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K. C. Das, M. Aouchiche,
P. Hansen

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Kinkar Ch. Das [a]

Mustapha Aouchiche [b]

Pierre Hansen [b]

[a] *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea*

[b] *GERAD & Department of Decision Sciences, HEC Montréal, Montréal (Québec), Canada, H3T 2A7*

kinkardas2003@googlemail.com
mustapha.aouchiche@gerad.ca
pierre.hansen@gerad.ca

Abstract: Let G be a graph of order n. The energy $\mathcal{E}(G)$ of a simple graph G is the sum of absolute values of the eigenvalues of its adjacency matrix. The Laplacian energy, the signless Laplacian energy and the distance energy of graph G are denoted by $LE(G)$, $SLE(G)$ and $DE(G)$, respectively. In this paper we introduce a distance Laplacian energy DLE and distance signless Laplacian energy $DSLE$ of a connected graph. We present Nordhaus-Gaddum type bounds on Laplacian energy $LE(G)$ and signless Laplacian energy $SLE(G)$ in terms of order n of graph G and characterize graphs for which these bounds are best possible. The complete graph and the star give the smallest distance signless Laplacian energy $DSLE$ among all the graphs and trees of order n, respectively. We give lower bounds on distance Laplacian energy DLE in terms of n for graphs and trees, and characterize the extremal graphs. Also we obtain some relations between DE, $DSLE$ and DLE of graph G. Moreover, we give several open problems in this paper.

Keywords: Distance eigenvalues, distance (signless) Laplacian eigenvalues, (signless) Lapplacian energy, distance energy, distance (signless) Laplacian energy

Résumé : Soit G un graph d'ordre n. L'énergie $\mathcal{E}(G)$ d'un graph simple G est la somme de des valeurs absolues des valeurs propres de sa matrice d'adjacenc. L'énergie du laplacien, l'énergie du laplacien sans signe et l'énergie de la matrice des distances sont notées $LE(G)$, $SLE(G)$ et $DE(G)$, respectivement. Dans le présent article, nous introduisons l'énergie du laplacien des distances DLE et l'énergie du laplacien sans signe des distances $DSLE$ d'un graph connexe. Nous présentons des bornes de la forme Nordhaus-Gaddum sur l'énergie du laplacien $LE(G)$ et sur l'énergie du laplacien sans signe $SLE(G)$ en fonction de l'ordre n de G, et charactériserons les graphes pour lesquels ces bornes sont atteintes. Le graphe complet et l'étoile donne la plus petite valeur de l'énergie du laplacien sans signe des distances $DSLE$, sur l'ensemble des graphes et l'ensemble des arbres d'ordre n, respectivement. Nous donnons des bornes inférieures sur l'énergie du laplacien des distances DLE en fonction de n pour les graphes et les arbres, et nous charactériserons les graphes extrêmes. Nous démontrons également quelques relations entre DE, $DSLE$ et DLE d'un graphe G. De plus, nous donnons plusieurs problèmes ouverts.

Mots clés : Valeurs propres des distances, valeurs propres du laplacien (sans signe) des distances, énergie du laplacien (sans signe), énergie du laplacien (sans signe) des distances

1 Introduction

In the present paper we consider simple, undirected and connected graphs. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$, $|E(G)| = m$. Also let d_i be the degree of the vertex $v_i \in V(G)$. For $v_i \in V(G)$, the set of adjacent vertices of the vertex v_i is denoted by $N_G(V_i)$. Given two vertices v_i and v_j in a connected graph G, $d_{ij} = d_G(v_i, v_j)$ denotes the distance (the length of a shortest path) between v_i and v_j. The diameter of a graph is the maximum distance between any two vertices of G. Let d be the diameter of G. The complement graph of a graph G is denoted by $\overline{G}$. The transmission $Tr(v_i)$ (or D_i or $D_i(G)$) of a vertex v_i is defined to be the sum of the distances from v_i to all other vertices in G, that is,

$$Tr(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j).$$

The average transmission is denoted by $t(G)$ and is defined by

$$t(G) = \frac{1}{n} \sum_{i=1}^{n} Tr(v_i).$$

The Wiener index $W(G)$ of a connected graph G is defined to be the sum of all distances in G, that is,

$$W(G) = \frac{1}{2} \sum_{v_i,\, v_j \in V(G)} d_G(v_i, v_j).$$

Let $A(G)$ be the *adjacency matrix* of G and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ denote the eigenvalues of $A(G)$. Sometimes, for convenience sake, we write $\lambda_i = \lambda_i(G)$. The *energy* of the graph G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This spectrum-based graph invariant has been much studied in both the chemical and the mathematical literatures. Its mathematical properties were extensively investigated, see the book [21], the recent articles [14, 18, 20, 22] and the references cited therein.

Let $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ be, respectively, the *Laplacian matrix* and the *signless Laplacian matrix* of the graph G, where $D(G)$ is the diagonal matrix of vertex degrees. The eigenvalues of $L(G)$ and $Q(G)$ will be denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_{n-1} \geq q_n$, respectively. Then the *Laplacian energy* and the *signless Laplacian energy* of G are defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \qquad \text{and} \qquad SLE = SLE(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|,$$

respectively. The Laplacian energy is nowadays reasonably well elaborated (see [11, 12, 14, 17, 21] and the references cited therein). For more results on the signless Laplacian energy $SLE(G)$, we refer readers to the references [1, 12, 13]. Moreover, $SLE(G) = LE(G)$ for any bipartite graph G.

The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is a symmetric real matrix with (i, j)-entry being $d_G(v_i, v_j)$ (or d_{ij}). The polynomial $P_D(\partial) = \det(\partial I - \mathcal{D}(G))$ is defined as the distance characteristic polynomial of the graph G. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_{n-1} \geq \partial_n$ be the distance spectra of G. The *distance energy* of a connected graph G was defined in [19] as

$$DE = DE(G) = \sum_{i=1}^{n} |\partial_i|.$$

Its mathematical properties were extensively investigated, see the recent articles [5, 19, 27, 28, 33] and the references cited therein.

Let $\mathcal{D}^L(G) = \mathcal{D}iag(Tr) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = \mathcal{D}iag(Tr) + \mathcal{D}(G)$ be, respectively, the *distance Laplacian matrix* [3, 7] and the *distance signless Laplacian matrix* [3, 6, 10] of the graph G, where $\mathcal{D}iag(Tr)$ denotes

the diagonal matrix of the vertex transmissions in G. The eigenvalues of $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ will be denoted by $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_{n-1}^L \geq \partial_n^L = 0$ and $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_{n-1}^Q \geq \partial_n^Q$, respectively. Then the *distance Laplacian energy* and the *distance signless Laplacian energy* of G are defined as

$$DLE = DLE(G) = \sum_{i=1}^{n} \left| \partial_i^L - t(G) \right| \qquad \text{and} \qquad DSLE = DSLE(G) = \sum_{i=1}^{n} \left| \partial_i^Q - t(G) \right|$$

respectively. The distance Laplacian energy of a graph G was first defined in [30], where several lower and upper bounds were obtained. As usual, we denote by $K_{1,\,n-1}$ the star, by K_n the complete graph, by P_n the path and by C_n the cycle. each on n vertices.

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present Nordhaus-Gaddum type bounds on Laplacian energy and signless Laplacian energy of graph G and characterize graphs for which these bounds are best possible. In Section 4, we show that the complete graph and the star give the smallest distance signless Laplacian energy $DSLE$ among all the graphs and trees of order n, respectively. In Section 5, we give some lower bounds on distance Laplacian energy DLE in terms of n for graphs and trees, and characterize the extremal graphs and trees. In Section 6, we obtain some relations between DE, $DSLE$ and DLE of graph G.

2   Preliminaries

In this section, we shall list some previously known results that will be needed in the proofs of our results in the next four sections.

We first recall the well-known and widely used Courant-Weyl inequalities.

Lemma 2.1 (Courant-Weyl; see, e.g., [25]) *For a real symmetric matrix M of order n, let $\rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M)$ denote its eigenvalues. If N_1 and N_2 are two real symmetric matrices of order n and if $N = N_1 + N_2$, then for every $i = 1, 2, \ldots, n$, we have*

$$\rho_i(N_1) + \rho_1(N_2) \geq \rho_i(N) \geq \rho_i(N_1) + \rho_n(N_2).$$

The next recalled result is a characterization of the smallest eigenvalue of a symmetric matrix, using the Rayleigh quotient.

Lemma 2.2 ([31]) *If A is a symmetric $n \times n$ matrix with eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ then for any $\mathbf{x} \in R^n$ ($\mathbf{x} \neq \mathbf{0}$),*

$$\mathbf{x}^T A \mathbf{x} \geq \rho_n \mathbf{x}^T \mathbf{x}. \tag{1}$$

Equality holds if and only if $\mathbf{x}$ is an eigenvector of A corresponding to the smallest eigenvalue ρ_n.

Ning et al. [23] proved a lower on the largest Laplacian eigenvalue of a graph in terms of order n and size m. It is stated in the following result:

Lemma 2.3 ([23]) *Let G be a graph of order n with m edges. Then*

$$q_1 \geq \frac{4m}{n}$$

with equality holding if and only if G is regular.

Cvetković et al. [9] obtained a relationship between the Laplacian and the signless Laplacian spectra of a regular graph on n vertices.

Lemma 2.4 ([9]) *Let G be a r-regular graph of order n. Then the signless Laplacian spectrum of graph G is given by*

$$SLS(G) = \{2r, \, 2r - \mu_{n-1}, \, 2r - \mu_{n-2}, \ldots, \, 2r - \mu_2, \, 2r - \mu_1\},$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ are the Laplacian eigenvalues of graph G.

Xing et al. [29] presented lower bound on distance signless Laplacian spectral radius $\partial_1^Q(G)$ of a connected graph G in terms of its average transmission $t(G)$, or equivalently, in terms of its order and its Wiener index $W(G)$ and order n.

Lemma 2.5 ([29]) *Let G be a connected graph of order n with wiener index $W(G)$. Then*

$$\partial_1^Q(G) \geq 2\,t(G) = \frac{4W(G)}{n}$$

with equality holding if and only if G is transmission regular.

The relation between Laplacian eigenvalues and distance Laplacian eigenvalues are given in the following result:

Lemma 2.6 ([3]) *Let G be a connected graph on n vertices with diameter $d(G) \leq 2$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ be the Laplacian eigenvalues of G. Then the distance Laplacian eigenvalues of G are $2n - \mu_{n-1}(G) \geq 2n - \mu_{n-2}(G) \geq \ldots \geq 2n - \mu_1(G) > \partial_n^L(G) = 0$. Moreover, for every $i = 1, 2, \ldots, n-1$, the eigenspaces corresponding to μ_i and to $2n - \mu_i$ are the same.*

For the statement of the next lemma, we need to recall the following definitions. Let M be a real and symmetric matrix of order n. Let $s_i(M)$, $i = 1, 2, \ldots, n$, be its singular values and $x_i(M)$, $i = 1, 2, \ldots, n$, its eigenvalues. Then $s_i(M) = |x_i(M)|$ for $i = 1, 2, \ldots, n$.

Lemma 2.7 ([16]) *Let X, Y, and Z be square matrices of order n, such that $X + Y = Z$. Then*

$$\sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y) \geq \sum_{i=1}^{n} s_i(Z)\,.$$

Equality holds if and only if there exists an orthogonal matrix P, such that PX and PY are both positive semi-definite.

For a graph G on n vertices and m edges with Laplacian spectrum $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, let ν $(1 \leq \nu \leq n-1)$ be the largest positive integer such that

$$\mu_\nu \geq \frac{2m}{n}\,.$$

The Laplacian energy of graph G presented in terms of ν:

Lemma 2.8 ([12]) *Let G be a graph of order n with m edges. Then*

$$LE(G) = \max_{1 \leq k \leq n} \left\{ 2\sum_{i=1}^{k} \mu_i - \frac{4m\,k}{n} \right\} = 2\sum_{i=1}^{\nu} \mu_i - \frac{4m\,\nu}{n}\,.$$

Chen et al. [8] proved a lower bound on Laplacian spread of a connected graph G and characterized the corresponding extremal graphs. The result is as follows:

Lemma 2.9 ([8]) *Let G be a connected non-complete graph of order $n > 2$. Then $\mu_1 - \mu_{n-1} \geq 2$ with equality holding if and only if $G \cong \overline{s\,K_2 \cup (n - 2s)\,K_1}$, where $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$.*

For a graph G on n vertices and m edges with signless Laplacian spectrum $q_1 \geq q_2 \geq \cdots \geq q_n$, let τ $(1 \leq \tau \leq n-1)$ be the largest positive integer such that

$$q_\tau \geq \frac{2m}{n}\,.$$

Lemma 2.10 ([12]) *Let G be a graph of order n with m edges. Then*

$$SLE(G) = \max_{1 \leq k \leq n} \left\{ 2\sum_{i=1}^{k} q_i - \frac{4m\,k}{n} \right\} = 2\sum_{i=1}^{\tau} q_i - \frac{4m\,\tau}{n}\,.$$

Denote by $T_{n-3,\,1}^1$, is a tree of order n such that the maximum degree is $n-2$. The next results gives the tree with the second smallest Wiener index over all trees with given order n.

Lemma 2.11 [15] *Let T $(\ncong K_{1,\,n-1})$ be a tree of order n. Then*

$$W(T) \geq n^2 - n - 2$$

with equality holding if and only if $T \cong T_{n-3,\,1}^1$.

The lower bound on Wiener index of a connected graph G is given in the following result:

Lemma 2.12 ([15]) *Let G be a connected graph of order n with wiener index W. Then*

$$W(G) \geq \frac{n(n-1)}{2}$$

with equality holding if and only if $G \cong K_n$.

3   On Laplacian energy and signless Laplacian energy of graphs

In 1956, Nordhaus and Gaddum gave lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement, in terms of the order of the graph.

Theorem 3.1 ([24]) *If G is a graph of order n*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1 \quad and \quad n \leq \chi(G) \cdot \chi(\overline{G}) \leq \frac{(n+1)^2}{4}.$$

Furthermore, these bounds are best possible for infinitely many values of n.

Since then, relations of a similar type are called Nordhaus-Gaddum inequalities and have been proposed for many other graph invariants, in several hundred papers. See [2] for a survey of such results.

In this section, we present Nordhaus-Gaddum type inequalities on Laplacian energy LE and signless Laplacian energy SLE of graph G and characterize graphs for which these bounds are best possible.

It is well known that the Laplcian spectrum and the signless Laplacian spectrum are same for bipartite graph, so one can easily see that the Laplacian energy LE and the signless Laplacian energy SLE are same for bipartite graphs. We first give a result on regular graph:

Theorem 3.2 *Let G be a r-regular graph of order n. Then*

$$
\begin{aligned}
(i) \qquad & LE(G) = SLE(G), \\
(ii) \qquad & LE(\overline{G}) = SLE(\overline{G}), \\
(iii) \qquad & LE(G) + LE(\overline{G}) = SLE(G) + SLE(\overline{G}).
\end{aligned}
$$

Proof. Since G is a regular graph, by Lemma 2.4, we have

$$LE(G) = \sum_{i=1}^{n} |\mu_i - r| = \sum_{i=1}^{n} |2r - \mu_i - r| = SLE(G).$$

Since G is a regular graph, we have $\overline{G}$ is also a regular graph. Hence $LE(\overline{G}) = SLE(\overline{G})$. From (i) and (ii), we get the required result in (iii). $\qquad\square$

Before stating our next result, we recall a known lower bound on $LE(G) + LE(\overline{G})$ of graphs G. Zhou et al. [32] gave the following lower bound in terms on n:

$$LE(G) + LE(\overline{G}) \geq 2(n-1)$$

with equality holding if and only if $G \cong K_n$ or $G \cong \overline{K}_n$. We now give a lower bound on $LE(G) + LE(\overline{G})$ in the following result:

Theorem 3.3 *Let G ($\not\cong K_n, \overline{K}_n$) be a graph of order n. Then*

$$LE(G) + LE(\overline{G}) \geq 2n \tag{2}$$

with equality holding if and only if $G \cong C_4$ or $G \cong P_3$.

Proof. Since $G \not\cong K_n, \overline{K}_n$, one can easily check that the inequality in (2) holds for $n \leq 4$ and the equality in (2) holds for C_4 and P_3. Otherwise, $n \geq 5$.

Let $\overline{m}$ and $\overline{\mu}_i$ be the number of edges in $\overline{G}$ and the i-th largest Laplacian eigenvalue of $L(\overline{G})$, respectively. By Lemma 2.8, we have

$$
\begin{aligned}
LE(G) + LE(\overline{G}) &\geq 2\Big(\sum_{i=1}^{n-2} \mu_i - \frac{2m(n-2)}{n}\Big) + 2\Big(\sum_{i=1}^{n-2} \overline{\mu}_i - \frac{2\overline{m}(n-2)}{n}\Big) \\
&= 2\Big(\frac{4m}{n} - \mu_{n-1}\Big) + 2\Big(\frac{4\overline{m}}{n} - \overline{\mu}_{n-1}\Big) \quad \text{as } \sum_{i=1}^{n-1}\mu_i = 2m, \ \sum_{i=1}^{n-1}\overline{\mu}_i = 2\overline{m} \\
&= 4(n-1) - 2(\mu_{n-1} + n - \mu_1) \quad \text{as } 2m + 2\overline{m} = n(n-1), \ \overline{\mu}_{n-1} = n - \mu_1 \\
&= 2n - 4 + 2(\mu_1 - \mu_{n-1}).
\end{aligned}
$$

If G is connected, then by Lemma 2.9, we have $LE(G) + LE(\overline{G}) \geq 2n$ as $G \not\cong K_n$. Otherwise, G is disconnected, that is, $\mu_{n-1} = 0$. Since $G \not\cong \overline{K}_n$, then G has at least one edge and hence $\mu_1 \geq 2$. Therefore

$$LE(G) + LE(\overline{G}) \geq 2n - 4 + 2\mu_1 \geq 2n.$$

The first part of the proof is done.

Suppose that equality holds in (2). First we assume that G is connected. Then by Lemma 2.9, we have $G \cong \overline{s\,K_2 \cup (n-2s)\,K_1}$, where $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, that is, $\overline{G} \cong s\,K_2 \cup (n-2s)K_1$. Therefore we must have

$$LE(\overline{G}) = \Big(2 - \frac{2s}{n}\Big)s + \frac{2s}{n}(n-s) = 4s - \frac{4s^2}{n} = \frac{8s}{n},$$

that is, $n = s + 2$, that is, $n \leq 4$ as $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, a contradiction.

Next we assume that G is disconnected. Then $\mu_1 = 2$ and hence $G \cong s\,K_2 \cup (n-2s)K_1$, where $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$. Similarly, as before, we can prove that $n \leq 4$, a contradiction. This completes the proof of the theorem. $\square$

We now give a lower bound on $SLE(G) + SLE(\overline{G})$ in terms of n of graph G and characterize the extremal graphs.

Theorem 3.4 *Let G be a graph of order n. Then*

$$SLE(G) + SLE(\overline{G}) \geq 2(n-1)$$

with equality holding if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

Proof. If $G \cong K_n$ or $G \cong \overline{K}_n$, then one can easily see that the equality holds. Otherwise, $G \not\cong K_n, \overline{K}_n$. In this case we have to prove that the inequality is strict. For this, let $\overline{m}$ and $\overline{q}_1$ be the number of edges in $\overline{G}$ and the largest eigenvalue of $Q(\overline{G})$, respectively. By Lemmas 2.3 and 2.10, we have

$$
\begin{aligned}
SLE(G) + SLE(\overline{G}) &\geq 2\Big(q_1 - \frac{2m}{n}\Big) + 2\Big(\overline{q}_1 - \frac{2\overline{m}}{n}\Big) \\
&= 2(q_1 + \overline{q}_1) - 2(n-1) \quad \text{as } 2m + 2\overline{m} = n(n-1) \\
&\geq 2\Big(\frac{4m}{n} + \frac{4\overline{m}}{n}\Big) - 2(n-1) = 2(n-1).
\end{aligned}
$$

The first part of the theorem is done.

Suppose that equality holds. Then by Lemma 2.3, both G and $\overline{G}$ are regular graphs. Since $G \ncong K_n, \overline{K}_n$, then by Theorems 3.2 and 3.3, we get

$$SLE(G) + SLE(\overline{G}) = LE(G) + LE(\overline{G}) \geq 2n > 2(n-1).$$

This completes the proof of the theorem. □
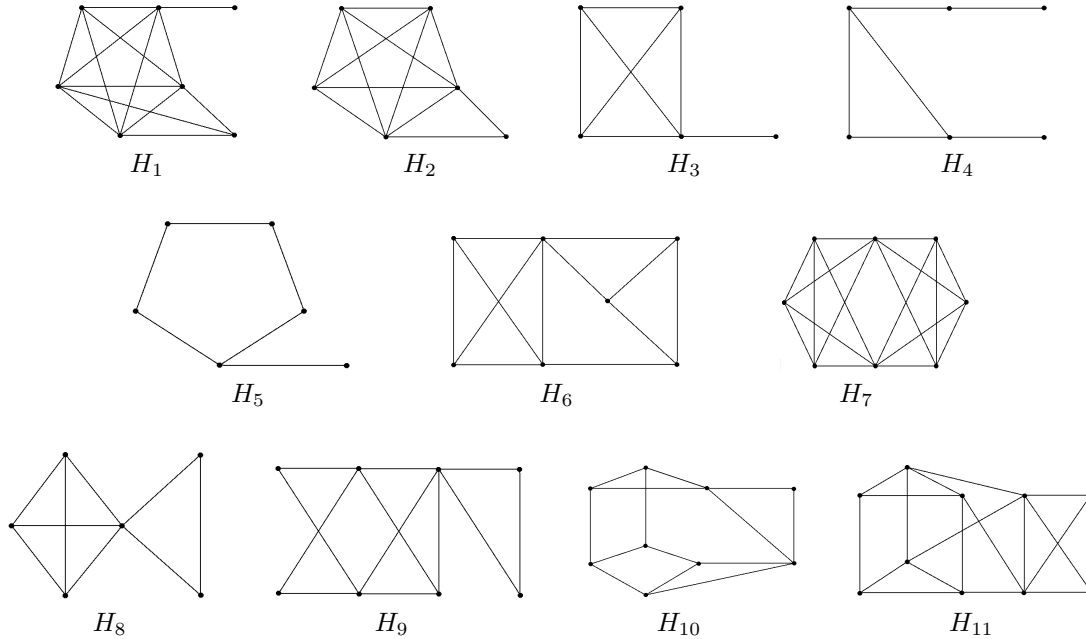
Figure 1: Graphs H_i, $i = 1, 2, \ldots, 11$.

Remark 3.5 *For both bipartite graph and regular graph, we have seen that $LE = SLE$. For the graphs $G = H_i$, $i = 1, 2, 3, 4$ (Figure 1) are satisfying $LE(G) > SLE(G)$; for $G = H_i$, $i = 5, 6, 7$ (Figure 1) are satisfying $LE(G) < SLE(G)$ and for $G = H_i$, $i = 8, 9, 10, 11$ (see, Figure 1) are satisfying $LE(G) = SLE(G)$. Hence $LE(G)$ and $SLE(G)$ are incomparable.*

Therefore it is interesting to get all the graphs divided into three classes in the following problem:

Problem 1 *Characterize all the graphs for which $LE(G) > SLE(G)$, $LE(G) < SLE(G)$ and $LE(G) = SLE(G)$.*

4 On distance signless Laplacian energy of graphs

In this section we prove that the complete graph and the star give the smallest distance signless Laplacian energy $DSLE$ among all the graphs and trees of order n, respectively.

First we prove that the complete graph gives the minimal distance signless Laplacian energy over the class of all connected graphs with fixed order n.

Theorem 4.1 *Let G be a connected graph of order n. Then*

$$DSLE(G) \geq 2(n-1)$$

with equality holding if and only if $G \cong K_n$.

Proof. Let σ be the largest positive integer such that $\partial_\sigma^Q(G) \geq t(G)$. We have

$$\sum_{i=1}^n \partial_i^Q(G) = \sum_{i=1}^n D_i = n\, t(G) = 2\, W(G).$$

Using this result with the definition of distance signless Laplacian energy, we have

$$\begin{aligned} DSLE(G) &= \sum_{i=1}^\sigma (\partial_i^Q(G) - t(G)) + \sum_{i=\sigma+1}^n (t(G) - \partial_i^Q(G)) \\ &= 2 \left(\sum_{i=1}^\sigma \partial_i^Q(G) - t(G)\,\sigma \right). \end{aligned}$$

First we have to prove that

$$DSLE(G) = 2 \left(\sum_{i=1}^\sigma \partial_i^Q(G) - t(G)\,\sigma \right) = 2 \max_{1 \leq k \leq n} \left(\sum_{i=1}^k \partial_i^Q(G) - t(G)\,k \right). \tag{3}$$

For $k > \sigma$, we have

$$\begin{aligned} \sum_{i=1}^k \partial_i^Q(G) - t(G)\,k &= \sum_{i=1}^\sigma \partial_i^Q(G) + \sum_{i=\sigma+1}^k \partial_i^Q(G) - t(G)\,k \\ &< \sum_{i=1}^\sigma \partial_i^Q(G) - t(G)\,\sigma \quad \text{as } \partial_i^Q(G) < t(G),\, i \geq \sigma + 1. \end{aligned}$$

Similarly, for $k \leq \sigma$,

$$\sum_{i=1}^k \partial_i^Q(G) - t(G)\,k \leq \sum_{i=1}^\sigma \partial_i^Q(G) - t(G)\,\sigma.$$

Hence this proves the result in (3). By Lemmas 2.5 and 2.12, from (3), we have

$$DSLE(G) \geq 2 \left(\partial_1^Q(G) - t(G) \right) \geq \frac{4W(G)}{n} \geq 2(n-1).$$

By Lemmas 2.5 and 2.12 with the above results, we conclude that $DSLE(G) = 2(n-1)$ if and only if $G \cong K_n$. $\qquad \square$

The distance signless Laplacian spectrum of the star $K_{1,\,n-1}$ [4] is

$$DSLS(K_{1,\,n-1}) = \left\{ \frac{5n - 8 \pm \sqrt{9n^2 - 32n + 32}}{2},\, \underbrace{2n - 5,\, 2n - 5, \ldots,\, 2n - 5}_{n-2} \right\}.$$

We have

$$t(K_{1,\,n-1}) = 2n - 4 + \frac{2}{n}.$$

From the definition of distance signless Laplacian energy, we have

$$DSLE(K_{1,\,n-1}) = \sqrt{9n^2 - 32n + 32} + \left(1 + \frac{2}{n} \right)(n-2) = n + \sqrt{9n^2 - 32n + 32} - \frac{4}{n}.$$

We now prove that the star gives the minimal distance signless Laplacian energy for any tree of order n.

Theorem 4.2 *Let T be a tree of order n. Then*

$$DSLE(T) \geq n + \sqrt{9n^2 - 32n + 32} - \frac{4}{n}$$

with equality holding if and only if $G \cong K_{1,\,n-1}$.

Proof. If $T \cong K_{1,\,n-1}$, then the equality holds. Otherwise, $T \ncong K_{1,\,n-1}$ and $n \geq 4$. Then by Lemma 2.11, we have

$$t(T) = \frac{2W(T)}{n} \geq 2n - 2 - \frac{4}{n}.$$

Using Lemma 2.5 with the above result and from (3), we get

$$
\begin{aligned}
DSLE(T) &\geq 2\left(\partial_1^Q(T) - t(T)\right) \\
&\geq 2\,t(T) \\
&\geq 2\left[2n - 2 - \frac{4}{n}\right] \\
&> 2\left[2n - 2.6 - \frac{2}{n}\right] \qquad \text{as } n \geq 4 \\
&> n + \sqrt{9n^2 - 32n + 32} - \frac{4}{n} = DSLE(K_{1,\,n-1}).
\end{aligned}
$$

This completes the proof. □

5 On distance Laplacian energy of graphs

In this section we give some lower bounds on distance Laplacian energy DLE in terms of n for graphs and trees, and characterize the extremal graphs.

Let σ' be the largest positive integer such that $\partial_{\sigma'}^L(G) \geq t(G)$. We have

$$\sum_{i=1}^{n} \partial_i^L(G) = \sum_{i=1}^{n} D_i = n\,t(G) = 2W(G).$$

Similar to the case of distance signless Laplacian energy of a graph G, we have

$$DLE(G) = 2\left(\sum_{i=1}^{\sigma'} \partial_i^L(G) - t(G)\,\sigma'\right) = 2 \max_{1 \leq k \leq n}\left(\sum_{i=1}^{k} \partial_i^L(G) - t(G)\,k\right). \tag{4}$$

By Lemma 2.12, we have $2W(G) \geq n(n-1)$. Using this result, from the above, we get

$$DLE(G) \geq 2\left(\sum_{i=1}^{n-1} \partial_i^L(G) - t(G)(n-1)\right) = 2(2W(G) - t(G)\,(n-1)) = \frac{4W(G)}{n} \geq 2(n-1).$$

From the above results, one can easily see that $DLE(G) = 2(n-1)$ if and only if $G \cong K_n$. Hence we have the following result:

Theorem 5.1 *Let G be a connected graph of order n. Then*

$$DLE(G) \geq 2(n-1)$$

with equality holding if and only if $G \cong K_n$.

Denoted by S_n, is a graph of order n such that the maximum degree $n-1$. That is, $K_{1,\,n-1} \subseteq S_n \subseteq K_n$.

Lemma 5.2 *Let S_n be a graph of order n. Then $\partial_{n-1}^L(S_n) = n$.*

Proof. It is well known that $\mu_1(S_n) = n$ as $\Delta(S_n) = n - 1$. Since $d(S_n) \leq 2$, by Lemma 2.6, we have $\partial_{n-1}^L(S_n) = 2n - \mu_1(S_n) = n$. □

We now give an upper bound on $\partial_{n-1}^L(G)$ of any connected graph G.

Theorem 5.3 *Let G be a connected graph of order n. Then*

$$\partial_{n-1}^L(G) \leq \frac{nD_n}{n-1}, \tag{5}$$

where D_n is the minimum transmission of graph G. Moreover, the equality holds in (5) if and only if $G \cong S_n$.

Proof. By Lemma 2.2, we have

$$\partial_{n-1}^L(G) \leq \frac{\sum_{1 \leq i < j \leq n} d_{ij} (x_i - x_j)^2}{\sum\limits_{i=1}^{n} x_i^2},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is any vector in R^n.

We choose $\mathbf{x} = \left(\underbrace{\frac{1}{\sqrt{n(n-1)}}, \ldots, \frac{1}{\sqrt{n(n-1)}}}_{n-1}, -\sqrt{\frac{n-1}{n}} \right)$, then from the above, we get

$$\partial_{n-1}^L(G) \quad \leq \quad \frac{\sum_{j=1}^{n-1} d_{jn} \left(\sqrt{\frac{n-1}{n}} + \frac{1}{\sqrt{n(n-1)}} \right)^2}{1} = \frac{n\,D_n}{n-1}.$$

The first part of the proof is done.

Suppose that equality holds in (5). Then $\mathbf{x} = \left(\underbrace{\frac{1}{\sqrt{n(n-1)}}, \ldots, \frac{1}{\sqrt{n(n-1)}}}_{n-1}, -\sqrt{\frac{n-1}{n}} \right)$ is an eigenvector

corresponding to the eigenvalue $\partial_{n-1}^L(G)$ of $\mathcal{D}^L(G)$. For $v_i \in V(G)$ such that $v_i v_n \in E(G)$, we have

$$\partial_{n-1}^L(G) \frac{1}{\sqrt{n(n-1)}} = D_i \frac{1}{\sqrt{n(n-1)}} - (D_i - d_{in}) \frac{1}{\sqrt{n(n-1)}} + d_{in} \sqrt{\frac{n-1}{n}},$$

that is,

$$\partial_{n-1}^L(G) = n\,d_{in} = n.$$

For $v_n \in V(G)$, we have

$$-\partial_{n-1}^L(G) \sqrt{\frac{n-1}{n}} = -D_n \sqrt{\frac{n-1}{n}} - D_n \frac{1}{\sqrt{n(n-1)}},$$

that is,

$$\partial_{n-1}^L(G) = \frac{n\,D_n}{n-1}.$$

From the above two results, we have $D_n = n - 1$. Then $v_n v_k \in E(G)$, $k = 1, 2, \ldots, n-1$ and hence $G \cong S_n$.

Conversely, one can easily see that the equality holds in (5) for S_n, by Lemma 5.2. $\square$

Now we introduce a graph transformation:

Transformation A. Suppose that T is a nontrivial tree of order n and $P_{d+1} : v_1 v_2 \ldots v_d v_{d+1}$ is a diametral path in T such that $D_T(v_1) \geq D_T(v_{d+1})$. Let T' be a tree of order n obtained from T by $T' = T - \{v_2 v_i : v_i \in N_T(v_2)\} + \{v_3 v_i : v_i \in N_T(v_2), i \neq 3\} + v_2 v_3$. The above referred trees have been illustrated in Figure 2.
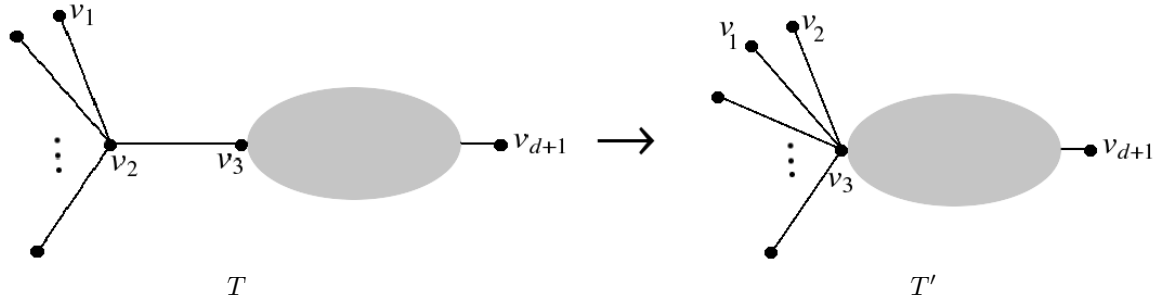
Figure 2: Trees T and T'.

Lemma 5.4 *Let T and T' be two trees of order $n > 5$ as shown in Figure 2. Then*

$$\sum_{i=1}^{n} \Big[D_i(T) - D_i(T') \Big] > \frac{n^2}{2(n-1)} \Big[D_n(T) - D_n(T') \Big],$$

where $D_n(T)$ and $D_n(T')$ are the minimum transmissions of tree T and T', respectively.

Proof. We assume that $S = \{v_k : v_2 v_k \in E(T), k \neq 3\} \cup \{v_2\}$ and $|S| = p$, (say). Let v_n be the vertex in T such that $D_n(T) = \min\{D_i(T) : v_i \in V(T)\}$. Then one can easily see that $v_n \in V \backslash S$. We have $D_i(T) = D_i(T') + n - p - 1$, $v_i \in S \backslash \{v_2\}$; $D_2(T) = D_2(T') - p + 1$; and $D_i(T) = D_i(T') + p - 1$, $v_i \in V \backslash S$. From this we conclude that the same vertex v_n gives the minimum $D_n(T')$ in T'. Since $D_T(v_1) \geq D_T(v_{d+1})$, we must have $p \leq n/2$. Now,

$$
\begin{aligned}
\sum_{i=1}^{n} \Big[D_i(T) - D_i(T') \Big] &= \sum_{v_i \in S} \Big[D_i(T) - D_i(T') \Big] + \sum_{V \backslash S} \Big[D_i(T) - D_i(T') \Big] \\
&= (n - p - 1)(p - 1) - (p - 1) + (p - 1)(n - p) \\
&= 2(p - 1)(n - p - 1).
\end{aligned}
$$

Moreover, $D_n(T) - D_n(T') = p - 1$ as $v_n \in V \backslash S$. Since

$$f(x) = 3n^2 - 4nx - 8n + 4x + 4, \ x \leq n/2$$

is a strictly decreasing function on $x \leq n/2$ and hence $f(x) \geq f(n/2) > 0$ (as $n > 5$), that is, $3n^2 - 4np - 8n + 4p + 4 > 0$. Using this result, one can easily see that

$$2(p - 1)(n - p - 1) > \frac{n^2}{2(n-1)}(p - 1),$$

which gives the required result. This completes the proof of the lemma. $\qquad \square$

The next lemma provides a lower bund on the average transmission of a tree, and will be used in the proof of our next theorem.

Lemma 5.5 *Let T be a tree of order $n > 4$. Then*

$$t(T) \geq 2n - 4 + \frac{2}{n} + \frac{n(D_n(T) - n + 1)}{2(n-1)}, \tag{6}$$

where $D_n(T)$ is the minimum transmission of tree T. Moreover, the equality holds in (6) if and only if $T \cong K_{1, n-1}$.

Proof. For $T \cong K_{1, n-1}$, then the equality holds in (6). For $T \cong P_5$ and $T \cong K_{2, 1}^1$, one can easily check that the inequality in (6) is strict. We therefore assume that $T \ncong K_{1, n-1}$, that is, $d(T) \geq 3$ and $n \geq 6$. In

this case we have to show that the inequality in (6) is strict. Let $P_{d+1} : v_1 v_2 \ldots v_d v_{d+1}$ be a diametral path in T such that $D_T(v_1) \geq D_T(v_{d+1})$. We transform T into another tree T' by **Transformation A**. Then by Lemma 5.4, we have

$$\sum_{i=1}^{n} \Big[D_i(T) - D_i(T') \Big] > \frac{n^2}{2(n-1)} \Big[D_n(T) - D_n(T') \Big].$$

If $T' \cong K_{1,\,n-1}$, then from the above result, one can easily see that

$$2W(T) - (2n^2 - 4n + 2) > \frac{n^2}{2(n-1)} \Big(D_n(T) - n + 1 \Big),$$

which gives the required result in (6). Otherwise, we continue the construction as follows. Let $P_{d+1} : v_1 v_2 \ldots v_d v_{d+1}$ be a diametral path in T' such that $D_{T'}(v_1) \geq D_{T'}(v_{d+1})$. Repeating the **Transformation A** sufficient number of times, we arrive at a tree in which diameter $d = 2$, i. e., we arrive at $K_{1,\,n-1}$. Thus

$$\sum_{i=1}^{n} \Big[D_i(T) - D_i(T') \Big] > \frac{n^2}{2(n-1)} \Big[D_n(T) - D_n(T') \Big],$$

$$\sum_{i=1}^{n} \Big[D_i(T') - D_i(T'') \Big] > \frac{n^2}{2(n-1)} \Big[D_n(T') - D_n(T'') \Big],$$

$$\ldots \ldots$$

$$\sum_{i=1}^{n} \Big[D_i(T^{k-1}) - D_i(K_{1,\,n-1}) \Big] > \frac{n^2}{2(n-1)} \Big[D_n(T^{k-1}) - D_n(K_{1,\,n-1}) \Big].$$

Adding the above inequalities, we have

$$\sum_{i=1}^{n} \Big[D_i(T) - D_i(K_{1,\,n-1}) \Big] > \frac{n^2}{2(n-1)} \Big[D_n(T) - D_n(K_{1,\,n-1}) \Big],$$

that is,

$$2W(T) - (2n^2 - 4n + 2) > \frac{n^2}{2(n-1)} \Big(D_n(T) - n + 1 \Big),$$

which gives the required result in (6). This completes the proof of the lemma. $\qquad\square$

We are now ready to give a lower bound on the distance Laplacian energy of tree of order n and characterize the extremal graphs.

Theorem 5.6 *Let T be a tree of order n. Then*

$$DLE(T) \geq 2 \left(3n - 8 + \frac{4}{n} \right) \tag{7}$$

with equality holding if and only if $G \cong K_{1,\,n-1}$.

Proof. For $T \cong K_{1,\,n-1}$, the distance Laplacian spectrum is

$$DLS(T) = \{ \underbrace{2n-1, \ldots, 2n-1}_{n-2}, \, n, \, 0 \}, \quad t(T) = 2n - 4 + \frac{2}{n}$$

and hence the equality holds in (7). For $T \cong P_4$, one can easily check that the inequality in (7) is strict. Otherwise, $T \not\cong K_{1,\,n-1}$ and $n \geq 5$. Using (5), from (4), we get

$$
\begin{aligned}
DLE(T) \;\; &\geq \;\; 2\left(\sum_{i=1}^{n-2} \partial_i^L(T) - t(T)\,(n-2)\right) \\
&= \;\; 2\left(2W(T) - \partial_{n-1}^L(T) - t(T)\,(n-2)\right) \\
&\geq \;\; 2\left(\frac{4W(T)}{n} - \frac{nD_n}{n-1}\right).
\end{aligned}
$$

Using the above result with (6), we get

$$
DLE(T) > 2\left(3n - 8 + \frac{4}{n}\right)
$$

as $T \not\cong K_{1,\,n-1}$ and $n \geq 5$, which is the required result in (7). This completes the proof. $\qquad\square$

6 Relations between $DSLE$, DLE and DE of graphs

Since all energies $DSLE$, DLE and DE are based on distances, their comparison is a question that arises naturally. Within that context we performed experiments and computations about wich we report in this section. A sample of our computational results is given in the tables below.

Table 1: **Distance signless Laplacian energy and Distance energy of** H_i**,** $i = 12,\ 13,\ 14,\ 15$**.**

G	$DSLE(G)$	$DE(G)$
H_{12}	42.326	31.024
H_{13}	39.055	31.572
H_{14}	22.029	22.446
H_{15}	21.74	22.771

Table 2: **Distance Laplacian energy and Distance energy of** H_i**,** $i = 16,\ 17,\ 18,\ 19$**.**

G	$DLE(G)$	$DE(G)$
H_{16}	38.258	36.634
H_{17}	71.263	47.163
H_{18}	15.5	16
H_{19}	16	17.092

From Table 1, we have $DSLE(H_i) > DE(H_i)$, $i = 12, 13$ (see, Figure 3) and $DSLE(H_i) < DE(H_i)$, $i = 14, 15$ (see, Figure 3). Therefore, our first conclusion is that $DSLE(G)$ and $DE(G)$ are incomparable.

From Table 2, we have $DLE(H_i) > DE(H_i)$, $i = 16, 17$ (see, Figure 3) and $DLE(H_i) < DE(H_i)$, $i = 18, 19$ (see, Figure 3). Thus, our second conclusion, $DLE(G)$ and $DE(G)$ are also incomparable.
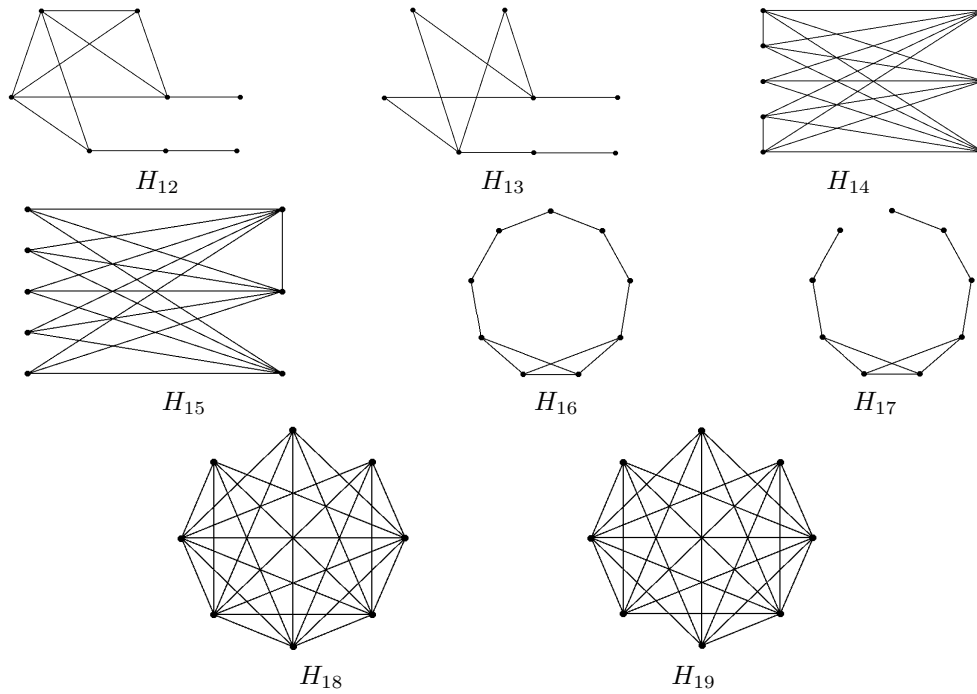
Figure 3: Graphs H_i, $i = 12, 13, \ldots, 19$.

Despite the above conclusions, when considering the three energies together, we have the following surprising result:

Theorem 6.1 *Let G be a graph of order n. Then*

$$DSLE(G) + DLE(G) \geq 2\,DE(G) \geq DSLE(G) - DLE(G).$$

Proof. We have $\mathcal{D}^Q(G) - \mathcal{D}^L(G) = 2\,\mathcal{D}(G)$, that is, $\left(\mathcal{D}^Q(G) - t(G)\,I\right) - \left(\mathcal{D}^L(G) - t(G)\,I\right) = 2\,\mathcal{D}(G)$. By Lemma 2.7, we get the left inequality. Moreover, we apply the same Lemma 2.7 on $\left(\mathcal{D}^Q(G) - t(G)\,I\right) = \left(\mathcal{D}^L(G) - t(G)\,I\right) + 2\,\mathcal{D}(G)$, we get the right inequality. This completes the proof of the theorem. $\square$

Remark 6.2 *From Theorem 6.1, we conclude that there is a graph G either $DSLE(G) > DE(G)$ or $DLE(G) > DE(G)$ or both.*

In order to improve the first inequality in Theorem 6.1 over the class of trees, we give two upper bounds on the second smallest distance Laplacian eigenvalue $\partial_{n-1}^L(T)$, of a tree T, in terms of n, p (no. of pendant vertices) and average transmission t.

Theorem 6.3 *Let T be a tree of order n with average transmission t. Then*

$$\partial_{n-1}^L(T) \leq t(T) - p + \frac{5p+2}{2n} + \frac{1}{2}\,,$$

where p is the number of pendant vertices in T.

Proof. Let v_n and v_{n-1} be the vertices in T such that D_n and D_{n-1} are the smallest and the second smallest transmission. First we have to prove that $v_n v_{n-1} \in E(T)$. For this we consider any pendant path $v_n v_{i_1} v_{i_2} \ldots v_{i_k}$ from vertex v_n to a pendant vertex v_{i_k}. For tree T, one can easily see that

$$D_r - D_q = n_q - n_r \quad \text{for} \quad v_r v_q \in E(T), \tag{8}$$

(n_r counts the number of vertices of T lying closer to the vertex v_r than to vertex v_q and n_q counts the number of vertices of T lying closer to the vertex v_q than to vertex v_r). Since D_n is minimum, for $v_n v_{i_1} \in E(T)$, we have $D_{i_1} \geq D_n$, that is, $n_n \geq n_{i_1}$. Since $v_n v_{i_1} v_{i_2} \ldots v_{i_k}$ is a pendant path in tree T, therefore we have $n_n \geq n_{i_1} > n_{i_2} > \cdots > n_{i_k}$. From (8), we conclude that $D_n \leq D_{i_1} < D_{i_2} < \cdots < D_{i_k}$. Hence $v_{n-1} \in N_T(v_n)$, that is, $v_n v_{n-1} \in E(T)$.

For any pendant vertex v_i and non-pendant vertex v_k such that $v_i v_k \in E(T)$, $D_i - D_k = n - 2$. For any non-pendant vertex v_k, $D_k - D_n \geq 1$, $k \neq n-1, n$. Now,

$$
\begin{aligned}
t - \frac{D_{n-1} + D_n + 2}{2} &= \frac{\sum\limits_{i=1}^{n} [2D_i - (D_{n-1} + D_n)]}{2n} - 1 \\
&= \frac{\sum\limits_{i=1}^{n-2} [2D_i - (D_{n-1} + D_n)]}{2n} - 1 \\
&\geq \frac{2p(n-2) + n - p - 2}{2n} - 1 = p - \frac{5p+2}{2n} - \frac{1}{2}.
\end{aligned}
$$

Therefore we have

$$
\frac{D_{n-1} + D_n + 2}{2} \leq t - p + \frac{5p+2}{2n} + \frac{1}{2}.
$$

Putting $\mathbf{x} = (0, 0, \ldots, 0, 1, -1)$ in (1), we get

$$
\partial_{n-1}^L \leq \frac{D_{n-1} + D_n + 2}{2} \quad \text{as } v_n v_{n-1} \in V(T).
$$

Using the above results, we have

$$
\partial_{n-1}^L \leq t - p + \frac{5p+2}{2n} + \frac{1}{2},
$$

which gives the required result. $\qquad\square$

From the proof of the above theorem, we get the following result:

Theorem 6.4 *Let G be a connected graph of order n. Then*

$$
\partial_{n-1}^L(G) \leq \min\left\{ \frac{D_i + D_j + 2\,d_{ij}}{2} : 1 \leq i < j \leq n \right\},
$$

where D_i is the i-th vertex transmission of graph G.

We now give a relation between $DSLE$, DLE and DE of any tree T, which is a stronger result then that of Theorem 6.1, but it is only applicable for trees.

Theorem 6.5 *Let T be a tree of order n. Then*

$$
DSLE(T) + DLE(T) \geq 2\,DE(T) + 2\left(p - \frac{5p+2}{2n} - \frac{1}{2} \right),
$$

where p is the number of pendant vertices in T.

Proof. It is well known that

$$
DE(T) = \sum_{i=1}^{n} |\partial_i(T)| = 2\,\partial_1(T), \tag{9}
$$

as there is exactly one positive distance eigenvalue of tree T.

We have $\mathcal{D}^Q = \mathcal{D}^L + 2\mathcal{D}$. By Lemma 2.1, we have $\partial_1^Q \geq 2\,\partial_1$. Using this result with Theorem 6.3, from (3), (4) and (9), we get

$$
\begin{aligned}
DSLE(T) + DLE(T) \;&\geq\; 2\left(\partial_1^Q - t\right) + 2\left(2W - \partial_{n-1}^L - t\,(n-2)\right) \\
&=\; 2\left(\partial_1^Q + t - \partial_{n-1}^L\right) \\
&\geq\; 4\,\partial_1 + 2\left(p - \frac{5p+2}{2n} - \frac{1}{2}\right) \\
&=\; 2\,DE(T) + 2\left(p - \frac{5p+2}{2n} - \frac{1}{2}\right).
\end{aligned}
$$

This completes the proof. $\hfill\square$

Comparing $DE(G)$, $DLE(G)$ and $DSLE(G)$ over the class of transmission regular graphs, we have the following result.

Proposition 6.6 *If G is transmission regular graph, then $DE(G) = DLE(G) = DSLE(G)$.*

Proof. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ be the distance eigenvalues of graph G. It is well known that for transmission regular graph G, the distance Laplacian spectrum and the distance signless Laplacian spectrum are

$$
DLS(G) = \{k - \partial_n,\, k - \partial_{n-1},\, \ldots,\, k - \partial_1\} \quad \text{and} \quad DSLS(G) = \{k + \partial_n,\, k + \partial_{n-1},\, \ldots,\, k + \partial_1\}.
$$

Using these results, one can easily see that $DE(G) = DLE(G) = DSLE(G)$. $\hfill\square$

Remark 6.7 *The converse of Proposition 6.6 is not necessarily true. For example, $G \cong \overline{C_3 \cup C_6}$, $G \cong \overline{C_4 \cup K_4}$ and $G \cong \overline{C_6 \cup 3\,K_2}$ satifying $DE(G) = DLE(G) = DSLE(G)$, but these graphs are not transmission regular.*

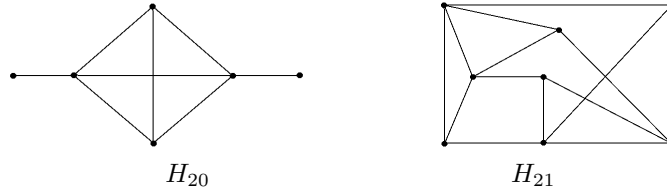**Example 2** The following two graphs are also satisfying $DLE(G) = DSLE(G)$.

$$H_{20} \qquad\qquad\qquad\qquad H_{21}$$

Figure 4: Graphs H_i, $i = 20,\, 21$.

The above observations lead to the statement of the following problem.

Problem 3 *Characterize all the graphs for which*

$$
DLE(G) = DSLE(G).
$$

Remark 6.8 *For complete graph K_n, we have*

$$
E(K_n) = LE(K_n) = SLE(K_n) = DE(K_n) = DLE(K_n) = DSLE(K_n) = 2(n-1).
$$

Also, the above remark leads to the statement of the following problem.

Problem 4 *Is there any connected graph G ($\ncong K_n$) such that $E(G) = LE(G) = SLE(G) = DE(G) = DLE(G) = DSLE(G)$?*

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