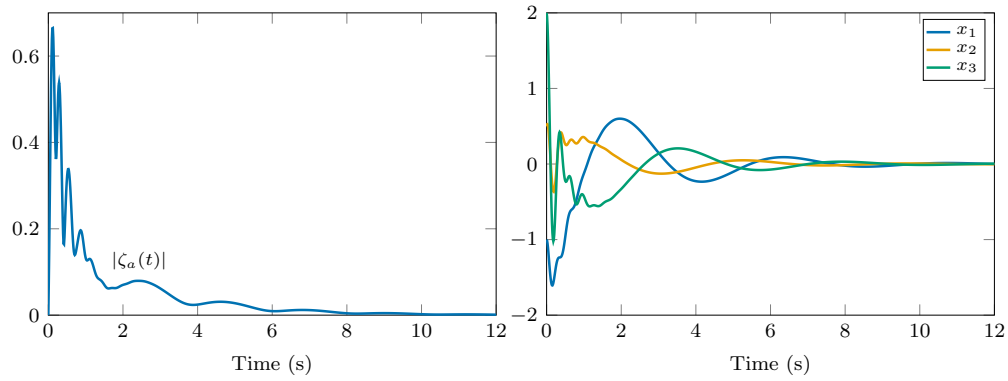
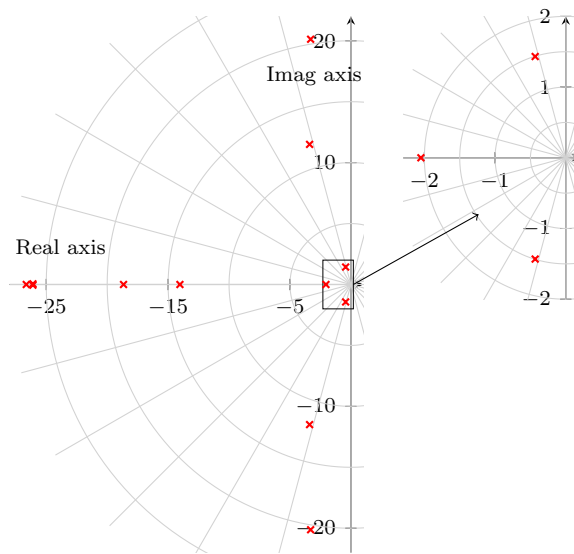
With each $r_i(0)$ chosen as the vector of diagonal entries of $M_i(0)$, the reduced-order filter (49) reproduces these entries exactly, while the off-diagonal entries of M evolve according to the stable autonomous dynamics. This construction reduces the cost of the offline filtering stage and is distinct from the reduced online controller (44).

6 Numerical results

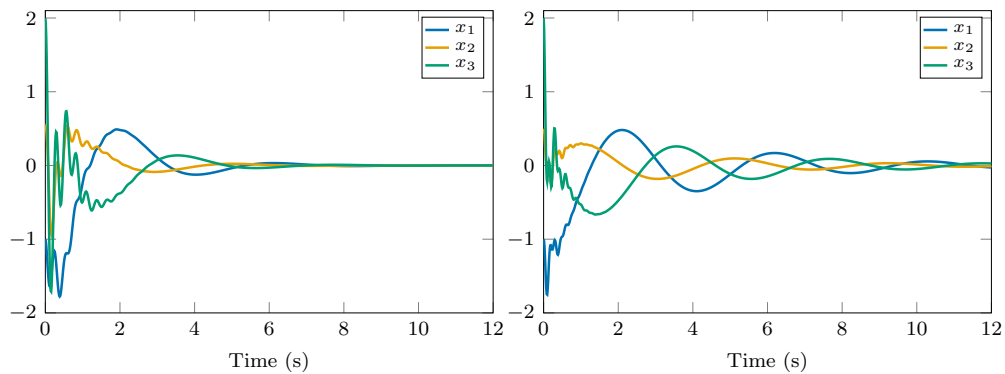
We illustrate the effectiveness of the proposed approach using the system in Example 1. Simulations are performed in MATLAB R2024b using YALMIP [17] and MOSEK [21]. The PBH test gives $n_c = 9$ for (A_e, B_e) . Accordingly, the input used for data collection is selected to be rich of order 9: $u = \text{col}(\sum_{k=1}^4 k \sin(5kt) + 0.1 \sin(25t), \sum_{k=1}^4 (5-k) \sin(7kt) + 0.1 \sin(35t))$, and data are collected over $[0, 2]$ s. We test both Algorithms 2 and 3. For the former, we use $N = 50$ uniformly spaced samples to construct the data matrices, whereas for the latter, we use the complete filtered trajectory over the same interval. All closed-loop simulations are initialized with $x(0) = \text{col}(-1, 0.5, 2)$ and $\xi(0) = 0$, and are run for 12 s.

Among the tested filter-pole choices, we select $\Lambda_\star = \text{diag}(-14, -25.2, -28)$, which provides a reasonable tradeoff between the resulting controller gain and the settling time. Algorithm 1 yields $r_a = 9$. The computed gain satisfies $\|K_a\|_F = 124.3$. Figs. 1 and 2 show the closed-loop responses and the plant-controller spectrum, respectively. The spectral abscissa is -0.4329 . The 2% settling time, defined as the first time after which $|x(t)| \leq 0.02|x(0)|$ holds, is 7.18 s.

To illustrate the effect of filter eigenvalues, we consider two further choices $\Lambda_c = \text{diag}(-8, -14.4, -16)$ and $\Lambda_m = \text{diag}(-24, -43.2, -48)$. Both cases yield $r_a = 9$. The resulting controller norms are $\|K_a\|_F = 174.5$ and 359.4, respectively, while the corresponding 2% settling times are 4.99 s and 10.72 s, as shown in Fig. 3.

Figure 1: Closed-loop responses for $\Lambda_\star$ Figure 2: Closed-loop spectrum for $\Lambda_\star$

We also implement the gain obtained from Algorithm 3. With $\Lambda_I = \text{diag}(-14, -28, -40)$, the decomposition again gives $r_a = 9$, and the computed gain satisfies $\|K_a^1\|_F = 1.38 \times 10^4$. The resulting plant-controller matrix has spectral abscissa -1.4636 , and its 2% settling time is 4.07 s; see Fig. 4.

Figure 3: Plant-state responses for Λ_c (left) and Λ_m (right)

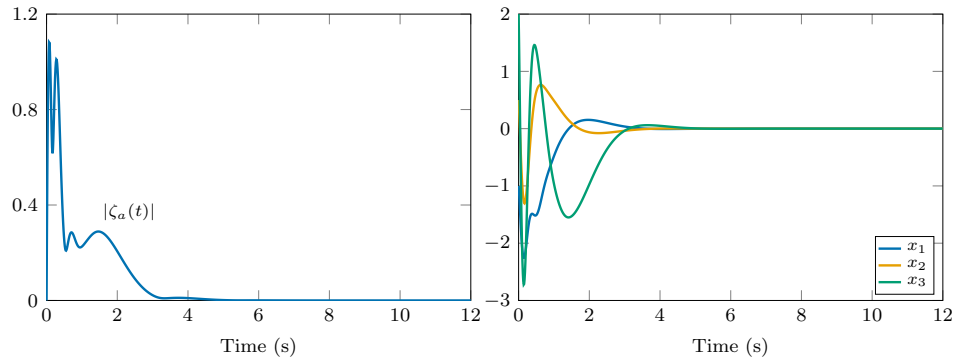


Figure 4: Closed-loop responses with the controller obtained from Algorithm 3

Finally, we empirically verify our approach with noisy trajectory measurements. At each simulation step t_j , we add independent zero-mean Gaussian samples to the measured output according to $y^m(t_j) = y(t_j) + v_j$, where $v_j \sim \mathcal{N}(0, 10^{-4}I_2)$. The noise is present both during offline data collection and during online closed-loop operation, and no process disturbance is added. A short segment of the measured output is shown in Fig. 5. We choose $\Lambda_n = \text{diag}(-6.6, -11.88, -13.2)$. For this case, Algorithm 1 gives $r_a = 12$. For noisy measurements, we test both Algorithms 2 and 3, with the resulting closed-loop spectral abscissas -0.6136 and -0.9401 , respectively. The closed-loop responses are presented in Fig. 6. In both cases, the plant state approaches and remains in a small neighborhood of the origin over the simulated horizon despite the continuing output noise. Since the analysis in this paper is noiseless, these simulations do not establish robustness under arbitrary noise conditions.

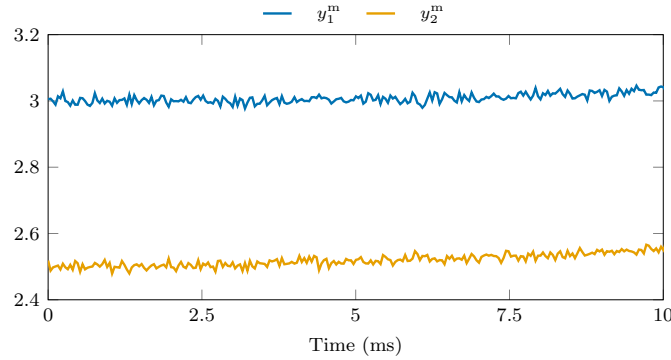


Figure 5: Measured output under additive Gaussian noise

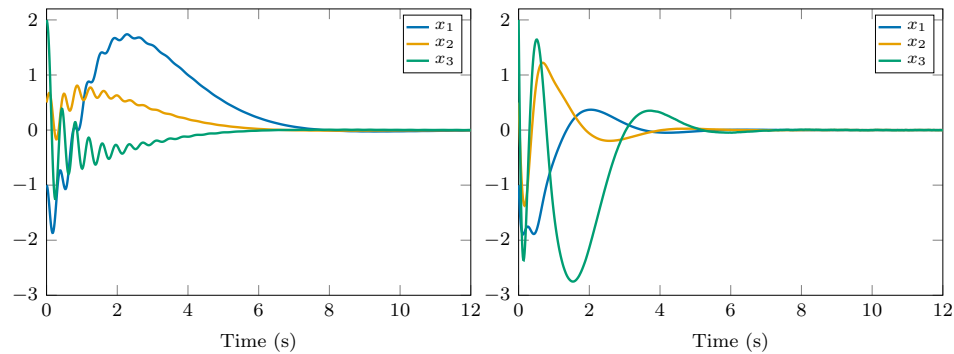


Figure 6: Closed-loop responses obtained with the sampled LMI (Algorithm 2, left) and the integral LMI (Algorithm 3, right) under noisy output measurements

7 Concluding remarks

We proposed an input-output data-driven stabilization approach for general LTI MIMO systems in continuous time without any restriction on the observability indices. Our key idea is to interpret Kreisselmeier's adaptive filter as an observer of a stabilizable canonical non-minimal realization of the plant. The complete filter is used to process the experimental data offline. Since the corresponding extended realization is generally stabilizable rather than controllable, we introduced a projected-residual decomposition that separates an excited stabilizable coordinate from stable autonomous modes. Exploiting the known input matrix of the resulting reduced data-compatible system, we derived an LMI that computes the feedback gain of a reduced-order dynamic output-feedback controller. We also provided input-design conditions ensuring the required excitation. Future work will study rigorous robustness guarantees in the presence of measurement noise and numerical rank uncertainty.

Appendix

A Proof of Lemma 2

Proof. The filter (5) can be written as

$$\dot{M} = \Lambda M + \begin{bmatrix} 0_{n \times pn} & u^\top \otimes I_n \end{bmatrix} + \begin{bmatrix} y^\top \otimes I_n & 0_{n \times mn} \end{bmatrix}.$$

We vectorize each term on the right-hand side. Since M has $\mu = (p + m)n$ columns, $\text{vec}(\Lambda M) = (I_\mu \otimes \Lambda) \text{vec}(M) = F\hat{z}$. For the input term, let $a := \text{col}(0_p, u) \in \mathbb{R}^{p+m}$, so that

$$\begin{bmatrix} 0_{n \times pn} & u^\top \otimes I_n \end{bmatrix} = a^\top \otimes I_n.$$

Writing $a = \sum_{i=1}^{p+m} a_i e_i$, where e_i is the i -th standard basis vector of $\mathbb{R}^{p+m}$, gives

$$\begin{aligned} \text{vec}(a^\top \otimes I_n) &= \sum_{i=1}^{p+m} a_i \text{vec}(e_i^\top \otimes I_n) \\ &= \sum_{i=1}^{p+m} a_i (e_i \otimes \text{vec}(I_n)) \\ &= (I_{p+m} \otimes \text{vec}(I_n)) a \\ &= \begin{bmatrix} 0_{pn^2 \times m} \\ I_m \otimes \text{vec}(I_n) \end{bmatrix} u = Gu. \end{aligned}$$

Similarly, applying the same identity to $\text{col}(y, 0_m)$ yields

$$\text{vec}\left(\begin{bmatrix} y^\top \otimes I_n & 0_{n \times mn} \end{bmatrix}\right) = \begin{bmatrix} I_p \otimes \text{vec}(I_n) \\ 0_{mn^2 \times p} \end{bmatrix} y = Ly.$$

Combining these identities and using $\hat{z} = \text{vec}(M)$ gives $\dot{\hat{z}} = F\hat{z} + Gu + Ly$, with F , G , and L as in (15). This proves (14). $\square$

B Proof of Proposition 2

Proof. Let (Q, Y) satisfy (31) and define $W := \mathbf{X}Q \succ 0$. Since $\dot{\mathbf{X}} - \mathfrak{B}\mathbf{U} = \mathfrak{A}\mathbf{X}$, the second inequality in (31) becomes $(\mathfrak{A}W + \mathfrak{B}Y) + (\mathfrak{A}W + \mathfrak{B}Y)^\top \prec 0$. With $K = YW^{-1}$, this is $(\mathfrak{A} + \mathfrak{B}K)W + W(\mathfrak{A} + \mathfrak{B}K)^\top \prec 0$, so $\mathfrak{A} + \mathfrak{B}K$ is Hurwitz.

Conversely, let K be any stabilizing gain. There exists $W = W^\top \succ 0$ such that the preceding Lyapunov inequality holds. Since $\text{rank } \mathbf{X} = n$, one can choose Q such that $\mathbf{X}Q = W$. Setting $Y := KW$ then gives $K = Y(\mathbf{X}Q)^{-1}$, and using $\dot{\mathbf{X}} - \mathfrak{B}\mathbf{U} = \mathfrak{A}\mathbf{X}$ shows that (Q, Y) satisfies (31). $\square$

In fact, $\text{rank } \mathbf{X} = n$ is the necessary and sufficient condition to stabilize all the systems within the data-compatible model set $\Sigma_D^{\mathfrak{B}} := \{\bar{\mathfrak{A}} \in \mathbb{R}^{n \times n} : \bar{\mathfrak{A}}\mathbf{X} = \dot{\mathbf{X}} - \mathfrak{B}\mathbf{U}\}$, which is shown below.

Lemma 4. There exists a matrix $K \in \mathbb{R}^{m \times n}$ such that $\bar{\mathfrak{A}} + \mathfrak{B}K$ is Hurwitz for every $\bar{\mathfrak{A}} \in \Sigma_D^{\mathfrak{B}}$ if and only if $\text{rank } \mathbf{X} = n$.

Proof. ($\Leftarrow$) Suppose that $\text{rank } \mathbf{X} = n$. The data-compatible set is the singleton $\Sigma_D^{\mathfrak{B}} = \{\mathfrak{A}\}$, because $\bar{\mathfrak{A}} = (\dot{\mathbf{X}} - \mathfrak{B}\mathbf{U})\mathbf{X}^\top(\mathbf{X}\mathbf{X}^\top)^{-1}$. Since $(\mathfrak{A}, \mathfrak{B})$ is stabilizable, a stabilizing gain K exists.

($\Rightarrow$) Suppose $\text{rank } \mathbf{X} < n$. Let $\mathfrak{A}_0 \in \Sigma_D^{\mathfrak{B}}$ be any compatible matrix. There exists a nonzero vector $l \in \mathbb{R}^n$ such that $l^\top \mathbf{X} = 0$. Choose $v \in \mathbb{R}^n$ such that $l^\top v = 1$. For any scalar $\alpha \in \mathbb{R}$, define $\mathfrak{A}_\alpha := \mathfrak{A}_0 + \alpha v l^\top$. Then $\mathfrak{A}_\alpha \mathbf{X} = \mathfrak{A}_0 \mathbf{X} + \alpha v l^\top \mathbf{X} = \dot{\mathbf{X}} - \mathfrak{B}\mathbf{U}$, thus $\mathfrak{A}_\alpha \in \Sigma_D^{\mathfrak{B}}$. By the formulation, we parametrized a set of data-compatible matrices as $\mathfrak{A}_\alpha \in \Sigma_D^{\mathfrak{B}}$. By fixing a candidate feedback gain K and taking the trace of the resulting closed-loop dynamics, we have: $\text{tr}(\mathfrak{A}_0 + \alpha v l^\top + \mathfrak{B}K) = \text{tr}(\mathfrak{A}_0 + \mathfrak{B}K) + \alpha$. Since α is arbitrary, if we select $\alpha > -\text{tr}(\mathfrak{A}_0 + \mathfrak{B}K)$, then the positive trace of the closed-loop matrix indicates that for any gain K , there exists $\mathfrak{A}_\alpha \in \Sigma_D^{\mathfrak{B}}$ that is not stabilized. Therefore, if $\text{rank } \mathbf{X} < n$, no common stabilizing controller exists. $\square$

C Proof of Proposition 3

Proof. Given the data $\{\mathbf{x}(t), \mathbf{u}(t), \dot{\mathbf{x}}(t)\}_{t \in [0, t_0]}$, the set of all systems that can generate these data is characterized by

$$\Sigma_2^{\mathfrak{B}} := \left\{ \bar{\mathfrak{A}} \in \mathbb{R}^{n \times n} : \begin{bmatrix} I_n & \bar{\mathfrak{A}} \end{bmatrix} \mathcal{N} \begin{bmatrix} I_n \\ \bar{\mathfrak{A}}^\top \end{bmatrix} = 0 \right\},$$

with

$$\mathcal{N} := - \int_0^{t_0} \begin{bmatrix} \dot{\mathbf{x}}(s) - \mathfrak{B}\mathbf{u}(s) \\ -\mathbf{x}(s) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}(s) - \mathfrak{B}\mathbf{u}(s) \\ -\mathbf{x}(s) \end{bmatrix}^\top ds.$$

Firstly, we show the equivalence between the feasibility of the LMI (32) and the IE condition of $\mathbf{x}$.

($\Leftarrow$) Assume that $\mathbf{x}$ is IE. Then its Gramian satisfies $\Sigma_{\mathbf{x}} := \int_0^{t_0} \mathbf{x}(s)\mathbf{x}(s)^\top ds \succ 0$. Let $\bar{\mathfrak{A}}_1, \bar{\mathfrak{A}}_2 \in \Sigma_2^{\mathfrak{B}}$. Then

$$(\bar{\mathfrak{A}}_1 - \bar{\mathfrak{A}}_2)\mathbf{x}(t) = 0 \implies (\bar{\mathfrak{A}}_1 - \bar{\mathfrak{A}}_2)\Sigma_{\mathbf{x}}(\bar{\mathfrak{A}}_1 - \bar{\mathfrak{A}}_2)^\top = 0.$$

From $\Sigma_{\mathbf{x}} \succ 0$, we have $\bar{\mathfrak{A}}_1 = \bar{\mathfrak{A}}_2$. Therefore, there is a unique element in $\Sigma_2^{\mathfrak{B}}$, i.e., $\Sigma_2^{\mathfrak{B}} = \{\mathfrak{A}\}$. Since $(\mathfrak{A}, \mathfrak{B})$ is stabilizable, there exist $\hat{P} \succ 0$, $\hat{Y} \in \mathbb{R}^{m \times n}$, and $\hat{\rho} > 0$ such that

$$\mathcal{L}(\mathfrak{A}, \hat{P}, \hat{Y}) \preceq -\hat{\rho}I_n \quad (50)$$

with $\mathcal{L}(\mathfrak{A}, \hat{P}, \hat{Y}) := \mathfrak{A}\hat{P} + \mathfrak{B}\hat{Y} + \hat{P}\mathfrak{A}^\top + \hat{Y}^\top \mathfrak{B}^\top$.

From $\dot{\mathbf{x}}(t) - \mathfrak{B}\mathbf{u}(t) = \bar{\mathfrak{A}}\mathbf{x}(t)$, one has

$$\mathcal{N} = - \begin{bmatrix} \mathfrak{A}\Sigma_{\mathbf{x}}\mathfrak{A}^\top & -\mathfrak{A}\Sigma_{\mathbf{x}} \\ -\Sigma_{\mathbf{x}}\mathfrak{A}^\top & \Sigma_{\mathbf{x}} \end{bmatrix},$$

and

$$\begin{aligned} (32) \quad & \iff \begin{bmatrix} \mathfrak{M} & -P - \mathfrak{A}\Sigma_{\mathbf{x}} \\ -P - \Sigma_{\mathbf{x}}\mathfrak{A}^\top & \Sigma_{\mathbf{x}} \end{bmatrix} \succeq 0 \\ & \iff \mathcal{L}(\mathfrak{A}, P, Y) + P\Sigma_{\mathbf{x}}^{-1}P \preceq -\rho I_n \end{aligned} \quad (51)$$

with the definition $\mathfrak{M} := \mathfrak{A}\Sigma_{\mathbf{x}}\mathfrak{A}^\top - \mathfrak{B}Y - Y^\top \mathfrak{B}^\top - \rho I_n$, where in the second equivalence we have used the Schur complement. Let $P = \alpha \hat{P}$, $Y = \alpha \hat{Y}$ with $\alpha > 0$. One has

$$\begin{aligned} \mathcal{L}(\mathfrak{A}, P, Y) + P\Sigma_{\mathbf{x}}^{-1}P &= \alpha \mathcal{L}(\mathfrak{A}, \hat{P}, \hat{Y}) + \alpha^2 \hat{P}\Sigma_{\mathbf{x}}^{-1}\hat{P} \\ &\preceq -\alpha \hat{\rho}I_n + \alpha^2 r I_n, \end{aligned} \quad (52)$$

where $r := \lambda_{\max}(\hat{P}\Sigma_{\mathbf{x}}^{-1}\hat{P})$, which is independent of the decision variable α . Therefore, for any $0 < \alpha < \hat{\rho}/r$ and $0 < \rho \leq \alpha\hat{\rho} - \alpha^2r$, the last inequality in (51) holds. Thus, (32) is feasible. Moreover, $YP^{-1} = \alpha\hat{Y}(\alpha\hat{P})^{-1} = \hat{Y}\hat{P}^{-1}$, so the scaling preserves the feedback gain.

($\Rightarrow$) Suppose that (P, Y, ρ) is feasible for the LMI (32). For every $\bar{\mathfrak{A}} \in \Sigma_2^{\mathfrak{B}}$, the congruence of $\mathcal{N}$ with $[I_n \ \bar{\mathfrak{A}}]$ is zero by definition. Hence, pre- and post-multiplying (32) by $[I_n \ \bar{\mathfrak{A}}]$ and its transpose gives, for every $\bar{\mathfrak{A}} \in \Sigma_2^{\mathfrak{B}}$,

$$-[I_n \ \bar{\mathfrak{A}}]\mathcal{M}(Y, P) \begin{bmatrix} I_n \\ \bar{\mathfrak{A}}^\top \end{bmatrix} = \mathcal{L}(\bar{\mathfrak{A}}, P, Y) \preceq -\rho I_n.$$

Therefore, with $K = YP^{-1}$, $(\bar{\mathfrak{A}} + \mathfrak{B}K)P + P(\bar{\mathfrak{A}} + \mathfrak{B}K)^\top \prec 0$, $\forall \bar{\mathfrak{A}} \in \Sigma_2^{\mathfrak{B}}$. Thus, $K = YP^{-1}$ stabilizes every system in $\Sigma_2^{\mathfrak{B}}$. Following similar arguments to Lemma 4, this is possible only if $\Sigma_{\mathbf{x}} \succ 0$, i.e., $\mathbf{x}$ is IE. $\square$

D Some useful results

Lemma 5 ([15]). For a controllable system (2) with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, let $u \in \mathcal{C}^n([t_1, t_2], \mathbb{R}^m)$. Then:

- (1) If u is rich of order n , the state x is IE over $[t_1, t_2]$;
- (2) If u is rich of order $n + 1$, $\text{col}(x, u)$ is IE on $[t_1, t_2]$. Namely, there exists $\epsilon > 0$ such that

$$\int_{t_1}^{t_2} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix}^\top ds \succ \epsilon I_{n+m}.$$

Proof. The first part is exactly [15, Lem. 1], and the second part is its minor extension. Suppose, by contradiction, that $\text{col}(x, u)$ is not IE on $[t_1, t_2]$. Then there exist vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, not both zero, such that

$$a^\top x(t) + b^\top u(t) \equiv 0, \quad \forall t \in [t_1, t_2]. \quad (53)$$

Define $z(t) := a^\top x(t)$. We can write the transfer function from u to z as $G_a(s) := a^\top (sI - A)^{-1}B = \chi_A^{-1}(s)N_a^\top(s)$, where $\chi_A(s) := \det(sI - A)$ and $N_a(s)$ is a polynomial vector satisfying $\deg\{N_a\} \leq n-1$. Therefore, from $\dot{x} = Ax + Bu$, $z = a^\top x$ we have $\chi_A(p)z(t) = N_a^\top(p)u(t)$, with the differential operator $p(\cdot) = \frac{d}{dt}(\cdot)$. Using $z(t) = a^\top x(t) = -b^\top u(t)$ yields

$$Q^\top(p)u(t) \equiv 0, \quad \forall t \in [t_1, t_2],$$

with $Q^\top(p) := N_a^\top(p) + \chi_A(p)b^\top$ and $\deg Q \leq n$.

We now show that $Q(s)$ is not identically zero for $s \in \mathbb{C}$. From (53), we note that a and b are not both zero, and thus consider two cases: (i) if $b \neq 0$, then the leading term of $Q^\top(s)$ is $s^n b^\top$, so $Q(s)$ cannot vanish identically; (ii) if $b = 0$ and $a \neq 0$, the controllability of the pair (A, B) implies $G_a(s) \not\equiv 0$, hence $N_a(s) \not\equiv 0$. Thus, in all cases, there exist vectors $q_0, \dots, q_n \in \mathbb{R}^m$, not all zero, such that $\sum_{i=0}^n q_i^\top u^{(i)}(t) \equiv 0$, $\forall t \in [t_1, t_2]$. This means that the components of $U_n := \text{col}(u, \dot{u}, \dots, u^{(n)})$ are linearly dependent on $[t_1, t_2]$, which contradicts $\int_{t_1}^{t_2} U_n(\tau)U_n^\top(\tau) d\tau > 0$, i.e., u is rich of order $n + 1$. Hence, $\text{col}(x, u)$ is IE on $[t_1, t_2]$. $\square$

[15, Lemma 1] shows that the input u being rich of order n implies that the state x —rather than $\text{col}(x, u)$ —is IE; moreover, in a controllable single-input systems, this condition is necessary and sufficient for x to be IE with a positive excitation lower bound uniform over all initial conditions. This is, indeed, an extension of the celebrated relation between persistency of excitation and sufficient richness [27]. A similar result can be found in [25, Prop. 1].

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