

Mean field games and control on large expander graphs

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Mean field games and control on large expander graphs

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Abstract : This paper investigates mean field games and control on sparse networks. In the case of large expander graphs, the limit topologies are analyzed using the graphexon framework, which characterizes both dense network limits and sparse connections. We prove that the sequence of empirical graphexon measures defined on finite graphs converges weakly to a limit graphexon measure on a continuous state space. Furthermore, the associated sequence of discrete averaging operators converges strongly to a continuous operator. These properties enable the formulation of a linear-quadratic mean field game in which each agent is identified by a spatial network label $\alpha \in X$ and only interacts with the neighborhood average defined by the operator $\mathcal{G}$ characterized by large expander graphs. In Section 5, algebraic conditions for the global asymptotic stability of the closed-loop system are established. The analysis identifies parameter thresholds that gives rise to a Turing-type topological instability, where the homogeneous mean state remains stable while the spatial deviation field diverges over the continuous spectrum of the limit operator.

Keywords: Mean Field Games (MFGs); Expander graphs; Schreier graphs; Linear-quadratic control

Résumé : Cet article étudie les jeux à champ moyen et le contrôle sur des réseaux clairsemés. Dans le cas des grands graphes expanseurs, les topologies limites sont analysées à l'aide du cadre des graphexons, qui caractérise à la fois les limites des réseaux denses et les connexions clairsemées. Nous prouvons que la suite de mesures empiriques de graphexons définies sur des graphes finis converge faiblement vers une mesure de graphexon limite sur un espace d'états continu. De plus, la suite associée d'opérateurs de moyenne discrets converge fortement vers un opérateur continu. Ces propriétés permettent la formulation d'un jeu à champ moyen linéaire-quadratique dans lequel chaque agent est identifié par une étiquette de réseau spatiale $\alpha \in X$ et n'interagit qu'avec la moyenne de voisinage définie par l'opérateur $\mathcal{G}$ caractérisé par de grands graphes expanseurs. Dans la section 5, des conditions algébriques pour la stabilité asymptotique globale du système en boucle fermée sont établies. L'analyse identifie des seuils de paramètres qui donnent lieu à une instabilité topologique de type Turing, où l'état moyen homogène reste stable tandis que le champ de déviation spatiale diverge sur le spectre continu de l'opérateur limite.

Mots clés : Jeux à champ moyen (JCM); graphes expanseurs; graphes de Schreier; contrôle linéaire-quadratique

1 Introduction

Mean Field Game (MFG) theory, introduced by Lasry and Lions [21, 22] and Huang, Caines, and Malhamé [18], provides a method to analyze large decision systems of interacting agents. By assuming exchangeability, the theory approximates the interaction between an individual and the population using a deterministic mean field. Traditional MFGs assume uniform, all-to-all interactions. To model heterogeneous network topologies, the graphon MFG framework was developed, replacing the uniform mean-field coupling with a limit object defined by a graphon [7, 16]. A graphon represents the limit of a sequence of dense graphs where the number of edges scales with the population size N [23].

A structural limitation of the standard graphon approach is its inability to describe sparse networks. Under the dense graphon scaling, the limit of a sparse graph sequence degenerates to zero, losing the topological information. To analyze limits of sparse graphs, some methods are developed. For example, Borgs et al. developed the theory of L^p graphons and graphings [4], focusing on bounded-degree graph sequences [36]. In parallel, the concept of local weak convergence (or Benjamini-Schramm convergence) provides a probabilistic framework for analyzing the local neighborhood structure of sparse graph sequences and their application to stochastic games [3, 20]. To provide a unified methodology handling both dense and sparse network limits, the graphexon and the Laplexion framework was introduced [5, 8, 9, 38]. A graphexon encodes the asymptotic edge distribution as a symmetric Borel measure on a product space. Specifically, the *graphexon measure* $\mathbf{G}$ admits a decomposition into an absolutely continuous component and a singular component [8]:

$$\mathbf{G}(dx, dy) = g(x, y)dx dy + \mathbf{N}(dx, dy). \quad (1)$$

In this representation, the term $g(x, y)dx dy$ is the absolutely continuous part corresponding to the dense graphon limit, while $\mathbf{N}(dx, dy)$ is the singular measure that captures the sparse network links.

In the context of large-scale engineering systems, the application of algebraically constructed graph sequences, such as Schreier and Cayley graphs, addresses the trade-off between communication sparsity and global connectivity. As expander graphs, these algebraic structures possess a bounded uniform degree while maintaining a large spectral gap [17]. The ideal realization of these constructions is found in Ramanujan graphs, which attain the asymptotically optimal spectral gap under the Alon-Boppana bound [26, 28]. This optimal algebraic connectivity (or Fiedler value) is desirable in distributed control: it governs the convergence speed of multi-agent algorithms, ensuring rapid state consensus and reliable information diffusion without the severe $\mathcal{O}(N^2)$ communication burden of dense networks [30].

Because these algebraic topologies dictate logical routing rather than physical spatial proximity, they are widely deployed as abstract overlay networks or rigid hardware interconnections in large-scale engineering. Classical implementations have utilized Cayley graphs to design symmetric, deadlock-free interconnection architectures for multiprocessor systems [1], as well as to construct scalable routing geometries in decentralized peer-to-peer networks [31]. More recently, the deterministic high-throughput properties of expander graphs have been applied in modern cloud data centers, such as the Margulis-expander-based Xpander architecture [35].

The framework of this paper addresses the limit problem of MFGs in sparse networks. We first construct the finite graph topology using a discrete approximation of a group action, utilizing Gabber-Galil-Margulis expander graphs [13, 17, 25]. Expander graphs are connected sparse graphs that maintain bounded degrees while exhibiting specific algebraic mixing properties [17]. In contrast to dense networks where the mean field interaction is captured by an integral operator over the state space (e.g., $\int_X g(\alpha, \beta)m(\beta)\nu(d\beta)$), the sparse limit topology in this paper is characterized by a graphexon operator $\mathcal{G}$ defined in Equation (7) which acts via measure-preserving spatial shifts rather than integration, dictating the spatial coupling in the agent dynamics.

The primary contribution of this work is extending Lubotzky's algebraic framework for finite expander graphs to the continuous graphexon limit. Building upon this framework, the paper extends

the study of Mean Field Games (MFGs) on sparse networks in two directions. First, the weak convergence of empirical measures for the underlying finite graph structure to a limit graphexon measure is established, and the strong L^2 -convergence of the associated discrete averaging operators to an ergodic continuous integration operator is proved. Second, the infinite-horizon discounted scalar linear-quadratic-Gaussian (LQG) MFG on this limit space is formulated, yielding a coupled system of forward-backward differential equations. The existence of a stabilizing decoupling operator and the exact algebraic boundaries for the stability is established, characterizing the Turing-type topological instability phenomenon.

The paper is organized as follows. Section 2 introduces finite Schreier expander graphs and discrete graphexons. Section 3 addresses continuous graphexon limits and the associated operator convergence. Section 5 formulates the infinite-horizon LQG graphexon mean field game and establishes the algebraic conditions for mean field stability and topological instability. Section 6 presents numerical simulations, followed by concluding remarks in Section 7.

2 Finite Schreier expander graphs and discrete graphexons

In this work, the connectivity of the finite graphs is generated by a discrete approximation of a group action, constructing a sequence of Schreier graphs [17].

Let (X, d) be a compact metric space equipped with its Borel σ -algebra $\mathcal{B}$ and a Borel probability measure ν , forming the probability space $(X, \mathcal{B}, \nu)$. Let $\|\cdot\|_{L^2}$ be the induced L^2 -norm. Denote by $L_0^2(X, \nu)$ the orthogonal complement of the constant functions in $L^2(X, \nu)$. Let $\|\cdot\|_{\mathcal{L}(L_0^2)}$ denote the induced operator norm restricted to $L_0^2(X, \nu)$. Let $N \in \mathbb{N}$ denote the resolution parameter.

Let Γ be an infinite discrete group generated by a finite symmetric set $S = \{s_1, \dots, s_K\}$. Let $\{\Gamma_N\}_{N=1}^\infty$ be a sequence of finite-index subgroups of Γ . The index set representing the finite population is defined as the left coset space $X_N \triangleq \Gamma/\Gamma_N$.

Let $M_N \triangleq |X_N|$ denote the cardinality of the index set. We assume $M_N \rightarrow \infty$ as $N \rightarrow \infty$. Let $\ell^2(X_N)$ be the space of real-valued functions on the index set equipped with the normalized inner product $\langle \phi, \psi \rangle_{\ell^2} = \frac{1}{M_N} \sum_{v \in X_N} \phi(v) \psi(v)$, and $\ell_0^2(X_N)$ be the subspace of functions orthogonal to the constant functions.

Let $\mathcal{P}_N = \{Q_v^N \subset X\}_{v \in X_N}$ be a measurable partition of the space X such that the measure of each cell is uniform, satisfying

$$\nu(Q_v^N) = \frac{1}{M_N}, \quad \forall v \in X_N.$$

We assume that the maximum diameter of the partition cells, denoted by δ_N , vanishes as $N \rightarrow \infty$ satisfying

$$\delta_N \triangleq \max_{v \in X_N} \sup_{x, y \in Q_v^N} d(x, y), \quad \lim_{N \rightarrow \infty} \delta_N = 0.$$

Let $\iota_N : X_N \rightarrow X$ be an embedding map that assigns a representative point to each index, satisfying $\iota_N(v) \in Q_v^N$. For any $x \in X$, let $i_N : X \rightarrow X_N$ denote the unique index such that $x \in Q_{i_N(x)}^N$.

Denote a sequence of K bijections on the index set by

$$S_N = (\sigma_{N,1}, \dots, \sigma_{N,K}) \tag{2}$$

where each $\sigma_{N,k}$ is defined by the left action of the generators in S on the coset space X_N

$$\sigma_{N,k}(g\Gamma_N) \triangleq (s_k g)\Gamma_N$$

for any coset $g\Gamma_N \in X_N$ with $g \in \Gamma$.

Example 1 ([17]). Consider the state space $X = \mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Let $\tilde{\Gamma}$ be the subgroup of $\text{SL}(2, \mathbb{Z})$ generated by the matrices

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Define the infinite discrete group $\Gamma \triangleq \mathbb{Z}^2 \rtimes \tilde{\Gamma}$ equipped with the group operation $(z_1, \tilde{T}_1)(z_2, \tilde{T}_2) = (z_1 + \tilde{T}_1 z_2, \tilde{T}_1 \tilde{T}_2)$. Define the sequence of finite-index subgroups $\Gamma_N \triangleq (N\mathbb{Z})^2 \rtimes \tilde{\Gamma}$. Let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{Z}^2$. Define the symmetric generating set $S = \{s_1, \dots, s_8\} \subset \Gamma$ of size $K = 8$ by

$$\begin{aligned} s_1 &= (0, T_1), & s_2 &= (e_1, T_1), \\ s_3 &= (0, T_2), & s_4 &= (e_2, T_2), \\ s_5 &= (0, T_1^{-1}), & s_6 &= (-T_1^{-1}e_1, T_1^{-1}), \\ s_7 &= (0, T_2^{-1}), & s_8 &= (-T_2^{-1}e_2, T_2^{-1}). \end{aligned}$$

The finite coset space $X_N \triangleq \Gamma/\Gamma_N$ is bijectively identified with the discrete index set $(\mathbb{Z}/N\mathbb{Z})^2$ with cardinality $M_N = N^2$. The embedding map $\iota_N : X_N \rightarrow \mathbb{T}^2$ assigns coordinates on the torus by $\iota_N(v) = \frac{v}{N} \bmod 1$. The discrete graph topology is governed by the sequence of affine bijections $S_N = (\sigma_{N,1}, \dots, \sigma_{N,8})$ acting on X_N given by

$$\begin{aligned} \sigma_{N,1}(v) &= T_1 v \bmod N, & \sigma_{N,2}(v) &= (T_1 v + e_1) \bmod N, \\ \sigma_{N,3}(v) &= T_2 v \bmod N, & \sigma_{N,4}(v) &= (T_2 v + e_2) \bmod N, \\ \sigma_{N,5}(v) &= T_1^{-1} v \bmod N, & \sigma_{N,6}(v) &= T_1^{-1}(v - e_1) \bmod N, \\ \sigma_{N,7}(v) &= T_2^{-1} v \bmod N, & \sigma_{N,8}(v) &= T_2^{-1}(v - e_2) \bmod N. \end{aligned}$$

Definition 1 (Property (τ) [24]). The group Γ has *Property (τ)* with respect to $\{\Gamma_N\}$ if there exists a uniform constant $\kappa > 0$ such that for any $N \geq 1$ and any $f \in \ell_0^2(X_N)$, the following bound holds:

$$\sum_{k=1}^K \|f \circ \sigma_{N,k} - f\|_{\ell^2}^2 \geq \kappa^2 \|f\|_{\ell^2}^2.$$

where κ is termed a *Kazhdan constant*.

Assumption 1. It is assumed that Γ has Property (τ) with respect to $\{\Gamma_N\}$.

The topology on the edge space of the large-scale multi-agent mean field system is characterized by an undirected K -regular graph $G_N = (X_N, E_N)$. The edge set E_N is defined by

$$E_N \triangleq \{\{v, \sigma_{N,k}(v)\} \mid v \in X_N, k \in \{1, \dots, K\}\}.$$

Remark 1. G_N constitutes a *finite Schreier graph* associated with the action of the group Γ on the coset space Γ/Γ_N generated by the symmetric set S . Schreier graphs establish a natural algebraic generalization of Cayley graphs. Specifically, in the degenerate case where Γ_N is the trivial identity subgroup, the coset space coincides with the group Γ itself, and the Schreier graph reduces to the Cayley graph of Γ with respect to S (see, e.g., [17, 25]).

Let $O_N : \ell^2(X_N) \rightarrow \ell^2(X_N)$ denote the normalized discrete *adjacency operator*, evaluated pointwise

$$[O_N f](v) \triangleq \frac{1}{K} \sum_{k=1}^K f(\sigma_{N,k}(v)).$$

Definition 2 ([17]). Let $\{G_N = (X_N, E_N)\}_{N=1}^\infty$ be a sequence of undirected, K -regular finite graphs. Let $\lambda_2(O_N)$ denote the second largest eigenvalue of O_N . The sequence $\{G_N\}_{N=1}^\infty$ constitutes a family of *expander graphs* if there exists a uniform constant $c_{\text{gap}} > 0$ such that the spectral gap satisfies

$$1 - \lambda_2(O_N) \geq c_{\text{gap}}, \quad \forall N \geq 1.$$

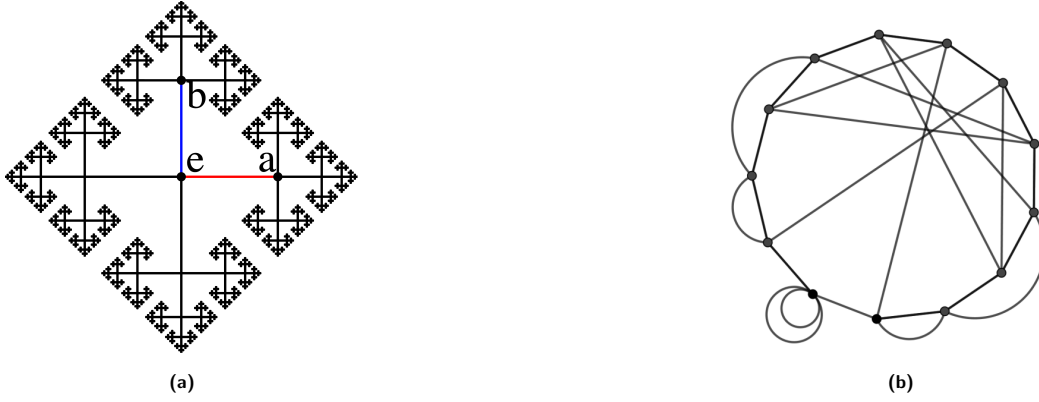


Figure 1: (a) A Cayley Graph with two generators (a, b) (Image source: Wikipedia); (b) A Schreier graph of the affine group acting on a finite field with 13 elements, picture from [32].

Lemma 1 (Theorem 4.3.2 in [24]). Under Assumption 1, the sequence $\{G_N\}_{N=1}^\infty$ constitutes a family of expander graphs, satisfying

$$1 - \lambda_2(O_N) \geq \frac{\kappa^2}{2K}, \quad \forall N \geq 1. \quad (3)$$

Remark 2. Property (τ) ensures the uniform connectivity of $\{G_N\}_{N=1}^\infty$. If Γ fails to satisfy Property (τ) with respect to $\{\Gamma_N\}$, the Kazhdan constant might vanish as $N \rightarrow \infty$. Consequently, the graph sequence loses its expander property.

Assumption 2. There exists a uniform constant $c_{abs} > 0$ such that

$$\|O_N\|_{\mathcal{L}(\ell_0^2)} = \max\{\lambda_2(O_N), |\lambda_{\min}(O_N)|\} \leq 1 - c_{abs}$$

for all $N \geq 1$.

Remark 3. This ensures that the spectrum is bounded away from -1 , which excludes configurations that exhibit or asymptotically approach a regular bipartite structure.

Define the edge space as $X \times X$ equipped with the metric $D((x_1, y_1), (x_2, y_2)) \triangleq d(x_1, x_2) + d(y_1, y_2)$.

Definition 3. The empirical graphexon measure $\mathbf{W}_N$ on the product space $X \times X$ is defined by

$$\mathbf{W}_N \triangleq \frac{1}{KM_N} \sum_{v \in X_N} \sum_{k=1}^K \delta_{(\iota_N(v), \iota_N(\sigma_{N,k}(v)))} \quad (4)$$

where $\delta_{(x,y)}$ denotes the Dirac measure at the point $(x, y) \in X \times X$.

Definition 4. The projection operator $P_N : L^2(X, \nu) \rightarrow \ell^2(X_N)$ is defined by the local cell average

$$[P_N f](v) \triangleq \frac{1}{\nu(Q_v^N)} \int_{Q_v^N} f(z) \nu(dz).$$

The extension operator $E_N : \ell^2(X_N) \rightarrow L^2(X, \nu)$ constructs a piecewise constant function from discrete values

$$[E_N \phi](x) \triangleq \sum_{v \in X_N} \phi(v) \mathbb{I}_{Q_v^N}(x).$$

Definition 5. The step-graphexon operator $\mathcal{G}_N : L^2(X, \nu) \rightarrow L^2(X, \nu)$ is the composition $\mathcal{G}_N \triangleq E_N \circ O_N \circ P_N$. For any $x \in X$, it is defined as

$$[\mathcal{G}_N f](x) = \frac{1}{K} \sum_{k=1}^K [P_N f](\sigma_{N,k}(i_N(x))). \quad (5)$$

3 Continuous graphexon limits and operator convergence

To guarantee the existence of a limit for the sequence of discrete graph interactions, we impose a uniform regularity condition on the sequences.

Assumption 3. There exists a uniform constant $L > 0$ such that for all $N \geq 1$, $k \in \{1, \dots, K\}$, and $u, v \in X_N$, the discrete mappings satisfy the Lipschitz condition on the embedded space

$$d(\iota_N(\sigma_{N,k}(u)), \iota_N(\sigma_{N,k}(v))) \leq Ld(\iota_N(u), \iota_N(v)).$$

Lemma 2. Under Assumption 3, there exists a sequence of bi-Lipschitz bijections $S_\infty \triangleq (\sigma_{\infty,1}, \dots, \sigma_{\infty,K})$ on X with uniform Lipschitz constant L , and a strictly increasing subsequence $\{N_j\}_{j=1}^\infty$, such that for each $k \in \{1, \dots, K\}$,

$$\lim_{j \rightarrow \infty} \sup_{x \in X} d(\iota_{N_j}(\sigma_{N_j,k}(i_{N_j}(x))), \sigma_{\infty,k}(x)) = 0.$$

Proof. Let $\mathcal{D} \subset X$ be a countable dense subset. Since X is compact, it follows from Cantor's diagonal argument that there exists a subsequence $\{N_j\}_{j=1}^\infty$ such that $f_{N_j,k}(x) \triangleq \iota_{N_j}(\sigma_{N_j,k}(i_{N_j}(x)))$ converge pointwise for all $x \in \mathcal{D}$ and all k . Denote the limit by $\sigma_{\infty,k}(x)$.

By Assumption 3 and $d(\iota_{N_j}(i_{N_j}(x)), x) \leq \delta_{N_j}$, for any $x, y \in \mathcal{D}$,

$$d(f_{N_j,k}(x), f_{N_j,k}(y)) \leq L(d(x, y) + 2\delta_{N_j}).$$

Taking $j \rightarrow \infty$ with $\delta_{N_j} \rightarrow 0$ shows that $\sigma_{\infty,k}$ is L -Lipschitz on $\mathcal{D}$, which extends to X . The symmetry of the generating set S gives the identical bound for the inverse mappings, establishing each $\sigma_{\infty,k}$ as a bi-Lipschitz bijection.

For uniform convergence, let $\epsilon > 0$ and let $y \in \mathcal{D}$ be a point in a finite ϵ -net of X closest to x . Then,

$$\begin{aligned} d(f_{N_j,k}(x), \sigma_{\infty,k}(x)) &\leq d(f_{N_j,k}(x), f_{N_j,k}(y)) \\ &\quad + d(f_{N_j,k}(y), \sigma_{\infty,k}(y)) + d(\sigma_{\infty,k}(y), \sigma_{\infty,k}(x)) \\ &\leq L(d(x, y) + 2\delta_{N_j}) + d(f_{N_j,k}(y), \sigma_{\infty,k}(y)) + Ld(x, y) \\ &= 2L(d(x, y) + \delta_{N_j}) + d(f_{N_j,k}(y), \sigma_{\infty,k}(y)). \end{aligned}$$

Taking the limit superior as $j \rightarrow \infty$, the last term vanishes uniformly over the finite net and $\delta_{N_j} \rightarrow 0$, leaving the bound $2L\epsilon$. Since ϵ is arbitrary, the convergence is uniform over X . $\square$

For all $A \in \mathcal{B}$, the limit mappings satisfy the measure-preserving property $\nu(\sigma_{\infty,k}^{-1}(A)) = \nu(A)$. Since the original generator set S is symmetric, S_∞ remains closed under inversion up to a permutation of its indices.

The discretization error associating the continuous mapping with its discrete approximation is quantified by

$$\epsilon_{N_j,k} \triangleq \max_{v \in X_{N_j}} d(\sigma_{\infty,k}(\iota_{N_j}(v)), \iota_{N_j}(\sigma_{N_j,k}(v))).$$

By Lemma 2, the maximum error vanishes asymptotically

$$\lim_{j \rightarrow \infty} \max_{k \in \{1, \dots, K\}} \epsilon_{N_j,k} = 0.$$

As $j \rightarrow \infty$, distinct discrete generators in S_{N_j} may converge to identical continuous mappings. Let $\tilde{S}_\infty$ denote the set comprising the distinct transformations from the sequence S_∞ , and let $\tilde{\Gamma}$ denote the finitely generated group constructed by $\tilde{S}_\infty$.

Since the underlying space X is compact, the product space $(X \times X, D)$ is also compact. By Prokhorov's theorem, the uniformly tight sequence of probability measures $\{\mathbf{W}_{N_j}\}_{j=1}^\infty$ is relatively compact in the topology of weak convergence, guaranteeing the existence of weakly convergent subsequential limits.

We therefore construct the continuous limit measure $\mathbf{W}$ on $X \times X$ associated with the limit sequence S_∞ by

$$\mathbf{W}(dx, dy) \triangleq \frac{1}{K} \sum_{k=1}^K \nu(dx) \delta_{\sigma_{\infty,k}(x)}(dy). \quad (6)$$

Within the graphexon framework [5, 8], any graphexon measure $\mathbf{G}$ on the product space $X \times X$ admits a Lebesgue decomposition

$$\mathbf{G}(dx, dy) = g(x, y) \nu(dx) \nu(dy) + \mathbf{N}(dx, dy)$$

where $g \in L^1(X \times X)$ represents the absolutely continuous component capturing dense connections, and $\mathbf{N}$ is the singular measure component capturing sparse connections.

For the limit measure $\mathbf{W}$ constructed above, the absolutely continuous component is $g(x, y) = 0$ since the measure is entirely supported on the graphs of the measure-preserving bijections $\sigma_{\infty,k}$. Consequently, the edge topology of the limit expander graph is purely sparse and is completely characterized by the singular component $\mathbf{N} = \mathbf{W}$.

Theorem 1. Under Assumption 3, the subsequence of empirical graphexon measures $\mathbf{W}_{N_j}$ converges weakly to the graphexon measure $\mathbf{W}$.

Proof. Let $\Psi \in C(X \times X)$ be a continuous test function. Since $X \times X$ is compact, Ψ is uniformly continuous. Let ω_Ψ denote its uniform modulus of continuity. The integration of Ψ with respect to $\mathbf{W}_{N_j}$ evaluates as a finite summation over the discrete atomic support

$$\int_{X^2} \Psi d\mathbf{W}_{N_j} = \frac{1}{KM_{N_j}} \sum_{k=1}^K \sum_{v \in X_{N_j}} \Psi(\iota_{N_j}(v), \iota_{N_j}(\sigma_{N_j,k}(v))).$$

The integration with respect to $\mathbf{W}$ expands over the uniform cells

$$\int_{X^2} \Psi d\mathbf{W} = \frac{1}{K} \sum_{k=1}^K \sum_{v \in X_{N_j}} \int_{Q_v^{N_j}} \Psi(x, \sigma_{\infty,k}(x)) \nu(dx).$$

For any $x \in Q_v^{N_j}$, the embedding distance satisfies $d(x, \iota_{N_j}(v)) \leq \delta_{N_j}$. Let L_τ denote the uniform maximum Lipschitz constant for all $\sigma_{\infty,k} \in S_\infty$. The total deviation D is bounded by

$$\begin{aligned} D((x, \sigma_{\infty,k}(x)), (\iota_{N_j}(v), \iota_{N_j}(\sigma_{N_j,k}(v)))) \\ &= d(x, \iota_{N_j}(v)) + d(\sigma_{\infty,k}(x), \iota_{N_j}(\sigma_{N_j,k}(v))) \\ &\leq \delta_{N_j} + d(\sigma_{\infty,k}(x), \sigma_{\infty,k}(\iota_{N_j}(v))) \\ &\quad + d(\sigma_{\infty,k}(\iota_{N_j}(v)), \iota_{N_j}(\sigma_{N_j,k}(v))) \\ &\leq \delta_{N_j} + L_\tau d(x, \iota_{N_j}(v)) + \epsilon_{N_j,k} \\ &\leq (1 + L_\tau) \delta_{N_j} + \epsilon_{N_j,k}. \end{aligned}$$

Applying the modulus of continuity ω_Ψ , the difference obeys the uniform bound

$$\left| \int_{X^2} \Psi d\mathbf{W}_{N_j} - \int_{X^2} \Psi d\mathbf{W} \right| \leq \omega_\Psi((1 + L_\tau) \delta_{N_j} + \epsilon_{N_j}).$$

It follows from $\lim_{j \rightarrow \infty} \delta_{N_j} = 0$ and $\lim_{j \rightarrow \infty} \epsilon_{N_j} = 0$ that the upper bound vanishes asymptotically. This establishes the weak convergence of the specified subsequence to $\mathbf{W}$. $\square$

Example 2. Following Example 1, as $N \rightarrow \infty$, the limit graphexon measure $\mathbf{W}$ takes the explicit singular form

$$\mathbf{W}(dx, dy) = \frac{1}{4} \sum_{T \in S_T} dx \delta_{(Tx \bmod 1)}(dy).$$

Definition 6. The *graphexon operator* $\mathcal{G} : L^2(X, \nu) \rightarrow L^2(X, \nu)$ is defined by, for any $m \in L^2(X, \nu)$,

$$[\mathcal{G}m](x) \triangleq \frac{1}{K} \sum_{k=1}^K m(\sigma_{\infty, k}(x)). \quad (7)$$

Remark 4. The operator $\mathcal{G}$ establishes the coupling mechanism for the multi-agent mean field system. In the classical dense network limit, the network aggregation of the mean state $m \in L^2(X, \nu)$ evaluated at a spatial label $\alpha \in X$ is governed by a Hilbert-Schmidt integral operator

$$\int_X W(\alpha, \beta) m(\beta) \nu(d\beta)$$

where W is the graphon kernel (see, e.g., [6, 14, 37]). For the sparse expander graph sequences considered in this work, the classical integral coupling degenerates. The graphexon operator $\mathcal{G}$ replaces the dense integral by Equation (7) which dictates how the local mean field interacts across the limiting sparse topology.

To preclude the limit network from fracturing into disjoint invariant subsets of positive measure, a scenario that would yield multiple principal eigenvalues and eliminate the spectral gap on $L_0^2(X, \nu)$, we require the limit group action to be *ergodic*.

Definition 7 ([39]). The group action of $\tilde{\Gamma}$ on X is *ergodic* if for any measurable set $E \in \mathcal{B}$ satisfying $\nu(\sigma_{\infty, k}(E) \triangle E) = 0$ for all $k \in \{1, \dots, K\}$ where $\triangle$ denotes the symmetric difference of sets, it holds that either $\nu(E) = 0$ or $\nu(E) = 1$.

Assumption 4. The action of the limit group $\tilde{\Gamma}$ on the space X is ergodic.

Example 3. Following Example 1, as $N \rightarrow \infty$, the embedded shift vectors scaled by e_j/N vanish. The continuous limit connections are governed by $S_\infty = (\sigma_{\infty, 1}, \dots, \sigma_{\infty, 8})$. For any $x \in \mathbb{T}^2$, the limit mappings converge pointwise to

$$\begin{aligned} \sigma_{\infty, 1}(x) &= \sigma_{\infty, 2}(x) = T_1 x \bmod 1, \\ \sigma_{\infty, 3}(x) &= \sigma_{\infty, 4}(x) = T_2 x \bmod 1, \\ \sigma_{\infty, 5}(x) &= \sigma_{\infty, 6}(x) = T_1^{-1} x \bmod 1, \\ \sigma_{\infty, 7}(x) &= \sigma_{\infty, 8}(x) = T_2^{-1} x \bmod 1. \end{aligned}$$

These limit transformations generate the linear subgroup $\tilde{\Gamma}$ defined in Example 1, which acts ergodically on $\mathbb{T}^2$, satisfying Assumption 4.

Lemma 3. The limit operator $\mathcal{G}$ is self-adjoint on $L^2(X, \nu)$ and its spectrum $\sigma(\mathcal{G})$ is real. Furthermore, under Assumption 4, the principal eigenvalue $\lambda = 1$ is simple, and its associated eigenspace is $\text{span}\{\mathbf{1}\}$.

Proof. For any $f, g \in L^2(X, \nu)$, the inner product is

$$\langle \mathcal{G}f, g \rangle = \frac{1}{K} \sum_{k=1}^K \int_X f(\sigma_{\infty, k}(x)) g(x) \nu(dx).$$

Applying the measure-preserving change of variables $y = \sigma_{\infty, k}(x)$ provides

$$\langle \mathcal{G}f, g \rangle = \int_X f(y) \left(\frac{1}{K} \sum_{k=1}^K g(\sigma_{\infty, k}^{-1}(y)) \right) \nu(dy).$$

Since S_∞ is closed under inversion up to a permutation of its indices, the summation over the inverse sequence is a rearrangement of the forward summation, which is $[\mathcal{G}g](y)$. Thus $\langle \mathcal{G}f, g \rangle = \langle f, \mathcal{G}g \rangle$, confirming that $\mathcal{G}$ is self-adjoint and $\sigma(\mathcal{G}) \subset \mathbb{R}$.

Suppose $f \in L^2(X, \nu)$ satisfies $\mathcal{G}f = f$. Taking the L^2 -norm yields

$$\|f\|_{L^2} = \|\mathcal{G}f\|_{L^2} \leq \frac{1}{K} \sum_{k=1}^K \|f \circ \sigma_{\infty, k}\|_{L^2} = \|f\|_{L^2}.$$

The strict convexity of $L^2(X, \nu)$ dictates that the triangle inequality reaches equality if and only if all functions $f \circ \sigma_{\infty, k}$ are identical almost everywhere up to a non-negative scalar, satisfying $f \circ \sigma_{\infty, k} = c_k(f \circ \sigma_{\infty, 1})$. Recall that each map $\sigma_{\infty, k}$ is measure-preserving, the L^2 -norms are conserved, forcing $c_k = 1$. Consequently, substituting $f \circ \sigma_{\infty, 1} = \dots = f \circ \sigma_{\infty, K}$ into the eigenvector equation establishes $f \circ \sigma_{\infty, 1} = f$. This implies f is $\tilde{\Gamma}$ -invariant. By Assumption 4, f must be constant ν -almost everywhere. Therefore, the eigenspace corresponding to $\lambda = 1$ is $\text{span}\{\mathbf{1}\}$, and its dimension is 1, establishing that the principal eigenvalue $\lambda = 1$ is simple. $\square$

Proposition 1. Under Assumption 3, for any $f \in L^2(X, \nu)$, the subsequence of operators $\{\mathcal{G}_{N_j}\}_{j=1}^\infty$ converges strongly to $\mathcal{G}$ in the L^2 -norm, satisfying

$$\lim_{j \rightarrow \infty} \|\mathcal{G}_{N_j} f - \mathcal{G}f\|_{L^2} = 0.$$

Proof. We first establish convergence on the dense subspace of Lipschitz continuous functions. Let $f : X \rightarrow \mathbb{R}$ be Lipschitz continuous with constant L_f . For a fixed $x \in X$, let $v = i_{N_j}(x)$. Define the pointwise evaluation error for $k \in \{1, \dots, K\}$ by

$$\Delta_k(x) \triangleq |[P_{N_j} f](\sigma_{N_j, k}(v)) - f(\sigma_{\infty, k}(x))|.$$

Let $u = \sigma_{N_j, k}(v)$. Expanding the projection operator P_{N_j} provides

$$\begin{aligned} \Delta_k(x) &\leq M_{N_j} \int_{Q_u^{N_j}} |f(z) - f(\sigma_{\infty, k}(x))| \nu(dz) \\ &\leq M_{N_j} \int_{Q_u^{N_j}} L_f d(z, \sigma_{\infty, k}(x)) \nu(dz). \end{aligned}$$

For any $z \in Q_u^{N_j}$, the metric bound observes

$$\begin{aligned} d(z, \sigma_{\infty, k}(x)) &\leq d(z, \iota_{N_j}(u)) + d(\iota_{N_j}(u), \sigma_{\infty, k}(\iota_{N_j}(v))) \\ &\quad + d(\sigma_{\infty, k}(\iota_{N_j}(v)), \sigma_{\infty, k}(x)). \end{aligned}$$

Because $z, \iota_{N_j}(u) \in Q_u^{N_j}$, one has $d(z, \iota_{N_j}(u)) \leq \delta_{N_j}$. The middle term is bounded by ϵ_{N_j} . Let L_τ denote the uniform maximum Lipschitz constant for all transformations $\sigma_{\infty, k} \in S_\infty$. Since both x and $\iota_{N_j}(v)$ reside in $Q_v^{N_j}$, the final term obeys $d(\sigma_{\infty, k}(\iota_{N_j}(v)), \sigma_{\infty, k}(x)) \leq L_\tau \delta_{N_j}$. Defining $C_{N_j} \triangleq L_f((1 + L_\tau)\delta_{N_j} + \epsilon_{N_j})$, it follows that $\Delta_k(x) \leq C_{N_j}$. The squared L^2 -norm of the operator difference is subsequently bounded by

$$\|\mathcal{G}_{N_j} f - \mathcal{G}f\|_{L^2}^2 \leq \int_X \left(\frac{1}{K} \sum_{k=1}^K \Delta_k(x) \right)^2 \nu(dx) \leq C_{N_j}^2.$$

It follows from $\lim_{j \rightarrow \infty} \delta_{N_j} = 0$ and $\lim_{j \rightarrow \infty} \epsilon_{N_j} = 0$ that C_{N_j} vanishes asymptotically. This confirms strong convergence for all Lipschitz continuous functions.

Furthermore, the projection P_{N_j} , discrete adjacency O_{N_j} , and extension E_{N_j} are contraction operators, guaranteeing the uniform bound $\|\mathcal{G}_{N_j}\|_{\mathcal{L}(L^2)} \leq 1$ for all j . Since the space of Lipschitz

continuous functions is dense in $L^2(X, \nu)$, the convergence on this dense subspace, combined with the a priori uniform bound of the operator norms, extends to strong operator convergence across the entire Hilbert space $L^2(X, \nu)$. $\square$

Proposition 2. Let Assumptions 2, 3, and 4 hold. Then the spectral radius of the limit graphexon operator $\mathcal{G}$ restricted to $L_0^2(X, \nu)$ satisfies $\|\mathcal{G}\|_{\mathcal{L}(L_0^2)} \leq 1 - c_{abs} < 1$.

Proof. By Proposition 1, the subsequence $\{\mathcal{G}_{N_j}\}_{j=1}^\infty$ converges strongly to $\mathcal{G}$ on $L^2(X, \nu)$ leading to that the operator norm is lower semi-continuous (see, e.g., [10]), i.e.,

$$\|\mathcal{G}\|_{\mathcal{L}(L_0^2)} \leq \liminf_{j \rightarrow \infty} \|\mathcal{G}_{N_j}\|_{\mathcal{L}(L_0^2)}.$$

By definition, the composition of the projection operator P_{N_j} and the extension operator E_{N_j} constitutes an isometry on the discrete cell structures and preserves the zero-mean property. Consequently, the operator norm of $\mathcal{G}_{N_j}$ restricted to $L_0^2(X, \nu)$ is bounded by the discrete operator norm of O_{N_j} restricted to $\ell_0^2(X_{N_j})$. By Assumption 2,

$$\|\mathcal{G}_{N_j}\|_{\mathcal{L}(L_0^2)} \leq \|O_{N_j}\|_{\mathcal{L}(\ell_0^2)} \leq 1 - c_{abs}.$$

Therefore,

$$\|\mathcal{G}\|_{\mathcal{L}(L_0^2)} \leq \liminf_{j \rightarrow \infty} \|\mathcal{G}_{N_j}\|_{\mathcal{L}(L_0^2)} \leq 1 - c_{abs} < 1.$$

By Lemma 3, $\mathcal{G}$ is self-adjoint. For bounded self-adjoint operators, the spectral radius coincides with the operator norm, ensuring that the spectrum of $\mathcal{G}$ on the zero-mean subspace is confined within the interval $[-(1 - c_{abs}), 1 - c_{abs}]$. $\square$

4 Graph construction on the flat torus

Recall the sequence of expander graphs on the two-dimensional torus $X = \mathbb{T}^2$ established in Examples 1 and 3. This sequence is generated using the discrete algebraic construction of the Gabber-Galil-Margulis expander [17].

Let O_N denote the normalized adjacency operator on $\ell^2(X_N)$ defined by the $K = 8$ discrete bijections from Example 1. The expander property of this discrete sequence is guaranteed by bounding its absolute spectral radius on the zero-mean subspace.

Lemma 4 ([13, 17]). There exists a spectral gap such that the adjacency operator O_N restricted to $\ell_0^2(X_N)$ satisfies

$$\|O_N\|_{\mathcal{L}(\ell_0^2)} = \max\{\lambda_2(O_N), |\lambda_{\min}(O_N)|\} \leq \frac{5\sqrt{2}}{8} \quad (8)$$

for any $N \geq 1$.

The limit graphexon operator $\mathcal{G} : L^2(\mathbb{T}^2, \nu) \rightarrow L^2(\mathbb{T}^2, \nu)$ generated by S_∞ evaluates

$$[\mathcal{G}f](x) = \frac{1}{4} (f(T_1x) + f(T_1^{-1}x) + f(T_2x) + f(T_2^{-1}x)).$$

Theorem 2. The limit graphexon operator $\mathcal{G}$ possesses a continuous real spectrum on $L_0^2(\mathbb{T}^2, \nu)$, i.e., there exists no non-zero square-integrable eigenfunction $f \in L_0^2(\mathbb{T}^2, \nu)$ satisfying $\mathcal{G}f = \lambda f$ for any $\lambda \in \mathbb{R}$. Furthermore, the spectral radius of $\mathcal{G}$ restricted to $L_0^2(\mathbb{T}^2, \nu)$ is bounded away from 1, satisfying $\|\mathcal{G}\|_{\mathcal{L}(L_0^2)} \leq \frac{\sqrt{3}}{2}$.

Proof. To analyze the spectral properties of the graphexon operator $\mathcal{G}$ on the continuous flat torus $\mathbb{T}^2$, we transfer the analysis to $\mathbb{Z}^2$ via the Fourier transform. We will first prove that $\mathcal{G}$ possesses a continuous spectrum. To prove this by contradiction, assume that there exists a real eigenvalue λ and a non-zero eigenfunction $f \in L_0^2(\mathbb{T}^2, \nu)$ satisfying $\mathcal{G}f = \lambda f$.

Let $\{e_n\}_{n \in \mathbb{Z}^2 \setminus \{0\}}$ denote the standard Fourier basis for $L_0^2(\mathbb{T}^2, \nu)$ defined by $e_n(x) = \exp(2\pi i n \cdot x)$ for $x \in \mathbb{T}^2$. The function f admits the Fourier series expansion $f(x) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} c_n e_n(x)$ with $\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} |c_n|^2 < \infty$. Let $c \in \ell^2(\mathbb{Z}^2 \setminus \{0\})$ defined point-wise by $c(n) \triangleq c_n$ with abuse of notation.

Let $S_T = \{T_1, T_1^{-1}, T_2, T_2^{-1}\}$. Substituting the Fourier expansion into the eigenvalue equation and utilizing the algebraic identity $e_n(Tx) = e_{T^\top n}(x)$ yields

$$\frac{1}{4} \sum_{T \in S_T} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} c_n e_{T^\top n}(x) = \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \lambda c_m e_m(x)$$

Applying the change of variables $m = T^\top n$ and observing that the generator set S_T is symmetric under inversion, matching the orthogonal basis coefficients reduces the functional equation to a discrete difference equation on the dual space $\mathbb{Z}^2 \setminus \{0\}$ given by

$$\frac{1}{4} \sum_{T \in S_T} c_{T^\top m} = \lambda c_m \quad (9)$$

for any $m \in \mathbb{Z}^2 \setminus \{0\}$. Define the operator $\mathcal{M} : \ell^2(\mathbb{Z}^2 \setminus \{0\}) \rightarrow \ell^2(\mathbb{Z}^2 \setminus \{0\})$ pointwise by $[\mathcal{M}c]_m \triangleq \frac{1}{4} \sum_{T \in S_T} c_{T^\top m}$, then Equation (9) is $\mathcal{M}c = \lambda c$.

Step 1. Let $S_T^\top = \{T^\top \mid T \in S_T\}$ and let $\tilde{\Gamma}^\top \subset \text{SL}(2, \mathbb{Z})$ denote the group generated by $S_T^\top$. Since $\tilde{\Gamma}^\top$ is a Zariski dense subgroup of $\text{SL}(2, \mathbb{R})$, its linear action on $\mathbb{R}^2$ admits no finite orbits other than the origin [24, 39].

Let $\mathcal{I}$ be a countable index set and let $\{m_i\}_{i \in \mathcal{I}} \subset \mathbb{Z}^2 \setminus \{0\}$. The dual space $\mathbb{Z}^2 \setminus \{0\}$ is partitioned into disjoint infinite orbits $\mathcal{O}_{m_i} = \{\gamma m_i \mid \gamma \in \tilde{\Gamma}^\top\}$ for $i \in \mathcal{I}$. Consequently, the Hilbert space admits the orthogonal direct sum decomposition $\ell^2(\mathbb{Z}^2 \setminus \{0\}) = \bigoplus_{i \in \mathcal{I}} \ell^2(\mathcal{O}_{m_i})$.

Hence, any sequence $c \in \ell^2(\mathbb{Z}^2 \setminus \{0\})$ can be uniquely represented as $c = \sum_{i \in \mathcal{I}} \psi^{(i)}$, where $\psi^{(i)} \in \ell^2(\mathcal{O}_{m_i})$ is the orthogonal projection of c onto the subspace $\ell^2(\mathcal{O}_{m_i})$. By the definition of $\mathcal{O}_{m_i}$, the subspace $\ell^2(\mathcal{O}_{m_i})$ is invariant under $\mathcal{M}$. Therefore, the eigenvalue equation $\mathcal{M}c = \lambda c$ holds if and only if $P_i \psi^{(i)} = \lambda \psi^{(i)}$ for all $i \in \mathcal{I}$, where $P_i : \ell^2(\mathcal{O}_{m_i}) \rightarrow \ell^2(\mathcal{O}_{m_i})$ denotes the restriction of the operator $\mathcal{M}$ to the invariant subspace $\ell^2(\mathcal{O}_{m_i})$.

For any fixed vector $m_0 \in \mathbb{Z}^2 \setminus \{0\}$, let $P \triangleq P_0$ and $\psi \triangleq \psi^0$ for simplicity. Then P coincides with the transition operator of the Schreier graph $G = (\mathcal{O}_{m_0}, E)$ induced by the action of $\tilde{\Gamma}^\top$ with edge set $E = \{(m, \gamma m) \mid m \in \mathcal{O}_{m_0}, \gamma \in S_T^\top\}$.

To determine the topological structure of G , Sanov's Theorem [11, 33] establishes that $\tilde{\Gamma}^\top$ is isomorphic to the non-abelian free group $\mathbb{F}_2$, i.e., $\tilde{\Gamma}^\top \cong \mathbb{F}_2$.¹ This implies that the Cayley graph of $\tilde{\Gamma}^\top$ with respect to $S_T^\top$ is a 4-regular tree T_4 (see, e.g., [24, Section 4.5]).

By definition, $G \cong T_4/H$ where $H = \{\gamma \in \tilde{\Gamma}^\top \mid \gamma m_0 = m_0\}$ is the stabilizer subgroup. Matrices in H possess an eigenvalue of 1, implying they are either unipotent or the identity, which makes H an abelian subgroup.

Given $\tilde{\Gamma}^\top \cong \mathbb{F}_2$, its abelian subgroups are cyclic (see, e.g., [27]). Thus, H is either trivial (containing only the identity matrix) or $H \cong \mathbb{Z}$ (generated by a single unipotent matrix). This indicates that, when H is trivial, G is a 4-regular tree ($G \cong T_4$); and when $H \cong \mathbb{Z}$, G possesses one finite core cycle, that is to say every node on this cycle acts as the root of an outward-extending infinite tree.

¹The non-abelian free group $\mathbb{F}_2$ is the group consisting of all finite reduced words generated by two distinct elements and their inverses, subject to no algebraic relations other than the trivial cancellations of adjacent inverse elements.

Step 2. To establish the absence of a point spectrum, we analyze the transition operator P on the possible configurations of the orbit graph G . For the trivial stabilizer case where $G \cong T_4$, the operator corresponds to the simple random walk on a 4-regular tree and therefore G admits a continuous spectrum (see, e.g., [12, Chapter 3]).

For the non-trivial configuration where $H \cong \mathbb{Z}$, the graph $G \cong T_4/H$ consists of a finite 2-regular core cycle, with each vertex on the cycle serving as the root of an infinite outward-extending regular tree branch. In this case, ψ is radial, i.e., its value depends solely on the graph distance from the core cycle, on all extending tree branches (see, e.g., [12, Chapter 3]).

Let $\psi_d \triangleq \psi(d)$ at distance $d \geq 0$ from the core cycle. Then $P\psi = \lambda\psi$ can be rewritten as

$$\frac{1}{4}(\psi_{d-1} + 3\psi_{d+1}) = \lambda\psi_d, \quad d \geq 1.$$

The ℓ^2 integrability condition on the branching trees requires $\sum_d 3^d |\psi_d|^2 < \infty$ dictating that the only admissible square-summable solution is $\psi_d = \alpha^d \psi_0$, subject to the bound $|\alpha| < 1/\sqrt{3}$. Substituting $\psi_d = \alpha^d \psi_0$ with $d = 1$ establishes $4\lambda = 3\alpha + \alpha^{-1}$.

Let A_C denote the unnormalized adjacency operator of the core cycle. Evaluating the eigenvalue equation for the vertices on the cycle yields

$$\frac{1}{4}(A_C\psi_0 + 2\psi_1) = \frac{1}{4}(A_C\psi_0 + 2\alpha\psi_0) = \lambda\psi_0.$$

Substituting $4\lambda = 3\alpha + \alpha^{-1}$ into the cycle equation reduces it to $A_C\psi_0 = (\alpha + \alpha^{-1})\psi_0$. This forces $\alpha + \alpha^{-1}$ to be an eigenvalue of A_C . Since the core cycle is a finite 2-regular graph, its eigenvalues, denoted by 2μ , are bounded in $[-2, 2]$, which implies $\mu \in [-1, 1]$. The algebraic relation $\alpha + \alpha^{-1} = 2\mu$ yields the interface quadratic equation $\alpha^2 - 2\mu\alpha + 1 = 0$. For any $\mu \in [-1, 1]$, the discriminant is non-positive, ensuring the roots satisfy $|\alpha| = 1$. This contradicts the normalizability constraint $|\alpha| < 1/\sqrt{3}$.

Consequently, the operator P admits no square-summable eigenfunctions on any orbit $\mathcal{O}_{m_0}$, forcing $c_m = 0$ for all m . Therefore, the continuous operator $\mathcal{G}$ admits no point spectrum on $L_0^2(\mathbb{T}^2, \nu)$.

Step 3. To bound the continuous spectrum, recall from previous arguments that $G \cong T_4/H$ and H is abelian, then H is amenable and the spectral radius of P on G coincides with that of T_4 ². By Kesten's theorem [19], the spectral radius on the 4-regular tree T_4 is $\frac{\sqrt{3}}{2}$. Thus, $\|P\|_{\mathcal{L}(\ell^2(\mathcal{O}_{m_0}))} = \frac{\sqrt{3}}{2}$ for every orbit. Therefore,

$$\|\mathcal{G}\|_{\mathcal{L}(L_0^2)} = \sup_{m_0} \|P\|_{\mathcal{L}(\ell^2(\mathcal{O}_{m_0}))} = \frac{\sqrt{3}}{2} \quad \square$$

Remark 5. The limit graphexon operator $\mathcal{G}$ defined by the expander graph sequence possesses a purely continuous spectrum. This differs from the limit operators derived from dense random graph sequences, such as the Erdős-Rényi (ER) or other graph sequences characterized by a graphon $W(x, y)$ which is a bounded measurable function associated with a compact integral operator. Compact operators on infinite-dimensional Banach spaces admit only point spectra and possess discrete spectrum [23].

5 Graphexon mean field games on expander graphs

To analyze the asymptotic behavior of massive populations interacting over the complex limit topologies, we formulate the graphexon mean field game on (X, ν) established in the preceding sections.

Each agent is identified by a spatial network label $\alpha \in X$, which dictates its limit position, and only interacts with the neighborhood average defined by the operator $\mathcal{G}$.

²A discrete group is amenable if it admits a finitely additive, left-invariant probability measure defined on the collection of all its subsets [2, Appendix G]. By the weak containment property of induced representations for amenable subgroups [2, Theorem F.3.5], the quasi-regular representation on $G = T_4/H$ is weakly contained in the regular representation on T_4 , which guarantees that the spectral radius of P on G coincides with that of T_4 (see [2, Proposition F.1.11]).

5.1 Graphexon mean field game model

Let $x : X \times [0, \infty) \rightarrow \mathbb{R}$ denote the state of the infinite population over time. For notational convenience, the shorthand $x_t^\alpha \triangleq x(\alpha, t)$ is utilized to represent the state of a representative agent located at the spatial network label $\alpha \in X$ at time $t \geq 0$.

Let $\mu_t(\cdot; \alpha)$ denote the mean field representing the state distribution of the agent at label α at time t . The local mean state $m_t(\alpha)$ is defined as $m_t(\alpha) \triangleq \int_{\mathbb{R}} y \mu_t(dy; \alpha)$.

Since the agents are distributed continuously over the compact metric space X , the global mean state $m_t(\cdot) \in L^2(X, \nu)$. The inner product on $L^2(X, \nu)$ is $\langle f, g \rangle_{L^2} \triangleq \int_X f(\alpha) g(\alpha) \nu(d\alpha)$ for any $f, g \in L^2(X, \nu)$.

Let the admissible control set $\mathcal{U}$ be the space of progressively measurable processes u^α satisfying the square-integrability condition $\mathbb{E} \int_0^\infty e^{-\gamma t} |u_t^\alpha|^2 dt < \infty$. The dynamics of the representative agent α are governed by the scalar stochastic differential equation

$$dx_t^\alpha = (ax_t^\alpha + bu_t^\alpha + c[\mathcal{G}m_t](\alpha))dt + \sigma dW_t^\alpha \quad (10)$$

where $a, b \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0\}$, and $\sigma > 0$ are constant parameters. For each agent α , W_t^α is an independent standard scalar Brownian motion.

The agents aim to minimize the infinite horizon discounted cost functional

$$J^\alpha(u^\alpha) = \frac{1}{2} \mathbb{E} \int_0^\infty e^{-\gamma t} [q(x_t^\alpha - \eta[\mathcal{G}m_t](\alpha))^2 + r(u_t^\alpha)^2] dt \quad (11)$$

where $\gamma > 0$, $q \geq 0$, $r > 0$, and $\eta \in \mathbb{R}$.

Let $k \triangleq b^2/r$. Let $\Pi > 0$ be the unique positive stabilizing solution to the scalar algebraic Riccati equation

$$2\Pi \left(a - \frac{\gamma}{2} \right) - k\Pi^2 + q = 0. \quad (12)$$

Let $a_c \triangleq a - k\Pi$ denote the closed-loop parameter. We define the network coupling factor as $\psi \triangleq c\Pi - q\eta$.

Lemma 5. The limit graphexon mean field game problem possesses a unique Nash solution if and only if the following coupled forward-backward ODE system admits a unique solution pair (m_t, S_t) in $L^2(X, \nu)$

$$\begin{cases} \dot{m}_t = (a_c \mathbf{I} + c\mathcal{G})m_t - kS_t, & m_0(\alpha) = \int_{\mathbb{R}} y \mu_0(dy; \alpha) \\ -\dot{S}_t = (a_c - \gamma)S_t + \psi \mathcal{G}m_t, & \lim_{t \rightarrow \infty} e^{-\gamma t} S_t = 0 \end{cases} \quad (13)$$

where $\mathbf{I}$ is the identity operator on $L^2(X, \nu)$. The initial mean state $m_0 \in L^2(X, \nu)$ is given. Furthermore, the optimal best response is given by the decentralized feedback law

$$u_t^\alpha = -r^{-1}b(\Pi x_t^\alpha + S_t(\alpha)). \quad (14)$$

Proof. The proof follows standard linear-quadratic mean field game literature (see e.g., [9, 15]). Applying the maximum principle to the local agent's optimal control problem yields the individual adjoint equation of S_t . $\square$

5.2 Algebraic conditions for mean field stability

Let E be the projection-valued measure defined on the Borel subsets of the real line, associated with the self-adjoint operator $\mathcal{G}$. The graphexon operator evaluates as $\mathcal{G} = \int_{\sigma(\mathcal{G})} \lambda dE(\lambda)$. Let the spectral radius of $\mathcal{G}$ restricted to $L^2(X, \nu)$ be ρ . The spectral domain is $\lambda \in [-\rho, \rho]$.

We seek a bounded linear operator $\mathbf{P}(\mathcal{G})$ on $L^2(X, \nu)$, defined as

$$\mathbf{P}(\mathcal{G}) = \int_{\sigma(\mathcal{G})} p(\lambda) dE(\lambda) \quad (15)$$

where $p(\lambda) \in \mathbb{R}$ acts as a spectral multiplier for each mode, ensuring

$$S_t = \mathbf{P}(\mathcal{G})m_t.$$

Differentiating this linear ansatz and substituting it into (13) requires the following Scalar Algebraic Riccati Equation (SARE) to hold for all $\lambda \in \sigma(\mathcal{G})$

$$-kp(\lambda)^2 + (2a_c - \gamma + \lambda c)p(\lambda) + \lambda\psi = 0. \quad (16)$$

Define $a_\gamma \triangleq a_c - \gamma/2$. The discriminant associated with the SARE is

$$\Delta(\lambda) \triangleq c^2\lambda^2 + 4(a_\gamma c + k\psi)\lambda + 4a_\gamma^2. \quad (17)$$

Theorem 3. Assume $c \neq 0$. The parameterized SARE admits real-valued solutions $p(\lambda)$ for all $\lambda \in [-\rho, \rho]$ if and only if the system parameters satisfy the following two algebraic conditions

1. $c^2\rho^2 + 4a_\gamma^2 \geq 4\rho|a_\gamma c + k\psi|$;
2. Let $\lambda^* = -2(a_\gamma c + k\psi)/c^2$. If $|\lambda^*| \leq \rho$, then $k\psi(2a_\gamma c + k\psi) \leq 0$.

Furthermore, if a solution $p(\lambda)$ renders the closed-loop state dynamics stable, it must be

$$p(\lambda) = \frac{2a_\gamma + \lambda c + \sqrt{\Delta(\lambda)}}{2k}. \quad (18)$$

Proof. The SARE admits real-valued solutions if and only if its discriminant $\Delta(\lambda) = c^2\lambda^2 + 4(a_\gamma c + k\psi)\lambda + 4a_\gamma^2$ satisfies $\Delta(\lambda) \geq 0$ for all $\lambda \in [-\rho, \rho]$.

As $c \neq 0$, $\Delta(\lambda)$ is a convex parabola. Non-negativity on the compact interval $[-\rho, \rho]$ is equivalent to the non-negativity of its values at the boundaries and its vertex (if contained in the interval):

1. $\Delta(\pm\rho) \geq 0$ is equivalent to $c^2\rho^2 + 4a_\gamma^2 \geq 4\rho|a_\gamma c + k\psi|$.
2. The global minimum is $\Delta(\lambda^*)$ at $\lambda^* = -2(a_\gamma c + k\psi)/c^2$. If $|\lambda^*| \leq \rho$, the condition $\Delta(\lambda^*) \geq 0$ simplifies to its own discriminant being non-positive: $16(a_\gamma c + k\psi)^2 - 16c^2a_\gamma^2 \leq 0$, which factors into $k\psi(2a_\gamma c + k\psi) \leq 0$.

Next, let $p_\pm(\lambda) = \frac{1}{2k}(2a_\gamma + \lambda c \pm \sqrt{\Delta(\lambda)})$. Substituting $2a_\gamma = 2a_c - \gamma$ into the closed-loop generator $\mathcal{A}_{cl}(\lambda) = a_c + \lambda c - kp(\lambda)$ yields $\mathcal{A}_{cl}^\pm(\lambda) = \frac{1}{2}(\gamma + \lambda c \mp \sqrt{\Delta(\lambda)})$.

Consider the negative branch $p_-(\lambda)$, which leads to $\mathcal{A}_{cl}^+(\lambda)$. At $\lambda = 0$, the stability condition $\mathcal{A}_{cl}^+(0) < 0$ requires $\gamma + \sqrt{4a_\gamma^2} < 0$. Since $\gamma > 0$ and $\sqrt{\Delta} \geq 0$, this is impossible.

For the positive branch $p_+(\lambda)$, the generator is $\mathcal{A}_{cl}^-(\lambda)$. Evaluating at $\lambda = 0$ gives $p_+(0) = (2a_\gamma + 2|a_\gamma|)/2k$. Since $a_\gamma < 0$, $p_+(0) = 0$, which is the unique solution consistent with the unperturbed system. $\square$

Let $\mathcal{A}_{cl}(\lambda) \triangleq \frac{1}{2}(\gamma + \lambda c - \sqrt{\Delta(\lambda)})$. Define $L : \mathbb{R} \rightarrow \mathbb{R}$ as

$$L(\lambda) \triangleq ((\gamma - a)c + kq\eta)\lambda - a_c(a_c - \gamma). \quad (19)$$

One can verify that $\mathcal{A}_{cl}(\lambda) < 0$ is equivalent with $L(\lambda) < 0$.

Proposition 3. Let $\Sigma \triangleq [-\rho, \rho] \cup \{1\}$ denote the spectrum of $\mathcal{G}$. Assume the algebraic conditions in Theorem 3 hold. Define the non-negative half-space $\Omega \triangleq \{\lambda \in \Sigma \mid \gamma + \lambda c \geq 0\}$, and let $\mathcal{E} \triangleq \{1, -\rho, \rho, -\gamma/c\} \cap \Omega$. Then the infinite-dimensional mean field system is globally asymptotically stable, that is, $\lim_{t \rightarrow \infty} \|m_t\|_{L^2} = 0$, if and only if $a_c < 0$ and, for every point $\lambda_e \in \mathcal{E}$, $L(\lambda_e) < 0$.

Proof. Global asymptotic stability requires the closed-loop generator to satisfy $\mathcal{A}_{cl}(\lambda) < 0$ for all spectral modes $\lambda \in \Sigma$. Substituting the positive branch solution $p(\lambda)$, this condition is equivalent to $\gamma + \lambda c < \sqrt{\Delta(\lambda)}$.

For any $\lambda \in \Sigma \setminus \Omega$, the term $\gamma + \lambda c < 0$ and $\mathcal{A}_{cl}(\lambda)$ is clearly less than zero.

For any $\lambda \in \Omega$, both sides of the inequality are non-negative. One can verify that $\mathcal{A}_{cl}(\lambda) < 0$ is equivalent with $L(\lambda) < 0$. The supremum of L is attained at the extreme points of Ω , which are contained in the finite set $\mathcal{E}$. Therefore, $L(\lambda) < 0$ holds for all $\lambda \in \Omega$ if and only if $L(\lambda_e) < 0$ for all $\lambda_e \in \mathcal{E}$.

Since $\gamma > 0$, the base mode $\lambda = 0$ is contained in Ω . The condition $L(0) = -a_c(a_c - \gamma) < 0$ requires $a_c < 0$ due to $a_\gamma = a_c - \gamma/2 < 0$, completing the proof. $\square$

5.3 Turing-type topological instability on the torus

In classical reaction-diffusion systems, *Turing instability* refers to the phenomenon where an equilibrium, which is stable without spatial coupling, becomes asymptotically unstable under spatial coupling [29, 34]. Within the present graphexon MFG framework, this mechanism is generalized to a topological setting over the continuous spectrum of the underlying expander graph structure.

Let $\lambda = 1$ denote the isolated principal eigenvalue of $\mathcal{G}$ associated with the mean state $\bar{m}_t$ defined by

$$\bar{m}_t \triangleq \int_X m_t(\alpha) \nu(d\alpha).$$

The spatial deviation field is defined accordingly by

$$e_t(\alpha) \triangleq m_t(\alpha) - \bar{m}_t.$$

The dynamic evolution of the error field on the zero-mean subspace $L_0^2(X, \nu)$ is governed by the closed-loop generator $\dot{e}_t = \mathcal{A}_{cl}(\mathcal{G})e_t$, where $\mathcal{A}_{cl}(\mathcal{G}) \triangleq \int_{-\rho}^{\rho} \mathcal{A}_{cl}(\lambda) dE(\lambda)$.

Let $\Sigma_0 = [-\rho, \rho]$ denote the continuous spectrum of $\mathcal{G}$ restricted to $L_0^2(X, \nu)$.

Definition 8 (Turing-Type Topological Instability). The graphexon mean field game system undergoes a *Turing-type topological instability* if the uncoupled local dynamics (i.e., $c = 0$) are stable, the average state $\bar{m}_t$ remains stable, i.e., $\mathcal{A}_{cl}(1) < 0$, whereas e_t diverges such that $\sup_{\lambda \in \Sigma_0} \mathcal{A}_{cl}(\lambda) > 0$.

To analyze the Turing-type instability, the uncoupled dynamics (i.e., $c = 0$) must be stable, which requires $a_c < 0$ from the proof of Proposition 3.

Define $\Theta \triangleq a_c(a_c - \gamma) > 0$ and the function $\Phi(c) \triangleq (\gamma - a)c + kq\eta$. Define the bifurcation thresholds

$$c^+ \triangleq \frac{\Theta/\rho - kq\eta}{\gamma - a}, \quad c^- \triangleq \frac{-\Theta/\rho - kq\eta}{\gamma - a}.$$

Lemma 6. The closed-loop generator $\mathcal{A}_{cl}(\lambda)$ is weakly monotonic with respect to $\lambda \in [-\rho, \rho]$. Consequently, the Turing-type instability condition is equivalent to $\max\{\mathcal{A}_{cl}(\rho), \mathcal{A}_{cl}(-\rho)\} > 0$.

Proof. One can easily verify that the equation $\frac{d}{d\lambda} \mathcal{A}_{cl}(\lambda) = 0$ is equivalent to $k\psi(2a_\gamma c + k\psi) = 0$. Thus, the derivative $\frac{d}{d\lambda} \mathcal{A}_{cl}(\lambda)$ maintains the same sign across the interior of the spectral interval $(-\rho, \rho)$. Therefore, the supremum of $\mathcal{A}_{cl}(\lambda)$ on $[-\rho, \rho]$ is attained at the spectral boundaries $\lambda \in \{-\rho, \rho\}$, establishing the equivalence. $\square$

Define the spatial instability manifold $\mathcal{I} \triangleq \mathcal{I}_0 \setminus \{0\}$ where $\mathcal{I}_0 \triangleq \{c \in \mathbb{R} \mid \max\{\mathcal{A}_{cl}(\rho), \mathcal{A}_{cl}(-\rho)\} > 0\}$.

Theorem 4. $\mathcal{I}_0$ is determined by the following domains:

1. If $\gamma > a$, then $\mathcal{I}_0 = \left(-\infty, \min\left(\frac{\gamma}{\rho}, c^-\right)\right) \cup \left(\max\left(-\frac{\gamma}{\rho}, c^+\right), \infty\right)$.
2. If $\gamma < a$, then $\mathcal{I}_0 = \left(-\frac{\gamma}{\rho}, c^+\right) \cup \left(c^-, \frac{\gamma}{\rho}\right)$.
3. If $\gamma = a$ and
 - (a) $\rho k |q\eta| \leq \Theta$, then $\mathcal{I}_0 = \emptyset$.
 - (b) $\rho k q\eta > \Theta$, then $\mathcal{I}_0 = \left(-\frac{\gamma}{\rho}, \infty\right)$.
 - (c) $\rho k q\eta < -\Theta$, then $\mathcal{I}_0 = \left(-\infty, \frac{\gamma}{\rho}\right)$.

Proof. By Lemma 6, spatial instability is equivalent to $\mathcal{A}_{cl}(\rho) > 0$ or $\mathcal{A}_{cl}(-\rho) > 0$.

The condition $\mathcal{A}_{cl}(\rho) > 0$ reduces to $\gamma + \rho c > \sqrt{\Delta(\rho)}$, which is satisfied if and only if $c > -\gamma/\rho$ and $\Phi(c) > \Theta/\rho$. By symmetry, $\mathcal{A}_{cl}(-\rho) > 0$ is satisfied if and only if $c < \gamma/\rho$ and $-\Phi(c) > \Theta/\rho$.

Case 1 ($\gamma > a$). The union of the solution sets directly yields the two disjoint rays for $\mathcal{I}_0$.

Case 2 ($\gamma < a$). The conditions resolve to $c \in (-\gamma/\rho, c^+)$ and $c \in (c^-, \gamma/\rho)$, respectively. Since $\Theta > 0$ and $\gamma - a < 0$, it follows that $c^+ - c^- = \frac{2\Theta/\rho}{\gamma-a} < 0$, which guarantees $c^+ < c^-$. The union of these domains generates the stated disjoint bounded intervals.

Case 3 ($\gamma = a$). Here, $\Phi(c) = kq\eta$. The quadratic constraints simplify to $\rho k q\eta > \Theta$ and $-\rho k q\eta > \Theta$. Because $\Theta > 0$, at most one inequality holds. If $\rho k |q\eta| \leq \Theta$, both inequalities fail, yielding $\mathcal{I}_0 = \emptyset$. If $\rho k q\eta > \Theta$, the first inequality holds, restricting instability to $c > -\gamma/\rho$, while the second fails unconditionally, yielding $\mathcal{I}_0 = \left(-\frac{\gamma}{\rho}, \infty\right)$. If $\rho k q\eta < -\Theta$, the second inequality holds, restricting instability to $c \leq \gamma/\rho$, while the first fails unconditionally, yielding $\mathcal{I}_0 = \left(-\infty, \frac{\gamma}{\rho}\right)$. $\square$

Define the mean state stability manifold $\mathcal{S}_1 \triangleq \{c \in \mathbb{R} \mid \mathcal{A}_{cl}(1) < 0\}$. Define the mean state bifurcation threshold $c^* \triangleq \frac{\Theta - kq\eta}{\gamma - a}$.

Theorem 5. The mean state stability manifold $\mathcal{S}_1$ is determined by the following domains.

1. If $\gamma > a$, then $\mathcal{S}_1 = (-\infty, \max(-\gamma, c^*))$.
2. If $\gamma < a$, then $\mathcal{S}_1 = (-\infty, -\gamma) \cup (c^*, \infty)$.
3. If $\gamma = a$, the stability depends on the parameter gap
 - (a) If $kq\eta < \Theta$, then $\mathcal{S}_1 = \mathbb{R}$.
 - (b) If $kq\eta \geq \Theta$, then $\mathcal{S}_1 = (-\infty, -\gamma)$.

Consequently, a Turing-type instability occurs when $c \in \mathcal{S}_1 \cap \mathcal{I}$ and $a_c < 0$.

Proof. The strict inequality condition $\mathcal{A}_{cl}(1) < 0$ reduces to satisfying either $c < -\gamma$ or $\Phi(c) < \Theta$. Evaluating this combined logic condition across the three cases defined by the sign of $\gamma - a$ directly yields the specified disjoint intervals. $\square$

To discuss the specific properties brought by the continuous spectrum of $\mathcal{G}$, we now restrict our analysis to the affine graphexon framework established in Section 4. To isolate the spatial profile of the instability from its exponential growth, we define the normalized deviation field

$$\tilde{e}_t(\alpha) \triangleq \frac{e_t(\alpha)}{\|e_t\|_{L^2}}.$$

In classical models with compact operators, the existence of discrete L^2 -eigenfunctions dictates that $\tilde{e}_t$ converges strongly to a stationary spatial pattern (i.e., $\tilde{e}_t \rightarrow \phi$ in L^2 for some eigenfunction ϕ). In contrast, the graphexon operator $\mathcal{G}$ in Section 4 possesses a continuous spectrum on $L_0^2(X, \nu)$ and admits no discrete eigenfunctions. The following theorem demonstrates that $\tilde{e}_t$ converges weakly to zero.

Assume the Turing-type instability condition holds such that $\mu_{\max} \triangleq \sup_{\lambda \in \Sigma_0} \mathcal{A}_{cl}(\lambda) > 0$. Let $\lambda^* \in \Sigma_0$ be the dominant spectral mode evaluating to $\mu(\lambda^*) = \mu_{\max}$.

Theorem 6. Assume that the initial spatial deviation $e_0 \in L_0^2(\mathbb{T}^2, \nu)$ is not orthogonal to the unstable spectrum in the sense that

$$\|E((\lambda^* - \epsilon, \lambda^* + \epsilon))e_0\|_{L^2} > 0 \quad \text{for any } \epsilon > 0.$$

Then, for any test function $\phi \in L_0^2(\mathbb{T}^2, \nu)$, it holds that $\lim_{t \rightarrow \infty} \langle \tilde{e}_t, \phi \rangle_{L^2} = 0$.

Proof. Let E denote the projection-valued measure associated with the self-adjoint Margulis operator $\mathcal{G}$. Because $\mathcal{G}$ possesses a continuous spectrum on $L_0^2(\mathbb{T}^2, \nu)$, the corresponding scalar spectral measure $\mu_\phi(\cdot) \triangleq \langle E(\cdot)\phi, \phi \rangle_{L^2}$ is non-atomic for any test function $\phi \in L_0^2(\mathbb{T}^2, \nu)$.

Let $I_\delta = (\lambda^* - \delta, \lambda^* + \delta) \cap \Sigma_0$ be a δ -neighborhood centered at the dominant spectral mode. Applying the orthogonal decomposition of the projection-valued measure yields the inner product relation

$$\langle \tilde{e}_t, \phi \rangle_{L^2} = \langle \tilde{e}_t, E(I_\delta)\phi \rangle_{L^2} + \langle \tilde{e}_t, E(I_\delta^c)\phi \rangle_{L^2}.$$

The energy component distributed over the complement set I_δ^c grows at an exponential rate bounded below the maximal rate $\mu_{\max}$. By the spectral measure assumption on e_0 , the total norm $\|e_t\|_{L^2}$ is exponentially dominated by the modes within I_δ . Consequently, under the continuous normalization factor $\|e_t\|_{L^2}$, the inner product term corresponding to I_δ^c vanishes asymptotically as $t \rightarrow \infty$.

By the Cauchy-Schwarz inequality, the absolute value of the limit supremum is bounded by $\|E(I_\delta)\phi\|_{L^2}$. Because the spectral measure is non-atomic, taking the limits yields $\lim_{\delta \rightarrow 0} \|E(I_\delta)\phi\|_{L^2} = \|E(\{\lambda^*\})\phi\|_{L^2} = 0$. This establishes the weak convergence of the normalized field to zero. $\square$

6 Numerical simulation

In this section, we provide a numerical simulation to illustrate the asymptotic consensus behavior and the onset of spatial instability. The finite agent network is constructed using a discrete Margulis expander graph on a 40×40 grid ($N = 1600$), generated by eight affine transformations. The fixed system parameters are $a = -1$, $b = 1$, $q = 2$, $r = 1$, $\gamma = 0.5$, and $\Gamma = 2$.

Based on the algebraic conditions established in Theorem 3, we examine two representative drift coupling values: c_{stable} from the strict stability region, and c_{turing} from the Turing instability region. In the latter regime, the global mean is stable ($L(1) < 0$), but the spatial stability condition is violated on the continuous spectrum $[-\rho, \rho]$. Initial states x_0^α are generated using a low-frequency two-dimensional sine wave with superimposed Gaussian noise. The closed-loop dynamics are computed over the interval $t \in [0, 3]$.

Figure 2 illustrates the spatial distribution of the agent states. For $c = c_{stable}$, the initial spatial perturbations decay over time, leading to a flat, homogeneous field. Conversely, for $c = c_{turing}$, the spatial field develops a fragmented pattern. As time progresses, energy transfers to the high-frequency spectral modes determined by the discrete Margulis group action. The resulting distribution exhibits rapid spatial oscillations between neighboring grid points, which is consistent with the theoretical property that the unstable modes of the graphexon operator lack smooth L^2 eigenfunctions.

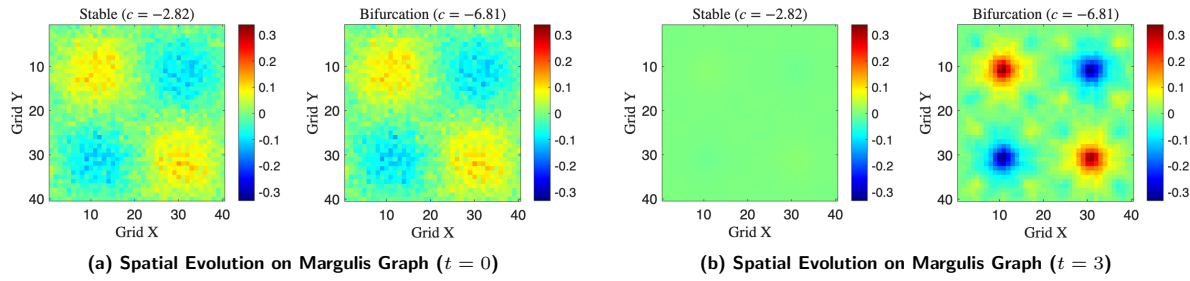


Figure 2: Spatial distribution of agent states on the 40×40 grid. Under c_{stable} , the system reaches a homogeneous spatial consensus. Under c_{turing} , the field fragments into high-frequency spatial oscillations.

7 Conclusion

This paper formulates and solves infinite-horizon discounted MFGs on expander graphs. The weak convergence of empirical graph measures to a limit graphexon measure is established and a scalar LQG MFG on the limit space is formulated. Furthermore, the exact algebraic stability boundaries and Turing-type topological instability is analyzed. A key finding is that the uniform spectral gap of expander graph sequence constructed in this paper is retained in the limit graph object. This guarantees a non-vanishing safe operating range for the coupling parameter c , confirming that system consensus and stabilization are attainable under sparse, constant-degree communication constraints.

Future research could incorporate the Laplace-Beltrami operator—induced by microscopic spatial Brownian motion—into the macroscopic forward-backward equations. This resulting reaction-diffusion-expander system will introduce high-frequency dissipation, maintaining spatial regularity for the large-scale multi-agent MFG systems.

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