

Capacity disclosure, market manipulation, and quantity competition: A two-stage game-theoretic analysis

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Capacity disclosure, market manipulation, and quantity competition: A two-stage game-theoretic analysis

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Abstract : This paper examines firm competition in markets where demand depends on firms' announced resource stocks or production capacities. We develop a two-stage model and characterize the conditions under which firms have incentives to manipulate the market when concealing information is costly. In the first stage, firms simultaneously announce their production capacities, which may differ from their actual capacities. In the second stage, they simultaneously choose the quantities supplied to the market. We characterize the unique subgame perfect equilibrium under two product regimes: (i) perishable goods, for which any unsold quantity has no salvage value, and (ii) goods for which unsold inventory retains a positive salvage value. Through analytical sensitivity analysis and numerical simulations, we identify the parametric conditions that determine the emergence of interior and corner equilibria, highlighting the threshold effects that govern firms' strategic behavior. Further, we analyze the case where there is no competition in the market to disentangle the effects of consumers' willingness to pay (WTP) dependence on stock availability from those arising from competition.

Keywords: Cournot competition; production capacity; concealing information; perishable product; subgame-perfect equilibrium

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1 Introduction

In any market, consumer buying behavior is influenced by information provided by firms through, for example, advertising and promotional activities, as well as by external sources such as social media influencers, reviews from previous buyers, and newspapers. The objective of this paper is to assess, in an imperfectly competitive market, the impact of a specific type of information, namely firms' announced production capacities, on consumers' maximum willingness to pay (WTP). We use the terms production capacity, stock of product (or resource), and product availability interchangeably.

The idea that variables controlled by a firm can influence consumers' WTP, and consequently demand, is not new. Investments in product quality and advertising are well known to increase the price consumers are willing to pay for a product. The literature has extensively confirmed this effect under different market structures (monopoly and oligopoly) and time horizons (static and dynamic), in both deterministic and stochastic settings; see the surveys on product quality by De Giovanni and Zaccour (2023) and on advertising models by Huang et al. (2012). Costly investments in advertising or product quality, including product greening, allow firms to differentiate their offerings from those of competitors, thereby softening price competition and increasing profits.

The impact of production capacity on WTP does not arise from product differentiation or positioning strategies; rather, it is related to product availability. Indeed, information indicating that production capacity is tight or that available quantities are limited can significantly affect market prices (Badia et al. (2021); Bischi et al. (2007); Laye and Laye (2008); Osborne and Pitchik (1986)). For example, the lower the production volume of a first-growth Bordeaux wine, the higher the price that connoisseurs and speculators may be willing to pay for it.

Production capacity, or the stock of a resource, is often privately held information. The question then becomes: under what circumstances would a firm voluntarily disclose this information to the market? The obvious answer is when doing so is advantageous. The next question is whether the firm should disclose the true value. To illustrate, consider the familiar situation in which an airline or an online platform states that “only x seats are left at this price.” Such information is intended to pressure consumers into purchasing immediately. Purchasing immediately or not depends on the extent to which consumers believe this claim and on their evaluation of future prices.

In this paper, we seek to answer the following two questions:

1. Under what conditions, is it in the best interest of competing firms to manipulate the market by understating their production capacities?
2. How the results would change if there is no competition?

To address these questions, we first consider a two-stage Cournot duopoly model. In the first stage, firms simultaneously announce their production capacities, which need not coincide with their actual capacities. In the second stage, they simultaneously choose the quantities to supply to the market. We determine the unique subgame perfect equilibrium and characterize the parameter conditions under which firms (i) truthfully announce their capacities, (ii) conceal their capacities to the greatest extent possible, or (iii) choose an interior announcement level. Following Daughety and Reinganum (2007), we suppose that manipulating or concealing information is costly. Next, we assume that there is one firm in the market, solve the corresponding optimization problem, and compare qualitatively the results to the duopoly case. This will enable us to disentangle the effects of consumers' WTP dependence on stock availability from those arising from competition.

When a parameter value is private information, one may adopt either a Bayesian or a robust approach to characterize equilibrium. In the Bayesian approach, the distribution of an uncertain parameter is assumed to be known and is used to define a Bayesian-Nash equilibrium (Harsanyi (1967-1968)). Although we do not examine whether firms are better off sharing private information with rivals or keeping it confidential, we note that Vives (1984) shows that the answer depends on the nature

of competition, namely whether firms compete in quantities (Cournot) or prices (Bertrand), as well as on whether their products are substitutes or complements. Interestingly, in our setting, quantities are strategic substitutes, whereas announced capacities are strategic complements.

In a robust approach, by contrast, the uncertain parameter is assumed to belong to a given set, and firms choose their actions while accounting for the worst-case realization of the unknown parameter (Crespi et al. (2017); Radi et al. (2024)). In our “naive” approach, each firm believes the announcement made by its rival and constructs its demand function accordingly. In a sense, the probability distribution in the Bayesian framework collapses to a single value, which corresponds to the worst-case realization in the robust framework.

Our assumption that concealing information is not free comes from signaling games where messages are costly¹ with, however, two caveats. First, the destination of the message is the market, not the rival. Second, without any loss of generality, we suppose that the signal is costly only when it conceals the true capacity.

Our contributions are as follows:

1. We develop a model of firm competition in a market sensitive to information about production capacity. To the best of our knowledge, this is the first attempt to analyze this effect within a duopoly framework.
2. We analytically characterize the equilibrium strategies of symmetric firms, namely their announced capacity (or resource stock) levels, in two cases: (i) when the product is perishable and unsold quantities have no value,² and (ii) when the remaining stock can be sold at a positive salvage value. Further, we analyze the case where there is no competition in the market.
3. We conduct a sensitivity analysis to assess the impact of parameter values on firms’ strategies and market prices. Numerical simulations clearly indicate the conditions under which each type of equilibrium (interior or corner) emerges for different parameter configurations.

The remainder of the paper is organized as follows. Section 2 presents the basic model. Section 3 characterizes the subgame perfect Nash equilibrium of the two-stage game and provides sensitivity analyses and numerical examples. In Section 4, we extend the model by first introducing a positive salvage value for the remaining stock and next by considering an absence of competition. Numerical examples and comparative statics with respect to the main parameters are also provided. Finally, Section 5 concludes and outlines possible directions for future research.

2 Model

Consider a two-stage game with two firms (players) producing a homogeneous product and selling it in the same market. Player $i = 1, 2$ owns a stock of product (or a capacity to produce) $S_i \geq 0$. In the first stage, each player announces its available stock $A_i \geq A_i^{\min}$, $i = 1, 2$, which may be not equal to S_i . The value $A_i^{\min}$ is the minimum availability that player i can announce. To keep it simple and without any loss of generality, we set $A_i^{\min} = 0$. At the second stage, given A_1 and A_2 , the players choose their quantity levels $q_1 \in [0, S_1]$ and $q_2 \in [0, S_2]$.

The total announced availability³ has an impact on demand; more precisely, on the choke price, with the larger this capacity, the lower consumers are willing to pay for the product (or resource). Let the inverse-demand be given by

$$P(Q) = 1 - \theta(A_1 + A_2) - (q_1 + q_2),$$

¹See, for example, Sobel (2020) for a general discussion of signaling games and Goltsman and Pavlov (2014) for an application to Cournot competition.

²For a game-theoretic analysis of a perishable product (fresh food), see, for example, Liu et al. (2025).

³We use availability, stock, and capacity interchangeably.

where $\theta \in [0, 1]$. The choke price is given by $1 - \theta(A_1 + A_2)$; if $A_1 + A_2 = 0$, or alternatively $\theta = 0$, then the maximum willingness to pay would be a constant and equal to 1. Note that to have a nonnegative price, we must have $q_1 + q_2 = Q \leq 1 - \theta(A_1 + A_2)$.

We assume that the production cost is zero, which should not affect qualitatively the results. We include a quadratic cost for hiding the information about the true stock, that is, $\tau(A_i - S_i)^2$, $\tau \geq 0, i = 1, 2$. The assumption that hiding information is costly, and increasing in the difference with the actual data, is common in signaling games (Sobel (2020); Goltsman and Pavlov (2014)).

The profit optimization problem of player $i, i = 1, 2$ is as follows:

$$\max_{0 \leq q_i \leq S_i, A_i \geq 0} \Pi_i = (1 - \theta(A_1 + A_2) - (q_1 + q_2)) q_i - \tau(A_i - S_i)^2.$$

Remark 1. In our formulation, any remaining stock has zero value, which is an appealing assumption when the product is perishable, e.g., fresh food product. In Section 4.1, we extend our model to account for a positive salvage value.

3 Equilibrium

To determine a subgame perfect Nash equilibrium, we start by solving the second-stage optimization problem given by

$$\max_{0 \leq q_i \leq S_i} \Pi_i = (1 - \theta(A_1 + A_2) - (q_1 + q_2)) q_i - \tau(A_i - S_i)^2.$$

At this stage the information structure is defined by the common knowledge of the pair (A_1, A_2) , and player i 's private knowledge of its actual capacity S_i , but not of the capacity $S_{3-i}, i = 1, 2$.

Introduce the Lagrangians:

$$L_i = (1 - \theta(A_1 + A_2) - (q_1 + q_2)) q_i - \tau(A_i - S_i)^2 + \lambda_i(S_i - q_i), i = 1, 2,$$

where λ_i is the Lagrange multiplier appended to the constraint $q_i \leq S_i, i = 1, 2$. The first-order equilibrium conditions are

$$\begin{aligned} \frac{\partial L_i}{\partial q_i} &= 1 - \theta(A_1 + A_2) - (2q_i + q_{3-i}) - \lambda_i = 0, i = 1, 2, \\ S_i - q_i &\geq 0, \quad \lambda_i \geq 0, \quad \lambda_i(S_i - q_i) = 0, i = 1, 2. \end{aligned}$$

It is easy to verify that the Lagrangians are concave. Indeed, we have $\frac{\partial^2 L_i}{\partial q_i^2} = -2 < 0$. Since the Jacobian matrix J_1 , that is,

$$J_1 = \begin{pmatrix} \frac{\partial^2 L_1}{\partial q_1^2} & \frac{\partial^2 L_1}{\partial q_1 \partial q_2} \\ \frac{\partial^2 L_2}{\partial q_2 \partial q_1} & \frac{\partial^2 L_2}{\partial q_2^2} \end{pmatrix}$$

is negative definite, the equilibrium at this stage is unique.

The first two equilibrium conditions lead to the following reaction functions:

$$q_i = \frac{1}{2} (1 - \theta(A_1 + A_2) - \lambda_i - q_{3-i}), \quad i = 1, 2,$$

which shows, as expected in a Cournot-type model, that the quantities are strategic substitutes, i.e., $\frac{\partial q_i}{\partial q_{3-i}} < 0, i = 1, 2$. Solving the first-order equilibrium conditions, we get

$$q_1 = \frac{1}{3} (1 - \theta(A_1 + A_2) - 2\lambda_1 + \lambda_2), \tag{1}$$

$$q_2 = \frac{1}{3} (1 - \theta(A_1 + A_2) - 2\lambda_2 + \lambda_1), \tag{2}$$

$$S_1 - q_1 \geq 0, \quad \lambda_1 \geq 0, \quad \lambda_1(S_1 - q_1) = 0, \quad (3)$$

$$S_2 - q_2 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_2(S_2 - q_2) = 0. \quad (4)$$

Note that (1) and (2) show that player i 's quantity is decreasing in both announced stocks. Substituting for the quantities in the inverse-demand function, we obtain the following price:

$$P = \frac{1}{3} (1 - \theta(A_1 + A_2) + \lambda_1 + \lambda_2).$$

Inserting the price and the quantities in the players' objective functions leads to the following first-stage optimization problems:

$$\begin{aligned} \max_{A_1 \geq 0} \Pi_1 &= \frac{1}{9} ((1 - \theta(A_1 + A_2) + \lambda_1 + \lambda_2) (1 - \theta(A_1 + A_2) - 2\lambda_1 + \lambda_2)) - \tau(A_1 - S_1)^2, \\ \max_{A_2 \geq 0} \Pi_2 &= \frac{1}{9} ((1 - \theta(A_1 + A_2) + \lambda_1 + \lambda_2) (1 - \theta(A_1 + A_2) - 2\lambda_2 + \lambda_1)) - \tau(A_2 - S_2)^2. \end{aligned}$$

The Lagrangians corresponding to the above optimization problems are given by

$$\begin{aligned} L_3 &= \frac{1}{9} ((1 - \theta(A_1 + A_2) + \lambda_1 + \lambda_2) (1 - \theta(A_1 + A_2) - 2\lambda_1 + \lambda_2)) - \tau(A_1 - S_1)^2 + \mu_1 A_1, \\ L_4 &= \frac{1}{9} ((1 - \theta(A_1 + A_2) + \lambda_1 + \lambda_2) (1 - \theta(A_1 + A_2) - 2\lambda_2 + \lambda_1)) - \tau(A_2 - S_2)^2 + \mu_2 A_2, \end{aligned}$$

where μ_i is the Lagrange multiplier appended to the nonnegativity constraint $A_i \geq 0, i = 1, 2$.

Let

$$X = \theta(\lambda_1 - 2\lambda_2 - 2) + 9\mu_1,$$

$$Y = \theta(\lambda_2 - 2\lambda_1 - 2) + 9\mu_2.$$

The first-order equilibrium conditions are

$$\frac{\partial L_3}{\partial A_1} = \frac{1}{9} (X + 2\theta^2(A_1 + A_2) - 18\tau(A_1 - S_1)) = 0, \quad (5)$$

$$\frac{\partial L_4}{\partial A_2} = \frac{1}{9} (Y + 2\theta^2(A_1 + A_2) - 18\tau(A_2 - S_2)) = 0, \quad (6)$$

$$\mu_1 A_1 = 0, \quad A_1 \geq 0, \quad \mu_1 \geq 0, \quad (7)$$

$$\mu_2 A_2 = 0, \quad A_2 \geq 0, \quad \mu_2 \geq 0. \quad (8)$$

Concavity of the profit function $\Pi_i(A_i, A_{3-i})$ requires that

$$\frac{\partial^2 L_3}{\partial A_1^2} = \frac{2}{9} (\theta^2 - 9\tau) \leq 0 \Leftrightarrow \theta^2 \leq 9\tau. \quad (9)$$

We assume throughout the paper that (9) holds.

Remark 2. To have uniqueness of the equilibrium at this stage and use the necessary and sufficient conditions provided in Rosen (1965),⁴ we require that the Jacobian matrix

$$J_2 = \begin{pmatrix} \frac{\partial^2 L_3}{\partial A_1^2} & \frac{\partial^2 L_3}{\partial A_1 \partial A_2} \\ \frac{\partial^2 L_4}{\partial A_2 \partial A_1} & \frac{\partial^2 L_4}{\partial A_2^2} \end{pmatrix} = \begin{pmatrix} \frac{2\theta^2}{9} - 2\tau & \frac{2\theta^2}{9} \\ \frac{2\theta^2}{9} & \frac{2\theta^2}{9} - 2\tau \end{pmatrix}$$

⁴Alternative sufficient conditions for uniqueness can be derived by requesting that the derivatives of the reaction functions $A_1(A_2)$ and $A_2(A_1)$ satisfy the following conditions:

$$\left| \frac{dA_1}{dA_2} \right| < 1 \quad \text{and} \quad \left| \frac{dA_2}{dA_1} \right| < 1.$$

It is easy to check that the above two conditions are equivalent to $\theta^2 < \frac{9}{2}\tau$, i.e., that is the same restriction on the parameter values in (10).

be negative definite, that is,

$$\left(\frac{2\theta^2}{9} - 2\tau\right) \left(\frac{2\theta^2}{9} - 2\tau\right) > \frac{4\theta^4}{81},$$

or equivalently

$$9\tau - 2\theta^2 > 0. \quad (10)$$

From the equilibrium conditions (5)–(6), we get the following reaction functions:

$$A_i = \frac{1}{2(9\tau - 2\theta^2)} (X + 2\theta^2 A_{3-i} + 18\tau S_i), \quad i = 1, 2,$$

which shows, under the concavity conditions, that A_1 and A_2 are strategic complements, i.e., $\frac{dA_i}{dA_{3-i}} \geq 0$ for $i = 1, 2$. Recall that in the second stage the interaction is of strategic substitutability.

Solving the above system, the equilibrium conditions (5)–(8) become

$$A_1 = \frac{(9\tau_2 - \theta^2)(X + 18\tau S_1) + \theta^2(Y + 18\tau S_2)}{18\tau(9\tau - 2\theta^2)}, \quad (11)$$

$$A_2 = \frac{(9\tau_1 - \theta^2)(Y + 18\tau S_2) + \theta^2(X + 18\tau S_1)}{8\tau(9\tau - 2\theta^2)}, \quad (12)$$

$$\mu_1 A_1 = 0, \quad A_1 \geq 0, \quad \mu_1 \geq 0, \quad (13)$$

$$\mu_2 A_2 = 0, \quad A_2 \geq 0, \quad \mu_2 \geq 0. \quad (14)$$

Inserting A_1 and A_2 in (1)–(2), we obtain the quantities as functions of the parameter values and the multipliers:

$$q_1 = \frac{1}{2\tau(9\tau - 2\theta^2)} (\tau\lambda_2 (6\tau - \theta^2) + 3\theta^2\lambda_1\tau - 3\theta\tau(\mu_2 + \mu_1) + 6\tau^2(1 - 2\lambda_1 - \theta S_1 - \theta S_2)),$$

$$q_2 = \frac{1}{2\tau(9\tau - 2\theta^2)} (\tau\lambda_1 (6\tau - \theta^2) + 3\theta^2\lambda_2\tau - 3\theta\tau(\mu_1 + \mu_2) + 6\tau^2(1 - 2\lambda_2 - \theta S_2 - \theta S_1)).$$

We focus on a symmetric equilibrium, i.e., $q_1 = q_2 = q$, $A_1 = A_2 = A$, and $S_1 = S_2 = S$. Consequently, the above conditions become

$$q = \frac{\lambda(6\tau + 2\theta^2) - 6\theta\mu + 6\tau(1 - 2\lambda - 2\theta S)}{2(9\tau - 2\theta^2)},$$

$$A = \frac{9\mu - 2\theta + 18S\tau - \theta\lambda + 2\theta\gamma}{2(9\tau - 2\theta^2)}.$$

Remark 3. The symmetry assumption is made to isolate the impact of the strategic choice of A_i , $i = 1, 2$, on the results. There are no conceptual difficulties in extending the model by letting τ and S to be player specific.

To fully characterize the equilibrium, we need to consider the four cases corresponding to the following complementarity conditions:

$$S - q \geq 0, \quad \lambda \geq 0, \quad \lambda(S - q) = 0,$$

$$\mu A = 0, \quad A \geq 0, \quad \mu \geq 0.$$

Also, for each case we check for the nonnegativity of demand.

Proposition 1. *The unique subgame perfect Nash equilibrium is defined as follows:*

1. For $\tau \in (\frac{2}{9}\theta^2, \frac{\theta}{3})$, and

(a) $S < \frac{1}{3}$, then

$$q = S, \quad A = 0,$$

(b) $\frac{1}{3} \leq S < \frac{\theta}{9\tau}$, then

$$q = \frac{1}{3}, \quad A = 0,$$

(c) $\frac{\theta}{9\tau} \leq S < \frac{1}{2\theta}$, then

$$q = \frac{3\tau(1-2\theta S)}{9\tau-2\theta^2} > 0, \quad A = \frac{9\tau S - \theta}{9\tau-2\theta^2} > 0.$$

2. For $\tau \geq \frac{\theta}{3}$, and

(a) $S < \frac{\theta}{\theta+6\tau}$, then

$$q = S, \quad A = 0,$$

(b) $\frac{\theta}{\theta+6\tau} \leq S < \frac{3\tau}{9\tau-2\theta^2+6\tau\theta}$, then

$$q = S, \quad A = \frac{(6\tau+\theta)S - \theta}{2(3\tau-\theta^2)} > 0,$$

(c) $\frac{3\tau}{9\tau-2\theta^2+6\tau\theta} \leq S < \frac{1}{2\theta}$, then

$$q = \frac{3\tau(1-2\theta S)}{9\tau-2\theta^2} > 0, \quad A = \frac{9\tau S - \theta}{9\tau-2\theta^2} > 0.$$

The corresponding equilibrium prices are

$$\begin{aligned} P_{1(a)} &= P_{2(a)} = 1 - 2S, & P_{1(b)} &= \frac{1}{3}, \\ P_{1(c)} &= P_{2(c)} = \frac{3\tau(1-2\theta S)}{9\tau-2\theta^2}, & P_{2(b)} &= \frac{3\tau - (6\tau(1+\theta) - \theta^2)S}{3\tau - \theta^2}. \end{aligned}$$

Proof. See Appendix A. □

The structure of the unique equilibrium is as follows: (i) If θ is larger than $1/2S$, then there is no equilibrium with positive quantities and $A = S$. Indeed, if both firms report truthfully their product availability, then the choke price is given $1 - 2\theta S$, and it is negative for $S > 1/2\theta$. Players are better off exiting the market. (ii) For S sufficiently low, the equilibrium strategy is to distort maximally the information by announcing zero capacity and leaking to the market the maximal possible quantity, i.e., $q = S$. (iii) For an intermediate value of S , and $\tau \in (\frac{2}{9}\theta^2, \frac{\theta}{3})$, then the announced capacity keeps its corner value $A = 0$, while q is set at the lower bound of S , i.e., $q = 1/3$. If τ is large enough ($\tau \geq \frac{\theta}{3}$), then q remains maximal ($q = S$), while the announced capacity is increased to $\frac{(6\tau+\theta)S - \theta}{2(3\tau-\theta^2)}$. The interior solution, that is, when both q and A are interior, is the same for all combinations of parameter values and it materializes when S is sufficiently high.

The price sensitivity in the corner cases is from $P = 1 - 2\theta A - 2q$. In cases 1(a) and 2(a), $A = 0$ and $q = S$, so $P = 1 - 2S$, with $\partial P/\partial S = -2$ and no dependence on τ or θ . In case 1(b), $A = 0$ and $q = 1/3$, so $P = 1/3$, which is locally insensitive to the parameters. The price sensitivity in the interior cases is given in the next proposition, and the corresponding case 2(b) result is given in Proposition 3. **Remark 4.** If $\theta = 0$, that is, the demand is independent of A , the unique (standard Cournot) equilibrium is then given by $q = 1/3$ and $A = S$, if $S \geq 1/3$; otherwise, $q = S$.

Proposition 2. In the interior equilibrium:

(i) The quantity is decreasing in S and τ , and varies non monotonically with θ .

(ii) The announced stock is increasing in S and τ , and varies non monotonically with θ .

(iii) The price is $P = 3\tau(1-2\theta S)/(9\tau-2\theta^2)$. A higher S or τ lowers the equilibrium price, whereas the effect of θ is positive for $S < 2\theta/(9\tau+2\theta^2)$, zero at that value, and negative for larger S .

Proof. See Appendix B. □

We make the following comments: (i) A higher hiding cost τ makes it more expensive for firms to deviate from truthful announcements. Consequently, they increase their announced capacity A towards the actual capacity S . This higher A reduces the choke price $1 - 2\theta A$, which in turn leads to a lower equilibrium quantity q . (ii) A higher available stock S leads to lower quantity. Given that we showed previously that q is decreasing in A , this result is in line with the one in (i). (iii) For S sufficiently small (respectively, high), the impact of θ on A is negative (respectively, positive). A larger θ damages more the choke price, which provides an incentive to increase A towards S , which has the double advantage of reducing the total hiding cost and the negative impact on the willingness to pay.

In Proposition 1, the equilibrium results in cases 1(a), 1(b), and 2(a) are independent of the three parameter values. In case 2(b), we have

$$q = S, \quad A = \frac{(6\tau + \theta)S - \theta}{2(3\tau - \theta^2)} > 0,$$

for $\tau > \frac{\theta}{3}$ and $\frac{\theta}{\theta + 6\tau} \leq S < \frac{3\tau}{9\tau - 2\theta^2 + 6\tau\theta}$.

Proposition 3. In case 2(b) in Proposition 1, A is increasing in S, τ , and it is decreasing in θ . In the same case, the equilibrium price is

$$P = \frac{3\tau - (6\tau(1 + \theta) - \theta^2)S}{3\tau - \theta^2},$$

which is decreasing in S and τ . Its sensitivity to θ is positive for $S < \theta/(3\tau + \theta^2 + \theta)$, zero at that value, and negative for larger S .

Proof. See Appendix C. □

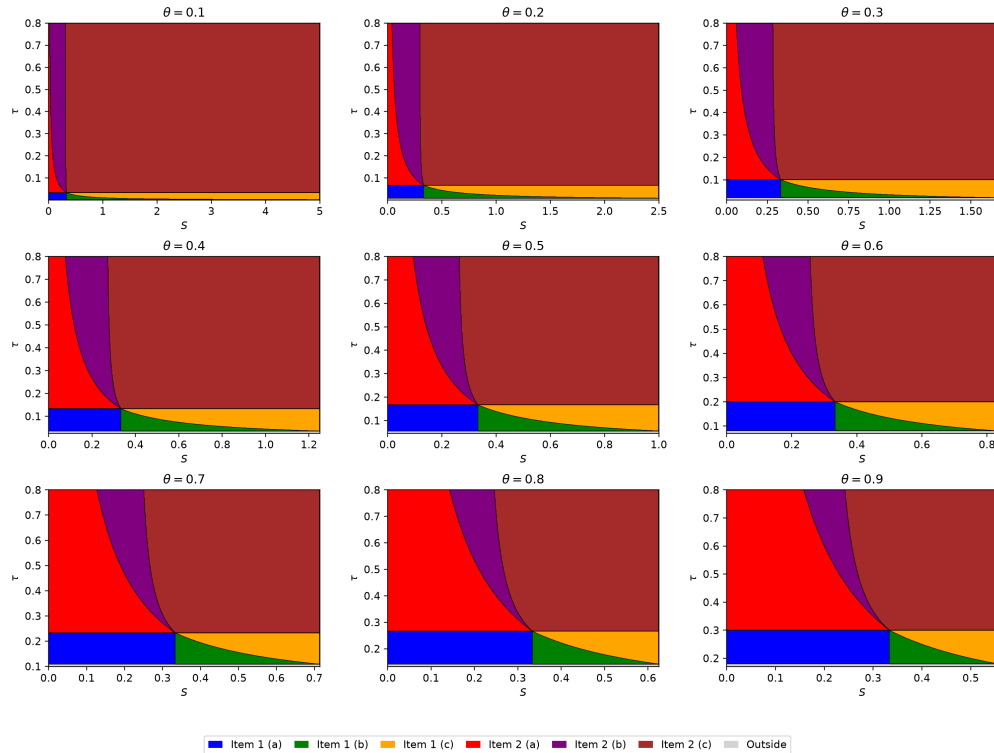


Figure 1: Characterization of items in Proposition 1 in coordinates (S, τ) for different values of $\theta = 0.1, \dots, 0.9$.

Figure 1 provides a visual representation of the regions described in Proposition 1 in the (S, τ) -space for 9 different values of θ ranging from 0.1 to 0.9. Note that the figures are drawn considering the restrictions $S \in (0, 1/2\theta)$ and $\tau > \frac{2}{9}\theta^2$. The region in the parameter space where the equilibrium is 1(a), 1(b), or 1(c) is smaller than when it is 2(a), 2(b), or 2(c). Further, the larger the value of θ , which means lower choke price, the smaller is the interior solution region and the larger the region where the players claim that they are out of stock ($A = 0$).

4 Extensions

We consider two extensions in this section. First, we extend our model by introducing a positive salvage value of the remaining stock, and next consider the case where there is no competition.

4.1 Valuing the remaining stock

Suppose that remaining quantities can generate revenues given by $\gamma(S_i - q_i)$, where $\gamma \in [0, 1]$, for $i = 1, 2$. Note that the value $\gamma(S_i - q_i)$ is determined endogenously. In this setup, Player $i, i = 1, 2$ maximizes its profit given by

$$\Pi_i = (1 - \theta(A_1 + A_2) - (q_1 + q_2))q_i - \tau(A_i - S_i)^2 + \gamma(S_i - q_i).$$

The following proposition characterizes the unique Nash equilibrium.

Proposition 4. *Suppose that $0 < \theta < 1$, $0 \leq \gamma < 1$, and $9\tau - 2\theta^2 > 0$. The unique symmetric subgame perfect Nash equilibrium is given by:*

1. For $\frac{2\theta^2}{9} < \tau \leq \frac{\theta}{3}$, and

(a) $0 \leq S < \frac{1-\gamma}{3}$, then

$$q = S, \quad A = 0,$$

(b) $\frac{1-\gamma}{3} \leq S < \frac{\theta(1-\gamma)}{9\tau}$, then

$$q = \frac{1-\gamma}{3}, \quad A = 0,$$

(c) $\frac{\theta(1-\gamma)}{9\tau} \leq S \leq \frac{1-\gamma}{2\theta}$, then

$$q = \frac{3\tau(1-\gamma-2\theta S)}{9\tau-2\theta^2} > 0, \quad A = \frac{9\tau S - \theta(1-\gamma)}{9\tau-2\theta^2} > 0.$$

2. For $\tau > \frac{\theta}{3}$, and

(a) $0 \leq S < \frac{\theta(1-\gamma)}{6\tau+\theta}$, then

$$q = S, \quad A = 0,$$

(b) $\frac{\theta(1-\gamma)}{6\tau+\theta} \leq S < \frac{3\tau(1-\gamma)}{9\tau-2\theta^2+6\theta\tau}$, then

$$q = S, \quad A = \frac{(6\tau+\theta)S - \theta(1-\gamma)}{2(3\tau-\theta^2)} > 0,$$

(c) $\frac{3\tau(1-\gamma)}{9\tau-2\theta^2+6\theta\tau} \leq S \leq \frac{1-\gamma}{2\theta}$,

$$q = \frac{3\tau(1-\gamma-2\theta S)}{9\tau-2\theta^2} > 0, \quad A = \frac{9\tau S - \theta(1-\gamma)}{9\tau-2\theta^2} > 0.$$

The corresponding equilibrium prices are

$$\begin{aligned} P_{1(a)} &= P_{2(a)} = 1 - 2S, & P_{1(b)} &= \frac{1+2\gamma}{3}, \\ P_{1(c)} &= P_{2(c)} = \gamma + \frac{3\tau(1-\gamma-2\theta S)}{9\tau-2\theta^2}, & P_{2(b)} &= \frac{3\tau(1-\gamma) - (6\tau(1+\theta) - \theta^2)S}{3\tau-\theta^2}. \end{aligned}$$

Proof. See Appendix B. □

As we can easily see, the structure of the equilibrium is similar to what we had in the perishable good case. Consequently, all the comments made before remain valid. The impact of the additional parameter γ is unambiguous. Increasing γ leads to a lower equilibrium quantity q and higher announced stock A . The intuition behind these results is as follows: A lower q leads to lower price P and a higher A diminishes both the negative effect of A on the willingness to pay and on the cost of hiding the information as A approaches S . Further, the lower q , the higher the salvage value given by $\gamma(S - q)$.

The corresponding price effects can be obtained from $P = 1 - 2\theta A - 2q$. In cases 1(a) and 2(a), $P = 1 - 2S$; in case 1(b), $P = (1 + 2\gamma)/3$. Thus, the former decreases in S , while the latter increases in γ .

In the interior cases 1(c) and 2(c), the equilibrium price decreases as the available stock or the hiding cost increases. Its response to θ is non-monotone: price increases for sufficiently low S and decreases for sufficiently high S . The effect of the salvage value depends on the sign of $3\tau - \theta^2$. In case 2(b), price decreases with the available stock, the hiding cost, and the salvage value, while its response to θ is again non-monotone, being positive for low S and negative for high S . The derivative calculations supporting these statements are provided in the proof of Proposition 4.

Figures 2–3 give a visual representation of the equilibrium described in the above proposition. The retained parameter values are as follows:

$$S \in (0, 1], \quad \tau \in [0, 1], \quad \theta \in \{0.1, 0.2, 0.3\}, \quad \gamma \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}.$$

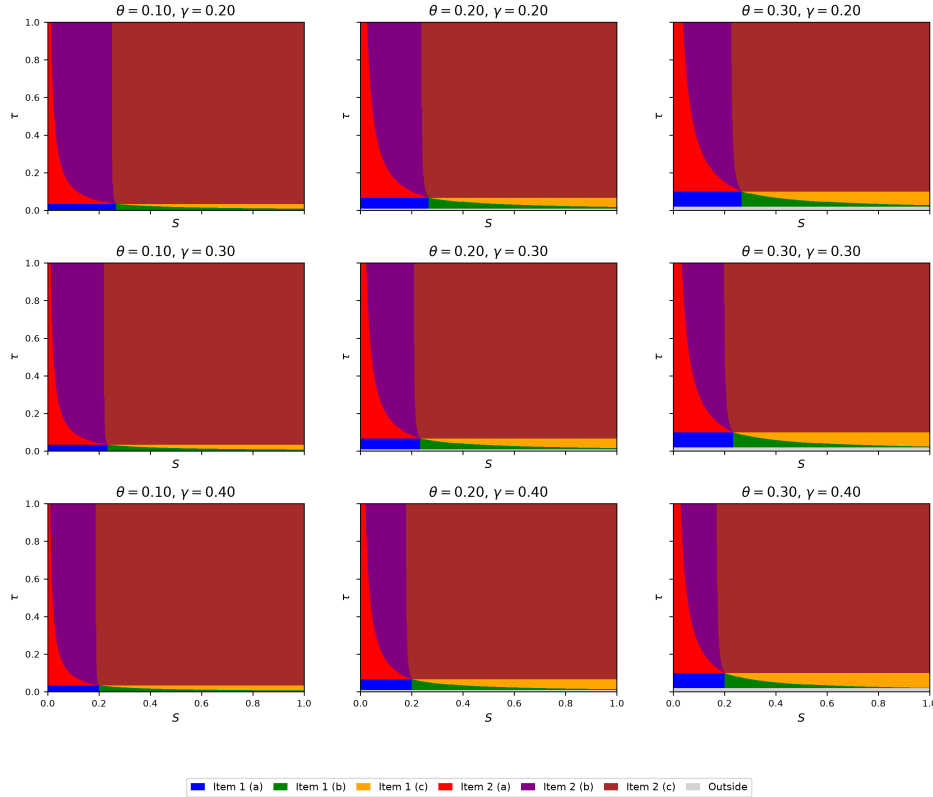


Figure 2: Characterization of items in Proposition 4 in coordinates (S, τ) for different values of $\theta = 0.10, 0.20, 0.30$ and $\gamma = 0.20, 0.30, 0.40$.

As expected in view of the analytical characterization of the equilibrium, Figures 2–3 look similar to Figure 1, with, however, one difference; depending on the value of θ , a large value of γ makes an equilibrium with positive quantities infeasible.

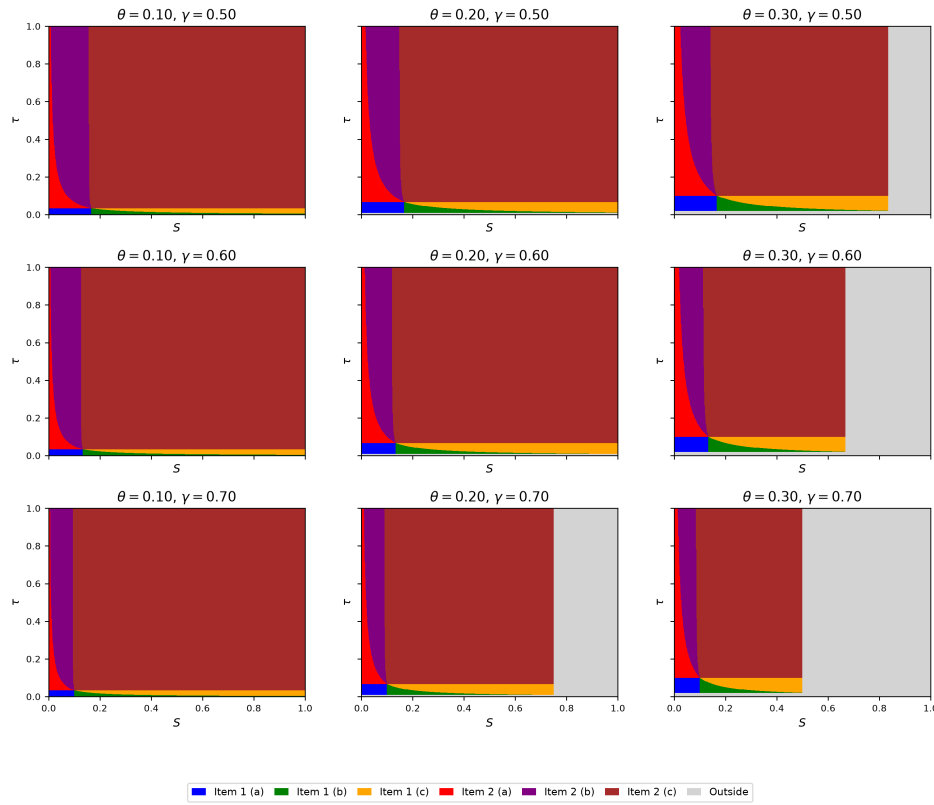


Figure 3: Characterization of items in Proposition 4 in coordinates (S, τ) for different values of $\theta = 0.10, 0.20, 0.30$ and $\gamma = 0.50, 0.60, 0.70$.

In the determination of the equilibrium, we do not impose any restriction on the relationship between the price P and the marginal value of the remaining stock. If we interpret the parameter γ as an end-of-season price that the players can get for their remaining stock, then it is plausible to consider $\gamma < P$. If γ is interpreted as the future price of a remaining unit of a non-perishable resource, then it might well be the case that $\gamma \geq P$. Figure 4 exhibits the regions where either one of the two inequalities is valid.

4.2 Monopoly case

Suppose that there is only one firm in the market. The following proposition characterizes the unique optimal solution.

Proposition 5. *Suppose that $0 < \theta < 1$ and $4\tau - \theta^2 > 0$. The unique solution is defined as follows:*

1. For $\tau \in \left(\frac{\theta^2}{4}, \frac{\theta}{2}\right]$, and

(a) $0 < S < \frac{1}{2}$, then

$$q = S, \quad A = 0,$$

(b) $\frac{1}{2} \leq S < \frac{\theta}{4\tau}$, then

$$q = \frac{1}{2}, \quad A = 0,$$

(c) $\frac{\theta}{4\tau} \leq S < \frac{1}{\theta}$, then

$$q = \frac{2\tau(1 - S\theta)}{4\tau - \theta^2} > 0, \quad A = \frac{4S\tau - \theta}{4\tau - \theta^2} \geq 0.$$

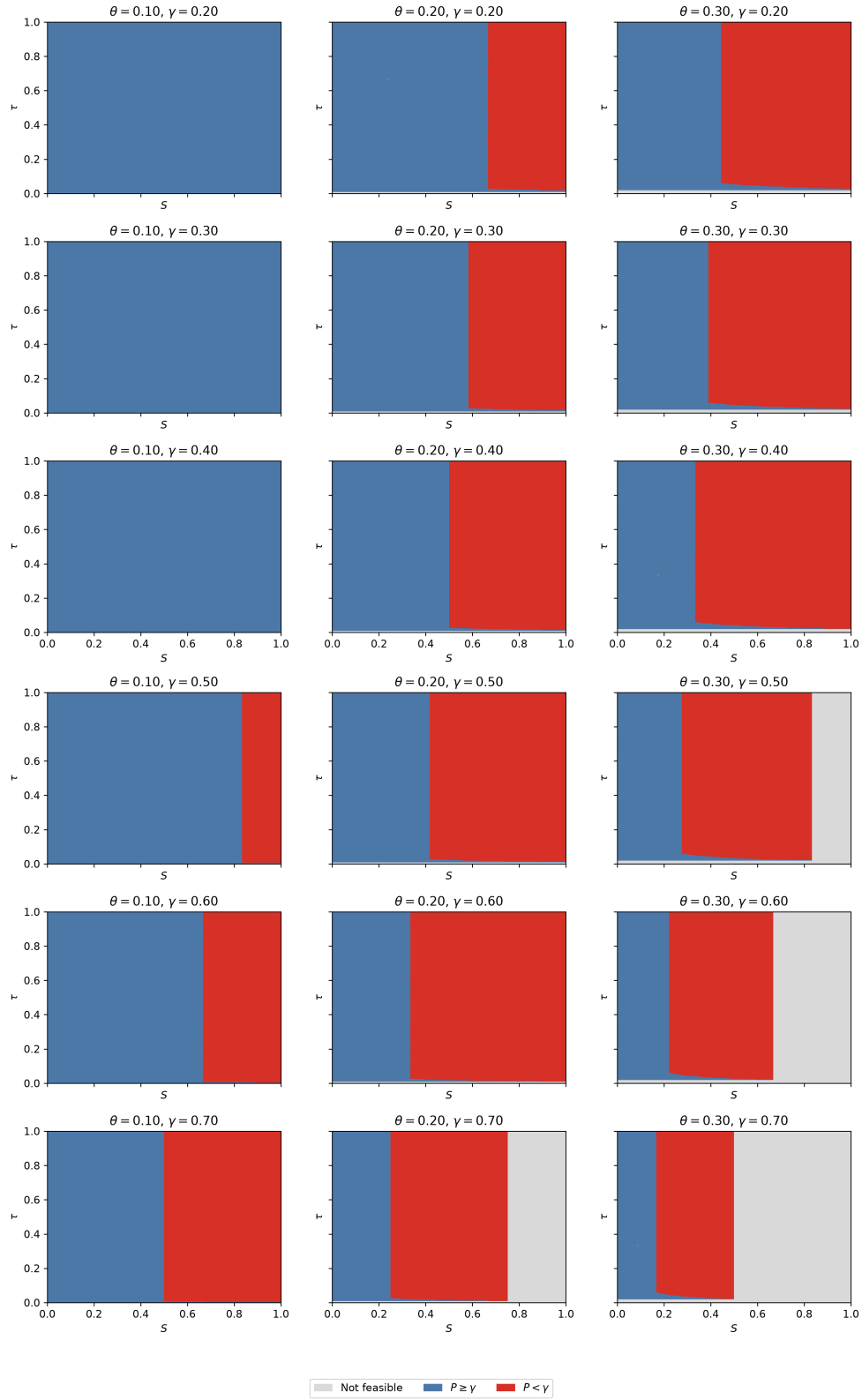


Figure 4: Price comparison in coordinates (S, τ) for different values of $\theta = 0.10, 0.20, 0.30$ and $\gamma = 0.20, \dots, 0.70$. Blue regions indicate $P \geq \gamma$, whereas red regions indicate $P < \gamma$.

2. For $\tau > \frac{\theta}{2}$, and

(a) $0 < S < \frac{\theta}{4\tau}$, then

$$q = S, \quad A = 0,$$

(b) $\frac{\theta}{4\tau} \leq S < \frac{2\tau}{4\tau - \theta^2 + 2\theta\tau}$, then

$$q = S, \quad A = \frac{4S\tau - \theta}{4\tau - \theta^2} \geq 0,$$

(c) $\frac{2\tau}{4\tau - \theta^2 + 2\theta\tau} \leq S < \frac{1}{\theta}$, then

$$q = \frac{2\tau(1 - S\theta)}{4\tau - \theta^2} > 0, \quad A = \frac{4S\tau - \theta}{4\tau - \theta^2} > 0.$$

The corresponding equilibrium prices are

$$\begin{aligned} P_{1(a)} &= P_{2(a)} = 1 - S, & P_{1(b)} &= \frac{1}{2}, \\ P_{1(c)} &= P_{2(c)} = \frac{2\tau(1 - S\theta)}{4\tau - \theta^2}, & P_{2(b)} &= \frac{4\tau - (4\tau + 4\theta\tau - \theta^2)S}{4\tau - \theta^2}. \end{aligned}$$

Proof. See Appendix E. □

Figure 5 provides a visual representation of the regions described in Proposition 5 in the (S, τ) -space for nine values of θ ranging from 0.1 to 0.9. The figure is drawn under the restrictions $S \in (0, 1/\theta)$ and $\tau > \theta^2/4$.

The results in the above proposition and Figure 5 lead to the straightforward conclusion that the structure of the solution is similar to what we obtained in the duopoly case. Indeed, as we can see, the larger the value of θ , the smaller is the interior solution region and the larger the region where the players claim that they are out of stock ($A = 0$). We conclude that the results, i.e., the equilibrium in the duopoly case and the optimal solution in the monopoly setup, are mainly driven by the demand specification rather than by the degree of competition.

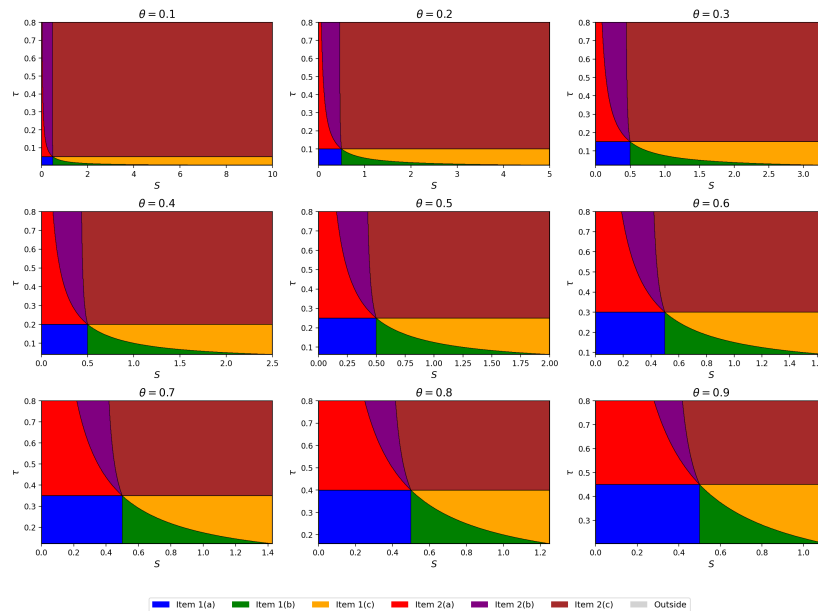


Figure 5: Characterization of the items in Proposition 5 in coordinates (S, τ) for different values of $\theta = 0.1, \dots, 0.9$.

5 Conclusions

We propose a model of firm competition when the market price is sensitive to the values of firms' available resources or product stocks. We analytically characterize the equilibrium strategies in a two-stage game, with firms first announcing their stock values and second choosing their production quantities. Numerical examples and analytical sensitivity analysis show when interior or corner-type equilibrium is reached and how the equilibrium strategies depend on main parameters of the game.

The following extensions are of interest: (i) Adopting an alternative approach for dealing with uncertainty, e.g. when firms may form an appropriate set of feasible values of the stock of the other player based on the information provided by this firm and make their decisions in a non-naive way. (ii) Considering asymmetric players. (iii) Extending the game to a dynamic setting, which would allow to account for reputation effects and learning. (iv) Analyzing when firms could tacitly collude in understating their capacities to raise the price.

A Proof of Proposition 1

Recall that

$$q = \frac{1}{2(9\tau - 2\theta^2)} (\lambda(6\tau - \theta^2) + 3\theta^2\lambda - 6\theta\mu + 6\tau(1 - 2\lambda - 2\theta S)),$$

$$A = \frac{1}{2(9\tau - 2\theta^2)} (3\theta\lambda + 9\mu - 2\theta + 18\tau S)$$

with the following complementarity conditions:

$$\begin{aligned} S - q &\geq 0, & \lambda &\geq 0, & \lambda(S - q) &= 0, \\ \mu A &= 0, & A &\geq 0, & \mu &\geq 0. \end{aligned}$$

We have four cases to consider:

Case 1. $\lambda = 0, \mu = 0$, or $q < S, A > 0$.

Case 2. $\lambda > 0, \mu = 0$, or $q = S, A > 0$.

Case 3: $\lambda = 0, \mu > 0$, or $q < S, A = 0$.

Case 4: $\lambda > 0, \mu > 0$, or $q = S, A = 0$.

Case 1. We have

$$q = \frac{3\tau(1 - 2\theta S)}{9\tau - 2\theta^2}, \quad A = \frac{9\tau S - \theta}{9\tau - 2\theta^2},$$

and, using $P = 1 - 2\theta A - 2q$, the equilibrium price is

$$P = \frac{1 - 2\theta A}{3} = \frac{3\tau(1 - 2\theta S)}{9\tau - 2\theta^2}.$$

The following inequalities must hold true:

$$0 \leq q < S, \quad A > 0,$$

which is equivalent to:

$$0 \leq \frac{3\tau(1 - 2\theta S)}{9\tau - 2\theta^2} < S, \quad 0 < \frac{9\tau S - \theta}{9\tau - 2\theta^2}.$$

Since $9\tau - 2\theta^2 > 0$, we have:

$$\theta^2 < 4.5\tau \Leftrightarrow \theta < 3\sqrt{0.5\tau}.$$

The system of inequalities takes the form:

$$S > \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau}, \quad S \leq \frac{1}{2\theta}, \quad S > \frac{\theta}{9\tau},$$

or equivalently:

$$\max \left\{ \frac{\theta}{9\tau}; \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} \right\} < S \leq \frac{1}{2\theta}.$$

Since

$$\frac{\theta}{9\tau} - \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} = \frac{(9\tau - 2\theta^2)(\theta - 3\tau)}{9\tau(9\tau - 2\theta^2 + 6\theta\tau)},$$

we have 2 cases: (i) If $\theta > 3\tau$, then $\frac{\theta}{9\tau} < S < \frac{1}{2\theta}$; (ii) If $\theta < 3\tau$, then $\frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} < S < \frac{1}{2\theta}$.

Case 2. $\lambda > 0$, $\mu = 0$, or $q = S$, $A > 0$.

We have:

$$q = S, \quad \lambda = \frac{3\tau(1 - 2\theta S)}{3\tau - \theta^2}, \quad A = \frac{(6\tau + \theta)S - \theta}{2(3\tau - \theta^2)}.$$

The corresponding equilibrium price is

$$P = 1 - 2S - 2\theta A = \frac{3\tau - [6\tau(1 + \theta) - \theta^2]S}{3\tau - \theta^2}.$$

For λ to be positive, its numerator and denominator must have the same sign. We have two cases: (i) $3\tau - \theta^2 > 0$ (i.e., $\tau > \theta^2/3$) and $S < \frac{1}{2\theta}$, or (ii) $3\tau - \theta^2 < 0$ (i.e., $\tau < \theta^2/3$) and $S > \frac{1}{2\theta}$. Similarly, for A to be positive, we must have either (i) $3\tau - \theta^2 > 0$ and $S > \frac{\theta}{6\tau + \theta}$, or (ii) $3\tau - \theta^2 < 0$ and $S < \frac{\theta}{6\tau + \theta}$. Further, the demand is nonnegative if, and only if, $S < \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau}$ if $3\tau - \theta^2 > 0$, and $S > \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau}$ if $3\tau - \theta^2 < 0$.

Combining these conditions, we get:

- If $3\tau - \theta^2 > 0$ (i.e., $\tau > \theta^2/3$), then we have

$$\frac{\theta}{6\tau + \theta} < S < \min \left\{ \frac{1}{2\theta}, \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} \right\} = \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau}.$$

The last equality follows from uniqueness condition. This interval makes sense only if

$$\frac{\theta}{6\tau + \theta} < \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau},$$

which happens if $\tau > \frac{\theta}{3}$.

- If $3\tau - \theta^2 < 0$ (i.e., $\tau < \theta^2/3$), then we have

$$\max \left\{ \frac{1}{2\theta}, \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} \right\} < S < \frac{\theta}{6\tau + \theta}.$$

From the uniqueness condition it follows that

$$\frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} < \frac{1}{2\theta}.$$

Consequently, we have

$$\frac{1}{2\theta} < S < \frac{\theta}{6\tau + \theta}.$$

These inequalities are valid only if

$$\frac{1}{2\theta} < \frac{\theta}{6\tau + \theta},$$

which happens iff $\tau < \frac{\theta^2}{3} - \frac{\theta}{6}$, and taking into account that $\frac{\theta^2}{3} - \frac{\theta}{6} < \frac{2}{9}\theta^2$, and uniqueness condition, we conclude that this case never materializes.

To wrap up, Case 2 occurs when $\tau > \frac{\theta^2}{3}$ and $\tau > \frac{\theta}{3}$ (which implies $\tau > \frac{\theta^2}{3}$) and

$$S \in \left(\frac{\theta}{6\tau + \theta}, \frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau} \right).$$

Case 3. $\lambda = 0$, $\mu > 0$ or $q < S$, $A = 0$.

From the equilibrium conditions, we get

$$q = \frac{1}{3}, \quad A = 0, \quad \mu = \frac{2\theta - 18\tau S}{9}.$$

The equilibrium price is therefore $P = 1/3$. The multiplier μ is positive if $S < \frac{\theta}{9\tau}$. Therefore, this case happens if

$$\frac{1}{3} < S < \frac{\theta}{9\tau},$$

and $\tau < \frac{\theta}{3}$.

Case 4. $\lambda > 0$, $\mu > 0$, or $q = S$, $A = 0$.

In this case, both constraints are binding, i.e., $q = S$ and $A = 0$, which requires that

$$S < \min \left\{ \frac{1}{3}, \frac{\theta}{6\tau + \theta}, \frac{1}{2} \right\} = \min \left\{ \frac{1}{3}, \frac{\theta}{6\tau + \theta} \right\}.$$

The equilibrium price is $P = 1 - 2S$, so it is decreasing in S and is independent of τ and θ in this corner case. We have 2 cases: (i) If $\tau < \frac{\theta}{3}$, then the above condition becomes $S < \frac{1}{3}$; (ii) $\tau > \frac{\theta}{3}$, then $S < \frac{\theta}{6\tau + \theta}$.

The above results are summarized in the statement of Proposition 1.

B Proof of Proposition 2

The derivatives of q with respect to the three parameters are given by

$$\begin{aligned} \frac{\partial q}{\partial S} &= -\frac{6\theta\tau}{9\tau - 2\theta^2} < 0, \quad \frac{\partial q}{\partial \tau} = \frac{6\theta^2(2S\theta - 1)}{(9\tau - 2\theta^2)^2} < 0 \text{ because } S < \frac{1}{2\theta}, \\ \frac{\partial q}{\partial \theta} &= -\frac{6\tau(2S\theta^2 - 2\theta + 9S\tau)}{(9\tau - 2\theta^2)^2}. \end{aligned}$$

For $\tau \in (\frac{2}{9}\theta^2, \frac{\theta}{3})$,

$$\frac{\partial q}{\partial \theta} \text{ is } \begin{cases} \text{positive for } S \in \left(\frac{\theta}{9\tau}, \frac{2\theta}{2\theta^2 + 9\tau} \right), \\ = 0 \text{ for } S = \frac{2\theta}{2\theta^2 + 9\tau}, \\ \text{negative for } S \in \left(\frac{2\theta}{2\theta^2 + 9\tau}, \frac{1}{2\theta} \right). \end{cases}$$

For $\tau \geq \frac{\theta}{3}$,

$$\frac{\partial q}{\partial \theta} \text{ is } \begin{cases} \text{positive for } S \in \left(\frac{3\tau}{9\tau - 2\theta^2 + 6\theta\tau}, \frac{2\theta}{2\theta^2 + 9\tau} \right), \\ = 0 \text{ for } S = \frac{2\theta}{2\theta^2 + 9\tau}, \\ \text{negative for } S \in \left(\frac{2\theta}{2\theta^2 + 9\tau}, \frac{1}{2\theta} \right). \end{cases}$$

The derivatives of A with respect to three parameters are given by

$$\begin{aligned} \frac{\partial A}{\partial S} &= \frac{9\tau}{9\tau - 2\theta^2} > 0, \quad \frac{\partial A}{\partial \tau} = \frac{9\theta(1 - 2S\theta)}{(9\tau - 2\theta^2)^2} > 0 \text{ because } S < \frac{1}{2\theta}, \\ \frac{\partial A}{\partial \theta} &= \frac{36S\tau\theta - 2\theta^2 - 9\tau}{(9\tau - 2\theta^2)^2}. \end{aligned}$$

For $\tau \in \left(\frac{2}{9}\theta^2, \frac{\theta}{3}\right)$,

$$\frac{\partial A}{\partial \theta} \text{ is } \begin{cases} \text{negative for } S \in \left(\frac{\theta}{9\tau}, \frac{9\tau+2\theta^2}{36\theta\tau}\right), \\ = 0 \text{ for } S = \frac{9\tau+2\theta^2}{36\theta\tau}, \\ \text{positive for } S \in \left(\frac{9\tau+2\theta^2}{36\theta\tau}, \frac{1}{2\theta}\right). \end{cases}$$

For $\tau \geq \frac{\theta}{3}$,

$$\frac{\partial A}{\partial \theta} \text{ is } \begin{cases} \text{negative for } S \in \left(\frac{3\tau}{9\tau-2\theta^2+6\tau\theta}, \frac{9\tau+2\theta^2}{36\theta\tau}\right), \\ = 0 \text{ for } S = \frac{9\tau+2\theta^2}{36\theta\tau}, \\ \text{positive for } S \in \left(\frac{9\tau+2\theta^2}{36\theta\tau}, \frac{1}{2\theta}\right). \end{cases}$$

The derivatives of P with respect to three parameters are given by

$$\begin{aligned} \frac{\partial P}{\partial S} &= -\frac{6\theta\tau}{9\tau-2\theta^2} < 0, & \frac{\partial P}{\partial \tau} &= -\frac{6\theta^2(1-2\theta S)}{(9\tau-2\theta^2)^2} < 0, \\ \frac{\partial P}{\partial \theta} &= \frac{6\tau(2\theta-2S\theta^2-9S\tau)}{(9\tau-2\theta^2)^2}. \end{aligned}$$

For $\tau \in \left(\frac{2}{9}\theta^2, \frac{\theta}{3}\right)$,

$$\frac{\partial P}{\partial \theta} \text{ is } \begin{cases} \text{positive for } S \in \left(\frac{\theta}{9\tau}, \frac{2\theta}{2\theta^2+9\tau}\right), \\ = 0 \text{ for } S = \frac{2\theta}{2\theta^2+9\tau}, \\ \text{negative for } S \in \left(\frac{2\theta}{2\theta^2+9\tau}, \frac{1}{2\theta}\right). \end{cases}$$

For $\tau \geq \frac{\theta}{3}$,

$$\frac{\partial P}{\partial \theta} \text{ is } \begin{cases} \text{positive for } S \in \left(\frac{3\tau}{9\tau-2\theta^2+6\tau\theta}, \frac{2\theta}{2\theta^2+9\tau}\right), \\ = 0 \text{ for } S = \frac{2\theta}{2\theta^2+9\tau}, \\ \text{negative for } S \in \left(\frac{2\theta}{2\theta^2+9\tau}, \frac{1}{2\theta}\right). \end{cases}$$

C Proof of Proposition 3

In case 2(b), the derivatives of A with respect to the three parameters are given by

$$\begin{aligned} \frac{\partial A}{\partial S} &= \frac{\theta+6\tau}{2(3\tau-\theta^2)}, \\ \frac{\partial A}{\partial \tau} &= \frac{3\theta(1-S-2S\theta)}{2(3\tau-\theta^2)^2} = \frac{3\theta \cdot f(S)}{2(3\tau-\theta^2)^2}, \\ \frac{\partial A}{\partial \theta} &= \frac{S(3\tau+\theta^2+12\theta\tau)-3\tau-\theta^2}{2(3\tau-\theta^2)^2} = \frac{g(S)}{2(3\tau-\theta^2)^2}. \end{aligned}$$

For $\theta \in (0, 1)$, we have $(3\tau-\theta^2) > (3\tau-\theta)$, which implies $\frac{\partial A}{\partial S} > 0$.

The denominators of $\frac{\partial A}{\partial \tau}$ and $\frac{\partial A}{\partial \theta}$ are positive, and we have

$$\begin{aligned} f\left(\frac{\theta}{\theta+6\tau}\right) &= \frac{2(3\tau-\theta^2)}{\theta+6\tau} > \frac{2(3\tau-\theta)}{\theta+6\tau} > 0, \\ f\left(\frac{3\tau}{9\tau-2\theta^2+6\tau\theta}\right) &= \frac{2(3\tau-\theta^2)}{9\tau-2\theta^2+6\tau\theta} > \frac{2(3\tau-\theta)}{9\tau-2\theta^2+6\tau\theta} > 0. \end{aligned}$$

As $f(S)$ is monotone ($f'(S) = -1-2\theta$), we conclude that $\frac{\partial A}{\partial \tau} > 0$. Further, we have

$$g\left(\frac{\theta}{\theta+6\tau}\right) = \frac{6\tau(\theta^2-3\tau)}{\theta+6\tau} < 0,$$

$$g\left(\frac{3\tau}{9\tau - 2\theta^2 + 6\tau\theta}\right) = \frac{18\tau^2(\theta - 1) + 2\theta^3(\theta - 3\tau)}{9\tau - 2\theta^2 + 6\tau\theta} < 0,$$

because $\theta < 1$ and $\theta < 3\tau$ for Case 2(b).

As $g(S)$ is monotone ($g'(S) = 3\tau + \theta^2 + 12\theta\tau$), we conclude that $\frac{\partial A}{\partial \theta} < 0$.

For the price in case 2(b), direct differentiation gives

$$\begin{aligned}\frac{\partial P}{\partial S} &= -\frac{6\tau(1 + \theta) - \theta^2}{3\tau - \theta^2} < 0, \\ \frac{\partial P}{\partial \tau} &= -\frac{3\theta^2(1 - S - 2S\theta)}{(3\tau - \theta^2)^2} < 0, \\ \frac{\partial P}{\partial \theta} &= -\frac{6\tau(S(3\tau + \theta^2 + \theta) - \theta)}{(3\tau - \theta^2)^2}.\end{aligned}$$

The sign of the last derivative changes at $S = \theta/(3\tau + \theta^2 + \theta)$, as stated in the proposition.

D Proof of Proposition 4

The equilibrium conditions for this case are:

$$A_i = \frac{(9\tau - \theta^2)(X + 18\tau S_i) + \theta^2(Y + 18\tau S_{3-i})}{18\tau(9\tau - 2\theta^2)}, \quad i = 1, 2 \quad (15)$$

$$\mu_i A_i = 0, \quad A_i \geq 0, \quad \mu_i \geq 0, \quad i = 1, 2. \quad (16)$$

Inserting A_1 and A_2 given by (15) into first-order conditions for q_i , $i = 1, 2$, we obtain the quantities as functions of the parameter values and the multipliers:

$$\begin{aligned}q_1 &= \frac{1}{2\tau(9\tau - 2\theta^2)} \left(\tau\lambda_2(6\tau - \theta^2) + 3\theta^2\lambda_1\tau - 3\theta\tau(\mu_2 + \mu_1) + 6\tau^2(1 - 2\lambda_1 - \gamma - \theta S_1 - \theta S_2) \right), \\ q_2 &= \frac{1}{2\tau(9\tau - 2\theta^2)} \left(\tau\lambda_1(6\tau - \theta^2) + 3\theta^2\lambda_2\tau - 3\theta\tau(\mu_1 + \mu_2) + 6\tau^2(1 - 2\lambda_2 - \gamma - \theta S_2 - \theta S_1) \right).\end{aligned}$$

In a symmetric equilibrium, i.e., for $q_1 = q_2 = q$ and $A_1 = A_2 = A$, the above conditions become:

$$q = \frac{\lambda(6\tau + 2\theta^2) - 6\theta\mu + 6\tau(1 - 2\lambda - \gamma - 2\theta S)}{2(9\tau - 2\theta^2)}, \quad (17)$$

$$A = \frac{9\mu - 2\theta + 18S\tau - \theta\lambda + 2\theta\gamma}{2(9\tau - 2\theta^2)}. \quad (18)$$

To fully characterize the equilibrium, we need to consider the four cases imbedded in the following complementarity conditions:

$$\begin{aligned}S - q &\geq 0, \quad \lambda \geq 0, \quad \lambda(S - q) = 0, \\ \mu A &= 0, \quad A \geq 0, \quad \mu \geq 0.\end{aligned}$$

Also, for each case we check for the nonnegativity of the price keeping in mind the uniqueness condition $9\tau - 2\theta^2 > 0$.

Case 1: $\lambda = 0$, $\mu = 0$, or $q < S$, $A > 0$.

Setting $\lambda = \mu = 0$ in Equations (17)–(18) gives

$$q = \frac{3\tau(1 - \gamma - 2\theta S)}{9\tau - 2\theta^2}, \quad A = \frac{9\tau S - \theta(1 - \gamma)}{9\tau - 2\theta^2}, \quad \text{and} \quad P = \frac{3\tau + 6\tau\gamma - 2\gamma\theta^2 - 6\tau\theta S}{9\tau - 2\theta^2}.$$

Since $9\tau - 2\theta^2 > 0$, the feasibility conditions are

$$\begin{aligned} 0 \leq q < S &\Leftrightarrow \frac{3\tau(1-\gamma)}{9\tau - 2\theta^2 + 6\theta\tau} < S \leq \frac{1-\gamma}{2\theta}, \\ A > 0 &\Leftrightarrow S > \frac{\theta(1-\gamma)}{9\tau}, \\ P \geq 0 &\Leftrightarrow S \leq \frac{2\gamma(3\tau - \theta^2) + 3\tau}{6\tau\theta}. \end{aligned}$$

Consequently,

$$\max \left\{ \frac{\theta(1-\gamma)}{9\tau}, \frac{3\tau(1-\gamma)}{9\tau - 2\theta^2 + 6\theta\tau} \right\} < S \leq \min \left\{ \frac{1-\gamma}{2\theta}, \frac{2\gamma(3\tau - \theta^2) + 3\tau}{6\tau\theta} \right\}.$$

Since the expressions under min operator satisfy: $\frac{2\gamma(3\tau - \theta^2) + 3\tau}{6\tau\theta} - \frac{1-\gamma}{2\theta} = \frac{\gamma(9\tau - 2\theta^2)}{6\tau\theta} \geq 0$, we obtain:

$$\min \left\{ \frac{1-\gamma}{2\theta}, \frac{2\gamma(3\tau - \theta^2) + 3\tau}{6\tau\theta} \right\} = \frac{1-\gamma}{2\theta}.$$

The difference between expressions in max operator is

$$\frac{\theta(1-\gamma)}{9\tau} - \frac{3\tau(1-\gamma)}{9\tau - 2\theta^2 + 6\theta\tau} = \frac{(1-\gamma)(\theta - 3\tau)(9\tau - 2\theta^2)}{9\tau(9\tau - 2\theta^2 + 6\theta\tau)},$$

where all multipliers other than $\theta - 3\tau$ are positive. Taking into account condition $9\tau - 2\theta^2 > 0$, we get two cases:

- If $\frac{2\theta^2}{9} < \tau \leq \frac{\theta}{3}$, then $\frac{\theta(1-\gamma)}{9\tau} < S \leq \frac{1-\gamma}{2\theta}$;
- If $\tau > \frac{\theta}{3}$, we have $\frac{3\tau(1-\gamma)}{9\tau - 2\theta^2 + 6\theta\tau} < S \leq \frac{1-\gamma}{2\theta}$.

Case 2: $\lambda > 0$, $\mu = 0$, or $q = S$, $A > 0$.

Substituting $q = S$ and $\mu = 0$ into (17) gives

$$\lambda = \frac{3\tau(1-\gamma) - (9\tau - 2\theta^2 + 6\theta\tau)S}{3\tau - \theta^2}. \quad (19)$$

Setting $\mu = 0$ in (18) and substituting λ defined by (19) gives

$$A = \frac{(6\tau + \theta)S - \theta(1-\gamma)}{2(3\tau - \theta^2)}.$$

Substituting for q ($q = S$) and A in the price gives

$$P = \frac{3\tau(1-\gamma) - [6\tau(1+\theta) - \theta^2]S}{3\tau - \theta^2},$$

which must be nonnegative.

Depending on the sign of expression $3\tau - \theta^2$, we examine two subcases within Case 2:

Case 2.1: $3\tau - \theta^2 > 0$.

In this case, the three feasibility conditions become

$$\begin{aligned} \lambda > 0 &\Leftrightarrow S < U_\lambda, \\ A > 0 &\Leftrightarrow S > L, \\ P \geq 0 &\Leftrightarrow S \leq U_P, \end{aligned}$$

where the three thresholds are

$$L := \frac{\theta(1-\gamma)}{6\tau + \theta}, \quad U_\lambda := \frac{3\tau(1-\gamma)}{9\tau - 2\theta^2 + 6\theta\tau}, \quad U_P := \frac{3\tau - \gamma\theta^2}{6\tau - \theta^2 + 6\theta\tau},$$

and

$$U_\lambda - L = \frac{2(1-\gamma)(3\tau-\theta)(3\tau-\theta^2)}{(6\tau+\theta)(9\tau-2\theta^2+6\theta\tau)}. \quad (20)$$

Since $3\tau - \theta^2 > 0$, the interval $L < S < U_\lambda$ is nonempty if and only if

$$3\tau - \theta > 0 \Leftrightarrow \tau > \frac{\theta}{3}.$$

For $0 \leq \theta < 1$, this condition automatically implies

$$3\tau - \theta^2 > 0 \quad \text{and} \quad 9\tau - 2\theta^2 > 0.$$

To compare the price and multiplier bounds, observe that

$$U_P - U_\lambda = \frac{(3\tau - \theta^2) [3\tau + 2\gamma(3\tau(1+\theta) - \theta^2)]}{(9\tau - 2\theta^2 + 6\theta\tau)(6\tau - \theta^2 + 6\theta\tau)}. \quad (21)$$

When $\tau > \theta/3$, every multiplier in (21) is positive. Hence $U_P > U_\lambda$ and, therefore,

$$S < U_\lambda \Rightarrow S < U_P,$$

so the nonnegative-price condition does not further restrict the feasibility interval in Case 2.

Case 2.2: $3\tau - \theta^2 < 0$. The system of inequalities is as follows:

$$\lambda > 0 \quad \Leftrightarrow \quad S > U_\lambda,$$

$$A > 0 \quad \Leftrightarrow \quad S < L,$$

$$P \geq 0 \quad \Leftrightarrow \quad S \geq U_P.$$

For $0 \leq \theta < 1$, the condition $3\tau - \theta^2 < 0$ implies that $3\tau - \theta < 0$ since $3\tau < \theta^2 \leq \theta$. Using (20), we obtain

$$U_\lambda > L.$$

However, Case 2.2 would require simultaneously $S > U_\lambda$ and $S < L$, but these inequalities are incompatible, and there are no feasible solutions for Case 2.2.

We exclude the point: $3\tau - \theta^2 = 0$ from our consideration since it will lead to $A < 0$, which is intractable.

Combining the preceding results, Case 2 exists if and only if $\tau > \frac{\theta}{3}$ and

$$\frac{\theta(1-\gamma)}{6\tau+\theta} < S < \frac{3\tau(1-\gamma)}{9\tau-2\theta^2+6\theta\tau}.$$

Case 3: $\lambda = 0$, $\mu > 0$, or $q < S$, $A = 0$.

From the symmetric equilibrium (17) and (18), we get

$$\mu = \frac{2\theta(1-\gamma) - 18\tau S}{9}, \quad q = \frac{-6\theta\mu + 6\tau(1-\gamma - 2\theta S)}{2(9\tau - 2\theta^2)},$$

which gives

$$q = \frac{1-\gamma}{3} > 0.$$

The conditions $\mu > 0$ and $q < S$ are respectively equivalent to

$$S < \frac{\theta(1-\gamma)}{9\tau}, \quad S > \frac{1-\gamma}{3}.$$

Therefore, we have

$$\frac{1-\gamma}{3} < S < \frac{\theta(1-\gamma)}{9\tau}.$$

This interval is nonempty if and only if

$$\frac{1-\gamma}{3} < \frac{\theta(1-\gamma)}{9\tau}.$$

Since $1-\gamma > 0$, this reduces to $3\tau < \theta$. Substituting for the quantity in the price, we get an expression:

$$P = \frac{1+2\gamma}{3},$$

which is always nonnegative.

Hence, the complete characterization of Case 3 is

$$\frac{2\theta^2}{9} < \tau < \frac{\theta}{3}, \quad \text{with} \quad \frac{1-\gamma}{3} < S < \frac{\theta(1-\gamma)}{9\tau}.$$

Case 4: $\lambda > 0$, $\mu > 0$, or $q = S$, $A = 0$.

Substituting $q = S$ and $A = 0$ into (17) and (18), we obtain:

$$\lambda = 1 - \gamma - 3S, \quad \mu = \frac{\theta(1-\gamma) - (6\tau + \theta)S}{3}.$$

Therefore, the following inequalities should be satisfied:

$$\begin{aligned} \lambda > 0 &\Leftrightarrow S < \frac{1-\gamma}{3}, \\ \mu > 0 &\Leftrightarrow S < \frac{\theta(1-\gamma)}{6\tau + \theta}. \end{aligned}$$

Since $A = 0$ and $q = S$, the price is $P = 1 - 2S$ and it is nonnegative because the condition $S < \frac{1-\gamma}{3}$ should be satisfied.

Consequently, we have

$$0 \leq S < \min \left\{ \frac{1-\gamma}{3}, \frac{\theta(1-\gamma)}{6\tau + \theta} \right\}.$$

To determine which multiplier condition is binding, compute

$$\frac{1-\gamma}{3} - \frac{\theta(1-\gamma)}{6\tau + \theta} = \frac{2(1-\gamma)(3\tau - \theta)}{3(6\tau + \theta)}.$$

Therefore,

- If $\tau < \theta/3$, then $\frac{1-\gamma}{3} < \frac{\theta(1-\gamma)}{6\tau + \theta}$.
- If $\tau > \theta/3$, then $\frac{\theta(1-\gamma)}{6\tau + \theta} < \frac{1-\gamma}{3}$.

Combining this comparison with the uniqueness condition, we obtain the final characterization of Case 4:

- If $\frac{2\theta^2}{9} < \tau \leq \frac{\theta}{3}$, then $0 \leq S < \frac{1-\gamma}{3}$.
- If $\tau > \frac{\theta}{3}$, then $0 \leq S < \frac{\theta(1-\gamma)}{6\tau + \theta}$.

For completeness, differentiating the price expressions obtained in the four cases verifies the price-sensitivity statements in the text. In the interior case, with $D = 9\tau - 2\theta^2$,

$$\frac{\partial P}{\partial S} = -\frac{6\theta\tau}{D}, \quad \frac{\partial P}{\partial \tau} = -\frac{6\theta^2(1-\gamma-2\theta S)}{D^2}, \quad \frac{\partial P}{\partial \theta} = \frac{6\tau[\theta(1-\gamma) - S(2\theta^2 + 9\tau)]}{D^2}, \quad \frac{\partial P}{\partial \gamma} = \frac{2(3\tau - \theta^2)}{D}.$$

In Case 2(b),

$$\frac{\partial P}{\partial S} = -\frac{6\tau(1+\theta) - \theta^2}{3\tau - \theta^2}, \quad \frac{\partial P}{\partial \theta} = -\frac{6\tau[S(3\tau + \theta^2 + \theta) - \theta(1-\gamma)]}{(3\tau - \theta^2)^2},$$

$$\frac{\partial P}{\partial \tau} = \frac{3\theta^2[S(1+2\theta) - (1-\gamma)]}{(3\tau - \theta^2)^2}, \quad \frac{\partial P}{\partial \gamma} = -\frac{3\tau}{3\tau - \theta^2}.$$

The corner prices are $1 - 2S$ in cases 1(a) and 2(a), and $(1 + 2\gamma)/3$ in case 1(b), as stated in the proposition. There is no feasible Case 4 equilibrium when $\theta = 0$. The overall results are collected in Proposition 4.

E Proof of Proposition 5

Let $0 < \theta < 1$, $\tau > 0$, and $S > 0$. The optimization problem of a monopolist is given by

$$\max_{0 \leq q \leq S, A \geq 0} \Pi = (1 - \theta A - q)q - \tau(A - S)^2.$$

We start by solving the second stage. Introduce the Lagrangian

$$L = (1 - \theta A - q)q - \tau(A - S)^2 + \lambda(S - q),$$

where λ is the Lagrange multiplier associated to the constraint $q < S$. The first-order optimality condition is given by

$$\frac{\partial L}{\partial q} = 1 - \theta A - 2q - \lambda = 0 \quad \Rightarrow \quad q = \frac{1 - \theta A - \lambda}{2}.$$

The second derivative $\frac{\partial^2 L}{\partial q^2} = -2 < 0$ shows that the problem is concave. The price is given by

$$P = 1 - \theta A - \frac{1 - \theta A - \lambda}{2} = \frac{1}{2}(1 - \theta A + \lambda).$$

The first-stage problem is as follows:

$$\max_{A \geq 0} \Pi_1 = \frac{1}{4}(1 - \theta A + \lambda)(1 - \theta A - \lambda) - \tau(A - S)^2,$$

and the Lagrangian is given by

$$L_1 = \frac{1}{4}(1 - \theta A + \lambda)(1 - \theta A - \lambda) - \tau(A - S)^2 + \mu A,$$

where μ is the Lagrange multiplier associate to the constraint $A \geq 0$. The optimality condition is

$$\frac{\partial L_1}{\partial A} = \frac{1}{2}A\theta^2 - \frac{1}{2}\theta + \mu - 2A\tau + 2S\tau = 0 \quad \Rightarrow \quad A = \frac{1}{4\tau - \theta^2}(-\theta + 2\mu + 4S\tau).$$

For concavity, we need

$$\frac{\partial^2 L_1}{\partial A^2} = \frac{1}{2}\theta^2 - 2\tau < 0 \Leftrightarrow 4\tau - \theta^2 > 0.$$

Substituting for A in q , we get

$$q = \frac{1 - \theta A - \lambda}{2} = \frac{1 - \theta \frac{1}{4\tau - \theta^2}(-\theta + 2\mu + 4S\tau) - \lambda}{2} = -\frac{2\theta\mu - 4\tau + 4\lambda\tau - \theta^2\lambda + 4S\theta\tau}{2(4\tau - \theta^2)}.$$

The system as function of parameters and multipliers is given by

$$\begin{aligned} A &= \frac{1}{4\tau - \theta^2}(-\theta + 2\mu + 4S\tau), \\ q &= -\frac{1}{2(4\tau - \theta^2)}(2\theta\mu - 4\tau + 4\lambda\tau - \theta^2\lambda + 4S\theta\tau). \end{aligned}$$

We have four cases to consider corresponding to the following complementarity conditions:

$$\begin{aligned} S - q &\geq 0, & \lambda &\geq 0, & \lambda(S - q) &= 0, \\ \mu A &= 0, & A &\geq 0, & \mu &\geq 0. \end{aligned}$$

Case 1: $S - q > 0$ and $A > 0$ ($\lambda = \mu = 0$). We have:

$$\begin{aligned} A &= \frac{4S\tau - \theta}{4\tau - \theta^2} > 0 \Leftrightarrow S > \frac{\theta}{4\tau}, \\ q = P &= \frac{2\tau(1 - S\theta)}{4\tau - \theta^2} > 0 \Leftrightarrow S < \frac{1}{\theta}, \\ \frac{\theta}{4\tau} &< S < \frac{1}{\theta}, \\ S - q &= S - \frac{2\tau(1 - S\theta)}{4\tau - \theta^2} > 0 \iff S > \frac{2\tau}{4\tau + 2\tau\theta - \theta^2}. \end{aligned}$$

Both candidate lower bounds must be smaller than the upper bound. Indeed,

$$\begin{aligned} \frac{\theta}{4\tau} < \frac{1}{\theta} &\iff 4\tau - \theta^2 > 0, \\ \frac{2\tau}{4\tau + 2\tau\theta - \theta^2} < \frac{1}{\theta} &\iff 4\tau - \theta^2 > 0. \end{aligned}$$

Thus, both inequalities hold under the concavity condition. Therefore,

$$\max \left\{ \frac{\theta}{4\tau}, \frac{2\tau}{4\tau + 2\tau\theta - \theta^2} \right\} < S < \frac{1}{\theta}.$$

To compare the two lower bounds, observe that

$$\frac{\theta}{4\tau} < \frac{2\tau}{4\tau + 2\tau\theta - \theta^2} \iff (2\tau - \theta)(4\tau - \theta^2) > 0.$$

Thus, taking into $4\tau - \theta^2 > 0$, the final characterization of Case 1:

- If $\frac{\theta^2}{4} < \tau \leq \frac{\theta}{2}$, then $\frac{\theta}{4\tau} < S < \frac{1}{\theta}$.
- If $\tau > \frac{\theta}{2}$, then $\frac{2\tau}{4\tau + 2\tau\theta - \theta^2} < S < \frac{1}{\theta}$.

Case 2: $S = q$ and $A > 0$ ($\lambda > 0, \mu = 0$).

$$\begin{aligned} A &= \frac{-\theta + 4S\tau}{4\tau - \theta^2} > 0 \Leftrightarrow S > \frac{\theta}{4\tau}, \\ S &= -\frac{1}{2(4\tau - \theta^2)} (-4\tau + 4\lambda\tau - \theta^2\lambda + 4S\theta\tau), \\ \lambda &= \frac{-8S\tau + 4\tau + 2S\theta^2 - 4S\theta\tau}{4\tau - \theta^2} > 0 \Leftrightarrow S < \frac{2\tau}{4\tau - \theta^2 + 2\theta\tau}, \\ P &= S + \lambda > 0, \\ \frac{\theta}{4\tau} &< S < \frac{2\tau}{4\tau - \theta^2 + 2\theta\tau}; \text{ holds true if } \frac{\theta}{2\tau} < 1. \end{aligned}$$

We must have $\frac{\theta}{4\tau} < \frac{2\tau}{4\tau - \theta^2 + 2\theta\tau}$, which finally equivalent to $\frac{\theta}{2\tau} < 1$.

The final characterization of Case 2:

- If $\tau > \frac{\theta}{2}$, then $\frac{\theta}{4\tau} < S < \frac{2\tau}{4\tau - \theta^2 + 2\theta\tau}$.

Case 3: $S - q > 0$ and $A = 0$ ($\lambda = 0, \mu > 0$). We have:

$$\begin{aligned} A = 0 &= \frac{4S\tau - \theta + 2\mu}{4\tau - \theta^2} \Leftrightarrow \mu = \frac{\theta - 4S\tau}{2} > 0 \text{ iff } S < \frac{\theta}{4\tau}, \\ q = P &= \frac{1}{2}. \end{aligned}$$

The condition $S - q > 0$ gives $S > 1/2$. Consequently,

$$\frac{1}{2} < S < \frac{\theta}{4\tau}.$$

This interval is nonempty only when $\frac{1}{2} < \frac{\theta}{4\tau}$, or equivalently, when $\theta > 2\tau$.

The final characterization of Case 3:

- If $\frac{\theta^2}{4} < \tau < \frac{\theta}{2}$, then $\frac{1}{2} < S < \frac{\theta}{4\tau}$.

Case 4: $S - q = 0$ and $A = 0$ ($\lambda > 0, \mu > 0$).

$$\begin{aligned} A=0 &= \frac{4S\tau - \theta + 2\mu}{4\tau - \theta^2} \Leftrightarrow \mu = \frac{\theta - 4S\tau}{2} > 0 \Leftrightarrow S < \frac{\theta}{4\tau}, \\ q=S &= -\frac{1}{2(4\tau - \theta^2)} \left(2\theta \frac{\theta - 4S\tau}{2} - 4\tau + 4\lambda\tau - \theta^2\lambda + 4S\theta\tau \right), \\ \lambda=1-2S &> 0 \Leftrightarrow S < \frac{1}{2}, \\ P=1-S &> 0. \end{aligned}$$

Combining the two upper bounds gives

$$0 < S < \min \left\{ \frac{1}{2}, \frac{\theta}{4\tau} \right\}.$$

To determine which multiplier condition is binding, compute $\frac{1}{2} - \frac{\theta}{4\tau} = \frac{4\tau - 2\theta}{8\tau} = \frac{2\tau - \theta}{4\tau}$. Therefore, If $\tau < \theta/2$, then $\frac{1}{2} < \frac{\theta}{4\tau}$, and if $\tau > \theta/2$, then $\frac{\theta}{4\tau} < \frac{1}{2}$.

The final characterization of Case 4:

- If $\frac{\theta^2}{4} < \tau \leq \frac{\theta}{2}$, then $0 < S < \frac{1}{2}$.
- If $\tau > \frac{\theta}{2}$, then $0 < S < \frac{\theta}{4\tau}$.

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