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Integer linear programming for a constant demand in redundancy allocation multistate series-parallel problem

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Abstract : We consider the problem of minimizing the linear cost of multistate homogeneous series-parallel system given the nonlinear reliability constraint on the system. We propose a simple 0-1 integer linear programming model and find optimal solutions for the test problems presented in previous research considering a constant demand corresponds to the maximum demand in the study period. The decision variables are the number of components in each subsystem, and the choice of components. The system has a finite number of performance levels varying from 0% (complete failure) to 100% (perfect function). Each level has a corresponding state probability. The system reliability is calculated using the universal generating function technique. Because of the complex nature of the problem, it is often solved by heuristics. By using an exact method, we are able to validate the solutions found by heuristics. The mathematical programming model has a relatively simple structure. It is implemented immediately with the help of a mathematical programming language and an integer linear programming software. Moreover, our method solves reasonable instances from the literature in just a few milliseconds.

Keywords : Redundancy allocation, series-parallel systems, multistate systems, universal generating function, 0–1 integer linear programming model

1 Introduction

We consider the redundancy allocation problem (RAP) of a series-parallel system (Figure 1). The system has a finite number of subsystems in series, and the failure of any subsystem implies the failure of the entire system. In each subsystem, multiple redundant components are used in parallel: the subsystem will function if at least one of its components is operational. The failure of a redundant component may however decrease the system performance. Redundant components have a cumulative effect on the overall performance.

The system is designed to achieve reliability and performance. However, while the redundant components contribute to this goal, they also increase the total cost. We wish to select a combination of the components that satisfies the system reliability and/or weight constraints while minimizing the total cost.

The complexity of the problem depends on the application. The RAP is generally an NP-hard combinatorial optimization problem [7]. The model is complex because many factors, such as allowing mixed components or taking into account new demand levels, impact the system reliability and performance. To solve optimally the problem, we have to develop simplifying assumptions (e.g., considering constant demand, restricting each subsystem to identical components and limiting each component function to two possible states: good or failed). The components of the system are characterized by their reliability, performance, and cost; they are chosen from the relevant items available in the market. We define the system reliability to be the ability to meet the customer's performance expectation. We apply a universal moment generating function (UMGF) to evaluate the reliability [15, 33].

In recent years, the RAP has been applied to energy production [38], telecommunications design [27], health [12], natural disasters [37], protection [18], and logistics and transportation [36]. The solution approaches include metaheuristics [1, 17, 24, 30]. In [30], the authors apply a combination of space partitioning, genetic algorithms (GAs), and tabu search (TS). The authors in [25] apply a TS–GA algorithm to optimize the nonhomogeneous redundancy of series-parallel multistate systems (MSS). In [29], the authors use space partitioning to solve two design optimization problems: the first is the expansion scheduling of series-parallel MSS, and the second is the RAP for series-parallel binary-state systems.

Other papers on MSS include [2, 3, 23, 26]. There have been several extensive reviews [13, 14, 21, 34]. GAs have been used [15, 16, 22] to find the minimal-cost configuration of a series-parallel MSS under reliability or availability constraints. Similarily, Lisnianski et al. [19] address the continuous-multistate system reliability in an extensive way. This reference offers an up-to-date overview of the latest developments in reliability theory for MSS. The authors in [35] give a very recent synthesis on reliability theory, focusing on concepts and methodologies. A good and extensive review of reliability literature can be found, for example, in [8, 11, 20, 31].

Exact optimization techniques are an alternative to metaheuristics. To the best of our knowledge, the existing exact approaches assume that the system has only two possible states: good or failed. This is unrealistic in many applications. For example, [39] uses column generation to solve the RAP for binary series-parallel systems, while [4] uses integer linear programming. For the same problem, Cao et al. [5] propose a decomposition-based approach while Caserta et al. [6] transform the RAP into a multiple choice knapsack problem and solve it to optimality via a branch and cut algorithm. Diallo et al. [9, 10] use a linearization method to efficiently solve some mixed integer nonlinear optimization problems in maintenance of complex binary multicomponent systems. In many cases, such as power systems reliability analysis and telecommunication systems reliability analysis, the states range from 0% (complete failure) to 100% (perfect functioning). MSS reliability modeling usually considers a finite set of performance levels. The multistate version of the RAP is more complex and has not been solved using exact methods.

In this paper, we propose a simple 0-1 integer linear programming model that provides an optimal solution for the multistate RAP. We show that this approach is efficient: it is relatively easy to understand and to implement using existing solvers. To the best of our knowledge, this is the first time that the multistate RAP with a UMGF has been solved using an exact method. This is the main contribution of this paper. We examine the performance of our method via extensive computational experiments on benchmark instances.

The remainder of this paper is organized as follows. In Section 2 we provide a description of the RAP for series-parallel MSS. The UMGF method is presented in Section 3, and our solution approach is introduced in Section 4. Section 5 presents the test problems and the results, and Section 6 provides concluding remarks.

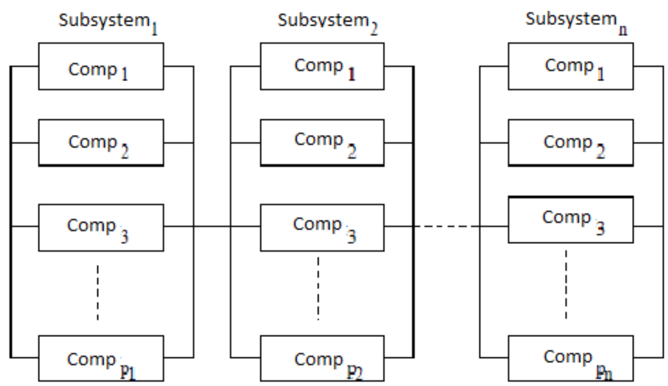

Figure 1: Example of series-parallel system.

2 RAP for series-parallel multistate systems

In this section, we present a description of the problem and its standard formulation. We begin with the necessary notation.

2.1 Notation

Variables	
X_{ij}	number of components of type j connected in parallel in subsystem i
Parameters	
N	number of series MSS subsystems
m_i	number of component choices available in the market for subsystem i, $i \in \{1, 2, ..., N\}$
$Max(X_{ij})$	maximum X_{ij} allowed (i.e., the upper bound on X_{ij})
X	vector: $(X_{ij})_{1 \leq i \leq N, 1 \leq j \leq m_i}$
R_0	specified minimum system reliability level
R_{ij}	binary-state reliability of component j used in subsystem i
C_{ij}	cost of component j in subsystem i
W_{ij}	nominal performance level of component j in subsystem i
D	required MSS performance
W	system capacity (system performance)
m	MSS state number, $m \in \{1, 2, ..., M\}$, where 1 is the worst and M is the best
W_m	MSS steady-state performance level associated with m

2.2 Description and assumptions

In each subsystem, a number of components are connected in parallel. The subsystems themselves are connected in series, so the MSS and its components can support multiple failures. The capacity or performance of the system is a function of the number and type of components used. Redundancy improves the reliability but increases the total cost. The goal is to select the items to use so that the total cost is minimized, subject to a multistate reliability constraint. For redundancy, we assume that the same type of components are used in each subsystem. That is, for each subsystem, we must determine the component type and the number of redundant components. Each component can have only two states: good or failed.

2.3 Mathematical formulation

The total system cost is the sum of the costs of the components. The cost of $Subsystem_i$ is $\sum_{j=1}^{m_i} C_{ij} X_{ij}$. Thus, the total cost (the objective function) is:

$$C(X) = \sum_{i=1}^{N} \sum_{j=1}^{m_i} C_{ij} X_{ij}. \tag{1}$$

The total cost may be a nonlinear function of X_{ij} to take into account price reductions [15, 28]. The problem can be formulated as follows:

$$\text{minimize } C(X) \tag{2}$$
$$\text{subject to } R(X) \geq R_0, \tag{3}$$
$$X_{ij} \in \{0, 1, ...Max(X_{ij})\} \qquad \forall i, 1 \leq i \leq N, \forall j, 1 \leq j \leq m_i \tag{4}$$

Constraints (3) enforce the reliability limits. Constraints (4) specify that, for each subsystem, the number of connected components is an integer that is at most a prespecified maximum value.

3 Reliability calculation for multistate systems

This section briefly summarizes estimations of MSS reliability. We use the universal z-transform technique [32], which has proven effective for large combinatorial optimization problems [15, 29]. This technique is also called the UMGF or simply the U-function or U-transform.

For MSS, the system capacity (performance) W must be determined and compared to some demand target D to assess the system reliability R. More precisely, R is defined as $Pr(W \geq D)$, and W is based on the performance of its components.

In power engineering, for example, $R(X)$ is often related to the loss of load probability (LOLP) index defined as $LOLP = 1 - R(X)$, this depends on X. LOLP is understood as the probability that the system cannot supply a given demand load. The demand is assumed to be constant.

3.1 Definition and properties of U-function

We now introduce the U-function and its properties.

Definition 3.1. The U-function of a discrete random variable W is a polynomial:

$$U(z) = \sum_{m=1}^{M} p_m z^{W_m}, \tag{5}$$

where W has M possible values and p_m is the probability that W is equal to W_m.

Definition 3.2. The reliability R is given [16, 32] by the probability

$$R = P[W \geq D] = \Phi\Big(U(z)z^{-D}\Big), \tag{6}$$

where Φ is a distributive operator defined by

$$\Phi(pz^{w}) = p1_{[w\geq 0]}. \tag{7}$$

Here $1_{[w\geq 0]}$ is an indicator function that is 1 if $w \geq 0$ and 0 otherwise. We have

$$\Phi\Big(\sum_{m=1}^{M} p_m z^{w_m}\Big) = \sum_{m=1}^{M} \Phi(p_m z^{w_m}). \tag{8}$$

The operator Φ satisfies Ushakov's four properties [32]:

1. $\Phi(pz^w) = pz^w$.
2. $\Phi(p_1 z^{w_1}, p_2 z^{w_2}) = p_1 p_2 z^{f(w_1,w_2)}$, where $f(w_1, w_2)$ is defined according to the system configuration.
3. $\Phi\Big(U_1(z), ..., U_k(z), U_{k+1}(z), ...U_n(z)\Big) = \Phi\bigg(\Phi\Big(U_1(z), ..., U_k(z)\Big), \Phi\Big(U_{k+1}(z), ..., U_n(z)\Big)\bigg)$ for any k.
4. $\Phi\Big(U_1(z), ..., U_k(z), U_{k+1}(z), ...U_n(z)\Big) = \Phi\Big(U_1(z), ..., U_{k+1}(z), U_k(z), ...U_n(z)\Big)$ for any k.

We now show that Eqs. (5)–(8) satisfy $P[W \geq D] = \sum_{W_m \geq D} p_m$. We have:

$$\begin{aligned} P[W \geq D] =& \Phi\Big(U(z)z^{-D}\Big) \\ =& \Phi\bigg(\sum_{m=1}^{M} p_m z^{W_m - D}\bigg) \\ =& \sum_{m=1}^{M} \Phi(p_m z^{W_m - D}) \\ =& \sum_{m=1}^{M} p_m 1_{[W_m - D \geq 0]} \\ =& \sum_{W_m \geq D} p_m. \end{aligned}$$

The operator Φ is used here to calculate the polynomial coefficients $U(z)$ by summing every term with $W_m \geq D$.

3.2 Series-parallel MSS reliability evaluation using U-functions

The series-parallel MSS reliability is obtained by applying the composition operators consecutively. We first calculate the U-function for a subsystem of components connected in parallel using the operator Φ over the U-function of each component. We then use the U-functions of the subsystems to obtain the reliability of the entire system.

The total performance of the parallel system is the sum of the performance of its components. The function to be used for Φ in Ushakov's second property is $f(w_1, w_2) = w_1 + w_2$. The U-function of $Subsystem_i$ containing X_i parallel components is:

$$U(z) = \Phi\Big(U_1(z), U_2(z), ..., U_{X_i}(z)\Big), \text{ where } f(w_1, w_2, ..., w_{X_i}) = \sum_{e=1}^{X_i} w_e \tag{9}$$

Therefore, for a pair of components connected in parallel:

$$\Phi\Big(U_1(z), U_2(z)\Big) = \Phi\left(\sum_{i=1}^{n} P_i z^{w_i}, \sum_{j=1}^{m} Q_j z^{w_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} P_i Q_j z^{w_i+w_j} \tag{10}$$

where n and m are the numbers of possible performance levels for these components. The operator Φ is simply a product of the individual U-functions. Thus, the U-function of $Subsystem_i$ is:

$$U_i(z) = \prod_{l=1}^{X_{ij}} U_{ijl}(z) \tag{11}$$

where $U_{ijl}(z)$ is the U-function of the l^{th} component of type j in the i^{th} subsystem containing X_{ij} parallel components. We assume that each component has only two states (nominal performance or total failure). For example, let l^{th} component of type j in subsystem i have capacity W_{ijl} and reliability R_{ijl}. Then, $Pr[W = W_{ijl}] = R_{ijl}$ and $Pr[W = 0] = 1 - R_{ijl}$. The UMGF has two terms:

$$U_{ijl}(z) = (1 - R_{ijl})z^0 + R_{ijl} z^{W_{ijl}}. \tag{12}$$

Given the individual U-function defined in Equation-(12), the U-function of $Subsystem_i$ with X_{ij} parallel components is:

$$U_i(z) = \Phi\Big(U_{ij1}(z), U_{ij2}(z), ..., U_{ijX_{ij}}(z)\Big) = \prod_{l=1}^{X_{ij}} \Big[(1 - R_{ijl})z^0 + R_{ijl} z^{W_{ijl}}\Big]. \tag{13}$$

Under the assumption that all the components are identical ($R_{ijl} = R_{ij}$ and $W_{ijl} = W_{ij} \quad \forall l$) , the U-function becomes:

$$U_i(z) = \Big[(1 - R_{ij})z^0 + R_{ij} z^{W_{ij}}\Big]^{X_{ij}} = \sum_{l=0}^{X_{ij}} \alpha_{il}(X_{ij}) z^{lW_{ij}}, \tag{14}$$

$$\alpha_{il}(X_{ij}) = \text{binm}(l, R_{ij}, X_{ij}) = \left[\frac{X_{ij}!}{l!(X_{ij} - l)!}\right] R_{ij}^l (1 - R_{ij})^{X_{ij}-l}. \tag{15}$$

To evaluate the probability that $Subsystem_i$ provides a performance level exceeding D, the operator Φ is applied to Equation (14) as follows:

$$\Phi\Big(U(z)z^{-D}\Big) = \sum_{lW_{ij} \geq D} \alpha_{il}(X_{ij}). \tag{16}$$

Using Equation (6), the reliability R_i for $Subsystem_i$ under demand D is given by:

$$R_i(X) = Pr[W \geq D] = \Phi\Big(U(z)z^{-D}\Big) = \sum_{lW_{ij} \geq D} \alpha_{il}(X_{ij}). \tag{17}$$

If N subsystems are connected in series, the system reliability $R(X)$ is the product of the subsystem reliabilities [15, 28]:

$$R(X) = \prod_{i=1}^{N} R_i(X). \tag{18}$$

4 Formulation and linearization

The RAP for a series-parallel MSS has a nonlinear reliability constraint: $R(X) \geq R_0$. Using Equation (18), we obtain

$$\prod_{i=1}^{N} R_i(X) \geq R_0, \tag{19}$$

or equivalently

$$\sum_{i=1}^{N} log\Big(R_i(X)\Big) \geq log(R_0). \tag{20}$$

Using Equation (17), we obtain

$$\sum_{i=1}^{N} log\left(\sum_{lW_{ij}\geq D} \alpha_{il}(X_{ij})\right) \geq log(R_0). \tag{21}$$

Let Y_{ijp} be a decision variable such that

$$Y_{ijp} = \begin{cases} 1 & \text{if component type } j \text{ is used } p \text{ times in subsystem } i; \\ 0 & \text{otherwise.} \end{cases}$$

The problem can then be reformulated as a linear 0-1 integer program:

$$\text{minimize } C(Y) = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\sum_{p=1}^{Max(X_{ij})} pC_{ij}Y_{ijp} \tag{P2}$$

$$\text{subject to } \sum_{i=1}^{N}\sum_{j=1}^{m_i}\sum_{p=1}^{Max(X_{ij})} a_{ijp}Y_{ijp} \geq \overline{R}_0, \tag{22}$$

$$\sum_{j=1}^{m_i}\sum_{p=1}^{Max(X_{ij})} Y_{ijp} = 1 \qquad \forall i, 1 \leq i \leq s, \tag{23}$$

$$Y_{ijp} \in \{0,1\} \qquad \forall i, 1 \leq i \leq s, \forall j, 1 \leq j \leq m_i, \forall p, 1 \leq p \leq Max(X_{ij}) \tag{24}$$

where

$$\overline{R}_0 = \log(R_0)$$

and

$$a_{ijp} = \log\left(\sum_{l=\left\lceil \frac{W_{ij}}{D}\right\rceil}^{p} \alpha_{il}(p)\right)$$

with $\lceil x \rceil$ being the smallest integer greater than or equal to x.

To solve (P2), we can use a standard 0-1 integer programming solver. There are many readily available packages such as IBM ILOG CPLEX, LINDO, and Xpress. For very large instances, a specialized algorithm can be used.

5 Test problems and numerical results

We use four design optimization problems (benchmarks) from the literature and two new instances to investigate our algorithm for (P2). These problems do not allow component mixing.

5.1 Notation for benchmarks

We denote each benchmark by *xxxa-(b/c)*, where *xxx* indicates the first three characters of the first author's name in the paper where the instance was introduced; a is the number of subsystems connected in series; (b/c) means that the number of component types ranges from b to c. The first three problems, lev4-(4/6), lev5-(4/9), and lis4-(7/11), were solved by GA and TS in [15, 16, 22, 28] assuming that the demand is represented as a piecewise cumulative load curve. The fourth, ouz6-(4/11), was solved by TS in [28], and the last two are new instances constructed as follows:

- ouz9-(4/9): merge of lev5-(4/9) and lev4-(4/6);
- ouz15-(4/11): merge of lev5-(4/9), lev4-(4/6), and ouz6-(4/11).

The algorithm is implemented in C^{++} using IBM ILOG CPLEX. The tests were performed on an Intel Core i7 at 2.8 GHz with 8 GB of RAM, running Linux.

5.2 Benchmark data and new instances

Table 1 gives a brief description of the four benchmarks. The new instances ouz9-(4/9) and ouz15-(4/11) are larger: there are nine subsystems for the first problem and fifteen subsystems for the second, with 4 to 11 component types. Tables A5 and A6 present the data.

Table 1: Information for each benchmark

Problem	Information
lev4-(4/6)	Data are given in Tables A1 in A. Four subsystems with 4–6 component types.
lev5-(4/9)	Data are given in Tables A2 in B. Five subsystems with 4–9 component types.
lis4-(7/11)	Data (reliability, cost, and nominal performance curve) are given in Tables A3 in C. Four subsystems with 7–11 component types.
ouz6-(4/11)	Data are given in Tables A4. Six subsystems with 4–11 component types.

5.3 Results

Table 2 gives the six optimal solutions. We set the demand to 100% and reliability index R_0 to 0.98, 0.99, and 0.999. Computational times are reported in Table 2 in the column "CPU". They did not exceed 5 ms. The second column gives the settings for R_0, and the third and fourth columns contain the optimal reliability and cost. The fifth and sixth columns contain the component type and the number of components used in each subsystem, e.g., the first instance uses one type-10 component in subsystem 1, two type-7 components in subsystem 2, five type-2 components in subsystem 3, and five type-2 components in subsystem 4.

Table 3 lists the cost obtained by the existing methods, mostly based on metaheuristics [15, 16, 28] assuming that the demand is represented as a piecewise cumulative load curve and is varied from 20% to 100%. The reliability index R_0 is seted to 99%. In most cases, to ensure 100% production satisfaction, the price to be added does not exceed 0,4 M$.

The cost evolution curves concerning demand have been drawn for the first two instances of the three problems 1 to 3. Figures 2 to 4 show these curves, which suggest a similarity to a linear trend. It's possible that this observation suggests a simplified relationship between demand and costs in these early scenarios. However, it's important to emphasize that further comprehensive analyses might be necessary to determine whether this linear trend persists in more intricate cases or for the remaining instances of the problems.

6 Conclusion

In this paper, the redundancy allocation problem (RAP) is formulated as an optimization problem for multistate homogeneous systems. A simple 0-1 integer linear programming model is proposed to solve RAP problems in series-parallel systems efficiently and exactly. In this model, we first calculate the system reliability using the universal generating function technique. We then linearize the reliability constraint through logarithmic transformation. Using a series-parallel system example from the literature, we compare the proposed approach with the test problems presented in previous research. We have found the optimal solution for each instance proposed in [28] and shown that the metaheuristics proposed in [15, 16, 28] are effective for this problem.

Because of the complex nature of the problem, we have only examined the series-parallel MSS structure in this paper and assuming that the demand is constant and the system and its components have binary states. In the future, it would be interesting to extend the method to nonhomogenous mutlistate series-parallel.

Table 2: Optimal solutions obtained by the exact method

Problem	R_0	$R(X)$	$C(X)$ ($M)	Component type	Number of components	CPU (sec)
lev4-(4/6)	0.980	0.9837	8.328	1,3,1,2	3,3,3,5	0.00
	0.990	0.9913	8.7320	1,3,1,1	3,3,4,5	0.01
	0.999	0.9992	10.674	1,3,1,2	4,4,4,6	0.01
lev5-(4/9)	0.980	0.9801	16.5710	2,5,2,9,2	2,6,3,6,1	0.02
	0.990	0.9904	17.0730	2,1,2,7,4	2,1,3,3,3	0.01
	0.999	0.9990	18.8270	7,3,1,7,4	6,3,2,4,4	0.01
lis4-(7/11)	0.980	0.9819	22.7063	10,7,2,2	1,2,5,5	0.01
	0.990	0.9912	24.3988	1,3,2,2	5,3,5,5	0.01
	0.999	0.9996	27.3987	1,3,2,3	6,4,6,7	0.01
ouz6-(4/11)	0.980	0.9802	11.5940	3,1,2,2,3,4	5,5,5,8,2,1	0.02
	0.990	0.9929	13.1610	3,1,2,2,3,4	4,4,5,7,2,2	0.01
	0.999	0.9992	16.6390	3,1,2,2,3,4	5,5,6,8,3,2	0.02
ouz9-(4/9)	0.980	0.9802	25.5440	2,5,2,9,3,1,3,1,2	2,6,3,6,3,3,3,3,5	0.02
	0.990	0.9902	26.4380	7,5,2,7,3,4,3,1,1	6,6,3,4,3,2,3,4,5	0.02
	0.999	0.9990	30.9880	7,5,1,9,4,1,1,1,1	7,7,2,7,4,5,8,5,6	0.02
ouz15-(4/11)	0.980	0.9801	39.0470	2,5,2,9,3,1,3,1,2,3,1,2,2,3,4	2,6,3,6,3,3,3,4,5,4,5,5,8,2,2	0.04
	0.990	0.9901	40.4130	2,5,2,7,3,4,1,1,1,3,1,2,2,3,4	2,6,3,4,3,2,7,4,5,5,5,5,8,2,2	0.05
	0.999	0.9990	50.1390	7,5,3,7,4,1,1,1,1,3,1,2,2,3,4	7,7,5,4,4,4,8,5,6,5,5,6,10,3,2	0.05

Table 3: Comparison of results

Problem	Exact method cost 100% demand satisfaction	metaheuristics cost demand varies from 20% to 100%	The price to add (M$)
lev4-(4/6)	8.732	8.328	0,404
lev5-(4/9)	17.073	17.050	0,023
lis4-(7/11)	24.398	24.305	0,094
ouz6-(4/11)	13.161	12.764	0,397

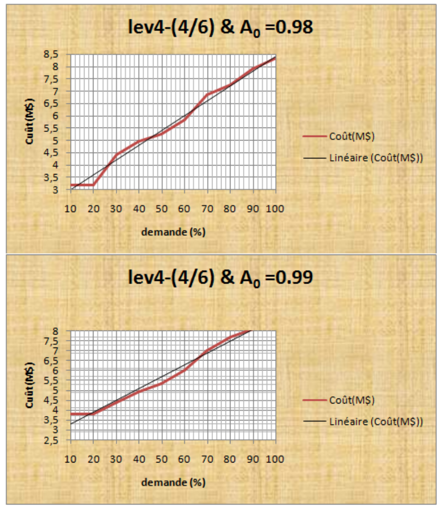

Figure 2: The evolution of costs according to the demand for lev4-(4/6)

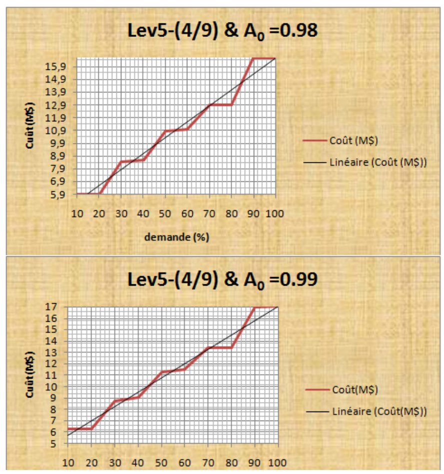

Figure 3: The evolution of costs according to the demand for lev5-(4/9)

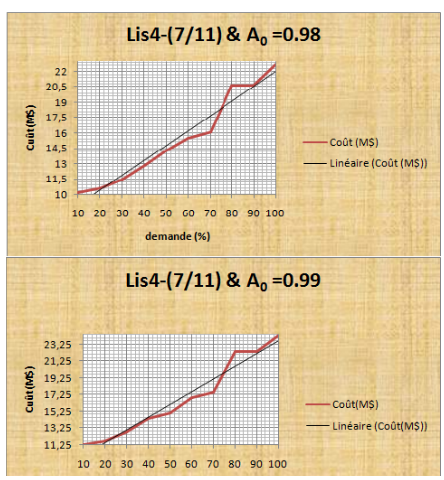

Figure 4: The evolution of costs according to the demand for lev4-(7/11)

Appendix

A Problem 1

Table A1: Data for components available in the market [16]

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
1	1	0.970	0.520	50
	2	0.964	0.620	80
	3	0.980	0.720	80
	4	0.969	0.890	100
	5	0.960	1.020	150
2	1	0.967	0.516	20
	2	0.914	0.916	50
	3	0.960	0.967	50
	4	0.953	1.367	75
3	1	0.959	0.214	60
	2	0.970	0.384	90
	3	0.959	0.534	180
	4	0.960	0.614	200
	5	0.970	0.783	200
	6	0.960	0.813	240
4	1	0.989	0.683	25
	2	0.979	0.645	25
	3	0.980	0.697	30
	4	0.960	1.190	70
	5	0.980	1.260	70

B Problem 2

Table A2: Data for components available in the market [15]

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
1	1	0.980	0.590	120
	2	0.977	0.535	100
	3	0.982	0.470	85
	4	0.978	0.420	85
	5	0.983	0.400	48
	6	0.920	0.180	31
	7	0.984	0.220	26
2	1	0.995	0.205	100
	2	0.996	0.189	92
	3	0.997	0.091	53
	4	0.997	0.056	28
	5	0.998	0.042	21
3	1	0.971	7.525	100
	2	0.973	4.720	60
	3	0.971	3.590	40
	4	0.976	2.420	20
4	1	0.977	0.180	115
	2	0.978	0.160	100
	3	0.978	0.150	91
	4	0.983	0.121	72
	5	0.981	0.102	72
	6	0.971	0.096	72
	7	0.983	0.071	55
	8	0.982	0.049	25
	9	0.977	0.044	25

Table A2: Data for components available in the market [15]

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
5	1	0.984	0.986	128
	2	0.983	0.825	100
	3	0.987	0.490	60
	4	0.981	0.475	51

C Problem 3

Table A3: Data for components available in the market [22]

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
1	1	0.990	1.117	25
	2	0.996	1.310	25
	3	0.995	1.903	35
	4	0.961	1.640	35
	5	0.993	2.122	50
	6	0.957	1.910	50
	7	0.942	1.722	50
	8	0.991	2.591	72
	9	0.951	2.001	72
	10	0.986	3.284	100
	11	0.979	3.095	100
2	1	0.967	4.010	40
	2	0.914	3.450	40
	3	0.960	4.350	55
	4	0.953	4.840	78
	5	0.920	4.210	78
	6	0.950	5.800	93
	7	0.948	6.550	110
3	1	0.967	0.636	25
	2	0.952	0.448	35
	3	0.973	0.868	35
	4	0.972	0.964	50
	5	0.949	0.678	50
	6	0.988	1.096	50
	7	0.966	1.358	72
	8	0.954	1.298	72
	9	0.945	1.810	100
4	1	0.987	0.614	12.5
	2	0.985	0.883	25
	3	0.961	0.745	25
	4	0.980	0.963	30
	5	0.958	0.885	30
	6	0.974	1.338	45
	7	0.982	1.445	45

D Problem 4

Table A4: Data for components available for problem 4 [28]

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
1	1	0.932	1.590	27.3
	2	0.998	0.515	27.7
	3	0.983	0.225	49.8
	4	0.927	3.220	52.5
	5	0.959	4.020	62.0
	6	0.955	4.270	66.4
	7	0.984	3.670	84.6

Table A4: Data for components available for problem 4 [28]

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} ($M)	Nominal performance W_{ij} (%)
	8	0.918	4.630	90.7
	9	0.939	1.010	97.0
	10	0.988	0.779	124
	11	0.984	3.130	129
2	1	0.989	0.050	35.9
	2	0.923	1.290	44.7
	3	0.900	0.204	51.4
	4	0.946	2.220	63.2
	5	0.917	0.872	68.8
	6	0.962	1.830	81.8
	7	0.994	0.294	82.0
	8	0.984	2.810	115
3	1	0.931	3.620	34.7
	2	0.950	0.475	41.0
	3	0.911	1.170	41.4
	4	0.956	0.793	43.6
	5	0.966	3.740	48.6
	6	0.992	4.590	59.6
	7	0.929	1.740	66.2
	8	0.968	1.720	91.9
	9	0.901	1.300	121
4	1	0.915	2.490	25.1
	2	0.908	0.078	28.8
	3	0.928	1.370	50.2
	4	0.944	4.470	129
5	1	0.908	1.550	34.9
	2	0.980	4.920	64.3
	3	0.964	2.650	108
	4	0.924	4.720	126
6	1	0.965	3.220	24.8
	2	0.927	2.890	47.3
	3	0.986	3.410	58.8
	4	0.983	1.920	107
	5	0.991	4.510	120
	6	0.954	4.580	125

E Problem 5

Table A5: Data for components available for problem 5

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} ($M)	Nominal performance W_{ij} (%)
1	1	0.980	0.590	120
	2	0.977	0.535	100
	3	0.982	0.470	85
	4	0.978	0.420	85
	5	0.983	0.400	48
	6	0.920	0.180	31
	7	0.984	0.220	26
2	1	0.995	0.205	100
	2	0.996	0.189	92
	3	0.997	0.091	53
	4	0.997	0.056	28
	5	0.998	0.042	21
3	1	0.971	7.525	100
	2	0.973	4.720	60
	3	0.971	3.590	40
	4	0.976	2.420	20

Table A5: Data for components available for problem 5

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
4	1	0.977	0.180	115
	2	0.978	0.160	100
	3	0.978	0.150	91
	4	0.983	0.121	72
	5	0.981	0.102	72
	6	0.971	0.096	72
	7	0.983	0.071	55
	8	0.982	0.049	25
	9	0.977	0.044	25
5	1	0.984	0.986	128
	2	0.983	0.825	100
	3	0.987	0.490	60
	4	0.981	0.475	51
6	1	0.970	0.520	50
	2	0.964	0.620	80
	3	0.980	0.720	80
	4	0.969	0.890	100
	5	0.960	1.020	150
7	1	0.967	0.516	20
	2	0.914	0.916	50
	3	0.960	0.967	50
	4	0.953	1.367	75
8	1	0.959	0.214	60
	2	0.970	0.384	90
	3	0.959	0.534	180
	4	0.960	0.614	200
	5	0.970	0.783	200
	6	0.960	0.813	240
9	1	0.989	0.683	25
	2	0.979	0.645	25
	3	0.980	0.697	30
	4	0.960	1.190	70
	5	0.980	1.260	70

F Problem 6

Table A6: Data for components available for problem 6

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
1	1	0.980	0.590	120
	2	0.977	0.535	100
	3	0.982	0.470	85
	4	0.978	0.420	85
	5	0.983	0.400	48
	6	0.920	0.180	31
	7	0.984	0.220	26
2	1	0.995	0.205	100
	2	0.996	0.189	92
	3	0.997	0.091	53
	4	0.997	0.056	28
	5	0.998	0.042	21
3	1	0.971	7.525	100
	2	0.973	4.720	60
	3	0.971	3.590	40
	4	0.976	2.420	20

Table A6: Data for components available for problem 6

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} (\$M)	Nominal performance W_{ij} (%)
4	1	0.977	0.180	115
	2	0.978	0.160	100
	3	0.978	0.150	91
	4	0.983	0.121	72
	5	0.981	0.102	72
	6	0.971	0.096	72
	7	0.983	0.071	55
	8	0.982	0.049	25
	9	0.977	0.044	25
5	1	0.984	0.986	128
	2	0.983	0.825	100
	3	0.987	0.490	60
	4	0.981	0.475	51
6	1	0.970	0.520	50
	2	0.964	0.620	80
	3	0.980	0.720	80
	4	0.969	0.890	100
	5	0.960	1.020	150
7	1	0.967	0.516	20
	2	0.914	0.916	50
	3	0.960	0.967	50
	4	0.953	1.367	75
8	1	0.959	0.214	60
	2	0.970	0.384	90
	3	0.959	0.534	180
	4	0.960	0.614	200
	5	0.970	0.783	200
	6	0.960	0.813	240
9	1	0.989	0.683	25
	2	0.979	0.645	25
	3	0.980	0.697	30
	4	0.960	1.190	70
	5	0.980	1.260	70
10	1	0.932	1.590	27.3
	2	0.998	0.515	27.7
	3	0.983	0.225	49.8
	4	0.927	3.220	52.5
	5	0.959	4.020	62.0
	6	0.955	4.270	66.4
	7	0.984	3.670	84.6
	8	0.918	4.630	90.7
	9	0.939	1.010	97.0
	10	0.988	0.779	124
	11	0.984	3.130	129
11	1	0.989	0.050	35.9
	2	0.923	1.290	44.7
	3	0.900	0.204	51.4
	4	0.946	2.220	63.2
	5	0.917	0.872	68.8
	6	0.962	1.830	81.8
	7	0.994	0.294	82.0
	8	0.984	2.810	115
12	1	0.931	3.620	34.7
	2	0.950	0.475	41.0
	3	0.911	1.170	41.4
	4	0.956	0.793	43.6
	5	0.966	3.740	48.6
	6	0.992	4.590	59.6
	7	0.929	1.740	66.2
	8	0.968	1.720	91.9
	9	0.901	1.300	121

Table A6: Data for components available for problem 6

$Subsystem_i$	Component type j	R_{ij}	Cost C_{ij} ($M)	Nominal performance W_{ij} (%)
13	1	0.915	2.490	25.1
	2	0.908	0.078	28.8
	3	0.928	1.370	50.2
	4	0.944	4.470	129
14	1	0.908	1.550	34.9
	2	0.980	4.920	64.3
	3	0.964	2.650	108
	4	0.924	4.720	126
15	1	0.965	3.220	24.8
	2	0.927	2.890	47.3
	3	0.986	3.410	58.8
	4	0.983	1.920	107
	5	0.991	4.510	120
	6	0.954	4.580	125

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