

Numerical evaluation of distributed wireless networks at mega scale

Supplementary Material

R. Khalvandi, B. Sansò

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Numerical evaluation of distributed wireless networks at mega scale

Supplementary Material

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Abstract : This document contains the complete geometric derivations and analytical proofs supporting the results stated and used in the main manuscript. Results that are used directly in the main manuscript are restated here solely to make their proofs self-contained. Auxiliary theorems, intermediate inequalities, and supporting geometric constructions are included because they form part of the logical proof chain. Unless explicitly redefined, the notation and assumptions are the same as in the main manuscript.

Guide to the supplement

The supplement is organized in the same logical order as the analysis in the main manuscript. Section S1 establishes the lattice geometry and the interference bounds. Section S2 provides the discrete-to-continuous results used for the normalization constant and expected hop count. Section S3 proves the boundary and geometric bounds used to extend the central-node analysis to the network as a whole. The two figures below reproduce the geometric configurations needed to read the proofs independently of the main manuscript.

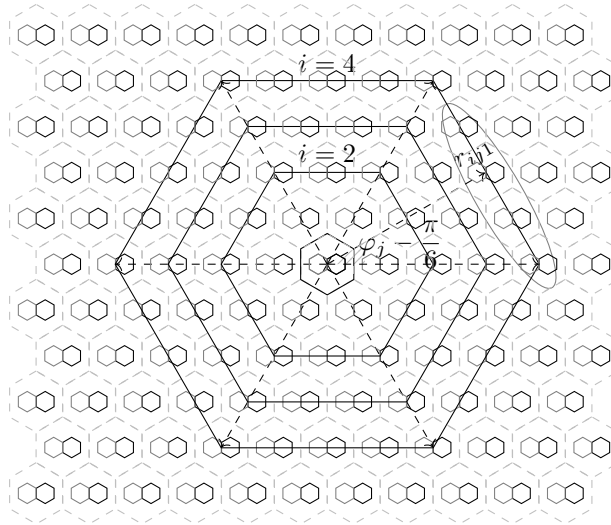


Figure S1: Reproduced from the main manuscript for completeness. Illustration of the interfering transmitters surrounding the central link and the sectorization of each interference ring in the hexagonal lattice. Concentric hexagons represent successive rings ($i = 2, 4, \dots$), and each ring is partitioned into 6 identical sectors. The dashed arrows indicate the distances r_{ij1} from the receiver of the central link to representative interferers in each ring. The shaded elliptical region highlights the range of angular deviations ($\varphi_j - \frac{\pi}{6}$) arising from the geometric offset of the hexagonal grid. This sectorized structure is used to derive closed-form expressions for the interference contributed by each ring in the analytical model.

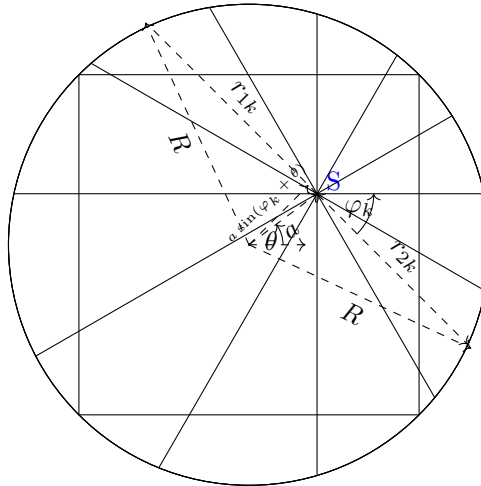


Figure S2: Reproduced from the main manuscript for completeness. Network sectoring for a general source node position. The source node S is located at the polar position (a, θ) with respect to the network center. Relative to S , the network can be divided into 6 pairs of sectors with radii (r_{1k}, r_{2k}) .

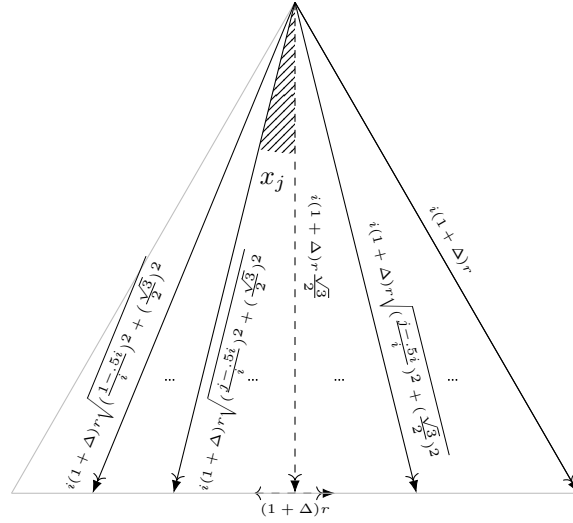


Figure S3: Distance computation for interferers in the i -th ring of a hexagonal lattice, shown over a single $\frac{\pi}{6}$ sector by symmetry. The ring index i determines the nominal radial distance $i(1 + \Delta)r$, while the index j enumerates nodes within the ring at different angular offsets on both sides of the symmetry axis. Nodes with $j < i/2$ and $j > i/2$ appear symmetrically and yield identical interference contributions. Distances are obtained by combining the vertical component $i(1 + \Delta)r \frac{\sqrt{3}}{2}$ with a horizontal offset proportional to $|j - \frac{i}{2}| (1 + \Delta)r$, where $1 \leq j \leq i$.

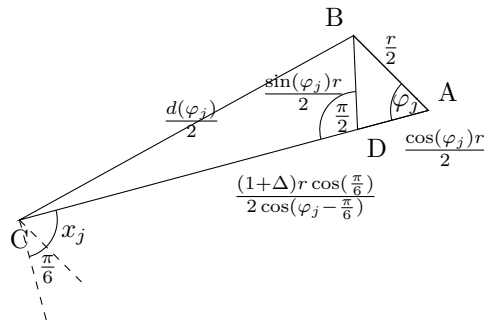


Figure S4: Geometric construction for computing distances to node j -th on the i -th ring within a hexagonal lattice sector. The angular index x_j (angle φ_j) parameterizes node positions, and the resulting distance $d(\varphi_j)$ is obtained from projections along the sector geometry.

S1 Geometric Characterization of Interfering Nodes and Interference Bounds

Theorem S1.1. Given that x_j is the angle between the perpendicular bisector of the sector and the line segment connecting the central node to node j (as shown in Fig. S3), where $-\frac{\pi}{6} < x_j \leq \frac{\pi}{6}$, and r is the distance between immediate neighbors, the distance between the central element to any element j is given by:

$$d_j^i = \frac{(1 + \Delta)ir \cos(\frac{\pi}{6})}{\cos(x_j)}. \quad (\text{S1})$$

Proof. The distance between the central element and element j , in a sector from ring i^{th} , defined as d_j^i , as depicted in Fig. S3, given by

$$d_j^i = i(1 + \Delta)r \sqrt{\left(\frac{j - 0.5i}{i}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}, \quad 1 \leq j \leq i. \quad (\text{S2})$$

Considering x_j as the angle shown in Fig. S3, we have

$$\cos(x_j) = \frac{i(1 + \Delta)r \frac{\sqrt{3}}{2}}{i(1 + \Delta)r \sqrt{\left(\frac{j - 0.5i}{i}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} = \frac{\sqrt{3}}{2 \sqrt{\left(\frac{j - 0.5i}{i}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}. \quad (\text{S3})$$

By substituting $\cos(x_j)$ from Equation (S3) into Equation (S2), we obtain:

$$d_j^i = \frac{(1 + \Delta)ir \cos(\frac{\pi}{6})}{\cos(x_j)}. \quad (\text{S4})$$

□

Theorem S1.2 (Transmitters-Receiver Distance). For a node j in the first sector, where $0 < \varphi \leq \frac{\pi}{3}$, and the i^{th} sector, the distance is calculated as:

$$r_j^i = r \sqrt{\left(\frac{i(1 + \Delta) \cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})} - \cos(\varphi_j)\right)^2 + \sin^2(\varphi)}. \quad (\text{S5})$$

Proof. From Theorem S1.1, if x_j is the angle between the perpendicular bisector of the sector and the segment connecting the central node to node j , where $-\frac{\pi}{6} < x_j \leq \frac{\pi}{6}$, the distance d_j^i is given by:

$$d_j^i = \frac{(1 + \Delta)ir \cos(\frac{\pi}{6})}{\cos(x_j)}. \quad (\text{S6})$$

For $i > 1$, as depicted in Fig. S4, we have:

$$\frac{r_j^i}{2} = \sqrt{\left(\frac{d_j^i}{2} - \frac{r}{2} \cos(\varphi_j)\right)^2 + \left(\frac{r}{2} \sin(\varphi)\right)^2}.$$

According to Fig. S4, the relationship between x_j and φ_j for $0 \leq \varphi_j \leq \frac{\pi}{3}$ is $x_j = \varphi_j - \frac{\pi}{6}$. Substituting d_j^i from Equation (S6), the distance r_j^i of any interfering transmitter node from the center receiver, for $0 < \varphi_j \leq \frac{\pi}{3}$, is

$$r_j^i = r \sqrt{\left(\frac{i(1 + \Delta) \cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})} - \cos(\varphi_j)\right)^2 + \sin^2(\varphi)}. \quad (\text{S7})$$

□

Theorem S1.3 (Distance to Interfering Transmitters). Consider a symmetric network arranged in a hexagonal lattice, as depicted in Fig. S1. For rings with $i \geq 2$, each ring is divided into six identical sectors, each containing i interfering transmitters. The distance from the central receiver to an interfering transmitter in ring i , sector k , at angle φ_j is given by:

$$r_{ijk} = r \psi(i, \varphi_j, k), \quad (\text{S8})$$

where

$$\psi(i, \varphi_j, k) = \left[\left(\frac{i(1 + \Delta) \cos(\frac{\pi}{6})}{\cos(\varphi_j - \frac{\pi}{6})} - \cos(\varphi_j + \frac{(k-1)\pi}{3}) \right)^2 + \sin^2(\varphi_j + \frac{(k-1)\pi}{3}) \right]^{\frac{1}{2}}. \quad (\text{S9})$$

Proof. In sector $k = 1$, the angle φ_j is determined by:

$$\cos(\varphi_j) = \frac{i - j}{2i \sqrt{(\frac{j - 0.5i}{i})^2 + (\frac{\sqrt{3}}{2})^2}}, \quad j = 1, 2, \dots, i. \quad (\text{S10})$$

For other sectors $k > 1$ with $\frac{(k-1)\pi}{3} \leq \varphi \leq \frac{k\pi}{3}$, the expressions are obtained by applying a phase shift of $\frac{(k-1)\pi}{3}$ to $\cos(\varphi_j)$ and $\sin(\varphi_j)$ in Equation (S5) of Theorem S1.2, and by factoring out $\psi(i, \varphi_j, k)$ as Equation (S9), Equation (S8) is achieved. $\square$

Theorem S1.4 ($\psi(i, \varphi, k)$ approximation). For any sectors of $1 \leq k \leq 6$ and $\varphi \in [0, \frac{\pi}{3}]$ the lower bound of the $\psi(i, \varphi, k)$ could be written as

$$\psi(i, \varphi, k) \geq (i(1 + \Delta) - C_k) \frac{\cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})}, \quad (\text{S11})$$

where $C_k \in \{\frac{1}{\sqrt{3}} + \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{\sqrt{3}} + \frac{1}{2}\}$.

Proof. We start with the expression:

$$\frac{\cos(\varphi - \frac{\pi}{6}) \cos(\varphi + \frac{(k-1)\pi}{3})}{\cos(\frac{\pi}{6})} = \frac{\cos(2\varphi - \frac{\pi}{6} + \frac{(k-1)\pi}{3}) + \cos(\frac{\pi}{6} + \frac{(k-1)\pi}{3})}{2 \cos(\frac{\pi}{6})}.$$

As shown in Table S1, the maximum of the expression for any sector is presented as C_k . Thus, we can write:

$$i(1 + \Delta) \frac{\cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})} - \cos(\varphi + \frac{(k-1)\pi}{3}) \geq i(1 + \Delta) \frac{\cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})} - C_k \frac{\cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})}.$$

Squaring both sides of the inequality, adding $\sin^2(\varphi)$, and then taking the square root, we obtain Equation (S12):

$$\begin{aligned} \psi(i, \varphi, k) &= (i(1 + \Delta) \frac{\cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})} - \cos(\varphi + \frac{(k-1)\pi}{3}))^2 + \sin^2(\varphi + \frac{(k-1)\pi}{3}) \\ &\geq ((i(1 + \Delta) - C_k)^2 \frac{\cos^2(\frac{\pi}{6})}{\cos^2(\varphi - \frac{\pi}{6})} + \sin^2(\varphi + \frac{(k-1)\pi}{3}))^{\frac{1}{2}} \\ &\geq (i(1 + \Delta) - C_k) \frac{\cos(\frac{\pi}{6})}{\cos(\varphi - \frac{\pi}{6})}. \quad (\text{S12}) \end{aligned}$$

$\square$

Table S1: Function extreme values over sectors.

K	Equation per sector	Max value (C_k)	Min value
1	$\frac{\cos(2\varphi - \frac{\pi}{6}) + \cos(\frac{\pi}{6})}{2 \cos(\frac{\pi}{6})}$	$\frac{1}{\sqrt{3}} + \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{\cos(2\varphi + \frac{\pi}{6}) + \cos(\frac{\pi}{2})}{2 \cos(\frac{\pi}{6})}$	$\frac{1}{2}$	$-\frac{1}{2}$
3	$\frac{\cos(2\varphi + \frac{\pi}{2}) + \cos(\frac{5\pi}{6})}{2 \cos(\frac{\pi}{6})}$	$-\frac{1}{2}$	$-(\frac{1}{\sqrt{3}} + \frac{1}{2})$
4	$\frac{\cos(2\varphi + \frac{5\pi}{6}) + \cos(\frac{7\pi}{6})}{2 \cos(\frac{\pi}{6})}$	$-\frac{1}{2}$	$-(\frac{1}{\sqrt{3}} + \frac{1}{2})$
5	$\frac{\cos(2\varphi + \frac{7\pi}{6}) + \cos(\frac{3\pi}{2})}{2 \cos(\frac{\pi}{6})}$	$\frac{1}{2}$	$-\frac{1}{2}$
6	$\frac{\cos(2\varphi + \frac{3\pi}{2}) + \cos(\frac{11\pi}{6})}{2 \cos(\frac{\pi}{6})}$	$\frac{1}{\sqrt{3}} + \frac{1}{2}$	$\frac{1}{2}$

Theorem S1.5 (Upper Bound of Interference). For a symmetric network arranged in a hexagonal lattice, the total interference from all rings beyond a threshold $i > \frac{\kappa}{1 + \Delta}$, where $\kappa \gg 1$, has an upper bound given by:

$$I(i > \frac{\kappa}{1 + \Delta}) < \frac{3Pg_0Q(\alpha_p)}{\pi r^{\alpha_p}} \sum_{k=1}^6 \chi(\Delta, \alpha_p, \kappa, C_k), \quad (\text{S13})$$

with:

$$\chi(\Delta, \alpha_p, \kappa, C_k) = \frac{Q(\alpha_p)(1 + \Delta)^{-1}(\alpha_p - 1)(\left\lceil \frac{\kappa}{1 + \Delta} \right\rceil - 0.5) - C_k(1 + \Delta)^{-2}}{\left[(1 + \Delta)(\left\lceil \frac{\kappa}{1 + \Delta} \right\rceil - 0.5) - C_k \right]^{\alpha_p - 1} (\alpha_p - 2)(\alpha_p - 1)}. \quad (\text{S14})$$

where:

- C_k are constants specific to each sector k provided in [Table S1](#),
- $Q(\alpha_p) = \int_0^{\frac{\pi}{3}} \frac{\cos^{\alpha_p}(\varphi - \frac{\pi}{6})}{\cos^{\alpha_p}(\frac{\pi}{6})} d\varphi$.

Proof. For $i \gg 1$, the sum over j and k in the interference $I(i)$ can be approximated using the Riemann integral. The discrete sum:

$$\frac{\pi}{3i} \sum_{k=1}^6 \sum_{j=1}^i \frac{i}{\psi(i, \varphi_j, k)^{\alpha_p}} \approx \sum_{k=1}^6 \int_0^{\frac{\pi}{3}} \frac{i}{\psi(i, \varphi, k)^{\alpha_p}} d\varphi, \quad (\text{S15})$$

where $\psi(i, \varphi, k)$ is as defined in [Equation \(S9\)](#).

According to geometric considerations (details in [Equation \(S11\)](#)), for the k^{th} sector, $\psi(i, \varphi, k)$ has a lower bound:

$$\psi(i, \varphi, k) \geq \frac{i(1 + \Delta) - C_k}{\cos(\frac{\pi}{6})} \cos(\varphi - \frac{\pi}{6}). \quad (\text{S16})$$

Substituting the lower bound into the integral, we obtain:

$$\begin{aligned} I(i) &< \frac{3Pg_0}{\pi r^{\alpha_p}} \sum_{k=1}^6 \int_0^{\frac{\pi}{3}} \frac{i}{\left[\frac{i(1 + \Delta) - C_k}{\cos(\frac{\pi}{6})} \cos(\varphi - \frac{\pi}{6}) \right]^{\alpha_p}} d\varphi \\ &= \frac{3Pg_0}{\pi r^{\alpha_p}} \sum_{k=1}^6 \frac{i}{[i(1 + \Delta) - C_k]^{\alpha_p}} \int_0^{\frac{\pi}{3}} \frac{\cos^{\alpha_p}(\varphi - \frac{\pi}{6})}{\cos^{\alpha_p}(\frac{\pi}{6})} d\varphi \end{aligned}$$

$$= \frac{3Pg_0Q(\alpha_p)}{\pi r^{\alpha_p}} \sum_{k=1}^6 \frac{i}{[i(1+\Delta) - C_k]^{\alpha_p}}, \quad (\text{S17})$$

where $Q(\alpha_p)$ is defined as:

$$Q(\alpha_p) = \int_0^{\frac{\pi}{3}} \frac{\cos^{\alpha_p}(\varphi - \frac{\pi}{6})}{\cos^{\alpha_p}(\frac{\pi}{6})} d\varphi. \quad (\text{S18})$$

To compute the upper bound of the total interference from rings beyond $i > \frac{\kappa}{1+\Delta}$, we consider the sum:

$$I(i > \frac{\kappa}{1+\Delta}) < \frac{3Pg_0Q(\alpha_p)}{\pi r^{\alpha_p}} \sum_{i=\left\lceil \frac{\kappa}{1+\Delta} \right\rceil}^{\infty} \sum_{k=1}^6 \frac{i}{[i(1+\Delta) - C_k]^{\alpha_p}}. \quad (\text{S19})$$

We define $l(r)$ as:

$$l(r) = \sum_{k=1}^6 \frac{r}{[r(1+\Delta) - C_k]^{\alpha_p}}. \quad (\text{S20})$$

Since $l(r)$ is a convex decreasing function for $r \geq \left\lceil \frac{\kappa}{1+\Delta} \right\rceil$ and $\kappa \gg C_k$, we can use [Theorem S2.7](#) for the upper bound:

$$\sum_{i=\left\lceil \frac{\kappa}{1+\Delta} \right\rceil}^{\infty} l(i) \leq \int_{\left\lceil \frac{\kappa}{1+\Delta} \right\rceil}^{\infty} l(r) dr. \quad (\text{S21})$$

Integrating $l(r)$, we get:

$$\int l(r) dr = C_0 - \sum_{k=1}^6 \frac{(1+\Delta)(\alpha_p-1)r - C_k}{(1+\Delta)^2 [r(1+\Delta) - C_k]^{\alpha_p-1} (\alpha_p-2)(\alpha_p-1)}, \quad (\text{S22})$$

where C is the constant of integration.

Therefore, the upper bound of $I(i > \frac{\kappa}{1+\Delta})$ is:

$$I(i > \frac{\kappa}{1+\Delta}) < \frac{3Pg_0}{\pi r^{\alpha_p}} \sum_{k=1}^6 \chi(\Delta, \alpha_p, \kappa, C_k), \quad (\text{S23})$$

where $\chi(\Delta, \alpha_p, \kappa, C_k)$ is as defined in [Equation \(S14\)](#).

This completes the proof. $\square$

S2 Series to Integral approximation

Theorem S2.6. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous, convex, and strictly decreasing function on the closed interval $[a, b]$. Let M be the mean value of f over $[a, b]$, i.e.,

$$M = \frac{1}{b-a} \int_a^b f(x) dx.$$

Then, if c is a point in (a, b) such that $f(c) = M$, c is located in the left half of the interval $[a, b]$, i.e., $c \leq \frac{a+b}{2}$.

Proof. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous, convex, and strictly decreasing function, and let M be the mean value of f over $[a, b]$:

$$M = \frac{1}{b-a} \int_a^b f(x) dx.$$

Suppose $c \in (a, b)$ satisfies $f(c) = M$. We aim to show $c \leq \frac{a+b}{2}$. Assume, for contradiction, that $c > \frac{a+b}{2}$. Since f is strictly decreasing, $f(x) > M$ for $x < c$ and $f(x) < M$ for $x > c$. Given f is convex, Jensen's inequality implies:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx = M.$$

However, if $c > \frac{a+b}{2}$, the integral would be weighted more towards lower function values, contradicting $f(c) = M$. Thus, c must satisfy $c \leq \frac{a+b}{2}$. □

Theorem S2.7. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous, convex, and strictly decreasing function on the closed interval $[a, b]$. Let M be the mean value of f over $[a, b]$, i.e.,

$$M = \frac{1}{b-a} \int_a^b f(x) dx.$$

Then, if c is a point in (a, b) such that $f(c) = M$, c is located in the left half of the interval $[a, b]$, i.e., $c \leq \frac{a+b}{2}$. For a convex decreasing function $f(x)$, $\sum_{i=a}^{\infty} f(i) \leq \int_{a-\frac{1}{2}}^{\infty} f(x) dx$

Proof. Based on [Theorem S2.6](#), for any interval $[i - \frac{1}{2}, i + \frac{1}{2}]$, $f(i) \leq \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} f(x) dx$. Therefore

$$\sum_{i=a}^{\infty} f(i) \leq \sum_{i=a}^{\infty} \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} f(x) dx \leq \int_{a-\frac{1}{2}}^{\infty} f(x) dx$$

□

Theorem S2.8 (P_0 series to integral). For $\alpha > 2$,

$$\sum_{i=1}^{\sqrt{\frac{n}{\pi}}} \frac{i}{(i+\beta)^\alpha} \geq \int_0^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr - \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha \beta^{\alpha-1}}.$$

Proof. Considering $f(r) = \frac{r}{(r+\beta)^\alpha}$, $\frac{d}{dr} f(r)$ is given by

$$\frac{d}{dr} f(r) = \frac{\beta + (1-\alpha)r}{(r+\beta)^{\alpha+1}}.$$

$f(r)$ has only one maximum at $r = \frac{\beta}{\alpha-1}$. For any $i \leq \left\lfloor \frac{\beta}{\alpha-1} \right\rfloor$:

$$\sum_{i=1}^{\left\lfloor \frac{\beta}{\alpha-1} \right\rfloor} f(i) = \sum_{i=1}^{\left\lfloor \frac{\beta}{\alpha-1} \right\rfloor} \frac{i}{(i+\beta)^\alpha} \geq \int_0^{\left\lfloor \frac{\beta}{\alpha-1} \right\rfloor} \frac{r}{(r+\beta)^\alpha} dr,$$

and for $i \geq \left\lfloor \frac{\beta}{\alpha-1} \right\rfloor$:

$$\sum_{i=\left\lfloor \frac{\beta}{\alpha-1} \right\rfloor+1}^{\sqrt{\frac{n}{\pi}}} f(i) = \sum_{i=\left\lfloor \frac{\beta}{\alpha-1} \right\rfloor+1}^{\sqrt{\frac{n}{\pi}}} \frac{i}{(i+\beta)^\alpha} \geq \int_{\left\lfloor \frac{\beta}{\alpha-1} \right\rfloor}^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr.$$

By summing up both cases, we have

$$\sum_{i=1}^{\lfloor \frac{\sqrt{n}}{\pi} \rfloor} \frac{i}{(i+\beta)^\alpha} \geq \int_0^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr - \int_{\lfloor \frac{\beta}{\alpha-1} \rfloor}^{\lfloor \frac{\beta}{\alpha-1} \rfloor + 1} \frac{r}{(r+\beta)^\alpha} dr.$$

Furthermore, $\int f(r) dr$ computed as

$$\int \frac{r}{(r+\beta)^\alpha} dr = -\frac{(r+\beta)^{1-\alpha}((\alpha-1)r+\beta)}{(\alpha-1)(\alpha-2)} + C, \quad (\text{S24})$$

that by using Equation (S24), we have

$$\int_{\lfloor \frac{\beta}{\alpha-1} \rfloor}^{\lfloor \frac{\beta}{\alpha-1} \rfloor + 1} \frac{r}{(r+\beta)^\alpha} dr \leq \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha \beta^{\alpha-1}}.$$

Consequently,

$$\sum_{i=1}^{\lfloor \frac{\sqrt{n}}{\pi} \rfloor} \frac{i}{(i+\beta)^\alpha} \geq \left(\int_0^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr - \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha \beta^{\alpha-1}} \right). \quad (\text{S25})$$

□

Theorem S2.9 (Approximate $\mathbf{E}(h)$ series). For $\alpha > 2$,

$$\sum_{i=2}^{\lfloor \frac{\sqrt{n}}{\pi} \rfloor} \int_0^{\frac{\pi}{6}} \frac{i^2}{\left(\frac{i \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi \leq \int_2^{\sqrt{\frac{n}{\pi}}} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr + \frac{(\alpha-2)^\alpha (2\sqrt{3} + \frac{4\pi}{3})}{3\beta^{\alpha-2} \alpha^\alpha}. \quad (\text{S26})$$

Proof. Considering the function $f(r, \varphi) = \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha}$,

$$\frac{df(r, \varphi)}{dr} = \frac{r((2-\alpha) \frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + 2\beta)}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^{\alpha+1}}$$

The maximum of $f(r, \varphi)$ occurs at $r_{\max} = \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})}$. $f(r, \varphi)$ is increasing for $r \leq \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})}$ and decreasing for higher values of r . We can thus write:

$$\sum_{i=2}^{\lfloor \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor} \int_0^{\frac{\pi}{6}} \frac{i^2}{\left(\frac{i \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi \leq \int_2^{\lfloor \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor + 1} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr, \quad (\text{S27})$$

and for $r \geq \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})}$, we have:

$$\sum_{\lfloor \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor + 1}^{\lfloor \frac{\sqrt{n}}{\pi} \rfloor} \int_0^{\frac{\pi}{6}} \frac{i^2}{\left(\frac{i \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi \leq \int_{\lfloor \frac{2\beta \cos(\varphi)}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor}^{\sqrt{\frac{n}{\pi}}} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr. \quad (\text{S28})$$

Summing up Equations (S27) and (S28), we get:

$$\sum_{i=2}^{\lfloor \frac{\sqrt{n}}{\pi} \rfloor} \int_0^{\frac{\pi}{6}} \frac{i^2}{\left(\frac{i \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi \leq$$

$$\left(\int_2^{\sqrt{\frac{n}{\pi}}} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr + \int_{\lfloor \frac{2\beta \cos(\frac{\pi}{6})}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor}^{\lfloor \frac{2\beta \cos(\frac{\pi}{6})}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor + 1} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr \right). \quad (\text{S29})$$

We then have:

$$\begin{aligned} \int_{\lfloor \frac{2\beta \cos(\frac{\pi}{6})}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor}^{\lfloor \frac{2\beta \cos(\frac{\pi}{6})}{(\alpha-2) \cos(\frac{\pi}{6})} \rfloor + 1} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr \\ \leq \int_0^{\frac{\pi}{6}} \frac{4\beta^2 \cos^2(\varphi) / \cos^2(\frac{\pi}{6})}{\left(\frac{2\beta}{\alpha-2} + \beta \right)^\alpha} d\varphi = \frac{(\alpha-2)^\alpha (2\sqrt{3} + \frac{4\pi}{3})}{3\beta^{\alpha-2} \alpha^\alpha}. \end{aligned} \quad (\text{S30})$$

It means the upper bound of the $\mathbf{E}(h)$ series could be given the below integral as

$$\sum_{i=2}^{\sqrt{\frac{n}{\pi}}} \int_0^{\frac{\pi}{6}} \frac{i^2}{\left(\frac{i \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi \leq \int_2^{\sqrt{\frac{n}{\pi}}} \int_0^{\frac{\pi}{6}} \frac{r^2}{\left(\frac{r \cos(\frac{\pi}{6})}{\cos(\varphi)} + \beta \right)^\alpha} d\varphi dr + \frac{(\alpha-2)^\alpha (2\sqrt{3} + \frac{4\pi}{3})}{3\beta^{\alpha-2} \alpha^\alpha}. \quad (\text{S31})$$

□

S3 Extreme values and upper bounds related to $\mathbf{E}(h)$

Theorem S3.10. By moving away from the corner of the network to the center on the line of $x = y$ as Figure A.14 of the main manuscript so that the minimum distance of the source node from all corners considered as $\kappa\beta$, then $\frac{P_0^{min}}{P_0^{\kappa\beta}}$ is given by

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} \geq \begin{cases} 1 - \frac{1 + \frac{1}{\pi}}{\kappa+1}, & \text{if } \alpha \geq 3 \\ 1 - \frac{0.75\alpha - 1.25 + \frac{1}{\pi}}{(\kappa+1)^{\alpha-2}}, & \text{if } \alpha \approx 3, \alpha < 3. \end{cases} \quad (\text{S32})$$

Proof.

The integral for P_0 can be rewritten as:

$$\begin{aligned} \int_1^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr &= \int_1^{\kappa\beta} \frac{r}{(r+\beta)^\alpha} dr + \int_{\kappa\beta}^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr \\ &= \frac{1}{(\alpha-1)(\alpha-2)} \left(\frac{\alpha-1+\beta}{(1+\beta)^{\alpha-1}} - \frac{(\alpha-1)\sqrt{\frac{n}{\pi}} + \beta}{(\sqrt{\frac{n}{\pi}} + \beta)^{\alpha-1}} \right) \end{aligned}$$

If we consider the $P_0^{\kappa\beta}$ as the P_0 constant for the point on the line $x = y$ that is at least $\kappa\beta$ away from all network corners, and call A as the effective area for the integral, we should have $P_0(\oint_A P(a) da + C(\alpha, \beta)) = 1$. Here, $P(a)$ is the interaction probability, and $C(\alpha, \beta)$ is the offset value of converting series to integral. On the other hand, for the P_0^{min} the area of integral is the $\int_0^{\sqrt{\frac{n}{\pi}}} \frac{r}{(r+\beta)^\alpha} dr \approx \int_0^\infty \frac{r}{(r+\beta)^\alpha} dr$. Therefore, we can write,

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} = \frac{\oint_A P(a) da + C(\alpha, \beta)}{\int_0^\infty \frac{r}{(r+\beta)^\alpha} dr + C(\alpha, \beta)}$$

As depicted in Figure A.14 of the main manuscript, the integral for the effective area of A could be written so that

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} = \frac{\frac{1}{\beta^{\alpha-2}} - \frac{(\alpha-2)(\alpha-1)^\alpha}{\alpha^\alpha \beta^{\alpha-1}} - \frac{3((\alpha-1)\kappa\beta + \beta)}{4((\kappa+1)\beta)^{\alpha-1}} + \frac{1}{\pi} \int_{\kappa\beta}^\infty \frac{\arcsin(\frac{\kappa\beta}{r})}{\pi(r+\beta)^\alpha} dr}{\frac{1}{\beta^{\alpha-2}} - \frac{(\alpha-2)(\alpha-1)^\alpha}{\alpha^\alpha \beta^{\alpha-1}}}$$

$$\approx 1 - \frac{3((\alpha - 1)\kappa + 1)}{4(\kappa + 1)^{\alpha-1}} + \frac{\beta^{\alpha-2}}{\pi} \int_{\kappa\beta}^{\infty} \frac{\arcsin(\frac{\kappa\beta}{r})r}{\pi(r + \beta)^{\alpha}} dr.$$

since always $\frac{P_0^{min}}{P_0^{\kappa\beta}} \leq 1$, always:

$$\frac{3((\alpha - 1)\kappa + 1)}{4(\kappa + 1)^{\alpha-1}} \geq \frac{\beta^{\alpha-2}}{\pi} \int_{\kappa\beta}^{\infty} \frac{\arcsin(\frac{\kappa\beta}{r})r}{\pi(r + \beta)^{\alpha}} dr.$$

Therefore, we can write:

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} = 1 - (1 - C) \frac{3((\alpha - 1)\kappa + 1)}{4(\kappa + 1)^{\alpha-1}}$$

where $0 \leq C \leq 1$. Now, we consider $f(\alpha) = \frac{3((\alpha-1)\kappa+1)}{4(\kappa+1)^{\alpha-1}}$, if we show $\frac{df}{d\alpha} \leq 0$ for all $\alpha \geq 3$, then by increasing α , $\frac{P_0^{min}}{P_0^{\kappa\beta}}$ increases. We have

$$\frac{df}{d\alpha} = \frac{3((1 - (\alpha - 1)^2)\kappa^2 + 2\kappa - \alpha + 1)}{4(\kappa + 1)^{\alpha}},$$

that shows that for any $\kappa \geq 1$ and $\alpha \geq 3$, $\frac{P_0^{min}}{P_0^{\kappa\beta}}$ is increasing with respect to α . Therefore, $\left. \frac{P_0^{min}}{P_0^{\kappa\beta}} \right|_{\alpha=3}$ is a lower bound for all $\alpha \geq 3$. Furthermore,

$$\frac{1}{\pi} \int_{\kappa\beta}^{\infty} \frac{\arcsin(\frac{\kappa\beta}{r})r}{\pi(r + \beta)^3} dr = \frac{\kappa^2}{\pi\beta} \left(\left(\frac{-\frac{\kappa \arcsin(\frac{\kappa x}{|x+\kappa|} + \frac{1}{|x+\kappa|})}{\sqrt{\kappa^2-1-\kappa^2}\sqrt{\kappa^2-1}} - \frac{\sqrt{1-\kappa^2}}{-\kappa^2 x + x - \kappa^3 + \kappa}}{2} - \frac{\arcsin(x)}{2(x + \kappa)^2} \right) \right|_{x=0}^{x=1} \right).$$

Could be shown the result of the integral if $\kappa \gg 1$, is approximated by

$$\frac{\beta}{\pi} \int_{\kappa\beta}^{\infty} \frac{\arcsin(\frac{\kappa\beta}{r})r}{(r + \beta)^3} dr = \frac{(0.5 - \frac{1}{\pi})\kappa - \frac{1}{\pi}}{(\kappa + 1)\kappa},$$

that if $\alpha \geq 3$;

$$\begin{aligned} \frac{P_0^{min}}{P_0^{\kappa\beta}} &\geq \frac{3((\alpha - 1)\kappa + 1)}{4(\kappa + 1)^{\alpha-1}} + \frac{(0.5 - \frac{1}{\pi})\kappa - \frac{1}{\pi}}{(\kappa + 1)\kappa} \\ &\geq 1 - \frac{3(2\kappa + 1)}{4(\kappa + 1)^2} + \frac{(0.5 - \frac{1}{\pi})\kappa - \frac{1}{\pi}}{(\kappa + 1)^2} \geq 1 - \frac{(1 + \frac{1}{\pi})\kappa + \frac{1}{\pi} + .75}{(\kappa + 1)^2}. \end{aligned}$$

Finally, the simplified ratio is given by

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} \geq 1 - \frac{(1 + \frac{1}{\pi})}{\kappa + 1}. \quad (\text{S33})$$

For $\alpha \leq 3$, also could be written that

$$\begin{aligned} \frac{P_0^{min}}{P_0^{\kappa\beta}} &\geq 1 - \frac{3((\alpha - 1)\kappa + 1)}{4(\kappa + 1)^{\alpha-1}} + \frac{\beta^{\alpha-3}(0.5 - \frac{1}{\pi})\kappa - \frac{1}{\pi}}{(\kappa + 1)\kappa} \\ &\geq 1 - \frac{3((\alpha - 1)\kappa + 1)}{4(\kappa + 1)^{\alpha-1}} + \frac{\beta^{\alpha-3}(0.5 - \frac{1}{\pi})\kappa - \frac{1}{\pi}}{(\kappa + 1)^2} \geq \frac{P_0^{min}}{P_0} \\ &\geq 1 - \frac{3((\alpha - 1)\kappa + 1) + ((\kappa + 1)\beta)^{\alpha-3}((0.5 - \frac{1}{\pi})\kappa - \frac{1}{\pi})}{4(\kappa + 1)^{\alpha-1}}. \end{aligned}$$

If $\alpha \approx 3$, $((\kappa + 1)\beta)^{\alpha-3} \approx 1$, then

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} \geq 1 - \frac{.75\alpha - 1.25 + \frac{1}{\pi}}{(\kappa + 1)^{\alpha-2}}. \quad (\text{S34})$$

Hence, for both cases we have

$$\frac{P_0^{min}}{P_0^{\kappa\beta}} \geq \begin{cases} 1 - \frac{1 + \frac{1}{\pi}}{\kappa + 1}, & \text{if } \alpha \geq 3 \\ 1 - \frac{0.75\alpha - 1.25 + \frac{1}{\pi}}{(\kappa + 1)^{\alpha-2}}, & \text{if } \alpha \approx 3, \alpha < 3. \end{cases} \quad (\text{S35})$$

□

Theorem S3.11. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a positive, radially decreasing function, i.e., $f(r, \vartheta) > 0$ and $\frac{\partial f}{\partial r} < 0$ for all sufficiently large r . Consider the integral of f over a square S centered at the origin with side length L .

Then, the integral satisfies the following bounds:

$$\sum_{k=1}^{12} \int_{\theta_k}^{\theta_k + \frac{\pi}{6}} \int_0^{R_k} f(r, \vartheta) r dr d\vartheta \leq \int_S f(r, \vartheta) dA \leq \int_C f(r, \vartheta) dA,$$

where:

- C is a circle centered at the origin with radius $R = \frac{L}{\sqrt{\pi}}$,
- The lower bound is obtained by dividing S into 12 sectors, each spanning an angle of $\frac{\pi}{6}$. Specifically, 8 sectors have radius $R_k = \frac{L}{2}$ and 4 sectors have radius $R_k = \frac{L}{\sqrt{3}}$.

Proof. Consider the square S centered at the origin with side length L . We partition S into 12 sectors, each with an angular span of $\frac{\pi}{6}$ radians.

Lower Bound:

- Among these 12 sectors, 8 sectors can be inscribed within the square with a radius $R = \frac{L}{2}$.
- The remaining 4 sectors are inscribed with a radius $R = \frac{L}{\sqrt{3}}$.

Since $f(r, \vartheta)$ is positive and radially decreasing, the integral of f over each sector is maximized when the radius R_k is as large as possible within the sector. Therefore, the sum of the integrals over these 12 sectors provides a lower bound for the integral over the entire square:

$$\int_S f(r, \vartheta) dA \geq \sum_{k=1}^{12} \int_{\theta_k}^{\theta_k + \frac{\pi}{6}} \int_0^{R_k} f(r, \vartheta) r dr d\vartheta.$$

We ensure that the combined area of these sectors is entirely contained within the square. Since f is positive and decreases with r , the integral over these sectors provides a conservative (lower) estimate of the integral over the entire square.

Upper Bound:

- Approximate the square S with a circle C centered at the origin such that both have the same area. The radius of this circle is $R = \frac{L}{\sqrt{\pi}}$, since $\text{Area}(C) = \pi R^2 = L^2$.
- For any point inside the square but outside the circle C , the radial distance r satisfies $r > \frac{L}{\sqrt{\pi}}$. Given that $f(r, \vartheta)$ is decreasing with r , the value of f at these points is less than or equal to f at corresponding points inside C .

The integral of f over the square is bounded above by the integral of f over the circle since any area outside the circle but inside the square contributes less to the integral than the corresponding area inside the circle. Hence, integrating over the circle gives an upper bound for the integral over the square.

$$\int_S f(r, \vartheta) dA \leq \int_C f(r, \vartheta) dA.$$

Combining both bounds, we obtain:

$$\sum_{k=1}^{12} \int_{\theta_k}^{\theta_k + \frac{\pi}{6}} \int_0^{R_k} f(r, \vartheta) r dr d\vartheta \leq \int_S f(r, \vartheta) dA \leq \int_C f(r, \vartheta) dA. \quad \square$$

Theorem S3.12. [Geometric relations of r_k] If we consider a point inside a circle of radius R , and divide the circle into 6 pair of sectors each of sectors covering $\frac{\pi}{6}$ concerning that point, then for any arbitrary point inside the circle

$$\sum_{k=1}^6 r_{1k}^2 + r_{2k}^2 = 12R^2$$

Proof. Consider a point inside a circle with radius R , and divide the circle into 12 sectors (6 pairs), each covering $\frac{\pi}{6}$. We aim to show that the sum of the squares of the distances from the node to the boundaries of these sectors is $12R^2$.

First, consider the distances r_{1k} and r_{2k} in a pair of sectors. as shown in [Fig. S2](#). For r_{1k} and r_{2k} , we have:

$$(r_{1k} + a \cos(\varphi_k + \theta))^2 + a^2 \sin^2(\varphi_k + \theta) = R^2,$$

and

$$(r_{2k} - a \cos(\varphi_k + \theta))^2 + a^2 \sin^2(\varphi_k + \theta) = R^2$$

By summing these equations, we obtain,

$$r_{1k}^2 + r_{2k}^2 + 2a^2 + 2(r_{1k} - r_{2k}) \cos(\varphi_k + \theta) = 2R^2.$$

Due to symmetry, we have:

$$r_{2k} - a \cos(\varphi_k + \theta) = r_{1k} + a \cos(\varphi_k + \theta),$$

which implies

$$r_{2k} - r_{1k} = 2a \cos(\varphi_k + \theta).$$

Substituting this back into our sum, we get

$$r_{1k}^2 + r_{2k}^2 = 2R^2 + 2a^2 \cos(2(\varphi_k + \theta)).$$

Since we have 6 pairs of such sectors, the total sum is

$$\sum_{k=1}^6 r_{1k}^2 + r_{2k}^2 = \sum_{k=1}^6 2R^2 + 2a^2 \cos(2(\varphi_k + \theta)).$$

As $\varphi_k \in \{0, \dots, \frac{(k-1)\pi}{6}, \dots, \frac{5\pi}{6}\}$, the sum of the cosines over a complete cycle averages out to zero, regardless of the values of a and θ , leaving us with:

$$\sum_{k=1}^6 r_{1k}^2 + r_{2k}^2 = 12R^2. \quad \square$$

Relationship to the main manuscript

The theorem statements and final analytical expressions used directly in the main manuscript remain in their corresponding main-text sections. This supplement provides their complete proofs and retains all auxiliary results needed by those proofs. The final geometric identity is included as an additional supporting result; its role should be cited explicitly in the main manuscript if it is used there.