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# Using symbolic calculations to determine largest small polygons

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**Abstract:** A small polygon is a polygon of unit diameter. The question of finding the largest area of small $n-$gons has been answered for some values of $n$. Regular $n-$gons are optimal when $n$ is odd and kites with unit length diagonals are optimal when $n = 4$. For $n = 6$, the largest area is a root of a degree 10 polynomial with integer coefficient having 4 to 6 digits. This polynomial was obtained through factorizations of a degree 40 polynomial with integer coefficients.

The present paper analyses the hexagonal and octogonal cases. For $n = 6$, we propose a new formulation which involves the factorization of a polynomial with integer coefficients of degree 14 rather than 40. And for $n = 8$, under an axial symmetry conjecture, we propose a methodology that leads to a polynomial of degree 344 with integer coefficients that factorizes into a polynomial of degree 42 with integer coefficients having 21 to 32 digits. A root of this last polynomial corresponds to the area of the largest small axially symmetrical octagon.

**Keywords:** Small polygons, planar geometry, disciminant

# 1 Introduction

The *diameter* of a polygon is the largest Euclidean distance between pairs of its vertices. A polygon is said to be *small* if its diameter equals one. The present work studies the problem of finding for a fixed value of $n \geq 3$, the small $n$-gon whose area is maximal. Regular polygons are optimal in the case where the $n$-gon is required to be equilateral [1], but are not always optimal in the non-equilateral case. Reinhard [11] showed almost 100 years ago that when $n$ is odd, the regular polygon is optimal but the situation gets more complicated when $n$ is even. For $n = 4$, any small quadrilateral for which both of its diagonals are unit-lenght and perpendicular are optimal with an area equal to $\frac{1}{2}$. In particular, the square with side length $\frac{1}{\sqrt{2}}$ is optimal. However, when $n \geq 6$ is even, the optimal $n$-gon is not the regular polygon as the area of the small regular $(n-1)$-sided polygon is larger than that of the small regular $n$-sided polygon [3]. The survey [2] presents a collection of similar extremal problems for polygons.

The diameter graph of a small $n$-gon is the graph composed of its $n$ vertices, and its edges connect pairs of vertices that are at unit-distance from each other. The diameter graph of the largest small hexagon was shown in 1975 by Graham [6] to be composed of a cycle of length 5 with a pending edge. The optimal hexagon has an axis of symmetry. In order to find its area, Jonhson and Graham [8] factor a polynomial of degree 40 into four irreducible polynomial of degree 10. On of these polynomials, say $p_6$, is such that one of its root is the value of the area of the largest small hexagon. Lazard [9] proposes alternate approaches, including two of them that factorize a polynomial of degree 24. One of its factor is the polynomial $p_6$ found by Johnson and Graham. In the present work, we use a complex-plane formulate the problem, which leads to a polynomial of degree 14 that factors into two irreducible polynomials: $p_6$ and another polynomial of degree 4.

For even values of $n \geq 6$, it was shown [5] that the diameter graph of the optimal polygon is composed of a cycle of length $n-1$ with a pending edge. Numerical solutions with 4 and 7 exact digits are proposed in [4] and [7] for the octagon. It has been conjectured [6, 7] that the optimal polygon has an axis of symmetry corresponding to the pending edge. This conjecture is verified for the quadrilateral [11], for the hexagon [12] and supported numerically for the octagon, decagon and dodecagon [7]. Lower and upper upper bounds on the optimal area are presented in [10] for larger values of $n$. The present work studies the octagon under the assumption of an axis of symmetry with respect to the pending edge of the diameter graph. Using symbolic calculations, we propose a way to derive a polynomial of degree 344, whose factorization produces a polynomial $p_8$ of degree 42 and for which one of its root is the largest small octagon with an axis of symmetry. The value of the optimal area is coherent with the bounds of Henrion and Messine [7].

The present paper is divided into two main sections. Section 2 is devoted to the hexagon, and Section 3 to the symmetrical octagon. A final section concludes with comments on larger $n$-gons.

# 2 The largest small hexagon

Symmetry of the largest small hexagon was proved by Yuan [12]. We show the same result using a different approach using a complex-plane representation, which is then used to find the optimal hexagon.

**Theorem 2.1** *The diameter graph of the largest small hexagon consists of a cycle of length* 5 *with one pending edge, and is axially symmetrical with respect to that edge.*

**Proof.** Foster and Szabo show that the diameter graph of largest small $n$-gon, where $n \geq 6$ is even, consists of a cycle of length $n-1$ with a pending edge. Let $P$ be the optimal small hexagon with diameter graph composed of a cycle of length 5 with one pending edge, and whose consecutive vertices in the complex plane are $z_1, z_3, z_5, z_2, z_4$ and $z_6$ as illustrated in Figure 1. Without any loss of generality, we fix the endpoints of the pending edge $z_5 = 0$ and $z_6 = i$ on the imaginary axis. Reinhardt showed that the pending edge is perpendicular to the line segment joining its neighboring vertices $z_1$ and $z_4$, which implies that $z_4 = -\bar{z}_1$. The area $A(P)$ of the hexagon can be decomposed as the sum of the areas of the triangle $z_5z_2z_4$, the quadrilateral $z_5z_4z_6z_1$ and the triangle $z_5z_1z_3$:

$$A(P) = \frac{1}{2}\operatorname{Im}(\bar{z}_4 z_2) + \operatorname{Re}(z_1) + \frac{1}{2}\operatorname{Im}(\bar{z}_3 z_1) \;=\; \operatorname{Re}(z_1) + \frac{1}{2}\operatorname{Im}\left(z_1(\bar{z}_3 - z_2)\right).$$

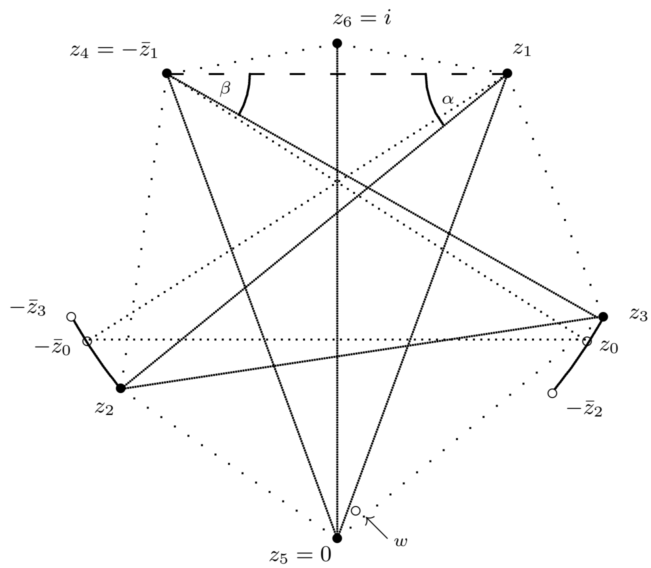


**Figure 1: Complex plane representation of cycle of length 5 with a pending edge**

Let $\alpha$ and $\beta$ denote the angles $\angle z_4z_1z_2$ and $\angle z_1z_4z_3$, respectively, as show in the figure. By contradiction, suppose that the polygon $P$ is not symmetrical with respect to the pending edge, i.e., suppose that $\alpha \neq \beta$. Consider a second hexagon $Q$ whose vertices are $z_1, z_0, z_5, -\bar{z}_0, z_4$ and $z_6$ where $z_0$ is the vertex at unit distance from $z_4$ such that the angle $\angle z_1z_4z_0$ is equal to $\frac{\alpha+\beta}{2}$. The diameter of this second hexagon is also equal to 1 because

$$\begin{aligned}
(|z_0 + \bar{z}_0|)^2 &= (2\operatorname{Re}(z_0))^2 \\
&= 4\left(\operatorname{Re}(-z_1) + \cos\frac{\alpha+\beta}{2}\right)^2 \\
&= 4\left(\operatorname{Re}(-z_1)^2 - 2\operatorname{Re}(z_1)\cos\frac{\alpha+\beta}{2} + \cos^2\frac{\alpha+\beta}{2}\right) \\
&\leq 4\left(\operatorname{Re}(-z_1)^2 - \operatorname{Re}(z_1)(\cos\alpha + \cos\beta) + \cos^2\frac{\alpha+\beta}{2}\right) \\
&= 4\operatorname{Re}(-z_1)^2 - 4\operatorname{Re}(z_1)(\cos\alpha + \cos\beta) + 2(1 + \cos\alpha\cos\beta - \sin\alpha\sin\beta) \\
&= (2\operatorname{Re}(-z_1) + \cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\
&= (|-2\operatorname{Re}(z_1) + \cos\alpha + \cos\beta + i(\sin\alpha - \sin\beta)|)^2 \\
&= (|z_3 - z_2|)^2 \;=\; 1.
\end{aligned}$$

The area $A(Q)$ of this second polygon is $A(Q) = \operatorname{Re}(z_1) + \operatorname{Im}(z_1\bar{z}_0)$ and satisfies

$$A(Q) - A(P) = \frac{1}{2}\operatorname{Im}\left(z_1(2\bar{z}_0 - \bar{z}_3 + z_2)\right) \;=\; \frac{1}{2}\operatorname{Im}(z_1\bar{w})$$

where $w = (z_0 - z_3) + (z_0 - (-z_2))$. The value $\frac{1}{2}\operatorname{Im}(z_1\bar{w})$ is strictly positive because it corresponds to the area of the triangle $z_5wz_1$ which leads to the contradiction that $A(P) < A(Q)$. $\square$

The next theorem shows a result of Graham, but with a different proof that uses the discriminant of a polynomial. Recall that the discriminant of a polynomial $p(x)$ is a polynomial function of the coefficients of $p(x)$, and the discriminant equals 0 if and only if $p(x)$ has a multiple root. In an optimization context, let $A$ be a critical value of a polynomial function $t(x)$ and let $x^*$ be a solution such that $A = f(x^*)$. Then $x^*$ is also a root of the polynomial function $t(x) - A$ and consequently, the discriminant of $t(x) - A$ is 0.

**Theorem 2.2** *The area $A$ of the largest small hexagon is the root of the polynomial*

$$\begin{aligned} p_6(A) \ &= 8\left(512\,A^{10} + 1024\,A^9 - 376\,A^8 - 3856\,A^7 + 2632\,A^6 + 18312\,A^5\right. \\ &\left.-27670\,A^4 + 154\,A^3 + 18058\,A^2 - 9811\,A + 1499\right) + 1 \end{aligned}$$

*that belongs to the interval* $[0.6, \frac{\pi}{4}]$.

**Proof.** The proof of Theorem 2.1 shows that the optimal hexagon is axially symmetrical with respect to the pending edge. The ordered vertices may be written as $z_1 = z, z_4 = -\bar{z}, z_2 = z + u, z_3 = -\bar{z} - \bar{u}, z_5 = 0$ and $z_6 = i$, where $z$ and $u$ are complex numbers such that $z\bar{z} = u\bar{u} = 1$. It follows that the following equation holds

$$\begin{array}{cccccc} 0 = & (z_1 - z_5) & +(z_2 - z_1) & +(z_3 - z_2) & +(z_4 - z_3) & +(z_5 - z_4) \\ = & z & +u & +1 & +\bar{u} & +\bar{z}. \end{array} \tag{1}$$

The area $A$ of the optimal hexagon may be written as

$$\begin{aligned} A &= \mathrm{Re}(z) - \mathrm{Im}\left(z(z+u)\right) \\ &= \frac{1}{2}\left(z + \bar{z}\right) + \frac{1}{2i}\left(z(z+u) - \bar{z}(\bar{z} + \bar{u})\right). \end{aligned}$$

Substituting $\bar{z}$ from (1) into the previous equation, multiplying both sides by $2iu$, and multiplying (1) by $zu$ leads to the equivalent question of finding the common roots of the pair of polynomial equations

$$\begin{aligned} p_{A,z}(u) &= -i\left(1 + u + u^2\right) + (u+1)\left((1+u)z + u^2 + u + 1)\right) - 2iuA \\ p_z(u) &= z^2u + zu^2 + zu + z + u. \end{aligned}$$

The variable $u$ is eliminated by finding the resultant $r_A(z)$ of the two polynomials $p_{A,z}(u)$ and $p_z(u)$, i.e., a polynomial on their coefficient which equals zero at the roots of both polynomials:

$$\begin{aligned} r_A(z) = &(-1-i)\,z^6 + (2\,iA - 1 + i)\,z^5 + (-2\,iA + 4\,A - 1 - i)\,z^4 \\ &+ \left(-4\,A^2 - 3\right)z^3 + (4\,A + 2\,iA - 1 + i)\,z^2 + (-2\,iA - 1 - i)\,z - 1 + i \end{aligned}$$

Finally, factoring the discriminant of $r_A(z)$ gives the polynomial of degree 14:[1]

$$\begin{aligned} d(A) = 4\ &\left(4096\,A^{10} + 8192\,A^9 - 3008\,A^8 - 30848\,A^7 + 21056\,A^6 + 146496\,A^5\right. \\ &\left.-221360\,A^4 + 1232\,A^3 + 144464\,A^2 - 78488\,A + 11993\right)(2\,A-1)^2\,(2\,A+1)^2. \end{aligned}$$

The main term of this polynomial is precisely the polynomial $p_6$ of degree 10 proposed by Graham, whose unique root contained in the interval $[0.6, \frac{\pi}{4}]$ (0.6 is a lower bound on the area of the small regular hexagon and $\frac{\pi}{4}$ is the area of the small circle) coincides with the area of the largest small hexagon: $A \approx 0.674981442930104703 69$. □

[1] These polynomials were obtained using the Maple symbolic calculation commands `factor(resultant(PAz(u), Pz(u), u))` and `factor(discrim(RA(z),z))`.

# 3 The largest small axially symmetrical small octagon

For the optimal axially symmetrical octagon, we use once again a complex plane formulation. Figure 2 illustrates the vertices of a small octagon, with a symmetry axis about the pending edge on the imaginary axis. The three complex numbers $z, u$ and $v$ satisfy $z\bar{z} = u\bar{u} = v\bar{v} = 1$. With this notation we can prove our main result.

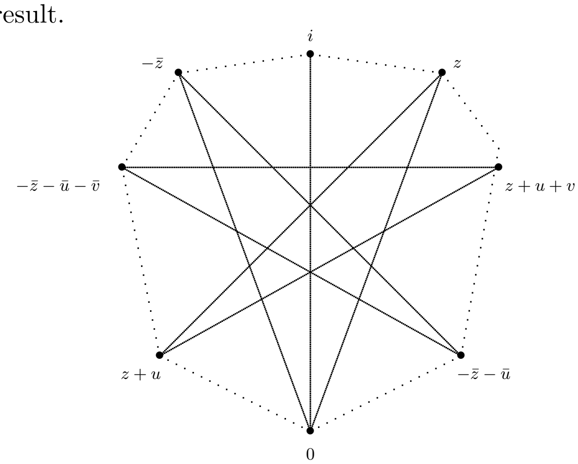


**Figure 2: Complex plane representation of a symmetrical cycle of length 7 with a pending edge**

**Theorem 3.1** *The area $A$ of the largest small axially symmetrical octagon is the root of the polynomial*

$$
\begin{aligned}
p_8(x) = {} & 1475739525896764129 28\, A^{42} - 442721857769029238784\, A^{41} \\
& + 260560260041147416576 0\, A^{40} + 7670386770149352931328\, A^{39} \\
& - 19803120195082488119296\, A^{38} - 90234644551552032833536\, A^{37} \\
& - 531709183791524869465702 4\, A^{36} - 1759404143063508465588633 6\, A^{35} \\
& + 2975839546270308157829939 2\, A^{34} + 2822072461197484761704038 40\, A^{33} \\
& + 3351032978877149042830213 12\, A^{32} - 1917928307706587784371240960\, A^{31} \\
& - 5240302758882335722850746368\, A^{30} + 4631615507099121446555746304\, A^{29} \\
& + 30114159874526648530622218240\, A^{28} - 7175008161182179668028030976\, A^{27} \\
& - 148064818635686576530703515648\, A^{26} - 42551878829792132053254275072\, A^{25} \\
& + 601318123428810231261639475200\, A^{24} + 332708870397989105275274002432\, A^{23} \\
& - 2358897389358876839124819509248\, A^{22} - 680235061366055307103034146816\, A^{21} \\
& + 7452392569346922858753860567040\, A^{20} - 1491865144134539091913264332800\, A^{19} \\
& - 15455347946546823025854527832064\, A^{18} + 9574865040443004381891485761536\, A^{17} \\
& + 20104198057699941048810876698624\, A^{16} - 20027080947914571766986403610624\, A^{15} \\
& - 16192270866005062836001824866304\, A^{14} + 23588130061203336356460301369344\, A^{13} \\
& + 8009206689639186621822611818496\, A^{12} - 17935820857956814364517526943744\, A^{11} \\
& - 2370238736752843325635609948160\, A^{10} + 9147034213711759916391887323136\, A^{9} \\
& + 367361764236902187872898865664\, A^{8} - 3078428637636379850280988117504\, A^{7} \\
& + 105551688808743610680134257 92\, A^{6} + 647330513128418259524157203072\, A^{5} \\
& - 23523528029439955698746202488\, A^{4} - 76143004877906320975709476552\, A^{3} \\
& + 5833707081723328603647313856\, A^{2} + 377304103834759651502100095 6\, A \\
& - 478425365462547737405343343
\end{aligned}
$$

*that belongs to the interval* $[0.7,\ \frac{\pi}{4}]$.

**Proof.** Under the notation of Figure 2, the area $A$ of the optimal small octagon may be written as

$$\begin{aligned} A &= \operatorname{Re}(z) - \operatorname{Im}\left((z+u)(z+u+v)\right) + \operatorname{Im}\left((\bar{z}+\bar{u}+\bar{v})z\right) \\ &= \frac{1}{2}(z+\bar{z}) + \frac{1}{2i}\left((\bar{z}+\bar{u}+z)(\bar{z}+\bar{u}+\bar{v}) - (z+u+\bar{z})(z+u+v)\right). \end{aligned}$$

For the octagon, the equivalent of Equation (1) is

$$0 = z + u + v - 1 + \bar{v} + \bar{u} + \bar{z}. \tag{2}$$

Substituting $\bar{z}$ from (2) into the equation for the area, and multiplying it by the value $2izu$, and multiplying (2) by $zuv$ gives the equivalent system of polynomial equations:

$$\begin{aligned} p_{A,z,u}(v) &= iu\left(z^2+1\right) + (u+z+z^2u)(1-z-u-v) - (z^2u+zu^2+u)(z+u+v) - 2izuA \\ p_{z,u}(v) &= z^2uv + zu^2v + zuv^2 - zuv + zu + zv + uv. \end{aligned}$$

In order to find the common roots, the first step consists in eliminating $v$ by finding the resultant $r_{A,z}(u)$ of the two polynomials $p_{A,z,u}(v)$ and $p_{z,u}(v)$:

$$\begin{aligned} r_{A,z}(u) = {} & \left(-z^2+z^3\right)u^6 + \left(-iz^4 + 2\,iz^3A - iz^2 - 5\,z^3 + 4\,z^4 + 3\,z^2 - 4\,z\right)u^5 \\ & + \big(-iz + 2\,iz^3A - 3\,iz^5 + 6\,iz^4A - iz^2 + 9\,z^3 + 3\,z - 15\,z^2 - iz^4 - 4\,iz^3 \\ & \quad -10\,z^4 + 2\,iz^2A + 5\,z^5 - 4\big)u^4 \\ & + \big(-4\,izA - 4\,z^3A^2 + 4\,iz^5A - 2\,iz^6 + 9\,z^2 - 10\,z + 9\,z^4 + 2\,iz^2 - 10\,z^5 \\ & \quad +2\,z^6 + 4\,z^2A - 23\,z^3 - 2\,iz^4 + 4\,z^4A + 2 + 2\,i\big)u^3 \\ & + \big(5\,z - 6\,iz^2A + 9\,z^3 + 4\,iz^3 - 4\,z^6 + 3\,iz - 10\,z^2 - 15\,z^4 - 2\,iz^3A + iz^4 \\ & \quad -2\,iz^4A + iz^5 + iz^2 + 3\,z^5\big)u^2 \\ & + \left(4\,z^2 - 2\,iz^3A - 4\,z^5 + iz^2 - 5\,z^3 + iz^4 + 3\,z^4\right)u + z^3 - z^4. \end{aligned}$$

Vanishing of this resultant leads to an unconstrained implicit equation for the area function of $z$ and $u$. The second step eliminates $u$ by finding the discriminant $d_A(z)$ with respect to $u$ of $r_{A,z}(u)$. The polynomials $(2z^4-4Az^3+4A^2z^2+3z^2-4Az+2)^2$ and $z^{10}$ can be factored out of the discriminant, resulting in a polynomial $d'_A(z)$ of degree 32 with respect to $z$ and of degree 10 with respect to $A$. Finally, the last step consists in eliminating the variable $z$ by finding the discriminant with respect to $z$ of $d'_A(z)$. This leads to a polynomial of degree 344 with integer coefficients. One of the factors of this discriminant is the the polynomial $p_8$ in $A$ given in the statement of the theorem, whose single root in the interval $[0.7, \frac{\pi}{4}]$ (0.7 is a lower bound on the area of the small regular octagon) is the area of the optimal symmetrical small octagon

$$A \approx 0.72686848275162676684.$$

□

# 4 Discussion

Using a complex-plane representation and using discriminants and resultants, we have found the same polynomial of degree 10 as Graham [6], but through the factorization of polynomial of lesser degree, and we have found a polynomial of degree 42 for the symmetrical octagon. The latter is the first analytical solution for the octagon, as only numerical approximations of the optimal area were known.

Computing time was negligible for the hexagon, and approximately 90 seconds for the octagon on a standard desktop computer, and 100 megabytes of storage was sufficient to perform the computations. We have applied the same methodology to the symmetrical decagon using a larger computer, but 100 gygabytes of memory was insufficient to compute the discriminant.

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