

# Graphs obtained by disjoint unions and joins of cliques and stable sets

A. Hertz

G–2023–59

December 2023

---

La collection *Les Cahiers du GERAD* est constituée des travaux de recherche menés par nos membres. La plupart de ces documents de travail a été soumis à des revues avec comité de révision. Lorsqu'un document est accepté et publié, le pdf original est retiré si c'est nécessaire et un lien vers l'article publié est ajouté.

**Citation suggérée** : A. Hertz (Décembre 2023). Graphs obtained by disjoint unions and joins of cliques and stable sets, Rapport technique, Les Cahiers du GERAD G– 2023–59, GERAD, HEC Montréal, Canada.

**Avant de citer ce rapport technique**, veuillez visiter notre site Web (<https://www.gerad.ca/fr/papers/G-2023-59>) afin de mettre à jour vos données de référence, s'il a été publié dans une revue scientifique.

---

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2023  
– Bibliothèque et Archives Canada, 2023

The series *Les Cahiers du GERAD* consists of working papers carried out by our members. Most of these pre-prints have been submitted to peer-reviewed journals. When accepted and published, if necessary, the original pdf is removed and a link to the published article is added.

**Suggested citation**: A. Hertz (December 2023). Graphs obtained by disjoint unions and joins of cliques and stable sets, Technical report, Les Cahiers du GERAD G–2023–59, GERAD, HEC Montréal, Canada.

**Before citing this technical report**, please visit our website (<https://www.gerad.ca/en/papers/G-2023-59>) to update your reference data, if it has been published in a scientific journal.

---

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Legal deposit – Bibliothèque et Archives nationales du Québec, 2023  
– Library and Archives Canada, 2023

# Graphs obtained by disjoint unions and joins of cliques and stable sets

Alain Hertz <sup>a</sup>

<sup>a</sup> *Département de mathématiques et de génie industriel, Polytechnique Montréal & GERAD Montréal, (Qc), Canada, H3T 1J4*

alain.hertz@gerad.ca

December 2023  
Les Cahiers du GERAD  
G–2023–59

Copyright © 2023 Hertz

---

Les textes publiés dans la série des rapports de recherche *Les Cahiers du GERAD* n'engagent que la responsabilité de leurs auteurs. Les auteurs conservent leur droit d'auteur et leurs droits moraux sur leurs publications et les utilisateurs s'engagent à reconnaître et respecter les exigences légales associées à ces droits. Ainsi, les utilisateurs:

- Peuvent télécharger et imprimer une copie de toute publication du portail public aux fins d'étude ou de recherche privée;
- Ne peuvent pas distribuer le matériel ou l'utiliser pour une activité à but lucratif ou pour un gain commercial;
- Peuvent distribuer gratuitement l'URL identifiant la publication.

Si vous pensez que ce document enfreint le droit d'auteur, contactez-nous en fournissant des détails. Nous supprimerons immédiatement l'accès au travail et enquêterons sur votre demande.

The authors are exclusively responsible for the content of their research papers published in the series *Les Cahiers du GERAD*. Copyright and moral rights for the publications are retained by the authors and the users must commit themselves to recognize and abide the legal requirements associated with these rights. Thus, users:

- May download and print one copy of any publication from the public portal for the purpose of private study or research;
- May not further distribute the material or use it for any profit-making activity or commercial gain;
- May freely distribute the URL identifying the publication.

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

**Abstract :** We consider the set of graphs that can be constructed from a one-vertex graph by repeatedly adding a clique or a stable set linked to all or none of the vertices added in previous steps. This class of graphs contains various well-studied graph families such as threshold, domishold, comishold and complete multipartite graphs, as well as graphs with linear clique-width at most 2. We show that it can be characterized by three forbidden induced subgraphs as well as by properties involving maximal stable sets and minimal dominating sets. We also give a simple recognition algorithm and formulas for the computation of the stability and domination numbers of these graphs.

**Keywords :** Disjoint union and join of graphs, cliques, stable sets, dominating sets

## 1 Introduction

The *disjoint union* of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  is the graph  $G = (V, E)$  with vertex set  $V = V_1 \cup V_2$  and edge set  $E = E_1 \cup E_2$ , while the *join* of  $G_1$  and  $G_2$  is the graph obtained from their disjoint union by linking each vertex of  $G_1$  to each vertex of  $G_2$ . The *complement*  $\overline{G}$  of a graph  $G$  is the graph with the same vertex as  $G$  and such that two distinct vertices are adjacent if and only if they are not adjacent in  $G$ . A set  $W$  of vertices is *stable* if no two vertices in  $W$  are linked by an edge, it is a *clique* if every two vertices in  $W$  are adjacent, and it is *dominating* if every vertex not in  $W$  is adjacent to at least one member of  $W$ . A *complete graph* is a graph whose vertex set is a clique, and an *empty graph* is the complement of a complete graph (i.e., its vertex set is stable). The *stability number*  $\alpha(G)$  of a graph  $G$  is the maximum size of a stable set in  $G$ , and the *domination number*  $\beta(G)$  of  $G$  is the smallest size of a dominating set in  $G$ . The *order* of a graph is its number of vertices. A graph is *H-free* if it does not contain  $H$  as induced subgraph. In what follows,  $P_n$  and  $K_n$  denote the path of order  $n$  and the complete graph of order  $n$ , respectively.

In this paper, we define and give various characterization of *junior* graphs (contraction of *join* and *union*) which are built by repeatedly performing joins and disjoint unions with empty and complete graphs.

**Definition 1.** A *junior* graph is a graph that can be built from the one-vertex graph by iteratively performing one of the following four operations:

- join with an empty graph;
- join with a complete graph of order at least 2;
- disjoint union with an empty graph;
- disjoint union with a complete graph of order at least 2.

Clearly, the class of junior graphs is hereditary (i.e., it is closed under induced subgraph containment) and complements of junior graphs are also junior graphs. Various well-studied graph families belong to the class of junior graphs. For example, *threshold* graphs [3] are constructed from the one-vertex graph by repeatedly applying one of the following two operations:

- join with the empty graph of order 1;
- disjoint union with the empty graph of order 1.

As another example, the *domishold* graphs [1] are constructed from the one-vertex graph by repeatedly applying one of the following two operations:

- join with the empty graph of order 1 or 2;
- disjoint union with the empty graph of order 1.

Additional junior graphs are *co-domishold* (complements of domishold) graphs, [2] *complete multipartite* graphs which are joins of empty graphs, and the *friendship* graphs which are the join of an empty graph of order 1 with the disjoint union of complete graphs of order 2.

As shown in [3] and [1], threshold and domishold graphs can be defined by properties on their stable and dominating sets, respectively. More precisely:

- a graph  $G = (V, E)$  is a threshold graph if there are a real number  $T$  and for each vertex  $v$  a real vertex weight  $\omega(v)$  such that a set  $S \subseteq V$  is stable if and only if  $\sum_{v \in S} \omega(v) \leq T$ .
- a graph  $G = (V, E)$  is a domishold graph if there are a real number  $T$  and for each vertex  $v$  a real vertex weight  $\omega(v)$  such that a set  $D \subseteq V$  is dominating if and only if  $\sum_{v \in D} \omega(v) \geq T$ .

The same authors also characterize threshold and domishold graphs by a list of forbidden induced subgraphs which are all depicted in Figure 1:

- a graph is a threshold graph if and only if it is  $(P_4, 2K_2, \overline{2K_2})$ -free;
- a graph is a domishold graph if and only if it is  $(P_4, 2K_2, \overline{2K_3}, \overline{K_3 + P_3}, \overline{2P_3})$ -free.

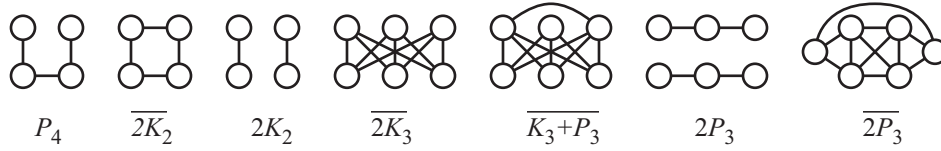


Figure 1: Some graphs.

In the next section, we show that junion graphs are characterized by properties on their maximal stable sets and minimal dominating sets and by a list of three forbidden induced subgraphs. More precisely, we show that a graph is a junion graph if and only if it is  $(P_4, 2P_3, \overline{2P_3})$ -free.

A graph is a *split* graph if its vertex set can be partitioned in an stable set and a clique. Threshold graphs form a subclass of split graphs. A graph is  $(P_4, \overline{2K_2}, 2P_3)$ -free if and only if it can be obtained from a threshold graph by replacing every vertex in the stable set of the split partition by a clique. These graphs are studied in [5]. They are junion graphs since  $\overline{2K_2}$  is a subgraph of  $\overline{2P_3}$ . They complement (i.e., the class of  $(P_4, 2K_2, \overline{2P_3})$ -free graphs) are also junion graphs and are characterized in [4] as those graphs with linear-clique width at most 2. A picture of the inclusion relations between the mentioned subclasses of junion graphs is given in Figure 2.

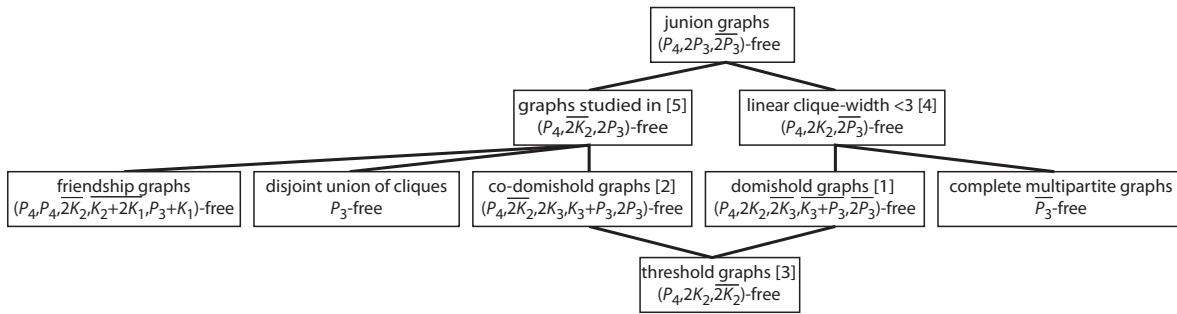


Figure 2: Subclasses of junion graphs.

## 2 Various characterizations of junion graphs

For a set  $W$ , an element  $u \in W$  and an element  $v \notin W$ , let  $W - u$  be the subset of  $W$  obtained by removing  $u$  and let  $W + v$  be equal to  $W \cup \{v\}$ . Also, let  $N_G(u)$  be the set of vertices adjacent to  $u$  in  $G$ . Consider the following property, where  $\mathcal{S}(G)$  is the set of maximal stable sets in  $G$ .

**Definition 2.** A graph  $G = (V, E)$  has Property  $P_S$  if and only if all pairs  $S_1, S_2$  of disjoint sets in  $\mathcal{S}(G)$  satisfy at least one of the following two conditions:

- (a) there are  $u_1 \in S_1$  and  $u_2 \in S_2$  such that both  $S_1 - u_1 + u_2$  and  $S_2 - u_2 + u_1$  belong to  $\mathcal{S}(G)$ .
- (b)  $S_1 \cup N_G(u_1) = V$  for all  $u_1 \in S_1$  or/and  $S_2 \cup N_G(u_2) = V$  for all  $u_2 \in S_2$ .

Similarly, let  $\mathcal{D}(G)$  be the set of minimal dominating sets in  $G$ .

**Definition 3.** A graph  $G$  has Property  $P_D$  if and only if all pairs  $D_1, D_2$  of disjoint sets in  $\mathcal{D}(G)$  satisfy at least one of the following two conditions:

- (a) both  $D_1 - u_1 + u_2$  and  $D_2 - u_2 + u_1$  belong to  $\mathcal{D}(G)$  for every two non-adjacent vertices  $u_1 \in D_1$  and  $u_2 \in D_2$ ,
- (b) there is  $u_1 \in D_1$  and  $u_2 \in D_2$  such that  $N_G(u_1) \cap D_2 = \{u_2\}$  and  $N_G(u_2) \cap D_1 = \{u_1\}$ .

We are now ready to state the theorem that gives equivalent characterizations for junion graphs.

**Theorem 1.** *The following statements are equivalent for all graphs  $G$ :*

- (i)  $G$  is a junion graph;
- (ii) all induced subgraphs  $G'$  of  $G$  satisfy Property  $P_S$ ;
- (iii) all induced subgraphs  $G'$  of  $G$  satisfy Property  $P_D$ ;
- (iv)  $G$  is  $(P_4, 2P_3, \overline{2P_3})$ -free.

**Proof.**

(i) $\Rightarrow$ (ii) and (iii). We proceed by induction on the number of vertices in  $G$ . Clearly, the graph of order 1 is a junion graph that satisfies Properties  $P_S$  and  $P_D$ . So assume that all junion graphs with at most  $n$  vertices satisfy Properties  $P_S$  and  $P_D$  and let  $G$  be a junion graph of order  $n + 1$ . Since the class of junion graphs is hereditary, it remains to prove that  $G$  satisfies  $P_S$  and  $P_D$ . Let  $O$  be the last operation performed to obtain  $G$ .

- If  $O$  is the disjoint union of a subgraph  $G'$  of  $G$  with an empty graph  $H$ , then all maximal stable sets and all minimal dominating sets of  $G$  contain all vertices of  $H$ . Hence, there are no pairs  $S_1, S_2$  of disjoint sets in  $\mathcal{S}(G)$  and no pairs  $D_1, D_2$  of disjoint sets in  $\mathcal{D}(G)$ , which means that  $G$  satisfies  $P_S$  and  $P_D$ .
- If  $O$  is the disjoint union of a subgraph  $G'$  of  $G$  with a complete graph  $H$  of order at least 2, then all sets in  $\mathcal{S}(G)$  and in  $\mathcal{D}(G)$  contain exactly one vertex of  $H$ . Hence,
  - if  $S_1, S_2$  are two disjoint maximal stable sets of  $G$ ,  $S_1$  contains a vertex  $u_1$  of  $H$  and  $S_2$  contains a vertex  $u_2 \neq u_1$  of  $H$ . So (a) of  $P_S$  is satisfied since both  $S_1 - u_1 + u_2$  and  $S_2 - u_2 + u_1$  belong to  $\mathcal{S}(G)$
  - if  $D_1, D_2$  are two disjoint minimal dominating sets of  $G$ ,  $D_1$  contains a vertex  $u_1$  of  $H$  and  $D_2$  contains a vertex  $u_2 \neq u_1$  of  $H$ . So (b) of  $P_D$  is satisfied since  $N_G(u_1) \cap D_2 = \{u_2\}$  and  $N_G(u_2) \cap D_1 = \{u_1\}$ .
- If  $O$  is the join of a subgraph  $G'$  of  $G$  with an empty graph  $H$ , then  $\mathcal{S}(G') \subset \mathcal{S}(G)$  and  $\mathcal{D}(G') \subset \mathcal{D}(G)$ . Let  $S_1, S_2$  be two disjoint sets in  $\mathcal{S}(G)$  and let  $D_1, D_2$  be two disjoint sets in  $\mathcal{D}(G)$ . If  $S_1$  and  $S_2$  do not contain any vertices of  $H$ , we know by induction hypothesis that (a) or/and (b) of  $P_S$  is satisfied. Similarly, (a) or/and (b) of  $P_D$  is satisfied if  $D_1$  and  $D_2$  do not contain any vertices of  $H$ . So assume, without loss of generality that  $S_1$  and  $D_1$  contain at least one vertex of  $H$ . Then
  - $S_1$  is the vertex set of  $H$  and (b) of  $P_S$  is satisfied since  $S_1 \cup N_G(u) = V$  for all  $u \in S_1$ .
  - if  $D_2$  contains no vertex of  $H$  then (a) of  $P_D$  is satisfied (since all  $u_1 \in D_1$  are adjacent to all  $u_2 \in D_2$ ); otherwise,  $D_1 = \{v_1, w_1\}$  and  $D_2 = \{v_2, w_2\}$  with  $v_1, v_2$  in  $H$  and  $w_1, w_2$  in  $G'$  which means that (a) of  $P_D$  is satisfied since both  $\{v_1, w_2\}$  and  $\{v_2, w_1\}$  belong to  $\mathcal{D}(G)$ .
- If  $O$  is the join of a subgraph  $G'$  of  $G$  with a complete graph  $H$  of order at least 2, then  $\mathcal{S}(G)$  and  $\mathcal{D}(G)$  are obtained from  $\mathcal{S}(G')$  and  $\mathcal{D}(G')$  by adding the singletons  $\{u\}$  for all vertices  $u$  in  $H$ . Let  $S_1, S_2$  be two disjoint sets in  $\mathcal{S}(G)$  and let  $D_1, D_2$  be two disjoint sets in  $\mathcal{D}(G)$ . As in the previous case, we can assume that  $S_1$  and  $D_1$  contain a vertex  $u$  of  $H$ , else (a) or/and (b) of  $P_S$  and  $P_D$  is satisfied. Then
  - $S_1 = \{u\}$  and (b) of  $P_S$  is satisfied since  $S_1 \cup N_G(u) = V$ .
  - $D_1 = \{u\}$  and (a) of  $P_D$  is satisfied since  $u$  is adjacent to all vertices in  $D_2$ .

(ii) and (iii) $\Rightarrow$ (iv). Suppose that  $G$  is a  $P_4$ ,  $2P_3$  or  $\overline{2P_3}$ . It is easy to verify that the black and grey maximal stable sets  $S_1$  and  $S_2$  depicted in Figure 3 (a), (b) and (c) show that  $G$  does not satisfy  $P_S$ . Also, the black and grey minimal dominating sets  $D_1$  and  $D_2$  depicted in Figure 3 (a), (b) and (d) show that  $G$  does not satisfy  $P_D$ .

(iv) $\Rightarrow$ (i). We proceed by induction on the order of  $G$ . The result is clearly valid for the graph of order 1. So assume it is valid for graphs of order at most  $n$  and let  $G$  be a  $(P_4, 2P_3, \overline{2P_3})$ -free graph of order  $n + 1$ . Since  $G$  has no induced  $P_4$ , it is well-known that either  $G$  or  $\overline{G}$  is disconnected (see for example [6]).

- If  $G$  is disconnected then at least one of its connected component is a complete graph  $H$ , else  $G$  contains a  $2P_3$  as induced subgraph. Hence,  $G$  is the disjoint union of a subgraph  $G'$  with an empty graph of order 1 or a complete graph of order at least 2.
- If  $\overline{G}$  is disconnected then at least one of its connected component  $H$  is a complete graph, else  $G$  contains a  $\overline{2P_3}$  as induced subgraph. Hence,  $G$  is the join of a graph  $G'$  with an empty graph of order 1 or a complete graph of order at least 2.

In both cases, the subgraph  $G'$  of  $G$  is  $(P_4, 2P_3, \overline{2P_3})$ -free and we know by induction hypothesis that  $G'$  is a junion graph. Hence,  $G$  is also a junion graph.  $\square$

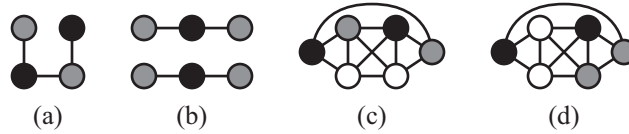


Figure 3: Pairs of maximal stable sets and minimal dominating sets in  $P_4$ ,  $2P_3$  and  $\overline{2P_3}$ .

### 3 Recognition algorithm and computation of the stability and domination numbers of junion graphs

We say that a connected component of a graph  $G$  is complete if it is a complete induced subgraph of  $G$ . It follows from Definition 1 that a graph  $G$  is a junion graph if and only if exactly one of  $G$  and  $\overline{G}$  is disconnected and has a complete connected component. This leads to the following recognition algorithm for junion graphs  $G$ .

#### Recognition of junion graphs

*Input:* a graph  $G$ .

1. If both  $G$  and  $\overline{G}$  are connected, then STOP :  $G$  is not a junion graph.
2. If  $G$  is connected then set  $H \leftarrow \overline{G}$ , else set  $H \leftarrow G$ .
3. If  $H$  has no complete connected component then STOP :  $G$  is not a junion graph.
4. Remove from  $H$  the vertices of all of its complete connected components.
5. If  $H$  is empty then STOP :  $G$  is a junion graph, else set  $H \leftarrow \overline{H}$ .
6. If  $H$  is connected then STOP :  $G$  is not a junion graph; else go to 3.

We finally show how to compute the domination and stability numbers of junion graphs. A vertex of a graph  $G$  is *universal* if it is adjacent to all other vertices of  $G$ . Clearly,  $\beta(G) = 1$  if  $G$  has a universal vertex and  $\beta(G) = 2$  if  $G$  is a connected junion graph with no universal vertex. Note that a junion graph has at most one non-complete connected component. So, let  $c(G)$  be the number of complete connected components of a graph  $G$ . If  $G$  is a junion graph, then

$$\beta(G) = \begin{cases} c(G) & \text{if all connected components of } G \text{ are complete,} \\ c(G)+1 & \text{if } G \text{ has a non-complete connected component with a universal vertex,} \\ c(G)+2 & \text{otherwise.} \end{cases}$$

Now, let  $o(G)$  be the largest order of a complete connected component of a graph  $G$  and let  $G^c$  be the graph obtained from  $G$  by removing all vertices of  $G$  that belong to a complete connected component of  $G$  or  $\overline{G}$ . Note that if  $G$  is a junion graph, then  $G^c$  is a proper subgraph of  $G$  since exactly one of

$G$  or  $\overline{G}$  has at least one complete connected component. The stability number  $\alpha(G)$  of a union graph  $G$  can easily be determined using the following recursive formula:

$$\alpha(G) = \begin{cases} \max\{\alpha(\overline{G}), \alpha(G^c)\} & \text{if } G \text{ is connected,} \\ c(G) + \alpha(G^c) & \text{otherwise.} \end{cases}$$

Note finally that since the complement of a union graph is a union graph, the largest size  $\omega(G) = \alpha(\overline{G})$  of a clique in a union graph  $G$  is also easy to compute.

## References

- [1] BENZAKEN, C., AND HAMMER, P. L. Linear separation of dominating sets in graphs. *Ann. Discrete Math.* 3, 1–10, 1978.
- [2] CHIARELLI, N., AND MILANIĆ, M. On a class of graphs between threshold and total domishold graphs. *Discrete Applied Mathematics* 195 (2015), 43–58. 12th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW 2013).
- [3] CHVÁTAL, V., AND HAMMER, P. L. Aggregation of inequalities in integer programming. In *Studies in Integer Programming*, P. Hammer, E. Johnson, B. Korte, and G. Nemhauser, Eds., vol. 1 of *Annals of Discrete Mathematics*. Elsevier, 1977, pp. 145–162.
- [4] GURSKI, F. Characterizations for co-graphs defined by restricted nlc-width or clique-width operations. *Discrete Mathematics* 306, 2 (2006), 271–277.
- [5] KARTHICK, T., AND MAFFRAY, F. Square-free graphs with no six-vertex induced path. *SIAM Journal on Discrete Mathematics* 33, 2 (2019), 874–909.
- [6] SEINSCHE, D. On a property of the class of n-colorable graphs. *Journal of Combinatorial Theory, Series B* 16, 2 (1974), 191–193.