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Price of independence for the dominating set problem

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Abstract: Let $\gamma(G)$ and $\iota(G)$ be the domination and independent domination numbers of a graph G, respectively. In this paper, we define the Price of Independence of a graph G as the ratio $\frac{\iota(G)}{\gamma(G)}$. Firstly, we bound the Price of Independence by values depending on the number of vertices. Secondly, we consider the question of computing the Price of Independence of a given graph. Unfortunately, the decision version of this question is Θ_2^p-complete. The class Θ_2^p is the class of decision problems solvable in polynomial time, for which we can make $O(\log(n))$ queries to an NP-oracle. Finally, we restore the true characterization of domination perfect gaphs, i.e. graphs whose the Price of Independence is always 1 for all induced subgraphs, and we propose a conjecture on futher problems.

Keywords: Domination, independent domination, forbidden induced subgraphs, computational complexity

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1 Introduction

1.1 Basic definitions and notations

We use standard notations and definitions, as you can find them in the reference book by Diestel [14]. Graphs are undirected and simple. V and E denote the vertex and the edge sets of a graph G. Given a vertex v, the set of adjacent vertices from v, i.e. its neighbors, is denoted by $N(v)$ while a vertex is *pendant* if it has only one neighbor. $\Delta(G)$ is the maximum degree in the graph G. For a given vertex set X, $G[X]$ denotes the subgraph of G induced by X. Moreover, for two given graphs G and H, G is called H-free if H does not appear as an induced subgraph of G. Therewith, we say that G is $(H_i)_{i=0}^{k}$-free when G is H_1-free, H_2-free, ... and H_k-free for some graphs $H_1, H_2, \ldots, H_k$. A *vertex cover* of a graph $G = (V, E)$ is a set C of vertices such that every edge in E has at least one endpoint in C. The minimum cardinality of a vertex cover in G, denoted by $\tau(G)$, is the *vertex cover number* and a vertex cover with such a cardinality is called *minimum*.

A *dominating set* of a graph $G = (V, E)$ is a set D of vertices such that every vertex $v \in V \setminus D$ has at least one neighbor in D. We denote by $\gamma(G)$ the minimum cardinality of a dominating set in the graph G and this value is called the *domination number* of G. A dominating set with such cardinality is called *minimum*. A set D of vertices is *stable* or *independent* if the subgraph induced by D contains no edge. An independent set X of a graph $G = (V, E)$ is *maximal* if for every vertex $v \in V \setminus X$, $X \cup \{v\}$ is not independent. A dominating set D of graph G is called *independent* if D is stable, or equivalently [4, 5], an independent dominating set is a maximal independent set. The *independent domination number* of a graph G, denoted by $\iota(G)$, is the minimum cardinality of an independent dominating set in G. Moreover, if the cardinality of an independent dominating set is minimum then this set is called *minimum*.

1.2 Previous works

The class of graphs such that the domination number and the independent domination number are equal for all induced subgraphs received a lot of attention in the last decades. Actually, Sumner and Moore [29] introduced the notion of *domination perfect graph*, as a graph G such that $\gamma(H) = \iota(H)$, for all induced subgraph H of G. Several authors [1, 6, 15, 19, 21, 24, 28, 29, 30] tried to find sufficient or necessary conditions to characterize this class of graphs. Sumner [28] stated that a graph is domination perfect if and only if $\gamma(H) = \iota(H)$ only for all induced subgraph H with $\gamma(H) = 2$, and supposed impossible to provide a finite list of forbidden induced subgraphs characterizing domination perfect graphs. Nevertheless, a first characterization with a list of 4 forbidden induced subgraphs was given by Zverovich and Zverovich [35]. However, Fulman [16] pointed out a counterexample. Then, another characterization with a list of 17 forbidden induced subgraphs was proposed again by Zverovich and Zverovich [36].

Theorem 1 (Zverovich and Zverovich [36]) *Let G be a graph. Then G is domination perfect if and only if G is $(G_i)_{i=1}^{17}$-free, where graphs G_i are depicted in Figure 1.*

Camby and Plein [10] claimed a failure in Theorem 1 and proposed a new characterization of domination perfect graphs.

Theorem 2 (Camby and Plein [10]) *Let G be a graph. Then G is domination perfect if and only if G is $(H_i)_{i=0}^{9}$-free, where graphs H_i are depicted in Figure 2.*

Zverovich [34] extended the concept of (domination) perfect graphs by considering the difference between two invariants bounded by a constant, instead of an equality of invariants. These classes are called k-bounded classes of dominant-independent perfect graphs. Zverovich found a characterization in terms of finite list of forbidden induced subgraphs for the k-bounded classes of independent-independent domination perfect graphs and the k-bounded classes of independent-domination perfect graphs. Moreover, he proposed the following conjecture : the k-bounded classes of independent domination-domination perfect graphs can be characterized by a finite list of forbidden induced subgraphs.

Naturally, several graph invariants were investigated for comparison. Cardinal and Levy [12, 22] introduced a new concept : the Price of Connectivity for the vertex cover problem. They defined it as the ratio

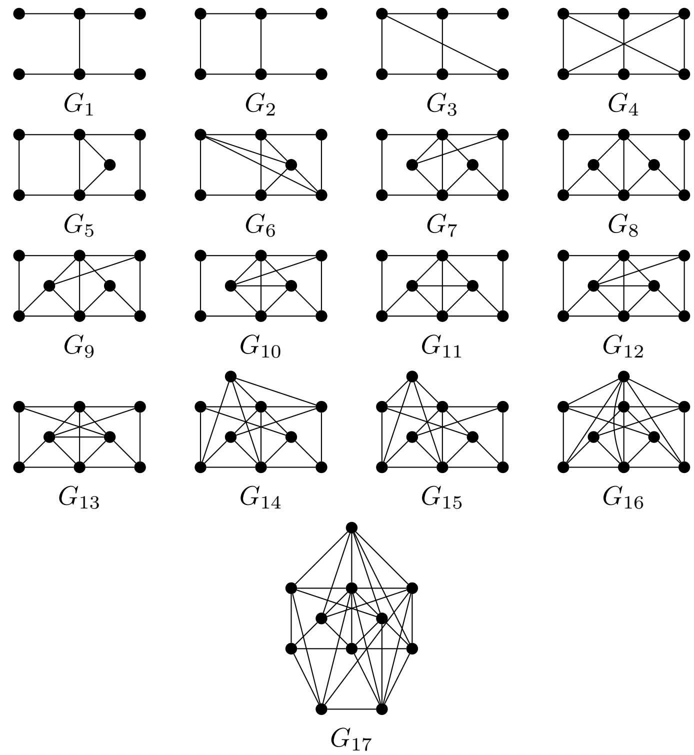

Figure 1: An illustration of graphs G_i, for $i = 1, \ldots 17$.

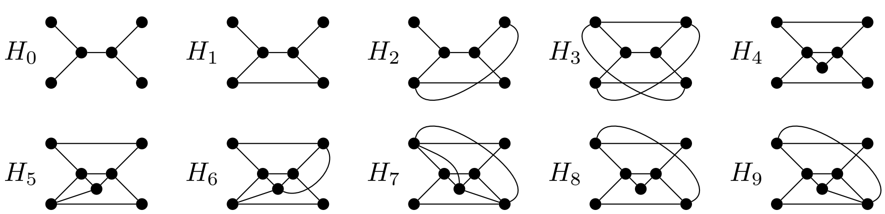

Figure 2: An illustration of graphs H_i, for $i = 0, \ldots 9$.

between the connected vertex cover number and the vertex cover number. Camby, Cardinal, Fiorini and Schaudt [9] studied this Price of Connectivity in terms of structural and computational complexity results while Camby and Schaudt [11] translated the notion to the domination problem and obtained similar works. Analogously, Belmonte, van 't Hof, Kamiński and Paulusma [2, 3] study the Price of Connectivity for feedback vertex set in hereditary graph classes whereas Hartinger, Johnson, Milanič and Paulusma [20] investigated the ratio for cycle transversals. Moreover, Chiarelli, Hartinger, Johnson, Milanič and Paulusma [13] designed polynomial-time algorithms for connected vertex cover, connected feedback vertex set and connected odd cycle transversal for certain classes of graphs, using the Price of Connectivity.

In this paper, we define the Price of Independence of any graph G as follows :

$$PoI(G) = \frac{\text{the independent domination number of } G}{\text{the domination number of } G} = \frac{\iota(G)}{\gamma(G)}.$$

Rad and Volkmann [31] obtained some upper bound on the Price of Independence, depending on the maximum degree :

Theorem 3 (Rad and Volkmann [31]) *Let G be a connected graph.*

- *If $3 \leqslant \Delta(G) \leqslant 5$, then $PoI(G) \leqslant \frac{\Delta(G)}{2}$.*
- *If $\Delta(G) \geqslant 6$, then $PoI(G) \leqslant \Delta(G) - 3 + \frac{2}{\Delta(G)-1}$.*

Moreover, they conjectured that

$$PoI(G) \leqslant \frac{\Delta(G)}{2} \text{ for all graphs } G. \quad (1)$$

Wang and Wei [32, 33] confirmed it in the class of trees and bipartite graphs while Goddard, Henning, Lyle and Southey [18] proved it in the class of cubic graphs. Furthermore, Southey and Henning [26] improved the result : $PoI(G) \leqslant \frac{4}{3}$ for connected cubic graphs, except for $K_{3,3}$. Furuya, Ozeki and Sasaki [17] pointed out a counterexample for (1) but they also showed that for every graph G, $PoI(G) \leqslant \Delta(G) - 2\sqrt{\Delta(G)} + 2$. However, for any value of $\Delta(G)$, $\Delta(G) - 2\sqrt{\Delta(G)} + 2 \geqslant \frac{\Delta(G)}{2}$. Besides, Bollobás and Cockayne [7] proved that, for $k \geqslant 3$, $PoI(G) \leqslant k-2$ for all $K_{1,k}$-free graphs G. Therefore, the following questions remain : is there other class of graphs in which the conjecture (1) is true? Is there another upper bound on the Price of Independence, depending possibly on the class of graphs?

In this paper, we find tight bounds on the Price of Independence, depending only on the number of vertices in the graph. Moreover, we prove that the following decision problem is Θ_2^p-complete : for every fixed rational number $r > 1$, given a n-vertex graph G such that $r \leqslant \frac{n}{4}$, is $PoI(G) \leqslant r$? Loosely speaking, it means that deciding whether the ratio of $\iota(G)$ and $\gamma(G)$ is bounded by some rational number r is as hard as computing both $\iota(G)$ and $\gamma(G)$ explicitly. Finally, we investigate the characterization of domination perfect graphs and further works.

2 Our contribution

2.1 Upper bound on the Price of Independence

Theorem 4 *Let G be a graph on $n \geqslant 4$ vertices. Then*

$$1 \leqslant PoI(G) \leqslant \frac{n}{4}.$$

Moreover, both bounds are tight.

Notice that, when $\Delta(G) \geqslant \frac{n}{2}$, then the above upper bound on the Price of Independence is better than $\frac{\Delta(G)}{2}$, i.e. the best known.

Proof. Let G be a graph. To prove the upper bound, we distinguish several cases depending on the value of $\gamma(G)$.

We assume that $\gamma(G) \geqslant 4$. Since $\iota(G) \leqslant n$, trivially we obtain that

$$PoI(G) \leqslant \frac{n}{4}.$$

If $\gamma(G) = 1$ then G contains a single dominating vertex, in particular an independent dominating set of cardinality only 1. Therefore, $PoI(G) = 1$.

We suppose that $\gamma(G) = 2$. Let $D = \{d_1, d_2\}$ be a minimum dominating set of G. If $d_1d_2 \notin E$, then D is an independent dominating set and $PoI(G) = 1$. Now, $d_1d_2 \in E$. We consider $N[d_i]$ the closed neighborhood of d_i, for $i = 1, 2$, i.e. $N[d_i] = N(d_i) \cup \{d_i\}$. Assume that $|N[d_1]| \leqslant \lceil n/2 \rceil$ and $|N[d_2]| \leqslant \lceil n/2 \rceil$. Because D is a dominating set, $V \subseteq N[d_1] \cup N[d_2]$. Observe that d_1 and d_2 appear in both sets $N[d_1]$ and $N[d_2]$. Hence $n \leqslant \lceil n/2 \rceil + \lceil n/2 \rceil - 2 \leqslant n - 1$, a contradiction.

So, without loss of generality, we have $|N[d_1]| > \lceil n/2 \rceil$. We apply a greedy algorithm to find an independent dominating set A in $G[V \setminus N[d_1]]$. Since $A \cup \{d_1\}$ is an independent dominating set of G,

$$\iota(G) \leqslant |A \cup \{d_1\}| < n - \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Hence

$$\iota(G) \leqslant \frac{n}{2}.$$

Accordingly,

$$PoI(G) \leqslant \frac{\frac{n}{2}}{2} = \frac{n}{4}.$$

Now, the last case is when $\gamma(G) = 3$. Let $D = \{d_1, d_2, d_3\}$ be a minimum dominating set of G. We need to prove that $\iota(G) \leqslant 3n/4$. Consider $N[d_i]$ the closed neighborhood of d_i. We assume that $|N[d_i]| \leqslant n/4$ for every $i = 1, 2, 3$. Since D is a dominating set of G,

$$V \subseteq \bigcup_{i=1}^{3} N[d_i],$$

then $n \leqslant 3n/4$, a contradiction. So, without loss of generality, we have that $|N[d_1]| > n/4$. Now, we apply a greedy algorithm to find an independent dominating set A in $G[V \setminus N[d_1]]$. Thus

$$\iota(G) \leqslant |A| + |\{d_1\}| < n - \frac{n}{4} + 1 = \frac{3n}{4} + 1,$$

this strict inequality implies the large desired inequality.

Moreover, the upper bound is tight. Indeed, the double star $S(k, k)$, graph on $n = 2k+2$ vertices obtained by adding an edge between the center of two stars $K_{1,k}$ (see Figure 3), satisfies the property : its domination number is 2 while its independence domination number is $k + 1 = \frac{n}{2}$. Thus, $PoI(S(k, k)) = \frac{n}{4}$.

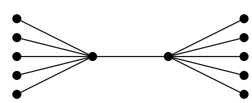

Figure 3: The double star $S(5, 5)$ on 12 vertices.

□

2.2 Complexity result

The class Θ_2^p, also denoted by $\mathrm{P}^{\mathrm{NP[log]}}$, is the class of decision problems solvable in polynomial-time by a deterministic Turing machine, that can make $\mathcal{O}(\log n)$ queries to a NP-oracle, where n is the size of the input. The following complexity result is inspired from [11].

Theorem 5 *Let $r > 1$ be a positive rational number. Given a graph G on n vertices such that $r \leqslant n/4$, the problem of deciding whether $\iota(G)/\gamma(G) \leqslant r$ is Θ_2^p-complete.*

Since $\iota(G)$ and $\gamma(G)$ can be computed by applying logarithmically an NP-oracle via a binary search, clearly the decision problem associated to the compute of the Price of Independence is in the class Θ_2^p. Theorem 5 means that deciding whether the ratio of $\iota(G)$ and $\gamma(G)$ is bounded by some constant r is as hard as computing both invariants explicitly, it remains true even if r is not part of the input.

Our reduction is from the decision problem whether $\tau(G) \geqslant \tau(H)$, for two given graphs G and H. The latter is known to be Θ_2^p-complete by Spakowski and Vogel [27].

Beforehand, we prove the two following lemmas.

Lemma 1 *Given a graph G with $n \geqslant 2$ vertices and $m > 0$ edges, one can construct in linear time a graph G' such that $\gamma(G') = n + \tau(G)$ and $\iota(G') = n + m$.*

Proof. For each vertex $v \in V(G)$, we associate $m+3$ vertices $v, v', v''_1, v''_2, \ldots, v''_{m+1}$ in $V(G')$, and for each edge $e \in E(G)$, we associate a vertex e of $V(G')$. So, we may consider $V(G)$ and $E(G)$ as subsets of $V(G')$. We define

$$E(G') = \bigcup_{e=uv \in E(G)} \{ue, ve\} \cup \bigcup_{v \in V(G)} \{vv', v'v''_1, v'v''_2, \ldots, v'v''_{m+1}\}.$$

Let D be a minimum dominating set of G'. Without loss of generality, we can suppose that $v' \in D$ for every $v \in V(G)$ because v' has many pendant vertices. If $e \in D$ for some $e = uv \in E(G)$, then $D \setminus \{e\}$ dominates all vertices of G' except e, since $u', v' \in D$. So, $(D \setminus \{e\}) \cup \{u\}$ is a minimum dominating set of G'. Accordingly, we may suppose that $D \cap E(G) = \emptyset$. In that case, $D \cap V(G)$ is a vertex cover of G, proving that $\gamma(G') = |D| \geqslant n + \tau(G)$.

Conversely, if C is a vertex cover of G then $\{v' | v \in V(G)\} \cup C$ is a dominating set of G', so $\gamma(G') \leqslant n + \tau(G)$. This gives the first equality.

For the second, clearly $\bigcup_{v \in V(G)} \{v'\} \cup E(G)$ is an independent dominating set of G', so $\iota(G') \leqslant n + m$.

Conversely, let I be a minimum independent dominating set of G'. Suppose that $v' \notin I$, for one $v \in V(G)$, then every pendant vertices $v''_1, v''_2, \ldots, v''_{m+1}$ must be in I. Moreover, for other vertex $u \in V(G) \setminus \{v\}$, we need at least one vertex in I to dominate u'. Thus, $\iota(G') = |I| \geqslant (m+1) + (n-1) = m + n$, proving the second equality. □

Lemma 2 *Given a graph G with n vertices and m edges, one can construct in linear time a graph G' such that $\gamma(G') = n+1$, $\iota(G') = n + 1 + \tau(G)$ and there exists a vertex belonging in every minimum dominating set and in every minimum independent dominating set.*

Proof. We construct G' by attaching two pendant vertices v_1 and v_2 to each vertex $v \in V(G)$, and adding a disjoint star $K_{1,s}$ of center x, with s arbitrarily linearly large.

Trivially, $V(G) \cup \{x\}$ is a dominating set of G', so $\gamma(G') \leqslant n+1$. On the other hand, let D be a minimum dominating set of G'. Without loss of generality, we may assume that for every vertex $v \in V(G)$, $v \in D$, since v has two pendant vertex. Moreover, we need one vertex in D to dominate the star. Thus $\gamma(G) = n+1$.

It remains to compute $\iota(G')$. Let C be a minimum vertex cover of G. Then $\{x\} \cup (V(G) \setminus C) \cup \{u_1, u_2 | u \in C\}$ is clearly an independent dominating set of G'. So $\iota(G') \leqslant n + 1 + \tau(G)$. Let I be a minimum independent dominating set of G'. Clearly, $I \cap V(K_{1,s}) = \{x\}$. The set $V(G) \cap I$ must be stable in G', especially in G, hence $V(G) \setminus I$ is a vertex cover of G. For every vertex $v \in V(G)$, either $v \in I$ or v_1 and v_2 belongs to I. In other words, for every vertex $v \in V(G) \setminus I$, v_1 and v_2 must be in I. So, $\iota(G') = |I| \geqslant 1 + n + \tau(G)$. Thus $\iota(G') = n + 1 + \tau(G)$.

Notice that the center x of the star $K_{1,s}$ is always in every minimum dominating set and in every minimum independent dominating set. □

Proof of Theorem 5. Let $r = r_1/r_2 > 1$ be a fixed rational number, with r_1 and r_2 positive numbers. It remains to prove the Θ_2^p-hardness. We reduce our problem from the Θ_2^p-complete decision problem of deciding, given two graphs G and H, whether $\tau(G) \geqslant \tau(H)$ [27]. Let (G, H) be an instance of the latter.

Step 1. We consider G_{r_2} the graph obtained by taking r_2 disjoint copies of G, and similarly for H_{r_1}. Let $n_G = |V(G)|$, $m_G = |E(G)|$ and $n_H = |V(H)|$. Clearly, $\tau(G_{r_2}) = r_2\tau(G)$, $|V(G_{r_2})| = r_2 n_G$ and $|E(G_{r_2})| = r_2 m_G$. Moreover, $\tau(H_{r_1}) = r_1\tau(H)$ and $|V(H_{r_1})| = r_1 n_H$.

Step 2. We apply Lemma 1 to G_{r_2} to get G'_{r_2} and we obtain

$$\begin{aligned} \gamma(G'_{r_2}) &= |V(G_{r_2})| + \tau(G_{r_2}) \\ &= r_2\tau(G) + r_2 n_G, \\ \iota(G'_{r_2}) &= |V(G_{r_2})| + |E(G_{r_2})| \\ &= r_2(n_G + m_G). \end{aligned}$$

Now, we apply Lemma 2 to H_{r_1} to get H'_{r_1} and we obtain

$$\begin{aligned}\gamma(H'_{r_1}) &= |V(H_{r_1})| + 1\\ &= r_1 n_H + 1,\\ \iota(H'_{r_1}) &= |V(H_{r_1})| + 1 + \tau(H_{r_1})\\ &= r_1 n_H + 1 + r_1\tau(H).\end{aligned}$$

Step 3. Let $\bar{r} = \lceil r \rceil$. Notice that $\bar{r}$ is a positive integer. We construct a new graph U by taking the disjoint union of $\bar{r}$ copies of G'_{r_2} and $\bar{r}$ copies of H'_{r_1}. By the construction of U,

$$\begin{aligned}\gamma(U) &= \bar{r}\gamma(G'_{r_2}) + \bar{r}\gamma(H'_{r_1})\\ &= \bar{r}(r_2\tau(G) + r_2 n_G) + \bar{r} r_1 n_H + \bar{r}\\ &= \bar{r} r_2 \tau(G) + \bar{r}(r_2 n_G + r_1 n_H + 1)\\ \iota(U) &= \bar{r}\iota(G'_{r_2}) + \bar{r}\iota(H'_{r_1})\\ &= \bar{r} r_2(n_G + m_G) + \bar{r}(r_1 n_H + r_1\tau(H) + 1)\\ &= \bar{r} r_1\tau(H) + \bar{r}(r_2(n_G + m_G) + r_1 n_H + 1).\end{aligned}$$

Step 4. Let

$$\begin{aligned}\varphi_1 &= r_2(n_G + m_G) + r_1 n_H + 1\\ \varphi_2 &= r_2 n_G + r_1 n_H + 1.\end{aligned}$$

Let $p = \max\{|\varphi_1 - (\bar{r}+1)\varphi_2|, |\varphi_2 - \varphi_1|\}$ and

$$\begin{aligned}a &= p((\bar{r}+1)r_2 - r_1) + (\varphi_1 - (\bar{r}+1)\varphi_2)\\ b &= p(r_1 - r_2) + (\varphi_2 - \varphi_1).\end{aligned}$$

By definition of p, $a \geqslant |\varphi_1 - (\bar{r}+1)\varphi_2|((\bar{r}+1)r_2 - r_1) + (\varphi_1 - (\bar{r}+1)\varphi_2) \geqslant |\varphi_1 - (\bar{r}+1)\varphi_2|((\bar{r}+1)r_2 - r_1 - 1)$ and $b \geqslant |\varphi_2 - \varphi_1|(r_1 - r_2) + (\varphi_2 - \varphi_1) \geqslant |\varphi_2 - \varphi_1|(r_1 - r_2 - 1)$. Since $r_1 > r_2$ and $(\bar{r}+1)r_2 > r_1$, then a and b are two non-negative integers. Furthermore, $a, b \in \mathcal{O}(\varphi_1 + \varphi_2)$.
Moreover, we can easily verify that

$$a + (\bar{r}+1)b + \bar{r}\varphi_1 = \bar{r}pr_1 \quad \text{and} \quad a + b + \bar{r}\varphi_2 = \bar{r}pr_2. \tag{2}$$

Finally, we construct a new graph U' from U as follows. Let P^a be the graph obtained from the induced path with vertex set $\{u_1, u_2, \ldots, u_a\}$ by attaching a pendant vertex to every member of $\{u_1, u_2, \ldots, u_a\}$. Let v be a vertex in U belonging in every minimum dominating set and in every minimum independent dominating set (such a vertex always exists, since $r_1 > 0$). Let P^b be the graph obtained from a clique with vertex set $\{v_1, v_2, \ldots, v_b\}$ by attaching $\bar{r}+1$ pendant vertices to every member of $\{v_1, v_2, \ldots, v_{b-2}\}$ and by attaching $2\bar{r} - 1$ pendant vertices to v_{b-1} and to v_b. (If $b = 1$ then P^b is the star $K_{1,\bar{r}+1}$ of center v_1). Let U' be the graph obtained from the disjoint union of U, P^a and P^b by putting an edge between v and v_1. The described procedure can be done in linear time in the size of the graph U, i.e. in the size of the input because $a, b \in \mathcal{O}(\varphi_1 + \varphi_2)$. By the construction of U', it follows that

$$\begin{aligned}\gamma(U') &= \gamma(U) + a + b\\ &= \bar{r} r_2\tau(G) + a + b + \bar{r}\varphi_2\\ &\overset{(2)}{=} \bar{r} r_2\tau(G) + \bar{r}pr_2\end{aligned}$$

and

$$\begin{aligned}\iota(U') &= \iota(U) + a + (\bar{r}+1)b\\ &= \bar{r} r_1\tau(H) + a + (\bar{r}+1)b + \bar{r}\varphi_1\\ &\overset{(2)}{=} \bar{r} r_1\tau(H) + \bar{r}pr_1.\end{aligned}$$

Since $r = r_1/r_2$, we have

$$\frac{\iota(U')}{\gamma(U')} = \frac{\bar{r}r_1\tau(H) + \bar{r}pr_1}{\bar{r}r_2\tau(G) + \bar{r}pr_2} = r\frac{\tau(H) + p}{\tau(G) + p}.$$

Accordingly, $\iota(U')/\gamma(U') \leqslant r$ if and only if $\tau(H) \leqslant \tau(G)$. This completes the proof.

□

2.3 Characterization of domination perfect graphs and further work

Camby and Plein [10] claimed that graphs H_5 and H_6 are counterexamples for Theorem 1 since $\iota(H_5) = 3 = \iota(H_6)$. However, it is incorrect as Figure 4 illustrates. Accordingly, the Camby-Plein's characterization of domination perfect graphs does not hold anymore. Notice that their algorithm to find an independent dominating set from a given dominating set is still valid in the class of $(H_i)_{i=0}^{9}$-free graphs. A natural question follows : is there a polynomial-time algorithm to transform a dominating set into an independent dominating set, without increasing its cardinality, in the class of $(G_i)_{i=1}^{17}$-free graphs ?

Figure 4: White vertices indicate a minimum independent dominating set in graphs H_5 and H_6.

As it is done for the Price of Connectivity, we define PoI-near-perfect graphs for a threshold t and critical graphs for the Price of Independence. A graph is *PoI-near-perfect* for the threshold t if $PoI(H) \leqslant t$ for all induced subgraphs H of G while a graph G is *critical* if for all proper induced subgraphs H of G, $PoI(H) < PoI(G)$. The former is a generalization of domination perfect graphs whereas the latter is used in the list of forbidden induced subgraphs for the characterization of PoI-near-perfect graphs.

We use the computer aided graph theory system GraphsInGraphs [8], called GIG, to find critical graphs up to 10 vertices. We retrieve them for the characterization of domination perfect graphs, as it is shown in Theorem 1. Moreover, due to GIG, we establish the following conjecture on PoI-near-perfect graphs for threshold $t = \frac{3}{2}$. Forbidden induced subgraphs are listed in Annex 1. Critical graphs are given in g6 format by McKay and Piperno [23]. Since GIG pointed out all critical graphs up to 10 vertices, the following conjecture is valid, except if there exist critical graphs on more vertices.

Conjecture 1 *Let G be a graph. The following assertions are equivalent :*

- *for every induced subgraph H of G, $PoI(H) \leqslant \frac{3}{2}$,*
- *G is $\mathcal{G}$-free, where the family $\mathcal{G}$ of 172 graphs is described in Annex 1.*

Annex 1

Table 1 gives critical graphs in g6 format [23] from family $\mathcal{G}$ in Conjecture 1. We found them by searching graphs G satisfying $PoI(G) > \frac{3}{2}$ and $PoI(H) \leqslant \frac{3}{2}$ for all proper induced subgraph H of G.

Table 1: Critical graphs for the characterization of PoI-near-perfect graphs with threshold $\frac{3}{2}$ in g6 format.

<table>
<tr><td>1</td><td>G??CZc</td><td>44</td><td>I@?A_WKgw</td><td>87</td><td>I_?_gxjfg</td><td>130</td><td>IAGOW }]Xw</td></tr>
<tr><td>2</td><td>G??MPk</td><td>45</td><td>I@??WYbSw</td><td>88</td><td>I`?@Ol]jW</td><td>131</td><td>IC?ha|]iw</td></tr>
<tr><td>3</td><td>G??ZKs</td><td>46</td><td>I@?COxfvG</td><td>89</td><td>I_?HOl]jW</td><td>132</td><td>IAGOZM^fo</td></tr>
<tr><td>4</td><td>G??xuK</td><td>47</td><td>I??R@qN^G</td><td>90</td><td>I?QO`Tfmg</td><td>133</td><td>I@`G`Ljdw</td></tr>
<tr><td>5</td><td>G?GTa[</td><td>48</td><td>I?GAKovvG</td><td>91</td><td>I?L@CLZlg</td><td>134</td><td>ICW?jEN\W</td></tr>
<tr><td>6</td><td>G?ClQk</td><td>49</td><td>I@?E?wnvG</td><td>92</td><td>I@OCWilTw</td><td>135</td><td>I?SqHUN[w</td></tr>
<tr><td>7</td><td>G?Kta[</td><td>50</td><td>I_?@Ol]jW</td><td>93</td><td>I?Q@gptiw</td><td>136</td><td>I@OXOlfew</td></tr>
<tr><td>8</td><td>G?OxuK</td><td>51</td><td>I_?GTpv\o</td><td>94</td><td>IGC?XL\lg</td><td>137</td><td>I@OOzKnfW</td></tr>
<tr><td>9</td><td>G?r@xw</td><td>52</td><td>I?@Hohxhw</td><td>95</td><td>IAG?g]trg</td><td>138</td><td>IPDIPKVuW</td></tr>
<tr><td>10</td><td>G?`zro</td><td>53</td><td>I?SoOLrbw</td><td>96</td><td>I@@GREV]g</td><td>139</td><td>I`H?oyfVg</td></tr>
<tr><td>11</td><td>Gs`zro</td><td>54</td><td>I?o@hgN|G</td><td>97</td><td>I@I?Wd\ww</td><td>140</td><td>IWCOYZRJw</td></tr>
<tr><td>12</td><td>H??E@KZ</td><td>55</td><td>I?SGHMZ\g</td><td>98</td><td>IGCOP\Vjg</td><td>141</td><td>Ig?WsMxXw</td></tr>
<tr><td>13</td><td>H??RC\x</td><td>56</td><td>I?@b?oW`w</td><td>99</td><td>IC?`Q|]jW</td><td>142</td><td>I_Kq?\Nkw</td></tr>
<tr><td>14</td><td>H?@HcLx</td><td>57</td><td>I??]?ozxW</td><td>100</td><td>I_GGKtv\o</td><td>143</td><td>I@DIcknYw</td></tr>
<tr><td>15</td><td>H??guLx</td><td>58</td><td>I@G?gZbvG</td><td>101</td><td>IQ?GOKzpw</td><td>144</td><td>I@GT\X^Vo</td></tr>
<tr><td>16</td><td>H??ZC\x</td><td>59</td><td>I?AQOplkw</td><td>102</td><td>I?L?jEL{ g</td><td>145</td><td>I_Kpc\rRw</td></tr>
<tr><td>17</td><td>H?CRZYr</td><td>60</td><td>IAG?gZb^G</td><td>103</td><td>I?OahqN^G</td><td>146</td><td>I?d_bAVYw</td></tr>
<tr><td>18</td><td>H_GOC\r</td><td>61</td><td>I??XT@V }G</td><td>104</td><td>IE@@XWZzG</td><td>147</td><td>I]?GOGzpw</td></tr>
<tr><td>19</td><td>H@?ISll</td><td>62</td><td>I?HOPeN{ g</td><td>105</td><td>IA_@XhLlg</td><td>148</td><td>I@QG`EjTw</td></tr>
<tr><td>20</td><td>H_?Glhj</td><td>63</td><td>IH?GoL\hw</td><td>106</td><td>IAGCWwvrW</td><td>149</td><td>IAGWXMZXw</td></tr>
<tr><td>21</td><td>H?DPSLx</td><td>64</td><td>I?AH`rJ\W</td><td>107</td><td>IC@@XhZjW</td><td>150</td><td>ICGHI|]hw</td></tr>
<tr><td>22</td><td>H?O_ze\</td><td>65</td><td>IGC__^M[w</td><td>108</td><td>I_?_wxjnG</td><td>151</td><td>IOP?X{}sw</td></tr>
<tr><td>23</td><td>H?Obc }]</td><td>66</td><td>I@AAWhlew</td><td>109</td><td>I_G?wxffW</td><td>152</td><td>I@DHIUVYw</td></tr>
<tr><td>24</td><td>H?HOs\r</td><td>67</td><td>IAGORK^xg</td><td>110</td><td>IQ?@Wxlfg</td><td>153</td><td>I@_Qywnhw</td></tr>
<tr><td>25</td><td>H@?iyyj</td><td>68</td><td>I?@HeUt]W</td><td>111</td><td>IAGKPlffW</td><td>154</td><td>I@`HOlZhw</td></tr>
<tr><td>26</td><td>HG?WtLZ</td><td>69</td><td>I?@PO|xfg</td><td>112</td><td>I@HG_ljfg</td><td>155</td><td>I@OYKsnXw</td></tr>
<tr><td>27</td><td>HG?Wuqf</td><td>70</td><td>I?AQP{}uW</td><td>113</td><td>I`?D\X^Vo</td><td>156</td><td>IQO[KpfMw</td></tr>
<tr><td>28</td><td>HG?YtKz</td><td>71</td><td>I?op_\F{ W</td><td>114</td><td>IG@POqF }G</td><td>157</td><td>IAGYXlfew</td></tr>
<tr><td>29</td><td>H@Aiyyj</td><td>72</td><td>IAC@XYVZW</td><td>115</td><td>I?DPRAlsw</td><td>158</td><td>I`H?wyfUw</td></tr>
<tr><td>30</td><td>H@Iayw^</td><td>73</td><td>IE?@XW^zG</td><td>116</td><td>I?Sq@EN[w</td><td>159</td><td>I`GXGtfew</td></tr>
<tr><td>31</td><td>HC_Zzx{</td><td>74</td><td>Ig?WoMxXw</td><td>117</td><td>I@GSQL]jW</td><td>160</td><td>Io?WrAF]G</td></tr>
<tr><td>32</td><td>I????cKwW</td><td>75</td><td>I??qoyN]W</td><td>118</td><td>I@CaQYVZg</td><td>161</td><td>I?ope?N{ W</td></tr>
<tr><td>33</td><td>I???GSopW</td><td>76</td><td>I?@HpYV]W</td><td>119</td><td>IH?GqL\hw</td><td>162</td><td>I]?GOKzpw</td></tr>
<tr><td>34</td><td>I??GE?jDw</td><td>77</td><td>I?@PpYN]W</td><td>120</td><td>I?L?jEtqw</td><td>163</td><td>IQQ@Gpfew</td></tr>
<tr><td>35</td><td>I???WWoow</td><td>78</td><td>I@OOrKnfW</td><td>121</td><td>IGCHaLNlW</td><td>164</td><td>IQQ@Gs{ ow</td></tr>
<tr><td>36</td><td>I???XOSow</td><td>79</td><td>Io?WsLf]W</td><td>122</td><td>I@OGhMjtW</td><td>165</td><td>II_HkGxpw</td></tr>
<tr><td>37</td><td>I??@Grdug</td><td>80</td><td>IQ??OK }rW</td><td>123</td><td>IACh?]VYw</td><td>166</td><td>I_OpdpNbw</td></tr>
<tr><td>38</td><td>I??AhYZ^G</td><td>81</td><td>I`??Oxffg</td><td>124</td><td>IGCHILZlW</td><td>167</td><td>IoCOZdlbw</td></tr>
<tr><td>39</td><td>I??OPHAeW</td><td>82</td><td>I@CaCEmVW</td><td>125</td><td>IGC_lT\Rw</td><td>168</td><td>I`_XIURZG</td></tr>
<tr><td>40</td><td>I??OPHBeW</td><td>83</td><td>I?GQSHxlg</td><td>126</td><td>IAGO\fLXw</td><td>169</td><td>ISP@xyN[w</td></tr>
<tr><td>41</td><td>I??Gc`JHw</td><td>84</td><td>IG?GpL\lg</td><td>127</td><td>I@P@G }]Yw</td><td>170</td><td>I@G\\X^Vo</td></tr>
<tr><td>42</td><td>I?GOONXxg</td><td>85</td><td>I@@D?xJnG</td><td>128</td><td>IAH@G }]Xw</td><td>171</td><td>I_KtbdNbw</td></tr>
<tr><td>43</td><td>I?KA?MurW</td><td>86</td><td>I?`?pLxlg</td><td>129</td><td>I@OPO|ffg</td><td>172</td><td>I?mtb| }^_</td></tr>
</table>

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