Les Cahiers du GERAD ISSN: 0711–2440

Approximating the strength of higher-level RLT inequalities in the first-level RLT space for polynomial 0-1 programs

I. Dan Dije, F. Djeumou Fomeni, L. C. Coelho

G–2025–68

October 2025

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2025
– Bibliothèque et Archives Canada, 2025

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Legal deposit – Bibliothèque et Archives nationales du Québec, 2025
– Library and Archives Canada, 2025

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Approximating the strength of higher-level RLT inequalities in the first-level RLT space for polynomial 0-1 programs

Ibrahim Dan Dije [a, b]

Franklin Djeumou Fomeni [a, b]

Leandro C. Coelho [b, c]

[a] *Department of Analytics, Operations and Information Technology, UQÀM, Montréal (Qc), Canada, H2X 3E7*

[b] *GERAD & CIRRELT, Montréal (Qc), Canada, H3T 1J4*

[c] *Department of Analytics, Operations and Information Technology & Canada Research Chair in Integrated Logistics, Université Laval, Québec (Qc), Canada, G1V 0A6*

dan_dije.ibrahim@courrier.uqam.ca
djeumou_fomeni.franklin@uqam.ca
leandro.coelho@fsa.ulaval.ca

October 2025
Les Cahiers du GERAD
G–2025–68

Abstract : The reformulation-linearization technique (RLT) is a well-established framework for generating hierarchies of linear programming (LP) relaxations for a wide range of optimization problems. Despite its theoretical power, the hierarchy is rarely applied beyond the first level, mainly due to the rapid growth in the number of variables and constraints. In this work, we introduce a framework that converts cutting planes derived from higher levels of the RLT hierarchy into cutting planes that involve only the variables from the first-level space. This is achieved by weakening higher-level inequalities into quadratic-space inequalities, using estimators of multilinear monomials obtained through set partitioning. These estimators are designed so that any inequality from RLT-k (for $k \geq 2$) can be reformulated as a valid inequality in quadratic space, i.e., within the scope of RLT-1. We also propose a polynomial-time separation algorithm to generate such inequalities. To assess the effectiveness of our approach, we implement it with $k = 3$ on three classes of test instances: the quadratic knapsack problem, the quadratic knapsack formulation of the dispersion problem, and the standard QLIB benchmark set. Across 523 tested instances, our computational results demonstrate that the proposed method substantially reduces the gap compared to benchmark techniques in the literature.

Keywords : Cutting planes; graph theory; mixed-integer nonlinear programming; polynomial optimization

Acknowledgements: This work was partly supported by the Canadian Natural Sciences and Engineering Research Council under grants 2021–03307 and 2025–03964. This support is greatly appreciated. We also thank the Digital Research Alliance of Canada for providing high-performance computing facilities.

1 Introduction

Consider a 0-1 polynomial programming problem given by:

$$\begin{aligned} \max\ & f(x) \\ \text{s.t.}\ & g^l(x) \geq 0,\ l = 1 \ldots L \\ & x_i \in \{0,1\},\ i \in N = \{1,\ldots,n\} \end{aligned} \tag{1}$$

where $f(x)$ could be a linear or polynomial objective function and $g^l(x)$ is a linear function. To solve those types of problems, exact methods begin with relaxation and then apply branching. Therefore, the tighter the relaxation, the easier it is to obtain an exact integer solution. Among the relaxation methods available, the Relaxation-Linearization Technique (RLT) provides a partial convex hull representation.

Following Sherali and Adams (1990), the reformulation-linearization technique (RLT) is a strategy for strengthening the linear relaxation of a broad class of optimization problems by reformulating and linearizing them through the introduction of auxiliary variables. Originally developed to address 0–1 polynomial programming problems (Adams and Sherali, 1986, 1990), such as those of the form (1), RLT has since been extended to a wide variety of settings, including continuous, integer, and global optimization problems (Sherali, 2007; Sherali and Tuncbilek, 1992; Sherali and Adams, 1998).

As the RLT level increases, the LP relaxation becomes stronger, but the number of auxiliary variables and constraints grows exponentially, which makes the resulting problem increasingly difficult to solve. Specifically, implementing RLT at a given level $k = 1,\ldots,n$ requires adding $\sum_{i=1}^{k}\binom{n}{i}$ distinct auxiliary variables (Sherali and Adams, 1990). Consequently, solving even the relaxation at relatively low levels can become computationally prohibitive for large-scale instances. As a result, most existing work in the literature restricts itself to first-level RLT, leaving the potential of higher levels largely unexplored. To address this, our approach aims to enhance first-level relaxations by generating inequalities valid in the space of higher-level RLT, then projecting them into the first-level space without introducing additional auxiliary variables. In effect, our method provides valid inequalities with only $\binom{n}{2} + n$ variables for any inequality originating from RLT-k.

A decade ago, Fomeni et al. (2015) introduced a recursive framework to approximate higher-level RLT inequalities within the first-level space. Their approach relies on progressively weakening inequalities from RLT-k to RLT-$(k-1)$, iterating until the inequalities lie in the RLT-1 space. While effective, this sequential reduction can incur substantial approximation losses and high separation complexity. Our contribution is to overcome these limitations by providing a direct mechanism for projecting inequalities from RLT-k to RLT-1. This one-step passage avoids the cumulative weakening inherent in recursive methods, thereby preserving the strength of higher-level inequalities while maintaining tractability. More precisely, our contribution starts with the definition of a graph structure for which we developed some set partitioning results. Given that any inequality in the space of RLT-k is the combination of multilinear monomials, we show that each of these monomial terms can be described by a unique sub-graph in the defined graph structure. Finally, we build estimators to each monomial terms based on the sub-graph structure by which it is described. This results in the introduction of a new family of inequalities valid for any 0-1 polynomial program.

To evaluate the effectiveness of the proposed framework, we conducted extensive computational experiments on three test classes: the quadratic knapsack problem, the quadratic knapsack formulation of the dispersion problem, and benchmark quadratic programming instances from the QLIB library (Furini et al., 2019). We compared our method against the RLT-1 relaxation as well as against inequalities generated by the recursive framework of Fomeni et al. (2015). The results show that our approach consistently produces tighter gaps than the latter, demonstrating its effectiveness.

The remainder of the paper is organized as follows. In Section 2, we present the literature review. In Section 3, we describe the estimators of the multilinear monomials that are generated from high space to the quadratic space. In Section 4, we give the class of k-star inequalities obtained from RLT-k.

Section 5 details the separation strategy. Finally, we present the results of extensive computational experiments in Section 6, followed by our conclusions in Section 7.

2 Literature review

In this section, we provide a literature review on the RLT techniques and cutting planes applied to polynomial 0–1 problems, continuous problems, and the existing techniques to improve RLT methods. The RLT given by Sherali and Adams (1990) in the context of a 0–1 linear program is presented as follows. Let us define a standard 0–1 LP as follows:

$$\min \quad \mathbf{c}^\top \mathbf{x} \tag{2}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{3}$$
$$\mathbf{x} \in \{0,1\}^n, \tag{4}$$

where $\mathbf{c} \in \mathbb{Q}^n$, $\mathbf{A} \in \mathbb{Q}^{m \times n}$, and $\mathbf{b} \in \mathbb{Q}^m$. The continuous relaxation of the problem is obtained by replacing the constraint $\mathbf{x} \in \{0,1\}^n$ by $\mathbf{x} \in [0,1]^n$.

Given an integer $1 \leq k \leq n$, the level-k RLT relaxation is constructed by first generating a set of valid inequalities with degree $k+1$ multilinear monomials, then linearizing each multilinear monomial in the resulting inequality by introducing new auxiliary variables. More formally, the reformulation phase can be detailed as follows:

- Construct a system of valid polynomial inequalities of degree $k+1$.
- Let $N = \{1, \ldots, n\}$, for any $S, T \subseteq N$ such that $S \cap T = \emptyset$, let us define the polynomial

$$J(S,T) = \prod_{i \in S} x_i \prod_{j \in T} (1 - x_j).$$

  By definition, $J(S,T) \geq 0$.
- For each linear inequality in the system (3), say $\mathbf{a}^\top \mathbf{x} \leq \beta$, and each disjoint pair $S, T \subseteq N$ satisfying $|S| + |T| = k$, the following inequality is a valid inequality:

$$J(S,T)(\beta - \mathbf{a}^\top \mathbf{x}) \geq 0.$$

The linearization phase then starts with the inequality $J(S,T)(\beta - \mathbf{a}^\top \mathbf{x}) \geq 0$, where one expands the left-hand side to obtain a weighted sum of distinct monomials. Each nonlinear monomial of the form $\prod_{i \in S} x_i$ is then replaced with a new binary variable x'_S, where $|S| \geq 2$ and $|S| \leq \min(k+1, n)$. Finally, each multilinear polynomial with degree larger than one is linearized by replacing it with the corresponding linear expression involving the new auxiliary variables.

Sherali and Adams (1994) proved that the strength of RLT relaxations increases as the level k increases. At level $k = n$, the relaxed polytope represents the convex hull of the initial 0–1 LP. The RLT is shown to be more efficient than some alternative methods, such as those due to Petersen et al. (1971) or Glover and Woolsey (1973). However, the number of variables and constraints grows exponentially with k, making it computationally expensive for large n. Despite this, RLT is recognized as one of the most effective approaches for solving 0–1 LPs. The RLT-1 is widely used, and many works provide methods for using cutting planes to enhance the RLT-1 relaxation. Among these, we mention the triangle inequalities Padberg (1989), which were used by Billionnet and Calmels (1996) in the quadratic knapsack problem, where they obtained a good upper bound for the optimal value of the linear relaxation. The triangle inequalities are used by Fomeni et al. (2015) to enhance the RLT-1 relaxation with an additional class of cutting planes. The semi-definite constraints can also be added to the RLT-1 relaxation (Parrilo, 2003) to make it tighter. Padberg (1989) studied the structure of the Boolean quadratic polytope and provided an important class of valid inequalities that define facets of the quadratic polytope. Then, the inequalities from Padberg (1989) could also be used to enhance the RLT-1 relaxation.

More related to our work, there is the framework developed a decade ago by Fomeni et al. (2015), which allows strengthening the RLT relaxation at any given level k with inequalities obtained from the space of RLT level k' for any $k' > k$. Indeed, within their framework, they demonstrate how to derive over- and under-estimators for any monomial of degree d using polynomials of degree $d-1$. Therefore, using these estimators, one can recursively estimate any inequality from the space of RLT level-k with inequalities that only use the variables from the space of RLT level-k' for any $k' < k$. It should be noted that the estimators provided by Fomeni et al. (2015) differ from those given by Tawarmalani and Sahinidis (2013), which are generally non-convex or non-concave.

More formally, for a multilinear monomial of the form $\prod_{i\in S} x_i$, where $|S| \geq 2$, Fomeni et al. (2015) presents a family of the best possible over- and under-estimators for monomials of degree $|S| - 1$, as follows.

Let $0 \leq x_i \leq 1$ for all $i \in S$. For any $T \subseteq S$ with $|T|$ odd, the following inequality holds:

$$\prod_{i\in S} x_i \leq \sum_{W\subset T} (-1)^{|W|} \prod_{i\in(S\setminus T)\cup W} x_i. \tag{5}$$

Similarly, for any $T \subseteq S$ with $|T|$ even, we have:

$$\prod_{i\in S} x_i \geq \sum_{W\subset T} (-1)^{|W|-1} \prod_{i\in(S\setminus T)\cup W} x_i. \tag{6}$$

Fomeni et al. (2015) prove these results by leveraging the multilinear nature of the variables and the bounds $x_i \in [0, 1]$. By expanding the products into sums of distinct monomials and systematically applying inequalities, the derived estimators are shown to respect the given constraints while tightly approximating the multilinear monomials. A special case of the resulting inequalities was even shown to be facets defining under certain conditions (Fomeni, 2016). Our proposed framework also uses estimators with the difference that our estimators are non-recursive, as they are obtained by projecting inequalities from the space of RLT level-k directly to the space of RLT-1. These estimators are obtained by exploiting a k-star-like graph.

3 Estimators for multilinear monomials

In this section, we provide the details of deriving quadratic space estimators that define lower-degree estimators of the multilinear monomials.

3.1 Set partition

In this section, we introduce a set partition to classify the estimators by using the notions of graphs and trees. Each partition will serve to categorize a family of estimators.

Let us consider an undirected complete graph $G = (N, E)$. We denote by $d(i)$ the degree of a node $i \in N$, and the set that contains the neighborhood of a node i is denoted by $N(i)$. For any subset, $S \subset N$, a path tree $T(S, E(S))$ is a graph induced by S, where all the vertices are connected in a single straight line.

Let $\mathcal{P}(S) = \{\mathcal{V} | \mathcal{V} \subseteq S\}$ be the power set of S, also let $F \subset S$ and $M \subset S$ be the sets of end nodes and internal nodes, respectively. Based on the structure of a path tree $T(S, E(S))$, we can define the collection ξ_k, with $k = 1, \ldots, 10$, that are mutually exclusive sets and such that $\bigcup_{k=1}^{10} \xi_k = \mathcal{P}(S)$. Each element of this collection is defined as follows.

- For $k = 1$:
$$\xi_1 = \Big\{ \mathcal{V} \subseteq S : \mathcal{V} \subseteq F \wedge |\mathcal{V}| = 1 \Big\}.$$
The set of subsets of S that contains only one endpoint of $T(S, E(S))$. The cardinality $|\xi_1| = 2$.
- For $k = 2$:
$$\xi_2 = \Big\{ \mathcal{V} \subseteq S : \mathcal{V} \subseteq F \wedge |\mathcal{V}| = 2 \Big\}.$$
The set of the subsets of S that contains all the endpoints of $T(S, E(S))$. The cardinality $|\xi_2| = 1$.
- For $k = 3$:
$$\xi_3 = \Big\{ \mathcal{V} \subseteq S : \mathcal{V} \subset M : |\mathcal{V}| = 1 \Big\}.$$
The set of subsets of S that contains at most one inner point of $T(S, E(S))$. The cardinality $|\xi_3| = |S| - 2$.
- For $k = 4$:
$$\xi_4 = \left\{ \mathcal{V} \subseteq S : \begin{array}{l} |\mathcal{V}| \geq 2, \\ |\mathcal{V} \cap F| = 1, \\ |\mathcal{V} \cap M| \geq 1, \\ \forall j \in \mathcal{V},\ \exists j' \in \mathcal{V} \text{ with } j' \in S(j) \end{array} \right\}.$$
The set of subsets S that contains at most one endpoint, and the possible combination of inner points that induce a subtree of $T(S, E(S))$. The cardinality $|\xi_4| = 2|S| - 4$.
- For $k = 5$:
$$\xi_5 = \Big\{ \mathcal{V} \subseteq S : |\mathcal{V}| = |S| \Big\}.$$
The set of subsets that contains all the elements of S. The cardinality $|\xi_5| = 1$.
- For $k = 6$:
$$\xi_6 = \Big\{ \mathcal{V} \subseteq S : |\mathcal{V}| = 0 \Big\}.$$
The set containing the empty set. The cardinality $|\xi_6| = 1$.
- For $k = 7$:
$$\xi_7 = \left\{ \mathcal{V} \subseteq S : \begin{array}{l} |\mathcal{V}| \geq 2, \\ |\mathcal{V} \cap F| = 1, \\ |\mathcal{V} \cap M| \geq 1, \\ \exists j, j' \in \mathcal{V} \text{ with } j' \notin S(j) \end{array} \right\}.$$
The set of subsets that contain at most one endpoint, and the possible combination of middle points, such that they do not induce a connected subtree of $T(S, E(S))$. The cardinality $|\xi_7| = 2^{|S|-1} - 2|S| + 2$.
- For $k = 8$:
$$\xi_8 = \left\{ \mathcal{V} \subseteq S : \begin{array}{l} \mathcal{V} \subseteq M, \\ |\mathcal{V}| \geq 2, \\ \forall j \in \mathcal{V},\ \exists j' \in \mathcal{V} \text{ with } j' \in S(j) \end{array} \right\}.$$
The set of all the subsets of the middle points that induce a path subtree of $T(S, E(S))$. The cardinality $|\xi_8| = ((|S| - 2)(|S| - 3))/2$.
- For $k = 9$:
$$\xi_9 = \Big\{ \mathcal{V} \subseteq S : \mathcal{V} \cap F = 2 \wedge (\mathcal{V} \setminus F) \subset M \Big\}.$$
The set of subsets that contains all the endpoints, and the possible combination of middle points, except M and $\emptyset$. The cardinality $|\xi_9| = 2^{|S|-2} - 2$.

- For $k = 10$:

$$\xi_{10} = \left\{ \mathcal{V} \subseteq S: \begin{array}{l} |\mathcal{V}| \geq 2, \\ \mathcal{V} \neq M, \\ \exists j, j' \in \mathcal{V}, \text{ with } j' \notin S(j) \end{array} \right\}.$$

The set of all the subsets of the middle points that do not induce a path subtree of $T(S, E(S))$. The cardinality,

$$|\xi_{10}| = \frac{2^{|S|-1} - (|S|-2)(|S|-3) - 2(|S|-2) - 2}{2}.$$

Theorem 1. *Consider an undirected complete graph $G(N, E)$, let $S \subset N$, $5 \leq |S| \leq |N| - 1$, and $T(S, E(S))$ be a path tree induced by S. Consider the defined subsets $\xi_1, \ldots, \xi_{10} \subseteq \mathcal{P}(S)$ we have that:*

$$\mathcal{P}(S) = \bigcup_{k=1}^{10} \xi_k \text{ and } \xi_k \cap \xi_{k'} = \emptyset, \ \forall k, k' \in \{1, \ldots, 10\} \text{ with } k \neq k'. \tag{7}$$

In other words, $\bigcup_{k=1}^{10} \xi_k$ forms a partition of the power set $\mathcal{P}(S)$.

Proof. The point is to prove firstly that the sets $\xi_k, k = 1, \ldots, 10$, are the mutually disjoint sets. For any $\mathcal{V} \subseteq S$, the following unique classification holds:

$$\mathcal{V} \in \begin{cases} \xi_1 & \text{if } |\mathcal{V}| = 1 \text{ and } \mathcal{V} \subseteq F, \\ \xi_2 & \text{if } \mathcal{V} = F, \\ \xi_3 & \text{if } |\mathcal{V}| = 1 \text{ and } \mathcal{V} \subseteq M, \\ \xi_4 & \text{if } |\mathcal{V} \cap F| = 1, \ |\mathcal{V} \cap M| \geq 1, \text{ and } \exists j, j' \in \mathcal{V}, \text{ with } j' \in S(j), \\ \xi_5 & \text{if } \mathcal{V} = S, \\ \xi_6 & \text{if } |\mathcal{V}| = 0, \\ \xi_7 & \text{if } |\mathcal{V} \cap F| = 1, \ |\mathcal{V} \cap M| \geq 1, \text{ and } \exists j, j' \in \mathcal{V}, \text{ with } j' \notin S(j), \\ \xi_8 & \text{if } \mathcal{V} \subseteq M, \ |\mathcal{V}| \geq 2, \text{ and } \exists j, j' \in \mathcal{V}, \text{ with } j' \in S(j), \\ \xi_9 & \text{if } \mathcal{V} \cap F = 2 \wedge (\mathcal{V} \setminus F) \subset M, \\ \xi_{10} & \text{if } \mathcal{V} \subseteq M, \ |\mathcal{V}| \geq 2, \text{ and } \exists j, j' \in \mathcal{V}, \text{ with } j' \notin S(j). \end{cases} \tag{8}$$

From (8), we have no overlaps; therefore, each $\xi_k, \ k = 1, \ldots, 10$, satisfies exactly one condition, which shows mutual exclusivity. We have now to prove that the union of $\xi_k, \ k = 1, \ldots, 10$ covers $\mathcal{P}(S)$. From (8), we have $\forall \mathcal{V} \in \mathcal{P}(S)$ there exists only one $\xi_k, \ k = 1, \ldots, 10$, such that $\mathcal{V} \in \xi_k$. Let us now show that, for any $\mathcal{V} \in \xi_k, \ k = 1, \ldots, 10$, we have $\mathcal{V} \in \mathcal{P}(S)$. By the structure of the path tree $T(S, E(S))$, the set $\mathcal{V}$ belongs to one of the following:

- Trivial subsets:
  - if $\mathcal{V} \in \xi_5$, then $\mathcal{V} = S \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,
  - if $\mathcal{V} \in \xi_6$, then $\mathcal{V} = \emptyset \Rightarrow \mathcal{V} \in \mathcal{P}(S)$.
- Single node subsets:
  - if $\mathcal{V} \in \xi_1$, then $\mathcal{V} \subseteq F, \ |\mathcal{V}| = 1 \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,
  - if $\mathcal{V} \in \xi_3$, then $\mathcal{V} \subseteq M, \ |\mathcal{V}| = 1 \Rightarrow \mathcal{V} \in \mathcal{P}(S)$.
- Multiple node's subset ($2 \leq |\mathcal{V}| \leq |S|$):
  - if $\mathcal{V} \in \xi_2$, then $\mathcal{V} \subseteq F, \ |\mathcal{V}| = 2 \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,
  - if $\mathcal{V} \in \xi_4$, then $\mathcal{V} \subseteq M \cup F, \ \mathcal{V} \cap F = 1$ and $\exists j, j' \in \mathcal{V}, \text{ with } j' \in S(j) \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,
  - if $\mathcal{V} \in \xi_7$, then $\mathcal{V} \subset M \cup F, \ \mathcal{V} \cap F = 1$ and $\exists j, j' \in \mathcal{V}, \text{ with } j' \notin S(j) \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,
  - if $\mathcal{V} \in \xi_8$, then $\mathcal{V} \subseteq M, \ |\mathcal{V}| \geq 2, \text{ and } \exists j, j' \in \mathcal{V}, \text{ with } j' \in S(j) \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,

– if $\mathcal{V} \in \xi_9$, then $\mathcal{V} \subset M \cup F,\ \mathcal{V} \cap F = 2 \Rightarrow \mathcal{V} \in \mathcal{P}(S)$,

– if $\mathcal{V} \in \xi_{10}$, then $\mathcal{V} \subset M,\ |\mathcal{V}| \geq 2$, and $\exists j, j' \in \mathcal{V}$, with $j' \notin S(j) \Rightarrow \mathcal{V} \in \mathcal{P}(S)$.

This proves that the sets $\xi_k, k = 1, \ldots, 10$ form a partition of the power set $\mathcal{P}(S)$. □

Corollary 1. *Let us consider a complete graph $G(N, E)$, let $S \subset N$, and an induced path tree $T(S, E(S))$. When $|S| = 3$, we only need the first six cases to characterize the power set of S, and when $|S| = 4$, we only need the first nine cases.*

Proof. We prove the corollary by checking the cardinalities of the sets.

- When $|S| = 3$: For all $k \in \{7, 8, 9, 10\}$, we have $|\xi_k| = 0$.
- When $|S| = 4$: For $k = 10$, $|\xi_{10}| = 0$.

Since a set is empty if and only if its cardinality is zero, we conclude that the sets ξ_k (for $k = 7, \ldots, 10$ when $|S| = 3$) and ξ_{10} (when $|S| = 4$) are empty. □

3.2 Quadratic space estimators

This section introduces the quadratic space estimators based on the previously defined partitions. Those estimators can be used to estimate the multilinear monomials. Each partition $\xi_k,\ k = 1, \ldots 10$ gives a class of estimators.

3.2.1 Definition

Consider the variables $x_1, \ldots, x_n \in \mathbb{R}$. Let $1 \leq i_1 < \ldots < i_k \leq n$, $\pi = x_{i_1}^{j_1} \ldots x_{i_k}^{j_k}$ is called a monomial with degree denoted by $\sum_{s=1}^{k} j_s$. The monomial π is multilinear, if $j_1 = j_2 = \ldots = j_k = 1$ or $\sum_{s=1}^{k} j_s = k$, i.e., π is linear in all its variables $x_{i_1}, \ldots, x_{i_k}$ (Chen and Fu, 2013).

Theorem 2. *Let $S \subset N$ with $|S| \geq 5$. Consider the partition $\bigcup_{k=1}^{10} \xi_k = \mathcal{P}(S)$. For any element $\xi_k^e \in \bigcup_{k=1}^{10} \xi_k, e = 1, \ldots, |\xi_k|,\ k = 1, \ldots, 10$ we define:*

$$N_{\xi_k^e} = \{i \in N \setminus S : i \in N(j) \text{ for all } j \in \xi_k^e\}.$$

We recall that the sets M and F denote, respectively, the sets of middle points and end points of the path tree $T(S, E(S))$ as defined previously. For each element of the partition ξ_k, let us define the multilinear monomial as follows.

$$x_i \left(\prod_{j \in S} x_j \right),\ \forall i \in N_{\xi_k^e},\ k = 1, \ldots, 10, \text{ and } e = 1, \ldots, |\xi_k|.$$

Therefore, we have the following under-estimators:

- *When $k = 1$:*

$$x_i \left(\prod_{j \in S} x_j \right) \geq - \sum_{j \in \xi_1^e \cup M} x_j + \sum_{(j,p) \in E(S)} x_j x_p + \sum_{j \in \xi_1^e} x_i x_j,\ \forall i \in N_{\xi_1^e}, e = 1, \ldots, |\xi_1|.$$

- *When $k = 2$:*

$$x_i \left(\prod_{j \in S} x_j \right) \geq - \sum_{j \in \{i\} \cup S \setminus (S(m) \cup \{m\})} x_j + \sum_{(j,p) \in E(S \setminus \{m\})} x_j x_p + \sum_{j \in \xi_2^e} x_i x_j,$$

$\forall i \in N_{\xi_2^e},\ \forall m \in \xi_2^e,\ e = 1, \ldots, |\xi_2|$. *The estimators can be computed using either endpoint $m \in \xi_2^e$, depending on which one is considered.*

- *When* $k = 3:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq -2\sum_{j\in\xi_3^e} x_j - \sum_{j\in M\setminus\xi_3^e} x_j + \sum_{(j,p)\in E(S)} x_jx_p + \sum_{j\in\xi_3^e} x_ix_j,\ \forall i \in N_{\xi_3^e}, e = 1,\dots,|\xi_3|.$$

- *When* $k = 4:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq -(|\xi_4^e|-1)x_i - \sum_{j\in M^4_{cut}(\xi_4^e)} x_j + \sum_{(j,p)\in E^4_{cut}(\xi_4^e)} x_jx_p + \sum_{j\in\xi_4^e} x_ix_j,\ \forall i \in N_{\xi_4^e}, e = 1,\dots,|\xi_4|.$$

Where $M^4_{cut}(\xi_4^e)$ *is the set of all* $j \in M$ *such that there exists a vertex* $j' \in S\setminus\xi_4^e$ *and* j *is adjacent to* j'*, with* $1 \leq |M^4_{cut}(\xi_4^e)| \leq |S| - 2$ *and* $E^4_{cut}(\xi_4^e)$ *defines edges from* $j \in S$ *to* $p \in S \setminus \xi_4^e$*, where* $1 \leq E^4_{cut}(\xi_4^e) \leq |S| - 2$.

- *When* $k = 5:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq -(|\xi_5^e|-1)x_i + \sum_{j\in\xi_5^e} x_ix_j,\ \forall i \in N_{\xi_5^e}, e = 1,\dots,|\xi_5|.$$

- *When* $k = 6:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq x_i - \sum_{j\in M} x_j + \sum_{(j,p)\in E(S)} x_jx_p - 1,\ \forall i \in N_{\xi_6^e}, e = 1,\dots,|\xi_6|.$$

- *When* $k = 7:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq - \sum_{j\in M\cup\xi_7^e} x_j + \sum_{(j,p)\in E(S)} x_jx_p + \sum_{j\in F\cap\xi_7^e} x_ix_j,\ \forall i \in N_{\xi_7^e}, e = 1,\dots,|\xi_7|.$$

- *When* $k = 8:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq -(|\xi_8^e|-1)x_i - \sum_{j\in M^8_{cut}(\xi_8^e)} x_j + \sum_{(j,p)\in E^8_{cut}(\xi_8^e)} x_jx_p + \sum_{j\in\xi_8^e} x_ix_j,\ \forall i \in N_{\xi_8^e}, e = 1,\dots,|\xi_8|.$$

Where $M^8_{cut}(\xi_8^e)$ *is the set of all* $j \in M$ *such that there exists a vertex* $j' \in S\setminus\xi_8^e$ *and* j *is adjacent to* j'*, with* $2 \leq |M^8_{cut}(\xi_8^e)| \leq |S| - 2$ *and* $E^8_{cut}(\xi_8^e)$ *defines edges from* $j \in \xi_8^e$ *to* $p \in S \setminus \xi_8^e$*, where* $2 \leq |E^8_{cut}(\xi_8^e)| \leq |S| - 2$.

- *When* $k = 9:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq -x_i - \sum_{j\in F\cup S(j)} x_j + \sum_{(j,p)\in E^9_{cut}(\xi_9^e)} x_jx_p + \sum_{j\in F} x_ix_j,\ \forall i \in N_{\xi_9^e}, e = 1,\dots,|\xi_9|.$$

Where $E^9_{cut}(\xi_9^e)$ *defines edges from* $j \in F \cup S(j)$ *to* $p \in S$*, where* $|E^9_{cut}(\xi_9^e)| = 4$.

- *When* $k = 10:$

$$x_i\left(\prod_{j\in S} x_j\right) \geq -2x_m - \sum_{j\in M\setminus m} x_j + \sum_{(j,p)\in E(S)} x_jx_p + x_ix_m,\ \forall i \in N_{\xi_{10}^e},\ \forall m \in \xi_{10}^e,\ e = 1,\dots,|\xi_{10}|,$$

We also have the following over-estimators:

- *When* $k = 1:$

$$x_i\left(\prod_{j\in S} x_j\right) \leq x_ix_m,\ \forall m \in \xi_1^e,\ \forall i \in N_{\xi_1^e}, e = 1,\dots,|\xi_1|.$$

- *When* $k = 2:$

$$x_i\left(\prod_{j\in S} x_j\right) \leq x_ix_m,\ \forall m \in \xi_2^e,\ \forall i \in N_{\xi_2^e}, e = 1,\dots,|\xi_2|.$$

- *When* $k = 3:$

$$x_i\left(\prod_{j\in S} x_j\right) \le x_i x_m,\ \forall m \in \xi_3^e,\ \forall i \in N_{\xi_3^e}, e = 1,\dots,|\xi_3|.$$

- *When* $k = 4:$

$$x_i\left(\prod_{j\in S} x_j\right) \ \le\ 1 - x_i - \sum_{j\in\zeta_4^e} x_j + \sum_{(j,p)\in E(\zeta_4^e)} x_j x_p + \sum_{j\in\zeta_4^e} x_i x_j,\ \forall \zeta_4^e \in E(\xi_4^e),\ \forall i \in N_{\xi_4^e}, e = 1,\dots,|\xi_4|.$$

- *When* $k = 5:$

$$x_i\left(\prod_{j\in S} x_j\right) \ \le\ 1 - x_i - \sum_{j\in\zeta_5^e} x_j + \sum_{(j,p)\in E(\zeta_5^e)} x_j x_p + \sum_{j\in\zeta_5^e} x_i x_j,\ \forall \zeta_5^e \in E(\xi_5^e),\ \forall i \in N_{\xi_5^e}, e = 1,\dots,|\xi_5|.$$

- *When* $k = 6:$

$$x_i\left(\prod_{j\in S} x_j\right) \ \le\ x_m x_n,\ \forall (m,n) \in E(S),\ \forall i \in N_{\xi_6^e}, e = 1,\dots,|\xi_6|.$$

- *When* $k = 7:$

$$x_i\left(\prod_{j\in S} x_j\right) \le x_i x_m,\ \forall m \in \xi_7^e,\ \forall i \in N_{\xi_7^e}, e = 1,\dots,|\xi_7|.$$

- *When* $k = 8:$

$$x_i\left(\prod_{j\in S} x_j\right) \ \le 1 - x_i - \sum_{j\in\zeta_8^e} x_j + \sum_{(j,p)\in E(\zeta_8^e)} x_j x_p + \sum_{j\in\zeta_8^e} x_i x_j,\ \forall \zeta_8^e \in E(\xi_8^e),\ \forall i \in N_{\xi_8^e}, e = 1,\dots,|\xi_8|.$$

- *When* $k = 9:$

$$x_i\left(\prod_{j\in S} x_j\right) \le x_i x_m,\ \forall m \in \xi_9^e,\ \forall i \in N_{\xi_9^e}, e = 1,\dots,|\xi_9|.$$

- *When* $k = 10:$

$$x_i\left(\prod_{j\in S} x_j\right) \le x_i x_m,\ \forall m \in \xi_{10}^e,,\ \forall i \in N_{\xi_{10}^e}, e = 1,\dots,|\xi_{10}|.$$

Proof. Here, we prove that the estimators we previously provided are valid using a disjunction argument. Let us consider $G(N, E)$ a complete graph, $S \subset N$, and $T(S, E(S))$ the induced path tree. Let $\bigcup_{k=1}^{10} \xi_k$ be a partition of the power set $\mathcal{P}(S)$ and $x_i \in \{0, 1\}$, $\forall i \in N$. We define the multilinear monomials

$$\pi_k^e = x_i\left(\prod_{j\in S} x_j\right),\ \forall i \in N_{\xi_k^e},\ k = 1,\dots,10,\ \text{and } e = 1,\dots,|\xi_k|.$$

and $U_k(.)$ and $O_k(.)$ as functions of under-estimators and over-estimators associated with a partition ξ_k. For the validity of $U_k(.)$ and, $O_k(.)$ we have to prove the following:

$$\pi_k^e \in [U_k(\pi_k^e), O_k(\pi_k^e)],\ \forall k = 1,\dots,10,\ \text{and } e = 1,\dots,|\xi_k|,$$

when

$$(x_i = 1 \land \prod_{j\in S} x_j = 1) \lor (x_i = 0 \land \prod_{j\in S} x_j = 1) \lor (x_i = 1 \land \prod_{j\in S} x_j = 0) \lor (x_i = 0 \land \prod_{j\in S} x_j = 0).$$

It should be noted that we only provide the main lines of the proof here for brevity. The full details can be found in Appendix A.1.

- Let $x_i = 1 \wedge \prod_{j \in S} x_j = 1$: In this case, we have $\pi_k^e = 1$, then we must have

$$U_k(\pi_k^e) \leq 1 \text{ and } O_k(\pi_k^e) \geq 1, \ \forall k = 1, \ldots, 10, \text{ and } e = 1, \ldots, |\xi_k|.$$

Here, we observe that each estimator $U_k(\pi_k^e)$ and $O(\pi_k^e)$ are constructed such that, when $x_i = 1, \ \forall i \in N \setminus S$ and $x_j = 1, \ \forall j \in S$, the left-hand side of all inequalities evaluates exactly to 1.

- Let $x_i = 0 \wedge \prod_{j \in S} x_j = 1$: The multilinear monomial is now $\pi_k^e = 0$. For the estimators $U_k(\pi_k^e)$ and $O_k(\pi_k^e)$, to be valid, we have to confirm that

$$U_k(\pi_k^e) \leq 0 \text{ and } O_k(\pi_k^e) \geq 0, \ \forall k = 1, \ldots, 10, \text{ and } e = 1, \ldots, |\xi_k|.$$

In this case, we observe that all under-estimators $U_k(\pi_k^e)$ evaluate to 0. This occurs because there exists a balance between the negative summation of x_j ($j \in S' \subseteq S$), and the positive summation of $x_j x_p$ ($(j,p) \in H \times H'$ where $H, H' \subseteq S$) in all inequalities, except when $k = 6$. For $k = 6$, the subtraction of 1 ensures the estimator becomes 0. Furthermore, since $x_i = 0$, all terms in the under-estimators involving x_i cancel. Regarding the over-estimators $O_k(\pi_k^e)$, all inequalities evaluate to 0 except when $k = 6$, $O_6(\pi_6^e) = 1$, this exception occurs because the corresponding over-estimator consists of a single term not involving x_i, but rather depending only on $x_j x_p$ ($(j,p) \in E(S)$).

- Let $x_i = 1 \wedge \prod_{j \in S} x_j = 0$: The multilinear monomial is $\pi_k^e = 0$, the validity of the estimators $U_k(\pi_k^e)$ and $O_k(\pi_k^e)$, holds when

$$U_k(\pi_k^e) \leq 0 \text{ and } O_k(\pi_k^e) \geq 0, \ \forall k = 1, \ldots, 10, \text{ and } e = 1, \ldots, |\xi_k|.$$

Here, all the under-estimators are evaluated to be less than or equal to 0, and the over-estimators are evaluated to be greater than or equal to 0. When $x_j = 0$ for all $j \in S$, for the under-estimators $U_k(\pi_k^e)$, all terms involving x_j vanish, the terms containing x_i are negative (except when $k = 6$). For $k = 6$, the only non zero terms are $x_i = 1$ and -1, yielding $U_6(\pi_6^e) = 0$. For the over-estimators $O_k(\pi_k^e)$, all x_j associated terms cancel and negative $-x_i$ terms cancel with the constant 1 (particularly for $k \in \{4, 5, 8\}$). Let us evaluate the case when $x_j = 0$ for $j \in S' \subset S$ and $x_j = 1$ for $j \in S \setminus S'$, all under-estimators $U_k(\pi_k^e)$ evaluate to either negative values or zero and all over-estimators $O_k(\pi_k^e)$ evaluate to values in the between 0 and 1.

- Let $x_i = 0 \wedge \prod_{j \in S} x_j = 0$: The multilinear monomial is $\pi_k^e = 0$, the validity of the estimators $U_k(\pi_k^e)$ and $O_k(\pi_k^e)$, holds when

$$U_k(\pi_k^e) \leq 0 \text{ and } O_k(\pi_k^e) \geq 0, \ \forall k = 1, \ldots, 10, \text{ and } e = 1, \ldots, |\xi_k|.$$

This case is obvious, as all the variables are considered to be zero, and when we have constant terms, for $U_k(\pi_k^e)$, the constant is negative (case $k = 6$, we have -1) and for $O_k(\pi_k^e)$, the constants are positive (case $k \in \{4, 5, 8\}$, we have 1). Then $U_k(\pi_k^e) \in \{-1, 0\}$ and $O_k(\pi_k^e) \in \{0, 1\}$.

The previous arguments prove the validity of the proposed estimators for all partitions $\xi_k, \ k = 1 \ldots, 10$. □

When we have a multilinear monomial that is not dependent on $i \in N \setminus S$, the quadratic space over-estimators of the monomial for $\forall j \in S \subset N$ are given as follows:

$$\prod_{j \in S} x_j \leq \min \{x_j x_p : \forall (j,p) \ \in E(S)\} \tag{9}$$

3.3 Compact form of the estimators

The under-estimators and over-estimators given in Theorem 2 can be written respectively in the following compact generalized form.

$$x_i \left(\prod_{j \in S} x_j \right) \ \geq \ \left(d_i^{+(k,e)} + \sum_{j \in A^{+(k,e)}} a_j^{+(k,e)} x_j + \sum_{j \in B^{+(k,e)}} b_{ij}^{+(k,e)} y_{ij} + \sum_{(j,p) \in C^{+(k,e)}} c_{jp}^{+(k,e)} y_{jp} \right). \tag{10}$$

$$x_i\Big(\prod_{j\in S} x_j\Big) \;\le\; \Big(d_i^{-(k,e)} + \sum_{j\in A^{-(k,e)}} a_j^{-(k,e)} x_j + \sum_{j\in B^{-(k,e)}} b_{ij}^{-(k,e)} y_{ij} + \sum_{(j,p)\in C^{-(k,e)}} c_{jp}^{-(k,e)} y_{jp}\Big). \qquad (11)$$

$\forall i \in N_{\xi_k^e}, k = 1,\ldots,10, e = 1,\ldots,|\xi_k|$, where:

- $A^+(k,e),\ A^-(k,e)$ are sets of indices for linear terms x_j,
- $B^+(k,e),\ B^-(k,e)$ are sets for quadratic terms $x_i x_j$,
- $C^+(k,e),\ C^-(k,e)$ are sets for quadratic terms $x_j x_p$,
- $a_j^+(k,e),\ a_j^-(k,e),\ b_{ij}^+(k,e),\ b_{ij}^-(k,e),\ c_{jp}^+(k,e),\ c_{jp}^-(k,e),\ d_i^+(k,e),\ d_i^-(k,e)$, are the coefficient terms.

For example, when $k = 1$, we have those sets of indices and coefficient terms:

$$\begin{array}{llll} A^+(1,e) = \xi_1^e \cup M, & B^+(1,e) = \xi_1^e, & C^+(1,e) = E(S), & d_i^+(1,e) = 0, \\ a_j^+(1,e) = [-1 : \text{if } j \in \xi_1^e \cup M], & b_{ij}^+(1,e) = [1 : \text{if } j \in \xi_1^e], & c_{jp}^+(1,e) = [1 : \text{if } (j,p) \in E(S)] & \end{array}$$

and

$$\begin{array}{llll} A^-(1,e) = \emptyset, & B^-(1,e) = \{m\}, & C^-(1,e) = \emptyset, & d_i^-(1,e) = 0, \\ a_j^-(1,e) = 0, & b_{ij}^-(1,e) = [1 :\ \text{if } j = m], & c_{jp}^-(1,e) = 0, & \end{array}$$

$\forall e = 1,\ldots,|\xi_1|$.

The compact form of the estimator given by (10) and (11) will be used in the next section to provide quadratic space inequalities. For the full description of the given sets, see Appendix A.2.

Example 1. To illustrate the generation of our estimators, let us consider an undirected complete graph $G(N,E)$. Let us take $S = \{s,u,w,t\} \subset N$, that induces a path tree $T(S,E(S))$ which can be illustrated as in Figure 1:

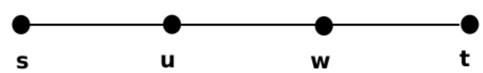

Figure 1: Path tree induced by the set S.

We have the set $F = \{s,t\}$ and the set $M = \{u,w\}$. When $k = 4$, we have the following set $\xi_4 = \{\{s,u\},\{t,w\},\{s,u,w\},\{t,u,w\}\}$. When we take the element $\xi_4^1 = \{s,u\} \in \xi_4$, the corresponding $N_{\xi_4^1}$ set can be written as:

$$N_{\xi_4^1} = \Big\{ i \in N \setminus S : i \in N(s) \wedge i \in N(u) \Big\}.$$

For any $i \in N_{\xi_4^1}$, the multilinear monomial $\pi_4^1 = x_i(x_s x_u x_w x_t)$, has the following under-estimator:

$$\pi_4^1 \ge -x_i - x_u - x_w + x_u x_w + x_w x_t + x_i x_s + x_i x_u.$$

And the over-estimator is:

$$\pi_4^1 \le 1 - x_i - x_s - x_u + x_s x_u + x_i x_s + x_i x_u,$$

where $E(\xi_4^1) = \{(s,u)\},\ \zeta_4^1 = (s,u)$.

Let us take another element $\xi_4^3 = \{s,u,w\} \in \xi_4$. The corresponding set $N_{\xi_4^3}$ is:

$$N_{\xi_4^3} = \Big\{ i \in N \setminus S : i \in N(s) \wedge i \in N(u) \wedge i \in N(w) \Big\}.$$

By using the corresponding formula for any $i \in N_{\xi_4^2}$, the multilinear monomial $\pi_4^1 = x_i(x_s x_u x_w x_t)$ has the following under-estimator

$$\pi_4^1 \geq -2x_i - x_w + x_w x_t + x_i x_s + x_i x_u + x_i x_w,$$

and over-estimators for any $\zeta_4^3 \in E(\xi_4^3) = \{(s,u),(u,w)\}$.

$$\pi_4^1 \leq 1 - x_i - x_s - x_u + x_s x_u + x_i x_s + x_i x_u,\ \zeta_4^3 = (s,u),$$

or

$$\pi_4^1 \leq 1 - x_i - x_u - x_w + x_u x_w + x_i x_u + x_i x_w,\ \zeta_4^3 = (u,w).$$

In this example, we give the corresponding quadratic space estimators of multilinear monomials that contain a product of five variables.

By the Theorem 2, we observe that by having a subset $N_{\xi_k^e},\ k = 1,\ldots,10,\ e = 1,\ldots,|\xi_k|$, we can construct estimators, which then allows to approximate the multilinear monomials directly from the space $|S|+1$ into the quadratic space. Unlike the estimators given in (Fomeni et al., 2015), which proceed recursively by going through all the intermediary spaces between $|S|+1$ to the quadratic space. For example, let us consider the following multilinear monomial.

$$x_i\Bigg(\prod_{j\in S} x_j\Bigg),\ \forall i \in N \setminus S$$

of the space $|S|+1$. Fomeni et al. (2015) generate, for the first iteration, estimators that contain the multilinear factors in the space $|S'| = |S|-1$ as follows:

$$x_i\Bigg(\prod_{j\in S} x_j\Bigg) \geq x_i\Bigg(\prod_{j\in S'} x_j\Bigg) + G(x),\ \forall i \in N \setminus S,$$

where $G(x)$ is a function of linear and quadratic terms, to reach the quadratic space, it required $|S|-1$ estimations. This could lead to significant information loss at each estimation step.

4 The family of k-Star inequalities

In this section, we provide the details of how inequalities generated at any level of the RLT hierarchy can be transformed into valid inequalities in the space of quadratic variables using the estimators given by Theorem 2. Let us consider the following quadratic polytope:

$$\mathcal{Q} = \Big\{(\mathbf{x},\mathbf{y}) \in \{0,1\}^{n+\binom{n}{2}} : (\boldsymbol{\alpha}^l)^T\mathbf{x} \leq \beta^l\ , y_{ij} = x_i x_j\ , \forall\, 1 \leq i,j \leq n,\ l = 1,\ldots,L\Big\} \tag{12}$$

where $\boldsymbol{\alpha}^l \in \mathbb{Q}^n,\ \beta^l \in \mathbb{Q}_+$. The relaxation of the set (12) is defined by

$$\mathcal{Q}_R = \Big\{(\mathbf{x},\mathbf{y}) \in [0,1]^{n+\binom{n}{2}} : (\boldsymbol{\alpha}^l)^T\mathbf{x} \leq \beta^l\ , y_{ij} = x_i x_j\ , \forall\, 1 \leq i,j \leq n,\ l = 1,\ldots,L\Big\}. \tag{13}$$

By using the RLT approach, one can generate exponentially large families of cutting planes for the relaxation of the quadratic polytope. For the RLT level 1, there are $2 \times n \times L$ possible inequalities with quadratic variables. Therefore, for RLT-1, we do not need to linearize as we work with a quadratic polytope. The inequalities we provide consist of tightening the RLT-1 relaxation. The following theorem presents a scheme for generating cutting planes from RLT-k to RLT-1 space variables.

Theorem 3. *Consider the complete graph $G = (N, E)$, let $S \subset N$ be the set of nodes that induce a path tree. The linear space variable x_i is associated with a node $i \in N$ and the quadratic space variable y_{ij} with an edge $(i, j) \in E$. For any inequality $l = 1, \dots, L$ from the polytope (13) given by*

$$\sum_{i \in N} \alpha_i^l x_i \leq \beta^l, \tag{14}$$

and for any $S \subset N$, with $3 \leq |S| \leq n-1$, let $N_{\xi_k^e} = N^+_{\xi_k^e} \cup N^-_{\xi_k^e}$ the subsets given by Theorem 2, where

$$N^+_{\xi_k^e} = \Big\{ i : \forall i \in N_{\xi_k^e} \wedge \alpha_i^l > 0 \Big\}, \quad N^-_{\xi_k^e} = \Big\{ i : \forall i \in N_{\xi_k^e} \wedge \alpha_i^l < 0 \Big\}.$$

Then, the following k-Star inequalities are valid to the quadratic polytope (13):

$$\begin{aligned}
\sum_{k=1}^{10} \sum_{e=1}^{|\xi_k|} \Bigg[& \sum_{i \in N^+_{\xi_k^e}} \alpha_i^l \Bigg(d_i^{+(k,e)} + \sum_{j \in A^{+(k,e)}} a_j^{+(k,e)} x_j + \sum_{j \in B^{+(k,e)}} b_{ij}^{+(k,e)} y_{ij} + \sum_{(j,p) \in C^{+(k,e)}} c_{jp}^{+(k,e)} y_{jp} \Bigg) \\
& + \sum_{i \in N^-_{\xi_k^e}} \alpha_i^l \Bigg(d_i^{-(k,e)} + \sum_{j \in A^{-(k,e)}} a_j^{-(k,e)} x_j + \sum_{j \in B^{-(k,e)}} b_{ij}^{-(k,e)} y_{ij} + \sum_{(j,p) \in C^{-(k,e)}} c_{jp}^{-(k,e)} y_{jp} \Bigg) \Bigg] \\
& \leq \Bigg(\beta^l - \sum_{j \in S} \alpha_j^l \Bigg) \min_{(j,p) \in E(S)} \{ y_{jp} \}, \quad \forall\, S \subset N.
\end{aligned}$$

Proof. Consider any inequality of the type $\alpha^T x \leq \beta$ from the polytope $\mathcal{Q}_R$. By RLT-k, for any $S \subset N$, such that $3 \leq |S| \leq n-1$ where $k = |S|$, the inequality $\prod_{j \in S} x_j \geq 0$ holds naturally. Furthermore, we can construct the following non-linear inequality, which corresponds to an RLT-k valid inequality:

$$\sum_{i \in N} \alpha_i x_i \Bigg(\prod_{j \in S} x_j \Bigg) \leq \beta \Bigg(\prod_{j \in S} x_j \Bigg) \; \forall S \subset N. \tag{15}$$

For $i \in N$ and $j \in S$ such that $i = j$, in the previous inequality for a given S,

$$x_i^2 \prod_{j \in S \setminus i} x_j = \prod_{j \in S} x_j.$$

Then

$$\sum_{i \in N \setminus S} \alpha_i x_i \Bigg(\prod_{j \in S} x_j \Bigg) + \sum_{i \in S} \alpha_i \prod_{j \in S} x_j \leq \beta \prod_{j \in S} x_j. \tag{16}$$

The expression (16) implies

$$\sum_{i \in N \setminus S} \alpha_i x_i \Bigg(\prod_{j \in S} x_j \Bigg) \leq \Bigg(\beta - \sum_{j \in S} \alpha_j \Bigg) \min_{(j,p) \in E(S)} \{ x_j x_p \}. \tag{17}$$

By letting $N \setminus S = \bigcup_{k=1}^{10} \bigcup_{e=1}^{|\xi_k|} N_{\xi_k^e}$ from sets partition the inequality (17) can be written as

$$\sum_{k=1}^{10} \sum_{e=1}^{|\xi_k^e|} \Bigg(\sum_{i \in N_{\xi_k^e}} \alpha_i x_i \Bigg(\prod_{j \in S} x_j \Bigg) \Bigg) \leq \Bigg(\beta - \sum_{j \in S} \alpha_j \Bigg) \min_{(j,p) \in E(S)} \{ x_j x_p \}, \; \forall S \subset N. \tag{18}$$

By replacing each multilinear monomial with the appropriate estimator shown in Section 3, we get the inequalities (15). □

4.1 Illustrative example: the 3-Star (SUT) inequality

We now use Theorem 3 to generate an example of cutting planes for any $S = \{s, u, t\} \subset N$. By using the set partition of Section 3.1, Theorem 2, and Corollary 1, we have the following required six subsets:

$$\xi_1 = \{\{s\}, \{t\}\}, \quad \xi_2 = \{\{s, t\}\}, \quad \xi_3 = \{\{u\}\},$$
$$\xi_4 = \{\{s, u\}, \{u, t\}\}, \quad \xi_5 = \{\{s, u, t\}\}, \quad \xi_6 = \{\emptyset\}.$$

where the elements of the sets $\xi_k^e \in \xi_k,\ k = 1, \ldots 6$ are the follows:

$$\xi_1^1 = \{s\}, \quad \xi_1^2 = \{t\}, \quad \xi_2^1 = \{s, t\},$$
$$\xi_3^1 = \{u\}, \quad \xi_4^1 = \{s, u\}, \quad \xi_4^2 = \{u, t\},$$
$$\xi_5^1 = \{s, u, t\}, \quad \xi_6^1 = \emptyset.$$

Then, the set

$$N \setminus \{s, u, t\} = N_{\xi_1^1} \cup N_{\xi_1^2} \cup N_{\xi_2^1} \cup N_{\xi_3^1} \cup N_{\xi_4^1} \cup N_{\xi_4^2} \cup N_{\xi_5^1} \cup N_{\xi_6^1}.$$

The family of 3-Star inequality is given by:

$$\sum_{k=1}^{6} \sum_{e=1}^{|\xi_k|} \Bigg[\sum_{i \in N^+_{\xi_k^e}} \alpha_i^l \Bigg(d_i^{+(k,e)} + \sum_{j \in A^{+(k,e)}} a_j^{+(k,e)} x_j + \sum_{j \in B^{+(k,e)}} b_{ij}^{+(k,e)} y_{ij} + \sum_{(j,p) \in C^{+(k,e)}} c_{jp}^{+(k,e)} y_{jp} \Bigg)$$
$$+ \sum_{i \in N^-_{\xi_k^e}} \alpha_i^l \Bigg(d_i^{-(k,e)} + \sum_{j \in A^{-(k,e)}} a_j^{-(k,e)} x_j + \sum_{j \in B^{-(k,e)}} b_{ij}^{-(k,e)} y_{ij} + \sum_{(j,p) \in C^{-(k,e)}} c_{jp}^{-(k,e)} y_{jp} \Bigg) \Bigg]$$
$$\leq \Bigg(\beta^l - \sum_{j \in S} \alpha_j^l \Bigg) \min_{(i,j) \in E(S)} \{y_{ij}\}, \quad \forall\ S \subset N.$$

The index sets and coefficients in the inequality (19), are given by the following Tables 1 and 2:

Table 1: Definitions of the index sets $A^{\pm}(k, e), B^{\pm}(k, e), C^{\pm}(k, e), d^{\pm}(k, e)$ for various (k, e).

(k,e)	$i \in N_{\xi_k^e}$	$A^+(k,e)$	$A^-(k,e)$	$B^+(k,e)$	$B^-(k,e)$	$C^+(k,e)$	$C^-(k,e)$	$d_i^+(k,e)$	$d_i^-(k,e)$
(1,1)	$N_{\xi_1^1}$	$\{s,u\}$	$\emptyset$	$\{s\}$	$\{s\}$	$\{(s,u),(u,t)\}$	$\emptyset$	0	0
(1,2)	$N_{\xi_1^2}$	$\{t,u\}$	$\emptyset$	$\{t\}$	$\{t\}$	$\{(s,u),(u,t)\}$	$\emptyset$	0	0
(2,1)	$N_{\xi_2^1}$	$\{i,t\}$	$\emptyset$	$\{s,t\}$	$\{\{s\} \vee \{t\}\}$	$\{(u,t)\}$	$\emptyset$	0	0
(3,1)	$N_{\xi_3^1}$	$\{u,\emptyset\}$	$\emptyset$	$\{u\}$	$\{u\}$	$\{(s,u),(u,t)\}$	$\emptyset$	0	0
(4,1)	$N_{\xi_4^1}$	$\{u,i\}$	$\{s,u\}$	$\{s,u\}$	$\{s,u\}$	$\{(u,t)\}$	$\{(s,u)\}$	0	1
(4,2)	$N_{\xi_4^2}$	$\{u,i\}$	$\{u,t\}$	$\{s,u\}$	$\{u,t\}$	$\{(s,u)\}$	$\{(u,t)\}$	0	1
(5,1)	$N_{\xi_5^1}$	$\{i\}$	$i \cup (\{s,u\} \vee \{u,t\})$	$\{s,u,t\}$	$\{s,u\} \vee \{u,t\}$	$\emptyset$	$\{(s,u) \vee (u,t)\}$	0	1
(6,1)	$N_{\xi_6^1}$	$\{i,u\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{(s,u),(u,t)\}$	$\{(s,u) \vee (u,t)\}$	-1	0

Table 2: Definition of the coefficients $a_j^{\pm}(k,e), b_{ij}^{\pm}(k,e), c_{jp}^{\pm}(k,e)$ for various (k,e).

(k,e)	$a_j^+(k,e)$	$a_j^-(k,e)$	$b_{ij}^+(k,e)$	$b_{ij}^-(k,e)$	$c_{jp}^+(k,e)$	$c_{jp}^-(k,e)$
(1,1)	-1 if $j \in \{s,u\}$	0	1 if $j = s$	1 if $j = s$	1 if $(j,p) \in \{(s,u),(u,t)\}$	0
(1,2)	-1 if $j \in \{t,u\}$	0	1 if $j = t$	1 if $j = t$	1 if $(j,p) \in \{(s,u),(u,t)\}$	0
(2,1)	-1 if $j \in \{i,t\}$	0	1 if $j \in \{s,t\}$	1 if $j \in \{\{s\} \vee \{t\}\}$	1 if $(j,p) = (u,t)$	0
(3,1)	-2 if $j = u$	0	1 if $j = u$	1 if $j = u$	1 if $(j,p) \in \{(s,u),(u,t)\}$	0
(4,1)	-1 if $j \in \{u,i\}$	-1 if $j \in \{s,u\}$	1 if $j \in \{s,u\}$	1 if $j \in \{s,u\}$	1 if $(j,p) = (u,t)$	1 if $(j,p) = (s,u)$
(4,2)	-1 if $j \in \{u,i\}$	-1 if $j \in \{u,t\}$	1 if $j \in \{s,u\}$	1 if $j \in \{u,t\}$	1 if $(j,p) = (s,u)$	1 if $(j,p) = (u,t)$
(5,1)	-2 if $j = i$	-1 if $j = i$ or $j \in \{s,u\} \vee \{u,t\}$	1 if $j \in \{s,u,t\}$	1 if $j \in \{s,u\} \vee \{u,t\}$	0	1 if $(j,p) \in \{(s,u) \vee (u,t)\}$
(6,1)	1 if $j = i$ or -1 if $j \in \{u\}$	0	0	0	1 if $(j,p) \in \{(s,u),(u,t)\}$	1 if $(j,p) \in \{(s,u) \vee (u,t)\}$

When we have to choose between multiple elements of the index set, for any chosen index, we have a valid estimator. For example, when $(k,e) = (6,1)$, $C^-(k,e)$ is either $\{(s,u)\}$ or $\{(u,t)\}$. After combining the terms, inequalities (19) are generalized as follows:

$$\begin{aligned}
&\sum_{i \in N_{\xi_1^1} \cup N_{\xi_2^1} \cup N_{\xi_4^1} \cup N^+_{\xi_4^2} \cup N_{\xi_5^1}} \alpha_i y_{is} + \sum_{i \in N_{\xi_3^1} \cup N_{\xi_4^1} \cup N_{\xi_4^2} \cup N_{\xi_5^1}} \alpha_i y_{iu} + \sum_{i \in N_{\xi_1^2} \cup N^+_{\xi_2^1} \cup N^+_{\xi_5^1} \cup N^-_{\xi_4^2}} \alpha_i y_{it} \\
&- \sum_{i \in N^+_{\xi_2^1} \cup N_{\xi_4^1} \cup N_{\xi_4^2} \cup N^-_{\xi_5^1}} \alpha_i x_i - \sum_{i \in N^+_{\xi_5^1}} 2\alpha_i x_i + \sum_{i \in N^+_{\xi_6^1}} \alpha_i x_i \\
&\le \alpha(N^+_{\xi_1^1} \cup N^-_{\xi_4^1} \cup N^-_{\xi_5^1}) x_s + \left[\alpha\left(N^+_{\xi_1^1} \cup N^+_{\xi_1^2} \cup N_{\xi_4^1} \cup N_{\xi_4^2} \cup N^-_{\xi_5^1} \cup N^+_{\xi_6^1}\right) + 2\alpha(N^+_{\xi_3^1})\right] x_u + \alpha(N^+_{\xi_1^2} \cup N^+_{\xi_2^1} \cup N^-_{\xi_4^2}) x_t \\
&- \alpha(N^+_{\xi_1^1} \cup N^+_{\xi_1^2} \cup N^+_{\xi_3^1} \cup N^-_{\xi_4^1} \cup N^+_{\xi_4^2} \cup N_{\xi_6^1}) y_{su} - \alpha(N^+_{\xi_1^1} \cup N^+_{\xi_1^2} \cup N^+_{\xi_3^1} \cup N^+_{\xi_4^1} \cup N^-_{\xi_5^1} \cup N^+_{\xi_6^1}) y_{ut} \\
&- \alpha(N^-_{\xi_4^1} \cup N^-_{\xi_4^2} \cup N^-_{\xi_5^1}) + \alpha(N^+_{\xi_6^1}) + (\beta - \alpha_s - \alpha_u - \alpha_t) \min\{y_{su}, y_{ut}\}, \; \forall \{s,u,t\} \subset N.
\end{aligned} \tag{19}$$

We observe that (19) can be split into two because of $\min\{y_{su},\ y_{ut}\}$. However, it is not necessary when performing a cutting plane. The cutting plane strategy is performed with a fractional solution, and the variables y_{su} and y_{ut} are fixed. Having the fractional solution, we select the variable that provides the minimum to evaluate the violation. This fact will be shown in the next section, where we will show how to separate inequalities (19) by performing a cutting plane.

5 Separation strategy

The separation problem involves generating valid inequalities that can cut off a given fractional solution without eliminating any feasible integral solution. Most separation routines found in the literature use the violation criterion or other metrics such that efficiency or parallelism of inequalities (Huang et al., 2022). In our case, we consider only the set of violated inequalities.

Given a fractional solution $(\mathbf{x}^*, \mathbf{y}^*)$ of the relaxed polytope $\mathcal{Q}_R$, the separation problem consists of finding violated valid inequalities of the family of the k-star inequalities. In other words, the separation problem consists of identifying for $l = 1, \dots, L$ the inequalities that belong to the family of (15) that are violated.

Let us consider an inequality $(\boldsymbol{\alpha}^l)^T \mathbf{x} \le \beta$ with $l = 1, \dots, L$, for any of $S \subset N$, there are $\binom{|N|}{|S|}$,with $k = |S|$, k-star inequalities that could be separated. For those inequalities, the RHS of (15) is constant, so the LHS can be maximized to find the most violated inequality. For each $i \in N \setminus S$, i is placed in one of the sets: $N_{\xi_k^e}$, with $k = 1, \dots, 10$ and $e = 1, \dots, |\xi_k|$ depending on which one of the estimators given by Theorem 2 is the largest if $\alpha_i^l > 0$ or the smallest if $\alpha_i^l < 0$. As for the RHS, the variable y is selected based on $\min_{(j,p)\in E(S)}\{y^*_{jp}\}$. At the end of this procedure, the indices $i \in N \setminus S$ are distributed in $N_{\xi^e_k}$, to have the most violated inequalities, which can be done in linear time. Therefore the separation routine runs in $\mathcal{O}(L|N|^{|S|+1})$ times. Computationally, the time can be reduced by implementing the separation using parallelism, as the steps of each $S \subset N$ can be computed independently for each $i \in N \setminus S$. In the next section, we use the example of 3-Star inequalities (19) to perform the numerical tests.

5.1 Separation of the 3-Star inequalities

To illustrate the separation method more explicitly, we rewrite inequalities (19) as follows:

$$\begin{aligned}
&\sum_{i\in N^+_{\xi^1_1}} \alpha_i(y_{is}+y_{su}+y_{ut}-x_s-x_u) + \sum_{i\in N^-_{\xi^1_1}} \alpha_i y_{is} + \sum_{i\in N^+_{\xi^2_1}} \alpha_i(y_{it}+y_{su}+y_{ut}-x_t-x_u) + \sum_{i\in N^-_{\xi^2_1}} \alpha_i y_{it}\\
&\quad + \sum_{i\in N^+_{\xi^1_2}} \alpha_i(y_{is}+y_{it}+y_{ut}-x_i-x_t) + \sum_{i\in N^-_{\xi^1_2}} \alpha_i y_{is} + \sum_{i\in N^+_{\xi^1_3}} \alpha_i(y_{iu}+y_{su}+y_{ut}-2x_u) + \sum_{i\in N^-_{\xi^1_3}} \alpha_i y_{iu}\\
&\quad + \sum_{i\in N^+_{\xi^1_4}} \alpha_i(y_{is}+y_{iu}+y_{ut}-x_u-x_i) + \sum_{i\in N^-_{\xi^1_4}} \alpha_i(y_{is}+y_{iu}+y_{su}-x_i-x_s-x_u+1)\\
&\quad + \sum_{i\in N^+_{\xi^2_4}} \alpha_i(y_{is}+y_{iu}+y_{su}-x_u-x_i) + \sum_{i\in N^-_{\xi^2_4}} \alpha_i(y_{iu}+y_{it}+y_{ut}-x_i-x_u-x_t+1)\\
&\quad + \sum_{i\in N^+_{\xi^1_5}} \alpha_i(y_{is}+y_{iu}+y_{it}-2x_i) + \sum_{i\in N^-_{\xi^1_5}} \alpha_i(y_{is}+y_{iu}+y_{su}-x_i-x_s-x_u+1)\\
&\quad + \sum_{i\in N^+_{\xi^1_6}} \alpha_i(y_{su}+y_{ut}+x_i-x_u-1) + \sum_{i\in N^-_{\xi^1_6}} \alpha_i y_{su}\\
&\quad\quad \le (\beta-\alpha_s-\alpha_u-\alpha_t)\min\{y_{su},\ y_{ut}\},\ \forall\{s,u,t\}\subset N.
\end{aligned} \tag{20}$$

For a given fractional solution (x^*, y^*) and any inequality of the form $\alpha^T x \le \beta$ from the polytope $\mathcal{Q}_R$, we have $\binom{n}{3}$ choices of $s, u, t \in N$. We will then place each $i \in N \setminus \{s, u, t\}$ in one of the sets

$$N^+_{\xi^1_1}, N^+_{\xi^2_1}, N^+_{\xi^1_2}, N^+_{\xi^1_3}, N^+_{\xi^1_4}, N^+_{\xi^2_4}, N^+_{\xi^1_5}, N^+_{\xi^1_6},$$
$$N^-_{\xi^1_1}, N^-_{\xi^2_1}, N^-_{\xi^1_2}, N^-_{\xi^1_3}, N^-_{\xi^1_4}, N^-_{\xi^2_4}, N^-_{\xi^1_5}, N^-_{\xi^1_6}$$

depending on which of the following LHS quantities are respectively the largest if $\alpha_i > 0$ or the smallest if $\alpha_i < 0$:

$$\max\left\{\begin{array}{l}
y^*_{si}+y^*_{su}+y^*_{ut}-x^*_s-x^*_u,\\
y^*_{it}+y^*_{su}+y^*_{ut}-x^*_t-x^*_u,\\
y^*_{is}+y^*_{it}+y^*_{su}-x^*_i-x^*_s,\\
y^*_{ui}+y^*_{su}+y^*_{ut}-2x^*_u,\\
y^*_{is}+y^*_{iu}+y^*_{ut}-x^*_u-x^*_i,\\
y^*_{iu}+y^*_{it}+y^*_{su}-x^*_u-x^*_i,\\
y^*_{is}+y^*_{iu}+y^*_{it}-2x^*_i,\\
y^*_{su}+y^*_{ut}+x^*_i-x^*_u-1.
\end{array}\right.
\quad \text{or} \quad
\min\left\{\begin{array}{l}
y^*_{is},\\
y^*_{it},\\
y^*_{is},\\
y^*_{iu},\\
y^*_{is}+y^*_{iu}+y^*_{su}-x^*_i-x^*_s-x^*_u+1,\\
y^*_{iu}+y^*_{it}+y^*_{ut}-x^*_i-x^*_u-x^*_t+1,\\
y^*_{is}+y^*_{iu}+y^*_{su}-x^*_i-x^*_s-x^*_u+1,\\
y^*_{su}.
\end{array}\right.$$

As stated in Section 4.1, we select the variable to be multiplied by the constant $(\beta - \alpha_s - \alpha_u - \alpha_t)$ by having the fractional solution by $\min\{y^*_{su}, y^*_{ut}\}$. After the sets are constructed, the separated inequalities can be added to $\mathcal{Q}_R$.

6 Computational experiments

In this section, we present and discuss the results of the numerical tests conducted to evaluate the quality of our proposed cutting planes framework. For these experiments, we focus on the optimality gaps that can be achieved by transforming inequalities from the RLT level-3 to inequalities in the RLT level-1 space. We also assess how our inequalities compare with the ones obtained using the recursive framework of Fomeni et al. (2015). We coded the cutting planes framework in the C language with compiler *gcc*12.2.0 wherein all LP problems are solved using CPLEX version 22.1.1 (IBM ILOG, 2023). All the computations are run on a computer equipped with 32 GB of RAM and a 3.00 GHz processor.

The test scheme works as follows:

- We solve a relaxation of each instance (see columns *Relaxation*).
- We separate the RLT-1 inequalities until no violations exist (see columns *RLT-1*).
- To the RLT-1 relaxation, we add the inequalities from the framework of Fomeni et al. (2015) (see the columns *RLT-1 + Fomeni et al. (2015)*)
- To the RLT-1 relaxation, we also add the violated inequalities from our proposed framework (see the columns *RLT-1 + SUT*).

6.1 Description of the instances

For these experiments, we have used a total of 523 instances, which include some standard QKP instances, the Knapsack version of dispersion problems, and some instances from the QPLIB library (Furini et al., 2019).

6.1.1 Standard QKP instances

The standard QKP instances, introduced by Gallo et al. (2009), are the most used ones for QKP. For those instances, each item's weight a_i is a random integer between 1 and 100, and the knapsack capacity is a randomly chosen integer between 50 and the total weights' sum $\sum_{i\in N} a_i$. The profit matrix's density is generated for each $\Delta \in \{25\%, 50\%, 75\%, 100\%\}$. We generated 10 instances for various values of $N \in \{20, 30, 40, 50, 60, 70, 80, 100\}$ and density probabilities. The optimal bounds are obtained for these instances using scripts that implement the exact method described in Caprara et al. (1999).

6.1.2 Dispersion Problem Instances

The dispersion problem is a known problem where k facilities are assigned to locations to maximize the distance between the selected locations. We generated knapsack versions of these variants (KP-{EXPO, GEO, WGEO, RAN}) where each location has a random weight in $\{1, \dots, 100\}$. These instances have also been used in (Fomeni, 2023; Fennich et al., 2024). We generate 10 instances of the knapsack version of the dispersion problem for each combination of size $n \in \{20, 30, 40, 50, 60\}$. We use Gurobi in Python to obtain optimal bounds of the dispersion problem instances.

6.1.3 QPLIB library instances

The QPLIB (Furini et al., 2019) hosts a collection of problem instances from the diverse class of general quadratic programming problems. The latest available version of QPLIB contains 319 discrete and 134 continuous instances with different characteristics. We use three instances formulated as a 0–1 quadratic programming problem, described by model (1). The selected instances are those whose constraint set is defined by polytope (12).

6.2 Results and analyses

The results of our computational experiments are shown in Table 3 (for the standard QKP instances), Table 4 (for the knapsack version of the dispersion problem instances), and in Table 5 (for the QPLIB instances). In these tables, we primarily show the optimality gaps and computational times for the initial LP relaxation, RLT-1 alone, and RLT-1 with improvement through the framework of Fomeni et al. (2015), and with our proposed framework. For our proposed framework, we present the results when all violated inequalities are used in the separation, as well as the results when we limit the separation to generate only the 100 most violated inequalities and the 10 most violated inequalities, respectively. The optimality gaps are calculated as the average relaxation gap in percentage, which is computed as

$$gap = \left(\frac{Upper\ Bound - Optimal\ Value}{Optimal\ Value} \right) \times 100.$$

It should be noted that the results shown in these tables are averages out of 10 instances, except for Table 5.

Table 3: Standard instances: RLT-1 enhancement.

Instances		**Relaxation**		**RLT-1**		**RLT-1+** Fomeni et al. (2015)		**RLT-1+SUT** All the violated		**RLT-1+SUT** 100 most violated		**RLT-1+ SUT** 10 most violated	
Δ	n	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time
25	20	4.903	0.006	3.291	0.011	2.960	0.041	**2.248**	0.081	2.251	0.077	2.252	0.130
	30	4.977	0.016	4.049	0.026	3.429	0.205	**1.721**	1.210	1.722	1.403	1.725	6.923
	40	3.524	0.033	2.800	0.055	2.462	0.455	**1.513**	1.924	1.514	1.638	1.516	8.476
	50	2.165	0.063	2.041	0.097	1.482	5.513	**0.741**	43.169	**0.741**	73.493	0.768	136.543
	60	4.021	0.141	2.411	0.251	1.888	5.250	**1.098**	32.573	1.099	27.627	1.103	145.043
	70	0.841	0.234	0.742	0.395	0.574	14.391	**0.360**	38.686	0.361	30.584	0.364	109.279
	80	2.515	0.284	2.056	0.517	1.655	47.220	0.904	269.820	**0.872**	152.028	0.948	247.963
	100	2.925	0.583	1.391	2.190	1.045	131.849	0.511	889.063	**0.472**	293.514	0.585	459.990
50	20	5.569	0.007	3.447	0.011	2.897	0.030	2.565	0.043	**2.564**	0.038	2.565	0.053
	30	5.907	0.018	2.438	0.031	2.055	0.106	**1.624**	0.290	1.625	0.252	1.626	0.618
	40	2.976	0.043	1.333	0.083	1.083	1.731	0.954	2.167	**0.953**	3.128	0.963	1.953
	50	3.197	0.077	1.045	0.152	0.954	0.367	**0.602**	2.001	0.603	1.115	0.603	3.503
	60	3.004	0.152	1.403	0.318	0.969	2.502	**0.611**	11.675	**0.611**	9.592	**0.611**	48.294
	70	3.813	0.282	1.951	0.760	0.988	10.864	**0.395**	165.144	0.396	54.393	0.403	161.861
	80	6.005	0.339	2.191	1.372	1.237	20.904	**0.406**	255.546	**0.406**	44.925	0.407	199.868
	100	2.755	0.612	0.891	3.800	0.460	48.758	0.261	1038.806	**0.194**	164.497	0.202	270.055
75	20	7.518	0.007	2.852	0.011	2.848	0.012	**2.823**	0.015	**2.823**	0.016	2.824	0.016
	30	5.789	0.019	1.416	0.035	1.339	0.156	**1.323**	0.651	**1.323**	1.205	**1.323**	2.935
	40	6.756	0.040	1.244	0.074	1.244	0.091	**1.243**	0.123	**1.243**	0.126	**1.243**	0.137
	50	2.920	0.081	0.964	0.202	0.940	1.898	**0.931**	13.882	0.941	11.551	0.936	100.608
	60	6.974	0.182	1.072	0.436	0.880	1.529	**0.681**	5.626	0.682	3.915	0.682	12.996
	70	4.689	0.278	0.688	0.698	0.675	1.769	**0.658**	3.869	**0.658**	2.821	0.659	3.594
	80	4.953	0.324	0.633	0.906	0.578	3.403	**0.509**	12.711	**0.509**	15.602	0.510	43.502
	100	4.422	0.656	0.173	3.116	0.173	6.890	**0.172**	5.144	**0.172**	5.219	**0.172**	4.951
100	20	5.563	0.007	2.696	0.011	2.388	0.083	2.309	0.221	2.305	0.271	**2.304**	0.717
	30	5.889	0.018	2.729	0.033	2.729	0.053	**2.721**	0.042	**2.721**	0.041	2.722	0.052
	40	6.824	0.049	**1.713**	0.087	**1.713**	0.134	**1.713**	0.103	**1.713**	0.111	**1.713**	0.106
	50	4.691	0.094	1.183	0.170	1.183	0.305	**1.181**	0.318	**1.181**	0.334	**1.181**	0.360
	60	6.199	0.208	1.181	0.440	1.180	0.827	**1.178**	0.685	1.178	0.752	1.178	0.806
	70	2.351	0.322	**0.551**	0.691	**0.551**	1.545	**0.551**	1.064	**0.551**	1.122	**0.551**	1.124
	80	2.638	0.517	**0.494**	1.287	**0.494**	1.858	**0.494**	1.744	**0.494**	1.786	**0.494**	1.488
	100	2.094	1.212	**0.201**	2.662	**0.201**	5.910	**0.201**	5.021	**0.201**	5.609	**0.201**	6.218

The results in Table 3 for the standard instances show that the cutting planes generated by our proposed framework can effectively strengthen the RLT level-1 relaxation. A comparison between our framework and the one proposed by Fomeni et al. (2015) reveals that our cutting planes are stronger in terms of the quality of the gaps obtained. For the four groups of instances, differentiated by the density of the profit matrix, our cutting planes consistently produce the lowest gaps. It can be observed that for the cases where the profit matrix is sparse ($\Delta = 25\%$, $\Delta = 50\%$), the gap obtained with our cutting planes can be significantly better than the gaps obtained with the cuts from Fomeni et al. (2015). The difference between the gaps of the two procedures is relatively narrow for the dense cases. Indeed, the quality of the RLT level-1 gaps for these instances indicates that these gaps are the tightest that can be achieved, unless one considers a much higher level of RLT relaxation.

For the dispersion problem instances shown in Table 4, it appears that for the family of the KP-RAN instances, both our framework and that of Fomeni et al. (2015) can only narrowly improve the optimality gap already obtained by the RLT level-1 relaxation. However, the improvement becomes more apparent for the families of KP-GEO, KP-WGEO, and KP-EXPO instances. For these instances, it is also evident that our framework allows for obtaining the best optimality gap, which can be up to 85% better than the gap of Fomeni et al. (2015) and up to 95% better than the gap of the RLT-1 relaxation. The results of the three instances from the QPLIB benchmark, shown in Table 5, also show the superiority of our cutting planes.

Table 4: Dispersion problem knapsack version: RLT-1 enhancement.

Instances		**Relaxation**		**RLT-1**		**RLT-1+** Fomeni et al. (2015)		**RLT-1+SUT** All the violated		**RLT-1+SUT** 100 most violated		**RLT-1+ SUT** 10 most violated	
Type	n	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time
	20	10.023	0.017	2.817	0.025	2.816	0.035	**2.809**	0.044	2.810	0.030	2.810	0.053
	30	6.273	0.044	1.848	0.072	1.847	0.085	**1.846**	0.100	**1.846**	0.064	**1.846**	0.125
KP-RAN	40	5.288	0.097	0.885	0.165	0.885	0.225	**0.884**	0.271	**0.884**	0.203	**0.884**	0.284
	50	4.568	0.199	0.892	0.356	0.891	0.467	**0.885**	0.816	**0.885**	0.582	**0.885**	1.128
	60	4.061	0.376	0.526	0.710	0.526	0.926	**0.526**	1.361	**0.526**	1.108	**0.526**	1.522
	20	22.670	0.018	2.719	0.036	2.549	0.140	**1.860**	0.284	**1.860**	0.265	**1.860**	0.849
	30	19.892	0.048	2.081	0.139	1.702	0.892	**1.042**	2.770	**1.042**	3.497	**1.042**	17.556
KP-GEO	40	19.497	0.102	2.296	0.365	1.784	3.809	**1.026**	18.612	**1.026**	34.214	**1.026**	197.194
	50	21.908	0.205	3.572	0.927	2.537	12.537	**1.309**	121.289	**1.309**	314.331	1.322	853.447
	60	20.224	0.340	2.943	1.923	2.012	29.871	**0.965**	427.642	0.966	979.283	1.021	2340.015
	20	37.648	0.013	9.237	0.035	8.135	0.189	**5.287**	0.770	**5.287**	0.548	**5.287**	2.486
	30	36.124	0.046	9.199	0.124	7.063	1.153	**3.876**	7.557	**3.876**	8.413	3.877	61.163
KP-WGEO	40	35.065	0.117	8.480	0.380	6.200	4.398	**3.384**	48.827	**3.384**	84.881	3.385	468.500
	50	35.069	0.199	9.663	0.846	6.546	15.616	**3.293**	266.357	**3.293**	440.415	3.326	962.557
	60	34.678	0.311	9.424	1.710	6.105	69.247	**3.137**	827.130	3.141	800.302	3.296	2307.885
	20	25.914	0.014	5.882	0.037	3.516	0.174	**1.827**	0.486	1.838	0.461	1.828	1.440
	30	30.875	0.046	14.124	0.129	7.482	2.207	**2.161**	17.249	2.236	23.181	2.163	224.744
KP-EXPO	40	31.882	0.096	16.593	0.289	8.558	10.785	**2.602**	196.982	2.660	379.806	2.747	1000.130
	50	32.575	0.203	18.161	0.682	9.622	60.340	**3.547**	945.130	3.747	1001.124	4.263	2294.464
	60	34.526	0.347	20.945	1.386	11.629	366.512	7.052	1113.479	**6.318**	1003.691	7.291	1000.983

Table 5: Benchmark instances: RLT-1 enhancement.

Instances		**Relaxation**		**RLT-1**		**RLT-1+** Fomeni et al. (2015)		**RLT-1+SUT** All the violated		**RLT-1+SUT** 100 most violated		**RLT-1+ SUT** 10 most violated	
Δ	ID	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time(s)	Gap (%)	Time
100	QPLIB 0067(n=80)	1.258	0.219	**1.092**	0.462	**1.092**	1.449	**1.092**	1.759	**1.092**	1.983	**1.092**	1.943
100	QPLIB 0633(n=75)	74.721	0.253	17.424	6.272	16.786	17.076	**15.342**	108.977	**15.342**	291.649	15.413	1001.110
100	QPLIB 3834(n=50)	73.529	0.055	11.750	1.532	11.191	2.512	**9.916**	8.943	**9.916**	12.257	**9.916**	82.526

In summary, the computational experiments demonstrate the effectiveness of the cutting planes generated by our proposed framework in strengthening the RLT level-1 relaxation. They also show the superiority of our cutting planes compared to those of Fomeni et al. (2015) in terms of the gaps obtained. In our computational experiments, we tested our framework in three settings: by generating all possible cuts, and by selecting only the 100 or 10 most violated cuts, respectively. For all three types of problem instances, no significant difference in terms of optimality gap was observed. However, the computational time is significantly reduced when we only consider the 100 most violated cuts.

While our framework requires longer computation times than the method of Fomeni et al. (2015), this is primarily due to the significantly larger number of inequalities generated per iteration. A key advantage of our method is its ability to identify violated inequalities in instances where the Fomeni et al. (2015) approach finds none, indicating that our separation routine explores a more extensive and effective family of cuts. We expect that by refining the separation strategy, our inequalities will have the potential to surpass those obtained by the method of Fomeni et al. (2015) in terms of computational time.

7 Conclusion

In this paper, we have introduced a new family of valid inequalities for the quadratic polytope, derived from higher-level Reformulation-Linearization Technique (RLT) principles. Those inequalities are novel in that they avoid the linearization of multilinear monomials, which typically complicates the relaxation process, because it needs additional variables and constraints; instead, we have provided estimators to represent multilinear monomials. The resulting estimators are shown to be valid and used to generalize the class of k-Star inequalities.

To demonstrate the effectiveness of the proposed k-Star inequalities, we have examined a specific case where $k = 3$, which we call the 3-Star inequalities, and implement a cutting-plane procedure to evaluate their impact by strengthening RLT-1 relaxation. The results we got are compared to the same level of inequalities generated using the estimation scheme from the literature. We observe that we have more inequalities separated in the case of our inequalities and better gaps for all cases. We apply sorting to select the 10 and 100 most violated inequalities at each cutting plane iteration to evaluate the impact on the running time. The computational experiments demonstrate that the provided family of inequalities can serve as support for the RLT-1 relaxation. Some possible separation strategies could help reduce computational time due to the large number of violated inequalities. Alternatively, an approach that involves adding our k-Star inequalities to the relaxation model without performing a cutting plane could also provide a tight relaxation bound.

Appendix

A.1 Partition proof details

Here are the details of the proof of Theorem 3.1. The estimators associated with partitions $\xi_k,\ k = 1,\dots,10$ hold.

- Let $x_i = 1 \wedge \prod_{j\in S} x_j = 1$, then we have
  - For all $i \in N_{\xi_1^e},\ e = 1,\dots,|\xi_1|$:

$$1 \geq -1-(|S|-2)+|S|-1+1 = 1,$$
$$1 \leq 1\times x_m = 1,\ m \in \xi_1^e.$$

  - For all $i \in N_{\xi_2^e},\ e = 1,\dots,|\xi_2|$:

$$1 \geq -1-(|S|-2)+|S|-2+2 = 1,$$
$$1 \leq 1\times x_m = 1,\ \forall m \in \xi_2^e.$$

  - For all $i \in N_{\xi_3^e},\ e = 1,\dots,|\xi_3|$:

$$1 \geq -2-(|S|-3)+|S|-1+1 = 1,$$
$$1 \leq 1\times x_m = 1,\ \forall m \in \xi_3^e.$$

  - For all $i \in N_{\xi_4^e},\ e = 1,\dots,|\xi_4|$: let the minimum cardinality $|\xi_4^e| = 2$, so $|M^4_{cut}(\xi_4^e)| = |S|-2$ and $|E^4_{cut}(\xi_4^e)| = |S|-2$.

$$1 \geq -1-(|S|-2)+|S|-2+2 = 1,$$
$$1 \leq 1-1-2+1+2 = 1,\ \forall \zeta_4^e \in E(\xi_4^e).$$

  - For all $i \in N_{\xi_5^e},\ e = 1,\dots,|\xi_5|$:

$$1 \geq -(|S|-1)+|S| = 1,$$
$$1 \leq 1-1-2+1+2 = 1,\ \forall \zeta_5^e \in E(\xi_5^e).$$

– For all $i \in N_{\xi_6^e},\ e = 1, \ldots, |\xi_6|$:

$$1 \geq 1 - (|S| - 2) + |S| - 1 - 1 = 1,$$
$$1 \leq x_m x_n = 1,\ \forall (m, n) \in E(S).$$

– For all $i \in N_{\xi_7^e},\ e = 1, \ldots, |\xi_7|$:

$$1 \geq -(|S| - 1) + |S| - 1 + 1 = 1$$
$$1 \leq 1 \times x_m = 1,\ \forall m \in \xi_7^e$$

– For all $i \in N_{\xi_8^e},\ e = 1, \ldots, |\xi_8|$: let the minimum cardinality $|\xi_8^e| = 2$, so $|M_{cut}^8(\xi_8^e)| = |S| - 2$ and $|E_{cut}^8(\xi_8^e)| = |S| - 2$.

$$1 \geq -1 - (|S| - 2) + (|S| - 2) + 2 = 1$$
$$1 \leq 1 - 1 - 2 + 1 + 2 = 1,\ \forall \zeta_8^e \in E(\xi_8^e)$$

– For all $i \in N_{\xi_9^e},\ e = 1, \ldots, |\xi_9|$: let the minimum cardinality $|\xi_9^e| = 3$, so $|E_{cut}^9(\xi_9^e)| = 4$.

$$1 \geq -1 - 4 + 4 + 2 = 1$$
$$1 \leq 1 \times x_m = 1,\ \forall m \in \xi_9^e$$

– For all $i \in N_{\xi_{10}^e},\ e = 1, \ldots, |\xi_{10}|$:

$$1 \geq -2 - (|S| - 3) + (|S| - 1) + 1 = 1,\ \forall m \in \xi_{10}^e$$
$$1 \leq 1 \times x_m = 1,\ \forall m \in \xi_{10}^e.$$

- Let $x_i = 0 \wedge \prod_{j \in S} x_j = 1$, then we have

  – For all $i \in N_{\xi_1^e},\ e = 1, \ldots, |\xi_1|$:

$$0 \leq -1 - (|S| - 2) + |S| - 1 + 0 = 0,$$
$$0 \geq 0 \times x_m = 0,\ m \in \xi_1^e.$$

  – For all $i \in N_{\xi_2^e},\ e = 1, \ldots, |\xi_2|$:

$$0 \leq -0 - (|S| - 2) + (|S| - 2) + 0 = 0,\ \forall m \in \xi_2^e,$$
$$0 \geq 0 \times x_m = 0,\ \forall m \in \xi_2^e.$$

  – For all $i \in N_{\xi_3^e},\ e = 1, \ldots, |\xi_3|$:

$$0 \geq -2 - (|S| - 3) + (|S| - 1) + 0 = 0,$$
$$0 \leq 0 \times x_m = 0,\ m \in \xi_3^e.$$

  – For all $i \in N_{\xi_4^e},\ e = 1, \ldots, |\xi_4|$: let the minimum cardinality $|\xi_4^e| = 2$, so $|M_{cut}^4(\xi_4^e)| = |S| - 2$ and $|E_{cut}^4(\xi_4^e)| = |S| - 2$.

$$0 \geq 0 - (|S| - 2) + (|S| - 2) + 0 = 0,$$
$$0 \leq 1 - 0 - 2 + 1 + 0 = 0,\ \forall \zeta_4^e \in E(\xi_4^e).$$

  – For all $i \in N_{\xi_5^e},\ e = 1, \ldots, |\xi_5|$:

$$0 \leq 0 + 0 = 0,$$
$$0 \geq 1 - 0 - 2 + 1 + 0 = 0,\ \forall \zeta_5^e \in E(\xi_5^e).$$

  – For all $i \in N_{\xi_6^e},\ e = 1, \ldots, |\xi_6|$:

$$0 \geq 0 - (|S| - 2) + (|S| - 1) - 1 = 0,$$
$$0 \leq x_n x_m = 1,\ \forall (m, n) \in E(S).$$

– For all $i \in N_{\xi_7^e},\ e = 1, \ldots, |\xi_7|$:

$$\begin{aligned} &0 \geq -(|S|-1) + |S| - 1 + 0 = 0, \\ &0 \leq 0 \times x_m = 0,\ \forall m \in \xi_7^e. \end{aligned}$$

– For all $i \in N_{\xi_8^e},\ e = 1, \ldots, |\xi_8|$: let the minimum cardinality $|\xi_8^e| = 2$, so $|M^8_{cut}(\xi_8^e)| = |S|-2$ and $|E^8_{cut}(\xi_8^e)| = |S| - 2$.

$$\begin{aligned} &0 \geq 0 - (|S|-2) + (|S|-2) + 0 = 0, \\ &0 \leq 1 - 0 - 2 + 1 + 0 = 0,\ \forall \zeta_8^e \in E(\xi_8^e). \end{aligned}$$

– For all $i \in N_{\xi_9^e},\ e = 1, \ldots, |\xi_9|$: let the minimum cardinality $|\xi_9^e| = 3$, so $|E^9_{cut}(\xi_9^e)| = 4$.

$$\begin{aligned} &0 \geq 0 - 4 + 4 + 0 = 0, \\ &0 \leq 0 \times x_m = 0,\ \forall m \in \xi_9^e. \end{aligned}$$

– For all $i \in N_{\xi_{10}^e},\ e = 1, \ldots, |\xi_{10}|$:

$$\begin{aligned} &0 \geq -2 - (|S|-3) + (|S|-1) + 0 = 0,\ \forall m \in \xi_{10}^e, \\ &0 \leq 0 \times x_m = 0,\ \forall m \in \xi_{10}^e. \end{aligned}$$

- Let $x_i = 1 \wedge \prod_{j \in S} x_j = 0$, then we have

  – For all $i \in N_{\xi_1^e},\ e = 1, \ldots, |\xi_1|$:

$$\begin{aligned} &\text{When } x_j = 0,\ \forall j \in \xi_1^e : \\ &0 \geq 0 - (|S|-2) + (|S|-2) + 0 = 0, \\ &0 \leq 1 \times x_m = 0,\ m \in \xi_1^e. \\ &\text{When } x_j = 0,\ \forall j \in S \setminus \xi_1^e : \\ &0 \geq -1 - 0 + 0 + 1 = 0, \\ &0 \leq 1 \times x_m = 1,\ m \in \xi_1^e. \end{aligned}$$

  – For all $i \in N_{\xi_2^e},\ e = 1, \ldots, |\xi_2|$:

$$\begin{aligned} &\text{When } x_j = 0,\ \forall j \in \xi_2^e : \\ &0 \geq -1 - (|S|-2) + (|S|-2) + 0 = -1, \\ &0 \leq 1 \times x_m = 0,\ m \in \xi_2^e. \\ &\text{When } x_j = 0,\ \forall j \in S \setminus \xi_2^e : \\ &0 \geq -1 - 0 + 0 + 1 = 0, \\ &0 \leq 1 \times x_m = 1,\ m \in \xi_2^e. \end{aligned}$$

  – For all $i \in N_{\xi_3^e},\ e = 1, \ldots, |\xi_3|$:

$$\begin{aligned} &\text{When } x_j = 0,\ \forall j \in \xi_3^e : \\ &0 \geq 0 - (|S|-3) + (|S|-3) + 0 = 0, \\ &0 \leq 1 \times x_m = 0,\ m \in \xi_3^e. \\ &\text{When } x_j = 0,\ \forall j \in S \setminus \xi_3^e : \\ &0 \geq -2 - 0 + 0 + 1 = -1, \\ &0 \leq 1 \times x_m = 1,\ m \in \xi_3^e. \end{aligned}$$

– For all $i \in N_{\xi_4^e},\ e = 1, \ldots, |\xi_4|$: let the minimum cardinality $|\xi_4^e| = 2$, so $|M_{cut}^4(\xi_4^e)| = |S| - 2$ and $|E_{cut}^4(\xi_4^e)| = |S| - 2$.

$$\text{When } x_j = 0,\ \forall j \in \xi_4^e :$$

$$0 \geq -1 - (|S| - 4) + (|S| - 3) + 0 = 0,$$
$$0 \leq 1 - 1 - 0 + 0 + 0 = 0,\ \zeta_4^e \in E(\xi_4^e).$$

$$\text{When } x_j = 0,\ \forall j \in S \setminus \xi_4^e :$$

$$0 \geq -1 - 2 + 1 + 2 = 0,$$
$$0 \leq 1 - 1 - 2 + 2 + 1 = 1,\ \zeta_4^e \in E(\xi_4^e).$$

– For all $i \in N_{\xi_5^e},\ e = 1, \ldots, |\xi_5|$:

$$\text{When } x_j = 0,\ \forall j \in \xi_5^e :$$

$$0 \geq -(|S| - 1) + 0 = -|S| + 1,$$
$$0 \leq 1 - 1 - 0 + 0 + 0 = 0,\ \zeta_4^e \in E(\xi_4^e).$$

$$\text{When } x_j = 0,\ \forall j \in S \setminus \xi_5^e$$

$$0 \geq -(|S| - 1) + (|S| - 1) = 0,$$
$$0 \leq 1 - 1 - 2 + 2 + 1 = 1,\ \zeta_5^e \in E(\xi_5^e).$$

– For all $i \in N_{\xi_6^e},\ e = 1, \ldots, |\xi_6|$:

$$\text{When } x_j = 0,\ \forall j \in \xi_6^e :$$

$$1 \geq 1 - (|S| - 2) + |S| - 1 - 1 = 1,$$
$$1 \leq x_m \times x_n = 1,\ \forall (m, n) \in E(S).$$

$$\text{When } x_j = 0,\ \forall j \in S \setminus \xi_5^e :$$

$$0 \geq 1 - 0 + 0 - 1 = 0,$$
$$0 \leq x_m \times x_n = 0,\ \forall (m, n) \in E(S).$$

– For all $i \in N_{\xi_7^e},\ e = 1, \ldots, |\xi_7|$:

$$\text{When } x_j = 0,\ \forall j \in \xi_7^e :$$

$$0 \geq -(|S| - |\xi_7^e| - 1) + (|S| - |\xi_7^e| - 1) + 0) = 0,$$
$$0 \leq 1 \times x_m = 0,\ \forall m \in \xi_7^e$$

$$\text{When } x_j = 0,\ \forall j \in S \setminus \xi_7^e :$$

$$0 \geq -|\xi_7^e| + |\xi_7^e| - 1 + 1 = 0,$$
$$0 \leq 1 \times x_m = 1,\ \forall m \in \xi_7^e.$$

– For all $i \in N_{\xi_8^e},\ e = 1,\dots,|\xi_8|$: let the minimum cardinality $|\xi_8^e| = 2$, so $|M_{cut}^8(\xi_8^e)| = |S|-2$ and $|E_{cut}^8(\xi_8^e)| = |S|-2$.

$$\begin{aligned}&\text{When } x_j = 0,\ \forall j \in \xi_8^e :\\ &0 \geq -1-(|S|-4)+0+0 = -1-|S|+4,\\ &0 \leq 1-1+0+0+0 = 0.\\ &\text{When } x_j = 0,\ \forall j \in S \setminus \xi_8^e :\\ &0 \geq -1-2+1+2 = 0,\\ &0 \leq 1-1-2+1+2 = 1.\end{aligned}$$

– For all $i \in N_{\xi_9^e},\ e = 1,\dots,|\xi_9|$: let the minimum cardinality $|\xi_9^e| = 3$, so $|E_{cut}^9(\xi_9^e)| = 4$.

$$\begin{aligned}&\text{When } x_j = 0,\ \forall j \in \xi_9^e :\\ &0 \geq -1-2+2+0 = -1,\\ &0 \leq 1 \times x_m = 0,\ \forall m \in \xi_9^e.\\ &\text{When } x_j = 0,\ \forall j \in S \setminus \xi_9^e :\\ &0 \geq -1-2+0+2 = -1,\\ &0 \leq 1 \times x_m = 1,\ \forall m \in \xi_9^e.\end{aligned}$$

– For all $i \in N_{\xi_{10}^e},\ e = 1,\dots,|\xi_{10}|$:

$$\begin{aligned}&\text{when } x_j = 0,\ \forall j \in \xi_{10}^e :\\ &0 \geq 0-(|S|-3)+(|S|-|\xi_{10}^e|-3) = -|\xi_{10}^e|,\ \forall m \in \xi_{10}^e,\\ &0 \leq 1 \times x_m = 0,\ \forall m \in \xi_{10}^e.\\ &\text{When } x_j = 0,\ \forall j \in S \setminus \xi_{10}^e :\\ &0 \geq -2-(|\xi_{10}^e|-1)+(|\xi_{10}^e|-2)+1 = -2,\\ &0 \leq 1 \times x_m = 1,\ \forall m \in \xi_{10}^e.\end{aligned}$$

- Let $x_i = 0 \wedge \prod_{j\in S} x_j = 0$, all the estimators hold.

A.2 Description of the sets for compact form of estimators

In this section, we provide the sets to write a compact form of the estimators.

- When $k = 1$:

  $A^+(1,e) = \xi_1^e \cup M,\quad B^+(1,e) = \xi_1^e,\quad C^+(1,e) = E(S),\quad d_i^+(1,e) = 0,$
  $a_{ij}^+(1,e) = [-1 : \text{if } j \in \xi_1^e \cup M],\quad b_{ij}^+(1,e) = [1 : \text{if } j \in \xi_1^e],\quad c_{jp}^+(1,e) = [1 : \text{if } (j,p) \in E(S)]$

  and

  $A^-(1,e) = \emptyset,\quad B^-(1,e) = \{m\},\quad C^-(1,e) = \emptyset,\quad d_i^-(1,e) = 0,$
  $a_{ij}^-(1,e) = 0,\quad b_{ij}^-(1,e) = [1 : \text{if } j = m],\quad c_{jp}^-(1,e) = 0$

  $\forall e = 1,\dots,|\xi_1|$.

- When $k = 2$:

  $A^+(2,e) = \{i\} \cup (S \setminus (S(m) \cup \{m\})),\quad B^+(2,e) = \xi_2^e,\quad C^+(2,e) = E(S \setminus \{m\}),\quad d_i^+(2,e) = 0,$
  $a_{ij}^+(2,e) = [-1 : \text{if } j = i \text{or } j \in S \setminus (S(m) \cup \{m\})],\quad b_{ij}^+(2,e) = [1 : \text{if } j \in \xi_2^e],\quad c_{jp}^+(2,e) = [1 : \text{if } (j,p) \in E(S \setminus \{m\})]$

and

$$A^-(2,e)=\emptyset,\quad B^-(2,e)=\{m\},\quad C^-(2,e)=\emptyset,\quad d_i^-(2,e)=0,$$
$$a_{ij}^-(2,e)=0,\quad b_{ij}^-(2,e)=[1:\text{if } j=m],\quad c_{jp}^-(2,e)=0$$

$\forall e=1,\ldots,|\xi_2|$.

- When $k=3$:

$$A^+(3,e)=\xi_3^e\cup(M\setminus\xi_3^e),\quad B^+(3,e)=\xi_3^e,\quad C^+(3,e)=E(S),\quad d_i^+(3,e)=0,$$
$$a_{ij}^+(3,e)=[-2:\text{if } j\in\xi_3^e,-1:\text{if } j\in M\setminus\xi_3^e],\quad b_{ij}^+(3,e)=[1:\text{if } j\in\xi_3^e],\quad c_{jp}^+(3,e)=[1:\text{if } (j,p)\in E(S)]$$

and

$$A^-(3,e)=\emptyset,\quad B^-(3,e)=\{m\},\quad C^-(3,e)=\emptyset,\quad d_i^-(3,e)=0,$$
$$a_{ij}^-(3,e)=0,\quad b_{ij}^-(3,e)=[1:\text{if } j=m],\quad c_{jp}^-(3,e)=0$$

$\forall e=1,\ldots,|\xi_3|$.

- When $k=4$:

$$A^+(4,e)=M^4_{\text{cut}}(\xi_4^e)\cup\{i\},\quad B^+(4,e)=\xi_4^e,\quad C^+(4,e)=E^4_{\text{cut}}(\xi_4^e),\quad d_i^+(4,e)=0,$$
$$a_{ij}^+(4,e)=[-1:\text{if } j\in M^4_{\text{cut}}(\xi_4^e),-(|\xi_4^e|-1):\text{if } j=i],\quad b_{ij}^+(4,e)=[1:\text{if } j\in\xi_4^e],\quad c_{jp}^+(4,e)=[1:\text{if } (j,p)\in E^4_{\text{cut}}(\xi_4^e)]$$

and

$$A^-(4,e)=\{i\}\cup\zeta_4^e,\quad B^-(4,e)=\zeta_4^e,\quad C^-(4,e)=E(\zeta_4^e),\quad d_i^-(4,e)=1,$$
$$a_{ij}^-(4,e)=[-1:\text{if } j=i,-1:\text{if } j\in\zeta_4^e],\quad b_{ij}^-(4,e)=[1:\text{if } j\in\zeta_4^e],\quad c_{jp}^-(4,e)=[1:\text{if } (j,p)\in E(\zeta_4^e)]$$

$\forall e=1,\ldots,|\xi_4|$.

- When $k=5$:

$$A^+(5,e)=\{i\},\quad B^+(5,e)=S,\quad C^+(5,e)=\emptyset,\quad d_i^+(5,e)=0,$$
$$a_{ij}^+(5,e)=[-(|S|-1):\text{if } j=i],\quad b_{ij}^+(5,e)=[1:\text{if } j\in S],\quad c_{jp}^+(5,e)=0$$

and

$$A^-(5,e)=\{i\}\cup\zeta_5^e,\quad B^-(5,e)=\zeta_5^e,\quad C^-(5,e)=E(\zeta_5^e),\quad d_i^-(5,e)=1,$$
$$a_{ij}^-(5,e)=[-1:\text{if } j=i,-1:\text{if } j\in\zeta_5^e],\quad b_{ij}^-(5,e)=[1:\text{if } j\in\zeta_5^e],\quad c_{jp}^-(5,e)=[1:\text{if } (j,p)\in E(\zeta_5^e)]$$

$\forall e=1,\ldots,|\xi_5|$.

- When $k=6$:

$$A^+(6,e)=M\cup\{i\},\quad B^+(6,e)=\emptyset,\quad C^+(6,e)=E(S),\quad d_i^+(6,e)=-1,$$
$$a_{ij}^+(6,e)=[-1:\text{if } j\in M,1:\text{if } j=i],\quad b_{ij}^+(6,e)=0,\quad c_{jp}^+(6,e)=[1:\text{if } (j,p)\in E(S)]$$

and

$$A^-(6,e)=\emptyset,\quad B^-(6,e)=\emptyset,\quad C^-(6,e)=\{(m,n)\},\quad d_i^-(6,e)=0,$$
$$a_{ij}^-(6,e)=0,\quad b_{ij}^-(6,e)=0,\quad c_{jp}^-(6,e)=[1:\text{if } (j,p)=(m,n)]$$

$\forall e=1,\ldots,|\xi_6|$.

- When $k=7$:

$$A^+(7,e)=M\cup\xi_7^e,\quad B^+(7,e)=M\cap\xi_7^e,\quad C^+(7,e)=E(S),\quad d_i^+(7,e)=0,$$
$$a_{ij}^+(7,e)=[-1:\text{if } j\in M\cup\xi_7^e],\quad b_{ij}^+(7,e)=[1:\text{if } j\in M\cap\xi_7^e],\quad c_{jp}^+(7,e)=[1:\text{if } (j,p)\in E(S)]$$

and

$$A^-(7,e)=\emptyset,\quad B^-(7,e)=\{m\},\quad C^-(7,e)=\emptyset,\quad d_i^-(7,e)=0,$$
$$a_{ij}^-(7,e)=0,\quad b_{ij}^-(7,e)=[1:\text{if } j=m],\quad c_{jp}^-(7,e)=0$$

$\forall e=1,\ldots,|\xi_7|$.

- When $k=8$:

$$A^+(8,e)=M^8_{\text{cut}}(\xi_8^e)\cup\{i\},\quad B^+(8,e)=\xi_8^e,\quad C^+(8,e)=E^8_{\text{cut}}(\xi_8^e),\quad d_i^+(8,e)=0,$$
$$a_{ij}^+(8,e)=[-1:\text{if } j\in M^8_{\text{cut}}(\xi_8^e),-(|\xi_8^e|-1):\text{if } j=i],\quad b_{ij}^+(8,e)=[1:\text{if } j\in\xi_8^e],\quad c_{jp}^+(8,e)=[1:\text{if } (j,p)\in E^8_{\text{cut}}(\xi_8^e)]$$

and

$$A^-(8,e)=\{i\}\cup\zeta_8^e,\quad B^-(8,e)=\zeta_8^e,\quad C^-(8,e)=E(\zeta_8^e),\quad d_i^-(8,e)=1,$$
$$a_{ij}^-(8,e)=[-1:\text{if } j=i,-1:\text{if } j\in\zeta_8^e],\quad b_{ij}^-(8,e)=[1:\text{if } j\in\zeta_8^e],\quad c_{jp}^-(8,e)=[1:\text{if } (j,p)\in E(\zeta_8^e)]$$

$\forall e=1,\ldots,|\xi_8|$.

- When $k = 9$:

$$A^+(9,e) = \{i\} \cup (F \cup S(j)), \quad B^+(9,e) = F, \quad C^+(9,e) = E^9_{\text{cut}}(\xi^e_9), \quad d^+_i(9,e) = 0,$$
$$a^+_{ij}(9,e) = [-1 : \text{if } j \in \xi^e_9 \cup \{i\}], \quad b^+_{ij}(9,e) = [1 : \text{if } j \in F], \quad c^+_{jp}(9,e) = [1 : \text{if } (j,p) \in E^9_{\text{cut}}(\xi^e_9)]$$

and

$$A^-(9,e) = \emptyset, \quad B^-(9,e) = \{m\}, \quad C^-(9,e) = \emptyset, \quad d^-_i(9,e) = 0,$$
$$a^-_{ij}(9,e) = 0, \quad b^-_{ij}(9,e) = [1 : \text{if } j = m], \quad c^-_{jp}(9,e) = 0$$

$\forall e = 1, \ldots, |\xi_9|$.

- When $k = 10$:

$$A^+(10,e) = \{m\} \cup (M \setminus \{m\}), \quad B^+(10,e) = \{m\}, \quad C^+(10,e) = E(S), \quad d^+_i(10,e) = 0,$$
$$a^+_{ij}(10,e) = [-2 : \text{if } j = m \in \xi^e_{10}, -1 : \text{if } j \in M \setminus \{m\}], \quad b^+_{ij}(10,e) = [1 : \text{if } j = m], \quad c^+_{jp}(10,e) = [1 : \text{if } (j,p) \in E(S)]$$

and

$$A^-(10,e) = \emptyset, \quad B^-(10,e) = \{m\}, \quad C^-(10,e) = \emptyset, \quad d^-_i(10,e) = 0,$$
$$a^-_{ij}(10,e) = 0, \quad b^-_{ij}(10,e) = [1 : \text{if } j = m], \quad c^-_{jp}(10,e) = 0$$

$\forall e = 1, \ldots, |\xi_{10}|$.

References

Warren P Adams and Hanif D Sherali. A tight linearization and an algorithm for zero-one quadratic programming problems. Management Science, 32(10):1274–1290, 1986.

Warren P Adams and Hanif D Sherali. Linearization strategies for a class of zero-one mixed integer programming problems. Operations Research, 38(2):217–226, 1990.

Alain Billionnet and Frédéric Calmels. Linear programming for the 0–1 quadratic knapsack problem. European Journal of Operational Research, 92(2):310–325, 1996.

Alberto Caprara, David Pisinger, and Paolo Toth. Exact solution of the quadratic knapsack problem. INFORMS Journal on Computing, 11(2):125–137, 1999.

Zhixiang Chen and Bin Fu. Approximating multilinear monomial coefficients and maximum multilinear monomials in multivariate polynomials. Journal of Combinatorial Optimization, 25(2):234–254, 2013.

Václav Chvátal and Peter L Hammer. Aggregations of inequalities. Studies in Integer Programming, Annals of Discrete Mathematics, 1:145–162, 1977.

M Eliass Fennich, Franklin Djeumou Fomeni, and Leandro C Coelho. A novel dynamic programming heuristic for the quadratic knapsack problem. European Journal of Operational Research, 319(1):102–120, 2024.

Franklin Djeumou Fomeni. A new family of facet defining inequalities for the maximum edge-weighted clique problem. Optimization Letters, 11:47–54, 2016.

Franklin Djeumou Fomeni. A lifted-space dynamic programming algorithm for the quadratic knapsack problem. Discrete Applied Mathematics, 335:52–68, 2023.

Franklin Djeumou Fomeni, Konstantinos Kaparis, and Adam N Letchford. Cutting planes for RLT relaxations of mixed 0–1 polynomial programs. Mathematical Programming, 151:639–658, 2015.

Franklin Djeumou Fomeni, Konstantinos Kaparis, and Adam N Letchford. A cut-and-branch algorithm for the quadratic knapsack problem. Discrete Optimization, 44:100579, 2022.

Fabio Furini, Emiliano Traversi, Pietro Belotti, Antonio Frangioni, Ambros Gleixner, Nick Gould, Leo Liberti, Andrea Lodi, Ruth Misener, Hans Mittelmann, et al. Qplib: a library of quadratic programming instances. Mathematical Programming Computation, 11:237–265, 2019.

Giorgio Gallo, Peter L Hammer, and Bruno Simeone. Quadratic knapsack problems. In Combinatorial optimization, pages 132–149. Springer, 2009.

Fred Glover and Eugene Woolsey. Further reduction of zero-one polynomial programming problems to zero-one linear programming problems. Operations Research, 21(1):156–161, 1973.

Zeren Huang, Kerong Wang, Furui Liu, Hui-Ling Zhen, Weinan Zhang, Mingxuan Yuan, Jianye Hao, Yong Yu, and Jun Wang. Learning to select cuts for efficient mixed-integer programming. Pattern Recognition, 123:108353, 2022.

IBM ILOG. IBM ILOG CPLEX Optimization Studio CPLEX User's Manual. IBM, 2023. URL https://www.ibm.com/docs/en/icos/22.1.1?topic=optimizers-users-manual-cplex.

Manfred Padberg. The boolean quadric polytope: some characteristics, facets and relatives. Mathematical programming, 45:139–172, 1989.

Pablo A Parrilo. Semidefinite programming relaxations for semialgebraic problems. Mathematical programming, 96: 293–320, 2003.

Clifford C Petersen et al. A note on transforming the product of variables to linear form in linear programs. Diskussionspapier, Purdue University, 1971.

Hanif D Sherali. Rlt: A unified approach for discrete and continuous nonconvex optimization. Annals of Operations Research, 149(1):185, 2007.

Hanif D Sherali and Warren P Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM Journal on Discrete Mathematics, 3(3):411–430, 1990.

Hanif D Sherali and Warren P Adams. A hierarchy of relaxations and convex hull characterizations for mixed-integer zero-one programming problems. Discrete Applied Mathematics, 52(1):83–106, 1994.

Hanif D Sherali and Warren P Adams. Reformulation-linearization techniques for discrete optimization problems. Handbook of Combinatorial Optimization: Volume1–3, pages 479–532, 1998.

Hanif D Sherali and Cihan H Tuncbilek. A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique. Journal of Global Optimization, 2:101–112, 1992.

Mohit Tawarmalani and Nikolaos V Sahinidis. Convexification and global optimization in continuous and mixed-integer nonlinear programming: theory, algorithms, software, and applications, volume 65. Springer Science & Business Media, 2013.

Douglas B. West. Introduction to Graph Theory. Prentice Hall, Upper Saddle River, NJ, 2nd edition, 2001. University of Illinois — Urbana.