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# Evaluation of counterparty credit risk under netting agreements

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**Abstract :** We investigate counterparty credit risk and credit valuation adjustments in portfolios including derivatives with early-exercise opportunities, under a netting agreement. We show that credit risk and netting agreements have a significant impact on the way portfolios are managed (that is, on options' exercise strategies) and, therefore, on the value of the portfolio and on the price of counterparty risk. We derive the value of a netted portfolio as the solution of a zero-sum, finite horizon, discrete-time stochastic game. We show that this dynamic-game interpretation can be used to determine the value of the reglementary capital charges required of financial institutions to cover for counterparty credit risk and we propose a numerical valuation method. Numerical investigations show that currently used numerical approaches can grossly misestimate the value of credit valuation adjustments.

**Keywords :** Game theory, finance, CVA, netting, options

**Résumé :** Cet article étudie le risque de crédit de contrepartie et l'ajustement réglementaire correspondant (CVA) pour des portefeuilles de produits dérivés avec possibilité d'exercice anticipé, lorsque les parties ont signé un accord de compensation (netting). Nous montrons que le risque de contrepartie et le netting ont un impact significatif sur la façon dont les portefeuilles sont gérés (c'est-à-dire sur les stratégies d'exercice des options) et, par conséquent, sur la valeur du portefeuille et sur le prix du risque de contrepartie. Nous obtenons la valeur du portefeuille à partir de la solution d'un jeu dynamique stochastique à somme nulle sur horizon fini. Nous montrons que cette interprétation de l'interaction entre les parties peut être utilisée pour déterminer la valeur des exigences réglementaires requises des institutions financières pour couvrir le risque de contrepartie et nous proposons une approche numérique. Nos expériences numériques montrent que les approches actuellement utilisées en pratique peuvent entraîner des erreurs importantes dans l'estimation du CVA.

**Mots clés :** Théorie des jeux, finance, CVA, netting, options

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# 1 Introduction

Over-the-counter (OTC) derivatives form a considerable portion of the securities market, with the Bank for International Settlements estimating the notional value of outstanding OTC derivatives to be in excess of $600 trillion as of June 2022 (see BIS (2022)). However, the absence of a centralized clearinghouse in the OTC market leads to *counterparty credit risk* (CCR), defined as the risk of incurring losses in OTC contracts, in the event of a counterparty defaulting on its payment obligations.

The 2008 financial crisis highlighted the systemic implications of CCR when the collapse of Lehman Brothers, a significant player in numerous OTC contracts, ignited a chain reaction through the global financial system, causing widespread instability. One of the responses of financial institutions and regulators to CCR was the extensive use of the *credit valuation adjustment* (CVA) as an instrument to mitigate counterparty credit risk (see Duffie and Singleton (2003), Bielecki and Rutkowski (2004), Brigo and Masetti (2005b) for early reviews on CVA).

The CVA can be interpreted as the market value of CCR. It is used to adjust the default-free value of a contract in order to obtain a fair value that accounts for the possible default of the counterparty. The use of the CVA has been advocated by the Basel Committee on Banking Supervision for the management of CCR; specifically, the Basel III Accord (BCBS (2011)) requires financial institutions to calculate a CVA risk capital charge, which is an amount of capital that banks must set aside to absorb potential losses from fluctuations of the CVA. Given that this adjustment can have a significant impact on financial statements and available investment capital, the precise and efficient estimation of the CVA is crucial for financial institutions.

Brigo and Masetti (2005b) give a general pricing formula for the CVA, which can be seen as a call option on the derivative value with a random maturity corresponding to the counterparty default date or, equivalently, as an expected discounted loss from counterparty default. While the computation of the CVA is generally straightforward for European-style derivatives (see, e.g., Klein (1996) and Lando (1998)), this is not the case for derivatives with early-exercise opportunities.

Simulation-regression approaches, commonly known as least-squares Monte Carlo (LSMC) methods, are extensively used in the financial industry to approximate the CVA (Brigo and Pallavicini (2007), Cesari et al. (2009), Gregory (2012)). Simulation-based approaches can be computationally intensive, especially when there are many stochastic factors or correlation structures, and alternative methods have been recently proposed to compute the CVA approximately or semi-analytically (see for instance Kim and Leung (2016), Brigo and Vrins (2018), Antonelli et al. (2022)).

However, none of these approaches consider contracts with early-exercise features. Typically, in practice, simulation-based methods disregard the impact of counterparty risk on the exercise mechanism, which is an unrealistic simplification that can lead to significant misestimations (Klein and Yang (2013), Breton and Marzouk (2018)).

In this paper, we address the estimation of the CVA of portfolios of contracts under a netting agreement, when these contracts can include early-exercise features. Netting agreements between two counterparties involved in multiple contracts are a widespread practice in the financial industry, primarily used to mitigate CCR. Netting refers to the understanding that, in the event of a default, all transactions between the two counterparties are to be netted and considered as a single transaction. As noted in Brigo and Masetti (2005a), the CVA of a netted portfolio is often smaller than the sum of individual CVAs. This result is predominantly driven by the offsetting positions within the portfolio, which reduce the overall exposure to CCR.

Considering netted portfolios raises the issue of both parties being subject to default risk. The *debt valuation adjustment* (DVA) is a measure of the CCR due to one party's own default risk (in a bilateral agreement, it is the negative of the CVA of the other party). The *bilateral valuation adjustment* (BVA), defined as the sum of the CVA and the DVA, accounts for the impact of both parties on the risk adjustment value.

The CVA or BVA of a netted portfolio can be readily obtained using analytical or Monte Carlo techniques when the contracts do not involve early-exercise opportunities (see, e.g., Brigo and Masetti (2005a) and Brigo and Pallavicini (2007)). Extending the approach of Brigo and Pallavicini (2007), Brigo et al. (2011) investigates the bilateral risk of netted portfolios, showing that the risk adjustment is akin to a long position in a put option and a short position in a call option on the residual net value of the portfolio at the relevant default times. Using a similar approach, an iterative evaluation procedure for the BVA of a netted portfolio is proposed in Durand (2010). More recently, Ballotta et al. (2019) introduces a structural approach to compute the CVA of a netted portfolio, discussing the impact of collateralization and wrong-way risk on the CVA value. The evaluation of the BVA of a portfolio for a party engaged in multiple netting sets with different defaultable counterparties is discussed in Burgard and Kjaer (2017) and Brigo et al. (2019), where it is shown that the value of a portfolio is not necessarily equal to the sum of the values of netting sets.

None of the above-mentioned works consider the possibility of counterparties in a netting agreement having early-exercise opportunities. As noted in Breton and Marzouk (2018), it is possible to adapt a dynamic programming approach using risk-adjusted exercise strategies to compute the CVA of a netted portfolio, provided that only one of the counterparties has exercise rights (see also Andersson and Oosterlee (2021) for an empirical study). However, when both counterparties involved in a netting agreement have early-exercise opportunities, exercise decisions by one of the parties impact the expected gains of both parties, adding another layer of complexity to the CVA computation.

In this paper, we introduce a novel approach to price the CVA of a netted portfolio of derivative securities with early-exercise features, based on a dynamic game interpretation of the interaction between the counterparties involved in a netting agreement.

In particular, we show that credit risk and netting agreements have a significant impact on the way portfolios are managed (that is, on options' exercise strategies) and, therefore, on the value of the portfolio and on the price of counterparty risk. We show that our dynamic-game interpretation can be used to determine the value of the CVA and of the BVA and we propose a numerical implementation, yielding the CVA (or BVA) of a netted portfolio of Bermudan options, at any point of time, as a function of the value of the underlying asset(s) and of the set of options that have not yet been exercised.

To the best of our knowledge, this paper is the first to investigate portfolio of derivatives under a netting agreement when both parties have exercise rights. Our investigation allows to better understand the impact of strategic interactions on the price of CCR and to derive general properties characterizing the efficient management of such portfolios.

The paper is organized as follows. Section 2 is a motivating example illustrating the strategic interactions between two parties involved in a netting agreement. Section 3 proposes a dynamic-game model for the computation of the CVA/BVA. Section 4 reports on numerical experiments, providing insight about the sensitivity of the CVA/BVA to various parameters and about the impact of netting and CCR on exercise strategies and on adjustment values. Section 5 is a conclusion.

## 2 Motivating example

This section discusses a simplified example, involving a portfolio of two options with a single exercise opportunity, in order to illustrate the strategic aspects underlying the model presented in Section 3.

Consider two parties, named $C_1$ and $C_2$, involved as counterparties in a portfolio of two options, identified by $O_1$ and $O_2$, where $C_i$ has a long position on $O_i$ and a short position on $O_{3-i}$, $i = 1, 2$. Since all the cashflows generated by this portfolio are from $C_1$ to $C_2$ or conversely, the value of the portfolio from the point of view of $C_1$ is the negative of its value from that of $C_2$. Assume that both parties can exercise their option at a single given date before maturity (the *decision date*).

Table 1 provides, at the decision date, the exercise and holding values of the two options, respectively denoted $e_i, i \in \{1,2\}$ and $h_i, i \in \{1,2\}$, in the absence of CCR. From these values, it is straightforward to conclude that both parties should hold. The value of the portfolio is then $v_1 = h_1 - h_2 = 1$ for $C_1$ (and $v_2 = -1$ for $C_2$).

**Table 1: Exercise and holding values of options $O_1$ and $O_2$ at the decision date from the respective viewpoints of their holders when both parties are risk free.**

| | $O_1$ | $O_2$ |
|---|---|---|
| Exercise value ($e_i$) | 8 | 6 |
| Holding value ($h_i$) | 10 | 9 |

Suppose that there is a $p_2 = 0.3$ probability that $C_2$ defaults, so that $C_1$ does not recover anything upon maturity of $O_1$. Table 2 shows the updated holding values, where the expected holding value $\hat{h}_1$ accounts for counterparty default risk. In that case, it becomes optimal for $C_1$ to exercise $O_1$, and the portfolio value for $C_1$ is now $v_1 = e_1 - h_2 = -1$. Since $C_1$ is the only party vulnerable to CCR, the CVA for $C_1$ is computed by deducting the value of the vulnerable portfolio from that of the corresponding risk-free portfolio, yielding a CVA of 2.

**Table 2: Exercise and holding values of options $O_1$ and $O_2$ at the decision date from the respective viewpoints of their holders when the default probability of $C_2$ is $p_2 = 0.3$.**

| | $O_1$ | $O_2$ |
|---|---|---|
| Exercise value ($e_i$) | 8 | 6 |
| Holding value adjusted for CCR ( $\hat{h}_i$) | $(1-p_2)h_1 = 7$ | $h_2 = 9$ |

Note that the exercise strategy is modified by the presence of CCR. Using the risk-free exercise strategy to evaluate this portfolio would result in the erroneous value of $\hat{h}_1 - h_2 = -2$ for the portfolio and a CVA of 3, a significant overestimation.

Now suppose that the portfolio is subject to a netting agreement, so that the contractual cash-flows are no longer independent. Assuming that the probability of default by $C_2$ is $p_2 = 0.3$, Table 3 contains the expected cash flows from $C_2$ to $C_1$ under a netting agreement in matrix form, where $\mathrm{E}_i$ *(resp. $H_i$)* represents the exercising *(resp. holding)* decision by Counterparty $i$.

**Table 3: Netted portfolio's value from the viewpoint of $C_1$ at the decision date according to the pair of decision by the counterparties. Risk-free holding and exercise values are provided in Table 1. Default probability of $C_2$ is $p_2 = 0.3$.**

| | $\mathrm{E}_2$ | $\mathrm{H}_2$ |
|---|---|---|
| $\mathrm{E}_1$ | $e_1 - e_2 = 2$ | $e_1 - h_2 = -1$ |
| $\mathrm{H}_1$ | $\hat{h}_1 - e_2 = 1$ | $(1-p_2)(h_1 - h_2) = 0.7$ |

Table 3 is a representation of a *zero-sum* matrix game where $C_1$, the row player, is the maximizer, and $C_2$, the column player, is the minimizer. The *security strategy* for $C_1$, maximizing the worst (smallest) outcome, is to hold, which guarantees an outcome of at least 0.7. Conversely, the security strategy for $C_2$, minimizing the worst (largest) outcome, is also to hold, which guarantees an outcome of at most 0.7. In that specific example, the security strategies yield the same expected outcome, so that each party's decision is the optimal response to the other's, and neither party has an incentive to depart from it, yielding a *Nash equilibrium.* In that case, the equilibrium strategy consists of holding both options; the equilibrium value of the netted portfolio is 0.7, yielding a CVA of 0.3.

The results from the three cases are summarized in Table 4, showing that both CCR and netting affect the exercise strategy of the party exposed to default risk. One also observes that netting does reduce the portfolio's CVA in this case.

**Table 4: Impact of CCR and of netting on exercise decisions and on the CVA. Risk-free holding and exercise values are provided in Table 1 and default probability of $C_2$ is $p_2 = 0.3$.**

| Case | Decisions $(C_1, C_2)$ | Portfolio value for $C_1$ | CVA |
|---|---|---|---|
| Risk-free | (H,H) | $h_1 - h_2 = 1$ | – |
| CCR without netting | (E,H) | $e_1 - h_2 = -1$ | 2 |
| CCR with netting | (H,H) | $(1 - p_2)(h_1 - h_2) = 0.7$ | 0.3 |

We now consider a case of bilateral counterparty risk by assuming a default probability of $p_1 = 0.25$ for $C_1$ and $p_2 = 0.15$ for $C_2$. Table 5 reports the adjusted holding values of the vulnerable options, without netting, at the decision date. In that case, the optimal decision is for each party to hold its option, and the value of the portfolio for $C_1$ is $V_1 = \hat{h}_1 - \hat{h}_2 = 1.75$.

**Table 5: Exercise and expected holding values of $O_1$ and $O_2$ at the decision date from the respective viewpoints or their holders when the default probabilities of $C_1$ and $C_2$ are respectively $p_1 = 0.25$ and $p_2 = 0.15$.**

| | $O_1$ | $O_2$ |
|---|---|---|
| Exercise value $(e_i)$ | 8 | $e_2 = 6$ |
| Holding value $(\hat{h}_i)$ | $(1 - p_2)h_1 = 8.5$ | $(1 - p_1)h_2 = 6.75$ |

Table 6 reports the expected cash flows from $C_2$ to $C_1$ in the presence of a netting agreement and bilateral counterparty risk. According to these values, the security strategy of $C_1$ is to exercise $O_1$ and that of $C_2$ is to hold $O_2$, guaranteeing in both cases a portfolio value of $v_1 = 1.25$. The strategy pair (E,H) is then a Nash equilibrium for the matrix game, and differs from the optimal strategies obtained when counterparties are not linked by a netting agreement.

**Table 6: Matrix-game representation of the netted portfolio's value from the viewpoint of $C_1$ at the decision date. Risk-free holding and exercise values are provided in Table 1. Default probabilities of $C_1$ and $C_2$ are respectively $p_1 = 0.25$ and $p_2 = 0.15$.**

| | $E_2$ | $H_2$ |
|---|---|---|
| $E_1$ | $e_1 - e_2 = 2$ | $e_1 - \hat{h}_2 = 1.25$ |
| $H_1$ | $\hat{h}_1 - e_2 = 2.5$ | $(1 - p_2)(h_1 - h_2) = 0.85$ |

Similarly to the CVA, the BVA is computed by subtracting the vulnerable portfolio's value from its non-vulnerable counterpart. The BVA can be negative or positive, depending on the two parties' relative vulnerability to CCR. The impact of a netting agreement on the exercise decisions of defaultable parties and on the BVA for this bilateral instance is summarized in Table 7. Again, one observes that netting affects the portfolio's risk adjustment value and decreases the BVA (in absolute value). Note that using either the risk-free or the risk-adjusted strategy to evaluate the netted portfolio value would result in a BVA of -0.75, a significant overestimation.

**Table 7: Impact of CCR and netting agreement on exercise decisions and BVA. Risk-free holding and exercise values are provided in Table 1. Default probabilities of $C_1$ and $C_2$ are respectively $p_1 = 0.25$ and $p_2 = 0.15$.**

| Case | Decisions $(C_1, C_2)$ | Portfolio value for $C_1$ | BVA |
|---|---|---|---|
| Risk-free | (H,H) | $h_1 - h_2 = 1$ | – |
| CCR without netting | (H,H) | $\hat{h}_1 - \hat{h}_2 = 1.75$ | $-0.75$ |
| CCR with netting | (E,H) | $e_1 - \hat{h}_2 = 1.25$ | $-0.25$ |

Finally, Table 8 reports an instance where the default probabilities for $C_1$ and $C_2$ are respectively $p_1 = 0$ and $p_2 = 0.4$. In that case, the security (maxmin) strategy of $C_1$ is to hold, guaranteeing a payoff of at least 0, while the security (minmax) strategy of $C_2$ is also to hold, guaranteeing a payoff of at most 0.6. The matrix game does not admit a Nash equilibrium in pure strategies since the minmax and maxmin values do not coincide.[1]

**Table 8: Matrix-game representation of the netted portfolio's value from the viewpoint of $C_1$ at the decision date. Risk-free holding and exercise values are provided in Table 1. Default probabilities of $C_1$ and $C_2$ are respectively $p_1 = 0$ and $p_2 = 0.4$.**

| | $\mathrm{E}_2$ | $\mathrm{H}_2$ |
|---|---|---|
| $\mathrm{E}_1$ | $e_1 - e_2 = 2$ | $e_1 - h_2 = -1$ |
| $\mathrm{H}_1$ | $\hat{h}_1 - e_2 = 0$ | $(1 - p_2)(h_1 - h_2) = 0.6$ |

In such a situation, we can propose various conjectures about the way the parties would act, which will affect the value of the portfolio. One plausible assumption is that each party adheres to its own security strategy. In the case of the matrix game represented in Table 8, both parties would then choose to hold their option, resulting in a portfolio value of $v_1 = 0.6$ for $C_1$ (and $v_2 = -0.6$ for $C_2$).

A second assumption is that parties adopt a mixed strategy, that is, they randomize their decision by choosing a probability to exercise their option at the decision date. This assumption is founded on a game-theoretical interpretation of managing the netted portfolio, as zero-sum matrix games always admit a Nash equilibrium in mixed strategies. For the game presented in Table 8, the equilibrium mixed strategy is for $C_1$ to exercise with a probability of $1/6$ and for $C_2$ to exercise with a probability of $20/45$. It is straightforward to check that, if $C_1$ exercises with a probability of $1/6$, $C_2$ cannot reduce the expected value of the portfolio below $0.\overline{3}$ (actually, the value of the portfolio is $0.\overline{3}$ whether $C_2$ exercises or holds). In the same way, if $C_2$ exercises with a probability of $20/45$, $C_1$ can not do better than an expected value of $0.\overline{3}$. Under this equilibrium mixed strategy, the portfolio value is then $v_1 = 0.\overline{3}$.

The three simple instances reported in this section illustrate the impact of CCR and netting on the exercise decisions of the parties and, therefore, on the value of the portfolio. In the absence of a netting agreement, each claim is examined individually to determine the optimal exercise strategy, where the holding value of each individual claim is adjusted to account for the possibility of loss upon default. Under a netting agreement, however, losses upon default are applied to the net value of the portfolio; specifically, upon default of $C_2$ *(resp.* $C_1$*)*, losses are only incurred if the net value of the portfolio claims is positive *(resp. negative)* for $C_1$, which can modify the adjustment value.[2] When, in addition, the portfolio includes claims with early-exercise features, netting can also impact the exercise strategy and, therefore, further modify the risk adjustment.

The following section proposes a general model to evaluate a portfolio of claims having early-exercise features under CCR and netting.

# 3 Model

We consider two parties ($C_1$ and $C_2$) involved in a portfolio of claims under a netting agreement, where the portfolio includes contractual payments in both directions (from $C_2$ to $C_1$ and from $C_1$ to $C_2$), possibly with early-exercise features. We assume that both parties have at least one early-exercise opportunity.

Since all cash flows from this portfolio is from one party to the other, its value from the perspective of one party is the negative of that of the other. In the sequel, the netted portfolio value is expressed from the perspective of $C_1$.

The essential feature of a netting agreement is the consolidation of contractual obligations upon default of one of the parties. Accordingly, the agreement and the portfolio cease to exist on the date

[1] The best response of $C_2$ when $C_1$ holds is to exercise.

[2] Clearly, a portfolio of claims should include both positive and negative cash flows (from the point of view of one party) for netting to have an impact on the expected payoffs to the counterparties.

of the first default event; at that date, the values of the claims are netted, and the result is recovered by $C_1$ (if positive) or $C_2$ (if negative). Since, on the date of the first default event, some claims can include optional rights, we will assume in this paper that, upon default, the value of the netted portfolio corresponds to the expected value of its future cash flows under a risk- and netting-adjusted exercise strategy.

## 3.1 Notation

To simplify the exposition, we assume that the portfolio is composed of $n = n_1 + n_2$ Bermudan options with distinctive features (maturity, exercise payoffs and dates, underlying asset), where $C_1$ holds the optional rights of the first $n_1$ options and $C_2$ holds the optional rights of the remaining $n_2$ options.[3] Let $t = 0$ denote the inception of the netting contract and $t = T$ the longest maturity among the $n$ options included in the portfolio. Denote by $(X_t)_{0\leq t\leq T}$ the (possibly multidimensional) process of the underlying risk factors, including the price process of the options' underlying assets. We assume that $(X_t)_{0\leq t\leq T}$ is a finite Markov process, where $(\mathcal{F}_t)_{0\leq t\leq T}$ is the filtration generated by $(X_t)_{0\leq t\leq T}$.

Let $\mathcal{T} = \{t_m, m = 0, 1, ..., M\}$ be a set of discrete *evaluation dates* that includes all possible exercise dates for all options in the portfolio, where $t_M \equiv T$. The notation $\mathbb{E}_m[\cdot]$ represents the expectation at date $t_m$, under the risk-neutral measure, conditional on no prior default and on the filtration $(\mathcal{F}_{t_m})$. For $j = 1, ..., n$, $F_{mj}(x)$ then denotes the exercise payoff of option $j$ at $(t_m, X_{t_m} = x)$ from the perspective of $C_1$, where $F_{mj}(x) = 0$ when exercise of option $j$ is not allowed at $t_m$.

Let $r$ denote the risk-free interest rate, assumed constant. To simplify the notation, we assume that evaluation dates are evenly distributed in $[0, T]$, so that the discount factor corresponding to a single time step $\Delta \equiv t_{m+1} - t_m, m = 0, ..., M - 1$, is given by $\beta \equiv e^{-r\Delta}$.

We denote by $\tau_i$ the stochastic default date (possibly infinite) of $C_i$ and by $\rho_i \in [0, 1]$ the deterministic recovery rate upon default by $C_i$, $i \in \{1, 2\}$. The recovery rate is applied to the netted portfolio value eventually recovered by $C_{3-i}$.

To compute the CCR valuation adjustment, one needs to compare the value of the vulnerable portfolio with that of a risk-free portfolio with the same characteristics. To this end, we introduce a state vector $b = (b_1, b_2)$ of binary variables indicating which options of the portfolio are still alive, that is, for $j = 1, ..., n$, $b_j = 1$ if Option $j$ has not yet been exercised or expired, whereas $b_j = 0$ indicates that Option $j$ no longer exists in the portfolio. At a given evaluation date $t_m$ where $X_{t_m} = x$ and given no prior default, let $\hat{V}_m(x, b)$ and $V_m(x, b)$ denote respectively the value of the vulnerable portfolio and that of the corresponding risk-free portfolio, under the risk-neutral measure.

Finally, the indicator function $1_A$ is defined by

$$1_A \equiv \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise,} \end{cases}$$

and, for a given $y \in \mathbb{R}$,

$$y^+ \equiv \max\{0, y\}$$
$$y^- \equiv \min\{0, y\}.$$

## 3.2 The risk-free portfolio

It is easy to show that netting has no impact on the optimal exercise of the individual options in a risk-free portfolio, so that

$$V_m(x, b) = \sum_{j=1}^{n} b_j V_{mj}(x) \tag{1}$$

[3] It is straightforward to adapt the model to the general case of derivatives with multiple contractual cash flows.

where, for $j = 1, ..., n$, $V_{mj}(x)$ is the value (from the perspective of $C_1$) of Option $j$ at $(t_m, X_{t_m} = x)$, under its holder's optimal exercise strategy, assuming Option $j$ has not been exercised yet. The risk-free value of the $n$ options satisfy the following recursive equations

$$V_{mj}(x) = \max\{F_{mj}(x); \beta\mathbb{E}_m[V_{m+1,j}(X_{t_{m+1}})]\} \text{ for } j = 1, ..., n_1 \text{ and } m < M \tag{2}$$

$$V_{mj}(x) = \min\{F_{mj}(x); \beta\mathbb{E}_m[V_{m+1,j}(X_{t_{m+1}})]\} \text{ for } j = n_{1+1}, ..., n \text{ and } m < M \tag{3}$$

$$V_{Mj}(x) = F_{Mj}(x) \text{ for } j = 1, ..., n. \tag{4}$$

## 3.3 The vulnerable netted portfolio

However, as shown in Section 2, netting can impact the exercise strategies of the vulnerable portfolio's claims, giving rise to a dynamic game interpretation for the value of the netted portfolio. We therefore proceed to characterize the payoffs and exercise strategies of the counterparties involved in a netting agreement in order to obtain the value of a netted portfolio of vulnerable options.

### 3.3.1 Exercise payoff

At a given evaluation date, let $a = (a_1, a_2)$ represent a vector of binary decisions with respect to each of the $n$ options, where, for $j = 1, ..., n$, option $j$ is exercised by its holder if $a_j = 1$. Note that feasible decision vectors satisfy $a \leq b$, and recall that $F_{mj}(x) = 0$ if exercise of option $j$ is not allowed at $t_m$. The *exercise payoff* $R_m(x, a)$ corresponding to a feasible action vector $a$ at $(t_m, X_{t_m} = x)$ is defined by

$$R_m(x, a) \equiv \sum_{j=1}^{n} a_j F_{mj}(x). \tag{5}$$

### 3.3.2 Holding value

The *holding value* $W_m(x, b)$ of the portfolio at $(t_m, X_{t_m} = x)$, given no prior default, is the expected value of all the remaining options in the netted portfolio, described by the vector $b$. Accordingly, using a recursive interpretation and assuming that the value of the vulnerable portfolio is known at the next evaluation date as a function of the state vector, the holding value is computed by considering the expected discounted value of the portfolio upon three mutually exclusive and collectively exhaustive events during the time interval until the next evaluation date, namely, survival of both parties, first default of $C_1$, or first default of $C_2$, given no prior default. We can then write

$$W_m(x, b) = W_m^0(x, b) + W_m^1(x, b) + W_m^2(x, b), \tag{6}$$

where $W_m^0(x, b)$, $W_m^1(x, b)$ and $W_m^2(x, b)$ correspond respectively to the holding value upon each of these three mutually exclusive events, defined as follows:

**Case 0:** Let $D_m^0 = 1_{t_{m+1} < \tau_2} 1_{t_{m+1} < \tau_1}$ indicate the event that both parties will survive until $t_{m+1}$. In this case, the holding value at $t_m$ is the discounted value of the portfolio value at the next evaluation date, yielding

$$W_m^0(x, b) = \beta\mathbb{E}_m\left[D_m^0 \hat{V}_{m+1}(X_{m+1}, b)\right]. \tag{7}$$

**Case 1:** Let $D_m^1 = 1_{t_m < \tau_1 \leq t_{m+1}} 1_{\tau_1 < \tau_2}$ indicate the event that $C_1$ is the first to default during the time interval $(t_m, t_{m+1}]$. In this case, if the expected value of the portfolio at $(t_m + 1, X_{t_{m+1}})$ is negative, $C_2$ will recover a portion $\rho_1$ of this (discounted) value at $\tau_1$; otherwise, $C_2$ will deliver the total of the portfolio's expected discounted value to $C_1$ at $\tau_1$. We then have

$$W_m^1(x, b) = \beta\mathbb{E}_m\left[D_m^1\left(\hat{V}_{m+1}(X_{t_{m+1}}, b)^+ + \rho_1\hat{V}_{m+1}(X_{t_{m+1}}, b)^-\right)\right]. \tag{8}$$

**Case 2:** Let $D_m^2 = 1_{t_m<\tau_2\le t_{m+1}}1_{\tau_2<\tau_1}$ indicate the event that $C_2$ is the first to default during the time interval $(t_m, t_{m+1}]$. Similarly to Case 1, if the expected value of the portfolio at the next evaluation date is positive, $C_1$ will recover a portion $\rho_2$ of it, otherwise $C_2$ will recover the total value, yielding

$$W_m^2(x,b) = \beta\mathbb{E}_m\left[D_m^2\left(\rho_2\hat{V}_{m+1}(X_{t_{m+1}},b)^+ + \hat{V}_{m+1}(X_{t_{m+1}},b)^-\right)\right]. \tag{9}$$

Using (7)–(9), Equation (6) reduces to

$$\begin{aligned}W_m(x,b) =&\beta\mathbb{E}_m\left[\left(1-(1-\rho_2)D_m^2\right)\hat{V}_{m+1}(X_{t_{m+1}},b)^+\right.\\&\left.+\left(1-(1-\rho_1)D_m^1\right)\hat{V}_{m+1}(X_{t_{m+1}},b)^-\right].\end{aligned} \tag{10}$$

It is important to note that the above characterization of the holding value implicitly assumes that, upon default, both parties agree on the value of the portfolio, that is, on the expected discounted value of its future cash flows. Obviously, the future cash flows of an option with early-exercise opportunities depend on the exercise strategy of its holder, and the value of an option is obtained by assuming an optimal exercise strategy. As shown in Section 2, the holder's exercise strategy should account for counterparty risk and for the impact of netting on its exposure.

#### 3.3.3 Security strategies

A *security strategy* for $C_1$ at $(t_m, X_{t_m} = x, b)$ prescribes a decision vector maximizing the outcome against all the possible decisions of the other party. The *lower value* of the portfolio at $(m, x, b)$ is defined by

$$V_m^{S1}(x,b) \equiv \max_{a_1\le b_1}\left\{\min_{a_2\le b_2}\left\{R_m(x,a)+W_m(x,b-a)\right\}\right\}, \tag{11}$$

where $b-a$ indicates the contracts remaining in the portfolio after the exercise decisions designated by the vector $a=(a_1,a_2)$. A security strategy for $C_1$ then satisfies

$$a_m^{S1}(x,b) \in \arg\max_{a_1\le b_1}\left\{\min_{a_2\le b_2}\left\{R_m(x,a)+W_m(x,b-a)\right\}\right\}. \tag{12}$$

In the same way, a security strategy for $C_2$ at $(t_m, X_{t_m} = x, b)$ is a decision vector $a_m^{S2}(x,b)$ minimizing the outcome against all the possible decisions of $C_1$. The *upper value* of the portfolio at $(m, x, b)$ is defined by

$$V_m^{S2}(x,b) \equiv \min_{a_2\le b_2}\left\{\max_{a_1\le b_1}\left\{R_m(x,a)+W_m(x,b-a)\right\}\right\}, \tag{13}$$

and a security strategy for $C_2$ satisfies

$$a_m^{S2}(x,b) \in \arg\min_{a_2\le b_2}\left\{\max_{a_1\le b_1}\left\{R_m(x,a)+W_m(x,b-a)\right\}\right\}. \tag{14}$$

Security strategies are *pure* strategies, of dimension $n_1$ for $C_1$ and $n_2$ for $C_2$. They indicate the vector of decisions (exercise or hold) corresponding to all the options in the portfolio held by each counterparty, as a function of $t_m$, $x = X_{t_m}$ and $b$. Note that the feasibility condition $a \le b$ ensures that options that are no longer alive cannot be exercised.

### 3.4 Equilibrium

At a given evaluation date $t_m$ where $X_{t_m} = x$ and the options still included in the portfolio are described by the vector $b$, if the lower value and the upper value of the portfolio coincide, the security strategies of the counterparties define a Nash equilibrium at $(m, x, b)$. In that case, it is reasonable to assume that the counterparties will use the strategy pair $(a_m^{S1}(x, b), a_m^{S2}(x, b))$ since neither party can improve its outcome by changing its strategy.[4] The value of the netted portfolio is then defined by

$$\hat{V}_m(x, b) \equiv V_m^{S1}(x, b) = V_m^{S2}(x, b). \tag{15}$$

If however the upper and lower values do not coincide at $(m, x, b)$, there exists no equilibrium in pure strategies at $(m, x, b)$, and the value of the portfolio is open to interpretation. As illustrated in Section 2, we propose three ways to determine the value of the netted portfolio in that case, based on plausible conjectures about the exercise strategies used by the counterparties.

### 3.4.1 Robust interpretation

In the first case, we assume that each counterparty uses its security strategy, a robust behavior avoiding the worst possible outcomes and ensuring that the value of the portfolio lies between its lower and upper values. The strategy pair used by the counterparties is then $a_m^S(x, b) \equiv (a_m^{S1}(x, b), a_m^{S2}(x, b))$ and the value of the portfolio is given by

$$\begin{aligned} \hat{V}_m(x, b) \equiv& R_m(x, (a_m^S(x, b)) + W_m(x, b - a_m^S(x, b)) \\ &\in \left[V_m^{S1}(x, b), V_m^{S2}(m, b)\right]. \end{aligned} \tag{16}$$

### 3.4.2 Mixed strategies

In the second case, we consider the possibility that counterparties randomize their decisions by choosing a probability distribution over the set of actions available to them. A *mixed* strategy for $C_i$, $i \in \{1, 2\}$ is a vector $z_i$ of dimension $2^{n_i}$ such that each element is in $[0, 1]$ and the elements sum to 1. The exercise payoff and holding value corresponding to a mixed strategy is the weighted average of the values corresponding to each of the $2^{n_i}$ pure strategy vectors available to counterparty $C_i$, denoted by $a_{i_k}$, $k = 1, ...2^{n_i}$. Accordingly, under a mixed strategy $z_1$, the exercise payoff of the netted portfolio at $(t_m, X_{t_m} = x)$ when $C_2$ uses the action vector $a_2$ is defined by

$$\tilde{R}_m(x, z_1, a_2) = \sum_{k=1}^{2^{n_1}} z_{1_k} R_m(x, a_{1_k}, a_2). \tag{17}$$

In the same way, under a mixed strategy $z_1$, the holding value of the netted portfolio at $(t_m, X_{t_m} = x)$ when $C_2$ uses the action vector $a_2$ is defined by

$$\tilde{W}_m(x, z_1, a_2) = \sum_{k=1}^{2^{n_1}} z_{1_k} W_m(x, b - (a_{1_k}, a_2)). \tag{18}$$

The exercise payoff and holding value of the netted portfolio corresponding to the use of a mixed strategy by $C_2$ are defined similarly.

Note that a Nash equilibrium in mixed strategies always exists.[5] The value of the portfolio

$$\hat{V}_m(x, b) \equiv v \tag{19}$$

can be obtained by solving the following linear program at $(m, x, b)$:

$$\max_{z,v} v \tag{20}$$

[4] To simplify the exposition, we assume in the sequel that the solutions to the optimization problems (11) and (13) are unique. Note that the portfolio value is well-defined even when this is not the case. The issue of multiple solutions is addressed in Section 4.

[5] Again, the equilibrium value is unique even though multiple equilibrium strategies may exist.

$$
\begin{aligned}
&\text{s.t.} \\
&v \le \tilde{R}_m(x, z, a_{2_l}) + \tilde{W}_m(x, z, a_{2_l}) \text{ for } l = 1, ...2^{n_2} && (21)\\
\sum_{k=1}^{2^{n_1}} &z_k = 1 && (22)\\
&z_k \ge 0, \quad k = 1, ..., 2^{n_1}. && (23)
\end{aligned}
$$

The equilibrium mixed strategy for $C_1$ is the vector $z \in \mathbb{R}^{2^{n_1}}$ solving (20)–(23). The equilibrium mixed strategy for $C_2$ is the vector of dual variables corresponding to the $2^{n_2}$ constraints (21).

#### 3.4.3 Conservative values

Finally, we consider the possibility that parties do not agree on the value of the portfolio, so that each party computes its own estimation of the value of the vulnerable portfolio, a conservative value corresponding to either the lower (for $C_1$) or the upper (for $C_2$) value, obtained using Equations (11) or (13), respectively.

To summarize, we propose three distinct assumptions about the behavior of the parties in a netted agreement, leading to four different ways to compute the value of a vulnerable portfolio, namely:

**A1** Parties agree on the value of the portfolio, which corresponds to a Nash equilibrium. In that scenario, parties use mixed strategies when a Nash equilibrium in pure strategies does not exist. The value of the portfolio is obtained using Equation (19).

**A2** Parties agree on a robust interpretation of the value of the portfolio. In that scenario, each party uses its security strategy, which is not necessarily in equilibrium, but guarantees that the value of the portfolio, obtained using Equation (16) lies between the upper and the lower value.

**A3** Parties do not agree on the value of the portfolio and use a conservative value obtained using Equations (11) for $C_1$ and Equation (13) for $C_2$.

Clearly, both counterparties should agree that the value of the portfolio lies between its lower and upper values. Note that Equations (16) and (19) satisfy this condition and yield the same result, corresponding to Equation (15), when the lower and the upper values coincide.

### 3.5 Computation of valuation adjustments

Given that the value of the vulnerable portfolio is a known function of $(x, b)$ at maturity,

$$\hat{V}_M(x, b) = \sum_{j=1}^{n} b_j F_{Mj}(x), \tag{24}$$

Equations (6)–(9) provide a backward recursive formulation to compute the holding value $W_m(x, b)$ at $t_m$ when the value of the vulnerable netted portfolio is known at $t_{m+1}$ as a function of the state vector $(x, b)$. Under Assumptions A1 or A2, the vulnerable portfolio value can then be obtained at $t_m$ using Equations (16) or (19), respectively. Note that the two equations yield the same value when the upper and lower values of the vulnerable portfolio coincide.

The BVA at $(t_m, X_m = x, b)$ is then given by the difference

$$\text{BVA}_m(x, b) = V_m(x, b) - \hat{V}_m(x, b). \tag{25}$$

When only one party is exposed to default risk, say $C_i$, the stochastic default time $\tau_i$ is set to $= \infty$ in Equations (7)–(9). The CVA at $(m, x, b)$ is then given by

$$\text{CVA}_m(x, b) = V_m(x, b) - \hat{V}_m(x, b). \tag{26}$$

When parties do not agree on the value of the vulnerable portfolio (Assumption A3), and, therefore, on the price of counterparty risk (BVA or CVA), each party will compute its own *conservative* value of the risk adjustment, yielding the conservative BVAs

$$\begin{aligned}\mathrm{BVA}_m^1(x,b) =& V_m(x,b) - V_m^{S1}(x,b)\\ \mathrm{BVA}_m^2(x,b) =& V_m(x,b) - V_m^{S2}(x,b).\end{aligned}$$

These conservative BVA values are likely to differ and to be higher (in absolute value) than the BVA computed using either the mixed strategy or the robust assumptions.

The general model proposed in this section provides an analytical characterization of the price of counterparty risk under various default risk models and various assumptions about the state process $(X_t)_{0\le t\le T}$, provided the expectations in (7)–(9) can be computed or approximated efficiently. In particular, it can accommodate both intensity-based and structural default models by including the risk factors (e.g., structural or exogenous variables) in the state vector.

However, while analytic, Equations (2)–(4) and (11), (13), (16) or (19) do not admit closed-form solutions in general and require some form of numerical approximation. In the numerical illustrations presented in the next section, we solve Equations (2 )–(4) and (11), (13), (16) or (19) on a set of grid points for the state vector $X$ and approximate the value of the portfolio using linear spline interpolation (see Breton and Frutos (2012)).

# 4 Numerical illustration

This section reports on numerical experiments addressing the sensitivity of counterparty risk to various parameters and the impact of netting and CCR on exercise strategies and adjustment values.

## 4.1 Base-case specification

We consider a portfolio consisting of $n = n_1+n_2$ Bermudan put options written on the same underlying asset, with possibly distinct strike prices denoted by $K_j$, $j = 1, ..., n$. All options have the same maturity $T = 1$ and 50 equally spaced exercise opportunities, which, along with the inception date, form the set $\mathcal{T}$. Counterparty $C_1$ and $C_2$ are in a netting agreement, where $C_1$ holds a long position on Options $1, ..., n_1$ and a short position on Options $n_1 + 1, ...n$ and $C_2$ holds the opposite position.

We assume that the underlying asset price process is described by a geometric Brownian motion, so that the price process under the risk-neutral measure is given by

$$X_t = X_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma B_t\right), \tag{27}$$

where $X_0$ is the asset price at inception, $\sigma$ is the volatility of the price process, and $B_t$ denotes a standard Brownian motion. The benchmark values characterizing the underlying asset process are reported in Table 9.

**Table 9: Benchmark values for the numerical experiments.**

| | Underlying asset | | |
|---|---|---|---|
| Parameters | $r$ | $X_0$ | $\sigma$ |
| Base value | 0.05 | 100 | 0.35 |

We use an intensity-based model of default and assume that parties' defaults are exogenous events governed by the first jump of independent Poisson processes with constant hazard intensities, denoted respectively by $\lambda_i, i = 1, 2$. Accordingly, the probability that counterparty $C_i$ defaults first during a time interval $\Delta$, given that it has not defaulted yet, is a constant given by

$$p_i \equiv \mathbb{E}_m \left[1_{t_m < \tau_i \le t_{m+1}} 1_{\tau_i < \tau_{3-i}}\right] = \frac{\lambda_i}{\lambda_1 + \lambda_2}(1 - \exp(-\Delta(\lambda_1 + \lambda_2))), \quad m = 1, ..., M-1, \ i \in \{1, 2\}. \tag{28}$$

In that case, Equation (10) simplifies to

$$W_m(x, b) = \beta \left((1 - p_1(1 - \rho_1))\mathbb{E}_m \left[\hat{V}_{m+1}(X_{t_{m+1}}, b)^+\right] + (1 - p_2(1 - \rho_2))\mathbb{E}_m \left[\hat{V}_{m+1}(X_{t_{m+1}}, b)^-\right]\right). \tag{29}$$

In our numerical investigations, we set $\rho_1 = \rho_2 = 0$. This last assumption is without loss of generality, since it is easy to see, using Equation (29), that setting $\hat{p}_i = p_i(1 - \rho_i)$ and $\hat{\rho}_i = 0$ for $i \in 1, 2$ yields an equivalent model.

Finally, note that if the following condition is satisfied

$$p_1(1 - \rho_1) = p_2(1 - \rho_2) \equiv s, \tag{30}$$

the holding value further simplifies to

$$W_m(x, b) = \beta(1 - s)\mathbb{E}_m \left[\hat{V}_{m+1}(X_{t_{m+1}}, b)\right]. \tag{31}$$

In this specific instance, where both parties are subject to the exact same counterparty default risk, it is easy to see that netting has no impact, so that it is optimal for each party to use its risk-adjusted exercise strategy, independently of the decisions of the other party.

## 4.2 Exercise strategies in a netted portfolio

Our first set of experiments is meant to illustrate the complexity of the evaluation of CCR in a netted portfolio, even when only one party is exposed to counterparty risk, or when the portfolio contains identical options.

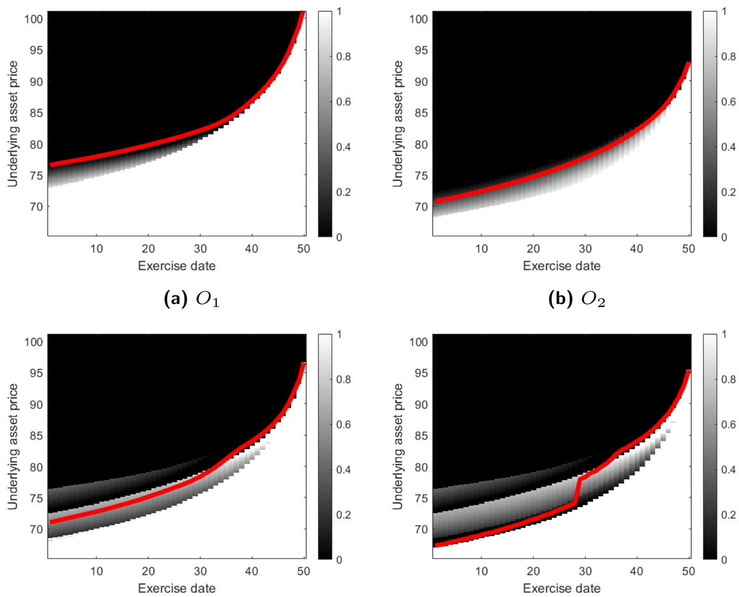
We first consider the portfolio of four Bermudan options described in Table 10.

**Table 10: Portfolio of four Bermudan put options with unilateral counterparty risk.**

| Option | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| Strike price $K$ | 110 | 100 | 105 | 103 |
| Holder | $C_1$ | | $C_2$ | |
| Default intensity $\lambda$ | 0 | | 0.5 | |

Figure 1 represents the exercise strategies of the four options over time, assuming that no option has been exercised yet. The red curve depicts the security strategy of the counterparty holding the option, that is, the exercise barrier as a function of the date and of the price of the underlying asset. Also represented on Figure 1 are the mixed strategies, that is, the probability of exercising the option (in grey) in the regions where the upper and lower values of the portfolio differ.

This example shows that there can exist significant regions for the underlying asset price, along the life of the portfolio, where there exists no equilibrium in pure strategies. It also shows that the exercise value of options under a netting agreement can exhibit non-smooth behavior, namely in regions where security strategies are not in equilibrium. It is noteworthy that the existence of a netting agreement can alter the exercise strategy of options even when their holder is not exposed to CCR (in this example, the payoff of Options $O_3$ and $O_4$, held by $C_2$, is deterministic).




**(a)** $O_1$ **(b)** $O_2$ **(c)** $O_3$ **(d)** $O_4$

**Figure 1: Equilibrium exercise strategies (security and mixed) of the four options described in Table 10 under a netting agreement. The white (resp. black) area is the region where the probability of exercising the option is equal to 1 (resp. 0). The red line is the security strategy of the option holder.**

We now consider a portfolio of four identical Bermudan put options, as described in Table 11. Note that, under a netting agreement, it may be optimal not to exercise identical options simultaneously, since the exercise decision depends on the composition of the portfolio. This is illustrated in Figure 2, where exercise barriers correspond to the parties' security strategies.

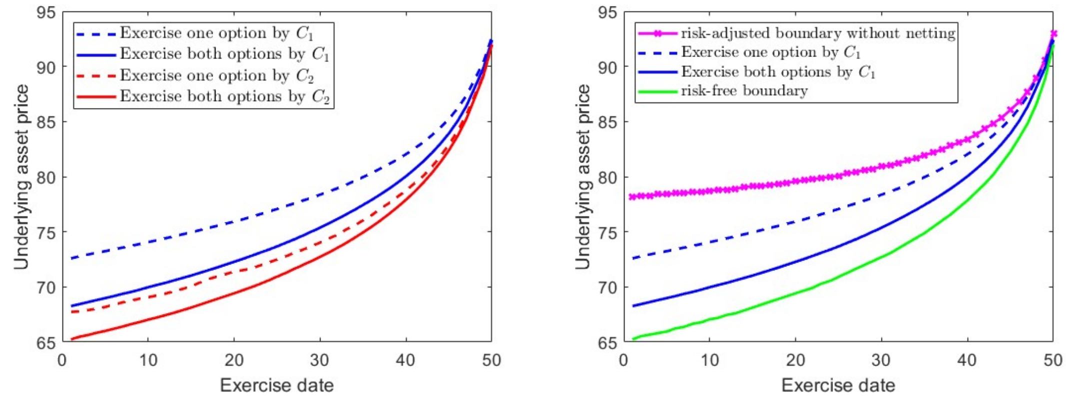
**Table 11: Portfolio of four identical Bermudan options with unilateral counterparty risk.**

| Option | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| Strike price $K$ | 100 | 100 | 100 | 100 |
| Holder | $C_1$ | | $C_2$ | |
| Default intensity $\lambda$ | 0 | | 0.5 | |

Under netting, there exists a region where both options are exercised, and a distinct region where only one of the two options is exercised, for the party exposed to counterparty risk as well as for the party not vulnerable to CCR. For $C_1$, who is exposed to CCR, both exercise barriers are higher ($C_1$ will exercise earlier) than in the risk-free case, but lower than when there is no netting agreement. $C_2$, who is not exposed to CCR, will exercise one of its two options at a higher price (earlier) than the risk-free barrier when involved in a netting agreement, while the exercise barrier for both options coincides with the risk-free barrier.

## 4.3 Impact of parameter values on exercise strategies

The next set of experiments illustrates the impact of CCR on the equilibrium strategies in a netted portfolio with bilateral risk. For ease of representation, we consider a portfolio of two options, and assume that parties use security strategies. Table 12 describes the base case parameters for this portfolio.




**(a) Exercise policies of both parties under netting.** **(b) Comparison of exercise policies for options held by $C_1$.**

**Figure 2: Illustration of the impact of netting and CCR on the exercise boundaries of four identical options as characterized in Table 11. Panel (a) shows the exercise boundaries of the four options under their respective owner's security strategies. Panel (b) compares the exercise boundaries for options held by $C_1$ with the risk-free boundary and with the risk-adjusted boundary when there is no netting agreement. Note that for the options held by $C_2$, the boundary for the exercise of both options coincides with the risk-free boundary.**

**Table 12: Portfolio of two Bermudan put options with bilateral counterparty risk.**

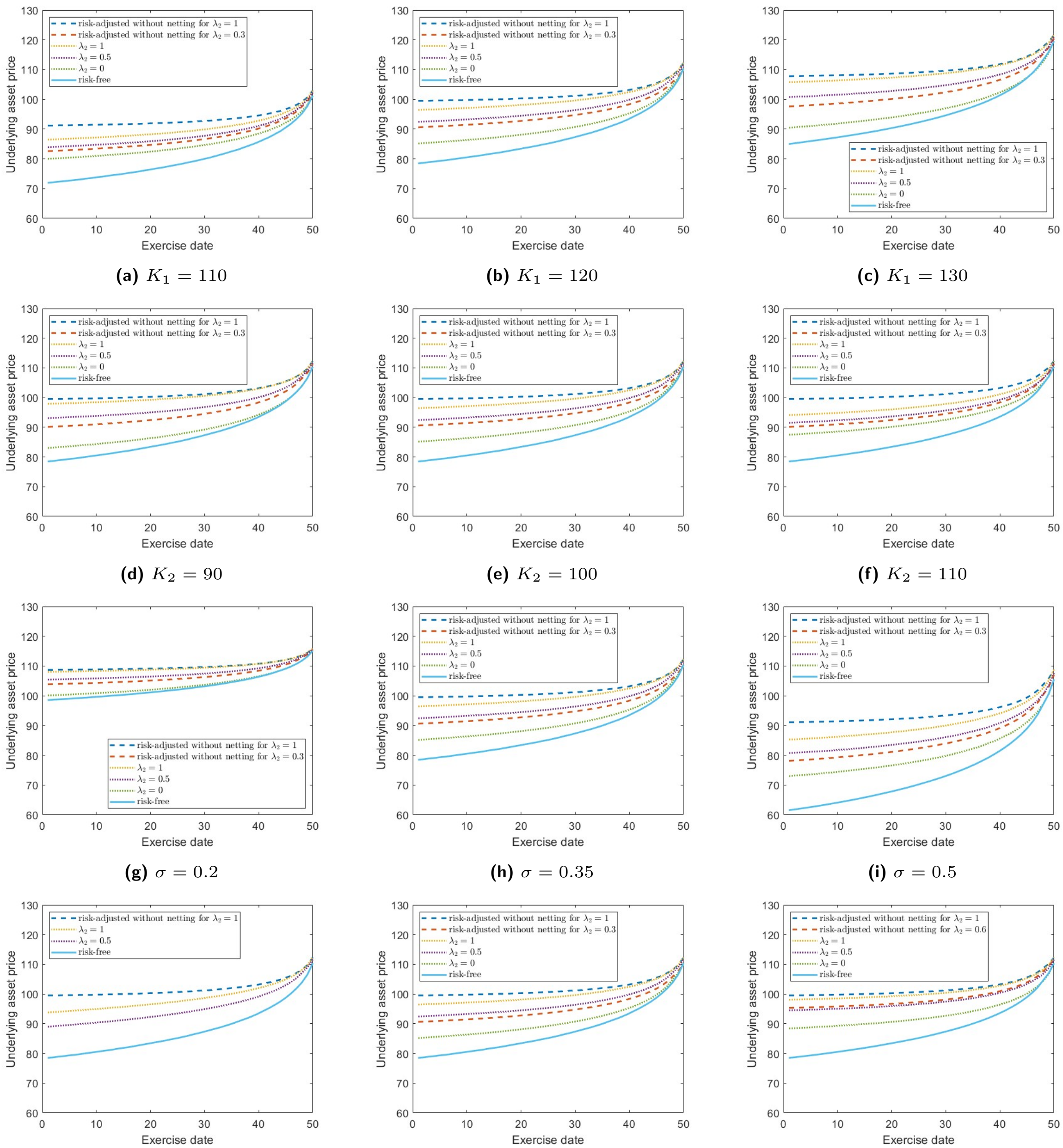
| | | |
|---|---|---|
| Option | $O_1$ | $O_2$ |
| Strike price $K$ | 120 | 100 |
| Holder | $C_1$ | $C_2$ |
| Default intensity $\lambda$ | 0.3 | 0.5 |

Figure 3 shows how the exercise strategy of Option 1, held by $C_1$, varies with the risk of default by $C_2$ ($\lambda_2$), the strike price of Option 1 ($K_1$), the strike price of Option 2 ($K_2$), the volatility of the underlying asset price ($\sigma$), and $C_1$'s own default risk ($\lambda_1$).

As expected, the exercise barrier rises (that is, the option is exercised earlier) with C. We also observe that the exercise barrier rises with decreasing $K_2$, which increases the value of the portfolio for $C_1$ and therefore its exposure to default risk, prompting $C_1$ to exercise its option earlier. The exercise barrier also rises with $\lambda_2$, which is an expected result since an increase in counterparty risk decreases the holding value of the portfolio. Notice that this impact of $\lambda_2$ on the exercise barrier becomes larger *(resp. smaller)* with increasing $K_1$ *(resp. $K_2$)*. An intriguing outcome is the observation that the option's exercise barrier rises with $\lambda_1$, that is, with the option holder's own default risk. The rationale is as follows: an increase in the risk of loss from $C_1$'s default leads to an earlier exercise of the option held by $C_2$, so that the benefit from netting mitigation for $C_1$ is reduced, which leads $C_1$ to exercise its option sooner. These observations, obtained by experimenting with a simple portfolio of two options, can be generalized to more complex situations. Risk and exposure have the expected impact on option holders' exercise barrier, while earlier exercise by one counterparty reduces the risk-mitigating effect of netting, inducing the other party to also exercise earlier.

However, while we observe in Figure 3 that $\sigma$ has a negative impact on the exercise boundary, this result cannot be generalized. An increase in the value of $\sigma$ affects all the options in the portfolio, so that a change in the volatility of the underlying asset affects both the total exposure and the value of individual options, possibly in opposite directions, so that the impact of a change in $\sigma$ on each option's exercise boundary cannot be predicted.

Figure 3 also compares the exercise barrier under netting with the exercise barriers of risk-free options. As expected, we find that the price at which vulnerable (put) options included in a netting portfolio are exercised is always higher than the risk-free exercise barrier.




**(a)** $K_1 = 110$ **(b)** $K_1 = 120$ **(c)** $K_1 = 130$

**(d)** $K_2 = 90$ **(e)** $K_2 = 100$ **(f)** $K_2 = 110$

**(g)** $\sigma = 0.2$ **(h)** $\sigma = 0.35$ **(i)** $\sigma = 0.5$

**(j)** $\lambda_1 = 0$ **(k)** $\lambda_1 = 0.3$ **(l)** $\lambda_1 = 0.6$

**Figure 3: Impact of changes in various parameters on the exercise boundary of Option $O_1$, held by $C_1$. Benchmark parameter values are reported in Tables 9 and 12.**

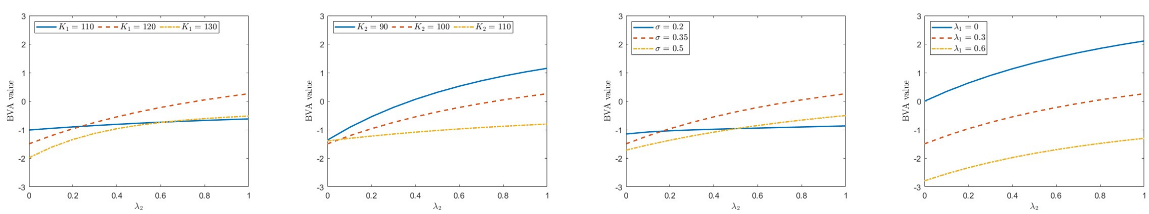
On the other hand, when Condition (30) is satisfied, netting has no impact, and therefore the equilibrium exercise barrier coincides with the risk-adjusted optimal exercise barrier. Examination of Equation (29) shows that netting benefits the party having the lowest probability of defaulting first (eventually adjusted for recovery). Accordingly, in our setting, the exercise barrier of vulnerable (put) options under netting is lower (*resp. higher*) than the corresponding risk-adjusted barrier for the counterparty having a lower (*resp. higher*) default probability, as can be observed on Figure 3.

Finally, one interesting conclusion from our investigation is the fact that options will be exercised in the order of their intrinsic value (or, equivalently, of their expected loss upon default). The reason is that, in a bilateral netting agreement, all vulnerable contracts held by one party are subject to the same level of risk. Consequently, multiple solutions may arise when deciding about an exercise strategy, for instance when a portfolio holds many identical options. Recall that this does not mean that identical options should be exercised together, but rather that any subset of those identical options can be exercised in the corresponding region.

### 4.4 Impact of parameter values on portfolio value and BVA/CVA

While the impact of model parameters on the exercise strategies in a netted portfolio can be deciphered, the situation is more complex when trying to predict how the value of a netted portfolio reacts to changes in parameter values that can affect subsets of options (e.g., common risk factors or underlying assets).

Moreover, since changes in parameter values also affect the the risk-free portfolio, the risk valuation adjustment (CVA or BVA), which is a difference, is even more unpredictable. Numerical investigations show that no general indication can be obtained concerning the sensitivity of risk valuation adjustments to changes in model parameters in a netted portfolio, even in the unilateral risk case, apart from the observation that an increase in default risk of the counterparty results in an increase of the risk valuation adjustment. This is illustrated in Figure 4, which shows the sensitivity of the CVA (BVA in panel (d)) for a portfolio of three options (see Table 13) to variations in the value of various parameters. These results are obtained under the assumption that parties use mixed strategies.




**(a) Variation of $K_1$** **(b) Variation of $K_2$** **(c) Variation of $\sigma$** **(d) Variation of $\lambda_1$**

**Figure 4: Sensitivity analysis of the CVA (BVA) at inception to counterparty risk ($\lambda_2$) for various values of $K_1$, $K_2$, $\sigma$, and $\lambda_1$ in the portfolio of three Bermudian options described in Table 13.**

**Table 13: Portfolio of three Bermudan put options with unilateral counterparty risk.**

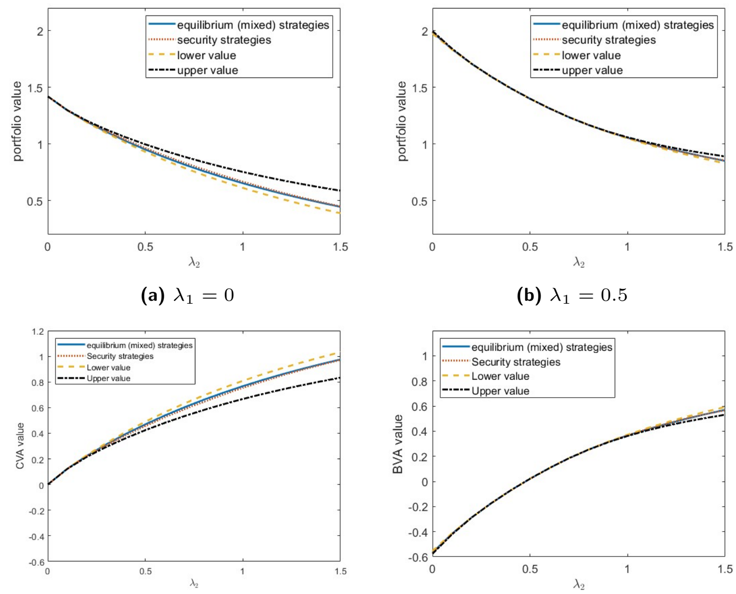
| Option | $O_1$ | $O_2$ | $O_3$ |
|---|---|---|---|
| Strike price $K$ | 120 | 100 | 90 |
| Holder | $C_1$ | $C_2$ | |
| Default intensity $\lambda$ | 0 | 0.3 | |

### 4.5 Behavioral assumptions

As indicated in Section 3.4, the dynamic-game model used to evaluate the price of CCR in a netted portfolio where both parties have optional rights can be used under distinct assumptions about the way to compute the value of a vulnerable portfolio. The three assumptions proposed in Section 3.4 lead to four different values for the BVA/CVA. This happens as soon as there exist at least one region of the state space, over the remaining horizon of the portfolio, where the the upper and lower values do not coincide. This is due to the fact that the portfolio value is an expectation of future cash flows, contingent to the exercise strategies of both parties.

Figure 5 plots the four portfolio and corresponding BVA values, computed under the three scenarios, for the netted portfolio described in Table 10.

One can observe that Assumption A3 leads to BVA values that differ among counterparties, who both overestimate their expected losses by assuming the worst possible outcomes when an equilibrium




**(a)** $\lambda_1 = 0$ **(b)** $\lambda_1 = 0.5$

**(c)** $\lambda_1 = 0$ **(d)** $\lambda_1 = 0.5$

**Figure 5: Comparison of the netted portfolio value (panels a and b) and CVA/BVA (panels c and d) at inception, as a function of $\lambda_2$, according to the assumption used to compute the portfolio value. The composition of the portfolio is described in Table 10.**

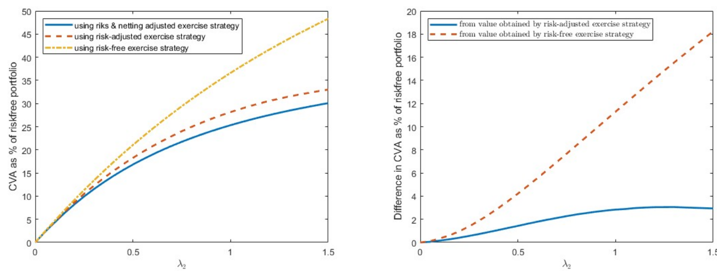
in pure strategies does not exist. The difference between these various interpretations of the price of CCR can be significant in some cases, namely when the region where the upper and lower values differ is extensive. Our numerical investigations show that this is likely to happen when the value of the portfolio approaches 0 (parties have commensurable claims) and/or when one party's default risk is substantially higher than that of the other. Recall that when Condition (30) is satisfied, the value of the portfolio is simply the sum of the individual options' values (netting has no impact), so that the four interpretations coincide, as seen in Figure 5 when $\lambda_2 = \lambda_1$.

## 4.6 Methodological choices

The conventional methodology used for assessing the risk adjustment value of derivatives with early-exercise features typically employs two successive steps. The exercise strategy is initially obtained for all derivatives, disregarding counterparty default risk. Subsequently, a Monte Carlo simulation of the default process and of the market risk factors is conducted in order to estimate the expected loss corresponding to the risk-free exercise strategy. As shown in Breton and Marzouk (2018), this two-step technique, while widely used in the industry, can lead to significant inaccuracies in the estimation of the CVA of individual derivative products. The numerical experiments presented above show that netting can further modify the exercise strategy of options, leading to unpredictable effects on the value of the CVA or BVA.

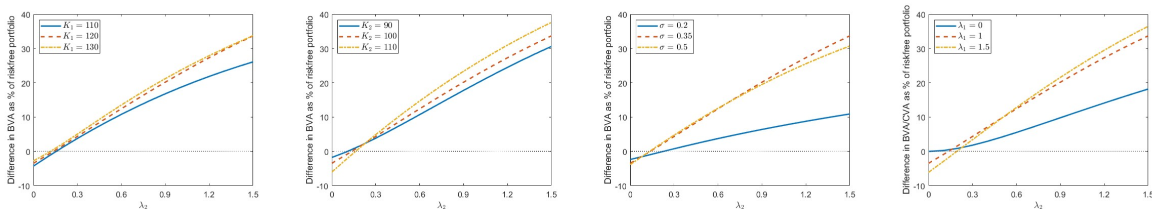
Our next set of experiments illustrates the misestimation that can arise when disregarding the impact of CCR and netting on the exercise strategies of derivatives. We use the simplest example of a portfolio of two Bermudan options written on the same underlying asset, described in Table 12, under the assumption that the parties can use mixed strategies.

For the unilateral-risk case, Figure 6 compares the CVA obtained using the risk-free, the risk-adjusted and the netting-adjusted exercise strategies, while in the bilateral-risk case, Figure 7 plots the difference, in percentage points, between the BVA obtained using the netting-adjusted strategies




**(a) CVA value** **(b) Difference**

**Figure 6: Difference between the CVA estimation when using the risk-free, risk-adjusted and netting-adjusted exercise strategy at inception, as a function of $\lambda_2$, for the portfolio of two Bermudan put options described in Table 12 when $\lambda_1 = 0$.**




**(a) Variation of $K_1$** **(b) Variation of $K_2$** **(c) Variation of $\sigma$** **(d) Variation of $\lambda_1$**

**Figure 7: Error in the estimation of the BVA when using the risk-free strategy for the portfolio of two Bermudan put options described in Table 12 when $\lambda_1 = 1$. Values are computed at inception as a function of $\lambda_2$, for various values of the strike prices, volatility, and $\lambda_1$.**

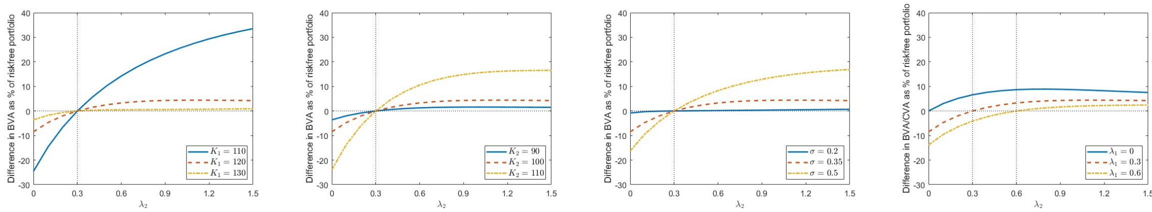
and the BVA obtained using the risk-free ones, as a function of the default intensity of the second party, for various values of the strike prices, the volatility, and the default intensity of the first party.

In the unilateral-risk case, using either the risk-free or the risk-adjusted strategy without accounting for the netting impact always leads to an overestimation of the CVA. This is because a risk-adjusted strategy results in exercising options earlier, thus reducing the expected loss upon default. Moreover, as explained in Section 4.3, netting also induces the vulnerable party to exercise its options earlier than under its risk-adjusted strategy, further reducing the price of CCR. However, in the bilateral case, the error in the estimation of the BVA can be positive or negative, depending on the relative parties' exposure to CCR, as can be observed from Figure 7.

## 4.7 Netting impact

We conclude this section by discussing and illustrating the netting impact, that is, the difference in the CVA or BVA of a netted portfolio with respect to a situation where parties are not in a netting agreement and correctly use a risk-adjusted strategy to evaluate the price of CCR. Figure 8 shows the sensitivity of the netting impact to various model parameters for the portfolio of put options described in Table 12. Note that the netting impact vanishes when Condition (30) is satisfied, that is, when the two parties have the same default intensity, irrespective of their relative exposure.

From a portfolio management perspective, one interesting interpretation of the netting impact is the evaluation of the difference in the portfolio value when parties do not adjust their exercise strategies to account for the impact of netting. As illustrated in Figure 8, this difference can be significant, especially when the probabilities of suffering losses from CCR differ substantially among the two counterparties.




**(a) Variation of $K_1$** **(b) Variation of $K_2$** **(c) Variation of $\sigma$** **(d) Variation of $\lambda_1$**

**Figure 8: Sensitivity of the netting impact on the CVA/BVA at inception as a function of $\lambda_2$ to various values of the strike prices, volatility, and initial asset price, for the portfolio of two options described in Table 12.**

# 5 Conclusion

In this paper, we examined the impact of netting on the price of counterparty credit risk. While it is well recognized that netting is an efficient risk-mitigation instrument when two counterparties participate in multiple bilateral contracts, the value of netting has not been investigated when these contracts include optional rights.

We have shown that the introduction of a netting agreement fundamentally changes the decision-making process related to the exercise strategies of optional rights in a netted portfolio because exercise decisions impact the expected payoffs of both counterparties in the agreement, giving rise to a stochastic zero-sum game between the two parties.

We proposed a model that allows to recursively compute the value of this game, which can then be compared with the value of a corresponding portfolio of risk-free contracts, thus allowing for the determination of the market price of CCR. Our dynamic-programming interpretation allows to characterize the CVA or BVA of a portfolio of contracts, for all possible values of the underlying market factors and all possible compositions of the portfolio, at all evaluation dates until maturity.

We numerically implemented this recursive algorithm and provided various experiments to illustrate how a netting agreement can modify the exercise strategy of both parties, even when one of the parties is not exposed to counterparty default risk. We also provided illustrations of the impact of changes in default probabilities, and other parameters affecting the parties exposure, on the exercise barrier of options within a netted portfolio.

Finally, our findings challenge traditional methodologies used to assess the risk adjustment value of netted portfolios, by showing that neglecting the impact of counterparty risk and netting on the exercise mechanism can lead to significant errors in the estimation of the CVA or BVA. The implicit signification of this last observation is that neglecting the impact of netting and of counterparty risk when managing a netted portfolio including optional rights leads to a decrease in expected value. Therefore, our investigation about the impact of CCR and netting on exercise barriers also provides valuable insight on how optional rights within netted portfolio should be managed by their holders.

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