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# Maximum likelihood estimation of first-passage structural credit risk models correcting for the survivorship bias

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**Abstract:** The survivorship bias in credit risk modeling is the bias that results in parameter estimates when the survival of a company is ignored. We study the statistical properties of the maximum likelihood estimator (MLE) accounting for survivorship bias for models based on the first-passage of the geometric Brownian motion. We find that if we neglect the survivorship bias, then the drift has a positive bias that may not disappear asymptotically. We show that correcting the survivorship bias by conditioning on survival in the likelihood function underestimates the drift. Therefore, we propose a bias correction method for non-iid samples that is first-order unbiased and second-order efficient. The economic impact of neglecting or miscorrecting for the survivorship bias is studied empirically based on a sample of more than 13,000 companies over the period 1980 through 2016 inclusive. Our results point to the important risk of misclassifying a company as solvent or insolvent due to biases in the estimates.

**Keywords:** survival bias, geometric Brownian motion, conditional estimation, default probability, inference, diffusion processes

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# 1 Introduction

Survivorship bias (or survival bias)<sup>1</sup> is the error that emerges when a study is solely based on a sample of observations that exclude failures. This bias is well documented in finance (e.g. Brown et al. (1995)), notably in performance studies (e.g. Brown et al. (1992) and Elton et al. (1996)) i.e. companies that go out of business are usually removed from the sample.

Quantifying a firm's solvency is a very important task in risk management. However, credit risk analysis and estimation could also be subject to survivorship bias because ignoring a company's survival during the observation period may overestimate its solvency. The aim of this paper is to analyze maximum likelihood estimation (MLE) of first-passage structural credit risk models accounting for survivorship. We show that maximum likelihood inference under survivorship is a non-trivial statistical problem and that parameter estimates that do not properly correct for survival could exhibit important biases with significant economic impacts.

It is common practice to assess credit risk of a company using its stock prices. Structural models are used for this purpose because they link credit risk to a company's capital structure (assets, liabilities, equity). The credit risk model proposed in Merton (1974) assumes that default only occurs at debt's maturity when assets are insufficient to repay debtholders. In this model, there is no survivorship bias because assets can take any value prior to debt's maturity. However, this default triggering mechanism is not realistic because creditors/investors will never let a debtor's solvency degrade to a point where they could lose significantly. First-passage structural credit risk models assume instead that default occurs the first time the asset value process crosses a deterministic barrier (see e.g. Black and Cox (1976)), thus allowing for default before maturity. In this context, a survivorship bias appears because observing the company's equity prices today is necessarily conditional upon its survival.

The literature in finance has largely focused on estimating the parameters of Merton (1974) because the capital structure is simple and yields closed-form expressions for equity prices. Examples of these works include Jones, Mason & Rosenfeld (1984), in which the authors set up a system of two equations and two unknowns based upon the initial equity value and stock volatility to back up the initial asset value and volatility. Although appealing due to its simplicity, Ericsson & Reneby (2004) have shown that it yields severe biases. Duan (1994, 2000) present an MLE-based estimation of the model in which the equity price is treated as a one-to-one transformation of the assets. Vassalou & Xing (2004) propose an alternative technique that uses an iterative algorithm to first find the asset volatility and subsequently the drift of the process – a procedure similar to the one employed in Moody's KMV model. As discussed in Duan et al. (2005), this technique yields estimates that are equivalent to the MLE of Duan (1994).

When estimating first-passage models, most authors have relied on the aforementioned techniques and have thus tended to neglect the survival bias (see e.g. Wong and Choi (2009), Dionne and Laajimi (2012), Afik et al. (2016)). There are, however, two noteworthy exceptions. In Duan et al. (2004), the authors explicitly account for the survivorship bias when estimating the parameters of a Merton-like model with multiple refinancing dates. Forte & Lovreta (2012) apply Duan et al. (2004) and investigate other approaches to infer the parameters of a first-passage model with credit default swap premiums. Unlike these authors, we focus on analyzing the statistical properties of the MLE for first-passage structural models conditional upon survival and present an extensive empirical study with a very large sample of companies to assess the impact of survivorship.

Our analysis is also related to the work of Li, Pearson and Poteshman (2004). In a class of diffusion processes, they derive a set of moments conditional on typical events (such as survival) for the purpose of GMM estimation. Whereas they briefly discuss the issue of the survivorship bias, they do not formally analyze properties of the maximum likelihood estimators nor quantify this bias empirically.

The broader problem that we investigate is the MLE of a geometric Brownian motion (GBM) conditional upon its infimum never crossing a fixed barrier. The first contribution of the paper is to analyze the impact of ignoring the survivorship bias on the MLE of the drift and diffusion of this process. We show that the survivorship bias has a non trivial impact on the estimate of the drift. We compute explicitly the size of the

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<sup>1</sup>We will use the terms "survivorship bias" and "survival bias" indistinguishably throughout this article.

bias and show that it is always positive (drift is overestimated). Also, we demonstrate that this bias only disappears asymptotically if the drift is positive.

Our second contribution is the characterization of the statistical behavior associated with the conditional MLE (i.e. conditional upon survival). We use this likelihood function to show that the final asset value is a sufficient statistic to estimate the drift, just like in the unconditional case. We use this characterization to show that the conditional MLE is biased downward, so numerical maximization of the likelihood function yields biased estimates of the drift.

The third contribution is a method to correct for the survivorship bias when estimating the drift. Standard bias correction methods such as the jackknife or bootstrap do not work in this context because observations are not independent nor identically distributed (iid). Based upon a series expansion of the expected parameter estimate (see e.g. Cox & Hinkley (1974)), we propose a debiasing method that is first-order unbiased and second-order efficient, which can be applied to correct the survivorship bias. This method adds to a rich literature in financial econometrics that analyzes and corrects some biases in the parameters of financial models (see e.g. Tang and Chen (2009), Yu (2012) and Bauer et al. (2012)).

Our fourth contribution is an extensive empirical study whose goal is to analyze the economic impact of ignoring or miscorrecting for survivorship bias in credit risk analyses. Based on a sample of 13,794 firms obtained from the intersection of CRSP and Compustat, we use monthly stock prices between 1979 and 2016 to estimate a credit risk model for each firm in the sample. We compare default probabilities and the proportion of risky companies over different business cycles between 1980 and 2016. We find that biases in the naive (unconditional) and conditional MLEs are significant economically. For example, during the recession of the early 1980s, the one-year default probability of a company is 25% when estimated with a debiased MLE, which is considerably higher than the 15% value obtained with the unconditional MLE, but lower than the 40% value obtained with the conditional MLE. These values illustrate how unconditional estimates are overly optimistic about the solvency of a company and conditional MLEs generally provide the opposite picture. Moreover, these effects are more important during economic recessions. The findings of this paper provide new evidence of how the survivorship bias can undermine risk management policies.

The remainder of the article is organized as follows. Section 2 presents the general framework and notation of the paper. In Section 3, we analyze the properties of the MLE assuming that the asset value is directly observed conditional on the firm's survival. Section 4 presents our credit risk modeling framework. Section 5 conducts an empirical study, illustrating the economic impact of ignoring or miscorrecting for the survivorship bias. Finally, Section 6 concludes.

## 2 Framework

In this section, we introduce the main concepts and notation of the paper. Let  $\underline{A} = \{A_t, t \geq 0\}$  be a stochastic process that represents the value of a company's assets over time. We assume that  $\underline{A}$  is a geometric Brownian motion (GBM). Written as a stochastic differential equation, the dynamics of  $\underline{A}$  is

$$dA_t = \mu A_t dt + \sigma A_t dW_t,$$

where  $\mu$  and  $\sigma$  are the drift and diffusion coefficients of the process. Moreover, the initial asset value  $A_0 := a_0$  is known and  $\{W_t, t \geq 0\}$  is a standard Brownian motion. Therefore, the solution of the process is such that  $\ln(A_t)$  is a standard Brownian motion (with drift) starting from  $\ln(a_0)$  i.e.

$$\ln(A_t) = \ln(a_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t.$$

The second component of the modeling framework is the default barrier  $L > A_0$ . This exogenous parameter represents the amount owed by the debtor or any quantity deemed to represent a solvency limit for the company. When the company's asset value falls below or reaches  $L$ , it is assumed that the company defaults and stops any economic activity.

Throughout the paper, we will often equivalently consider the stochastic process  $\underline{Z} = \{Z_t, t \geq 0\}$  where

$$Z_t = \ln(A_t/L).$$

Therefore,  $\underline{Z}$  is a Brownian motion with drift  $\nu = \mu - \frac{1}{2}\sigma^2$ , diffusion  $\sigma$ , and initial value of

$$z_0 := Z_0 = \ln(A_0/L) > 0.$$

The process  $\underline{Z}$  is similar to the distance-to-default metric popular in credit risk modeling.<sup>2</sup>

We are interested in studying the stochastic process  $\underline{A}$  subject to never crossing a barrier  $L$ . Therefore, let  $I_T^A$  be the minimum value attained by  $\underline{A}$  over the time interval  $[0, T]$  i.e.

$$I_T^A = \inf\{A_t, 0 \leq t \leq T\}$$

or equivalently

$$I_T^Z = \inf\{Z_t, 0 \leq t \leq T\}.$$

We refer to the event  $I_T^A > L$  (or  $I_T^Z > 0$ ) as the survival event, which is a fundamental part of first-passage structural credit risk models (see Section 4). In this setting, it becomes clear that selecting a company at time  $T$  is subject to survival.

### 3 Parameter estimation under survivorship

This section analyzes the survivorship bias of maximum likelihood estimates of  $\nu$  and  $\sigma$  subject to survivorship. Assume first that we can directly observe the conditional process

$$\underline{Z} | I_T^Z > 0$$

at discrete time points  $0 = t_0 < t_1 < t_2 < \dots < t_n = T$ , where  $n$  is the number of observations in the time interval  $[0, T]$ . For simplicity, assume time intervals have the same length i.e. that  $h := t_i - t_{i-1} = T/n, \forall i = 1, 2, \dots, n$ .

#### 3.1 Naive MLE

We first study parameter estimates of the process  $\underline{Z}$  that ignore any survivorship consideration. These estimates are referred to as the naive MLE (NMLE) and they are denoted by  $\hat{\nu}^N$  and  $\hat{\sigma}^N$ . They are obtained from the realizations of

$$R_i = Z_{t_i} - Z_{t_{i-1}} = \ln(A_{t_i}/A_{t_{i-1}}),$$

which correspond to the log-return of asset values over the time interval  $[t_{i-1}, t_i]$ . Since  $R_i$  is normally distributed with mean  $\nu h$  and variance  $\sigma^2 h$ , the NMLE of  $\nu$  and  $\sigma$  are calculated by directly maximizing the joint unconditional normal p.d.f.  $f_{R_1, R_2, \dots, R_n}(r_1, r_2, \dots, r_n)$ .

##### 3.1.1 Drift coefficient

The NMLE of  $\nu$  is given by

$$\hat{\nu}^N = \frac{1}{T} \ln(A_T/A_0) = \frac{1}{T} (Z_T - Z_0). \quad (1)$$

Since this estimate does not account for the survivorship event, there is a bias inherited in its computation. The following proposition establishes this bias.

<sup>2</sup>The common distance-to-default metric is usually normalized by the drift and diffusion (see Vassalou and Xing, 2004).

**Proposition 1** *The NMLE of  $\nu$  has a positive bias on a finite horizon  $T$ . It is given by*

$$\mathbb{E}[\hat{\mu}^N | I_T^A > L] - \mu = \mathbb{E}[\hat{\nu}^N | I_T^Z > 0] - \nu = \frac{2z_0}{T \Pr(I_T^A > L)} \exp\left(\frac{-2z_0\nu}{\sigma^2}\right) \Phi\left(\frac{\nu T - z_0}{\sigma\sqrt{T}}\right),$$

where the survival probability is

$$\Pr(I_T^A > L) = \Pr(I_T^Z > 0) = \Phi\left(\frac{\nu T + z_0}{\sigma\sqrt{T}}\right) - \exp\left(\frac{-2z_0\nu}{\sigma^2}\right) \Phi\left(\frac{\nu T - z_0}{\sigma\sqrt{T}}\right) \quad (2)$$

and  $\Phi$  is the normal c.d.f. (Proof shown in Appendix B.1)  $\square$

Since  $z_0 > 0$ , Proposition 1 shows that ignoring survivorship in the computation of the MLE always overestimates the drift of  $\underline{A}$ . In the context of credit risk modeling, this bias has important repercussions since default probabilities (based on these estimates) would be underestimated. To illustrate the size of this bias numerically, Table 1 shows  $\mathbb{E}[\hat{\mu}^N | I_T^A > L]$  (i.e. the conditional expected value of  $\hat{\mu}^N$ ) for various combinations of  $A_0$ ,  $\mu$  and  $T$ , providing different default probability levels. For instance, over a 1-year horizon, the default probability for the pair ( $\mu = -0.1, A_0 = 110$ ) is 85.34% and it decreases as we go further to the right and/or bottom in each panel of the table. In the most solvent case, the default probability is close to 0%.

**Table 1: Conditional expected value of  $\hat{\mu}^N$  for various values of  $\mu$  and  $A_0$**

$\mu - A_0$	1-year horizon					10-year horizon				
	110	150	200	250	300	110	150	200	250	300
-0.1	0.2739	0.0455	-0.0642	-0.0929	-0.0987	0.1089	0.0791	0.0528	0.0334	0.0183
-0.05	0.2926	0.0733	-0.0231	-0.0452	-0.0492	0.1213	0.0920	0.0668	0.0487	0.0349
0	0.3127	0.1032	0.0198	0.0031	0.0005	0.1370	0.1085	0.0848	0.0685	0.0565
0.05	0.3342	0.1352	0.0643	0.0520	0.0503	0.1567	0.1293	0.1078	0.0938	0.0841
0.1	0.3574	0.1694	0.1101	0.1012	0.1001	0.1812	0.1552	0.1365	0.1254	0.1182
0.15	0.3821	0.2057	0.1569	0.1507	0.1501	0.2108	0.1866	0.1712	0.1631	0.1585
0.2	0.4085	0.2441	0.2047	0.2004	0.2000	0.2454	0.2234	0.2114	0.2061	0.2035
0.25	0.4366	0.2843	0.2531	0.2503	0.2500	0.2843	0.2645	0.2557	0.2525	0.2512
0.3	0.4664	0.3263	0.3020	0.3001	0.3000	0.3265	0.3089	0.3027	0.3010	0.3004

One-year (ten-year) horizon shown in the left (right) panel ( $T = 1, 10$ ). Other parameters:  $\sigma = 0.3$  and  $L = 100$ .

First observe how the survivorship bias translates into a systematic overestimation of the drift. As the default probability increases (decreasing  $\mu$  and/or  $A_0$ ), so does the bias size. Over a short time horizon (columns 2 to 6), the survivorship bias is very small whenever  $A_0$  is very large. This is the case since the survival probability is significantly high for all values of  $\mu$  considered (e.g. with  $A_0 = 300$  and  $\mu = -0.1$  the default probability is 0.13%).

When the time horizon increases (columns 7 to 11), the default probability is driven by the drift of the process. For instance, with  $\mu < \frac{1}{2}\sigma^2$ , the GBM is pulled down over time no matter what is the initial asset value, thus the firm will default with certainty. This behavior is observed in the right panel of Table 1 as the bias on the drift increases even if  $A_0$  is very large. The following proposition characterizes this case.

**Proposition 2** *The NMLE of  $\nu$  is not consistent. Indeed, when  $T \rightarrow \infty$ :*

1.  $\hat{\nu}^N \rightarrow \nu$  if  $\nu \geq 0$ .
2.  $\hat{\nu}^N \rightarrow 0$  if  $\nu < 0$ .

(Proof shown in Appendix B.2)  $\square$

Proposition 2 shows that as  $T \rightarrow \infty$ , the survivorship bias will only disappear when  $\mu \geq \frac{1}{2}\sigma^2$  ( $\nu \geq 0$ ). Otherwise, the GBM is pulled down by the drift but the conditioning event (survivorship) pushes the process upward. These are two opposing forces and the process ultimately approaches a Brownian excursion.

We now illustrate the economic impact of ignoring the survivorship bias. According to a 2015 study by Standard & Poor's,<sup>3</sup> the 1-year default probability of non investment-grade bonds is in the range of 1%

<sup>3</sup>See Table 21 of S&P (2016)



to 30%. Based on values presented in Table 1, if we fix  $A_0 = 150$ , the true default probabilities are in the range of non investment-grade. Let us consider a fictitious company whose asset drift is  $\mu = 0.05$  (or  $\nu = 0.05 - \frac{1}{2}\sigma^2 = 0.005$ ). The true default probability for this firm will be 17%, which corresponds to a rating somewhere between B and CCC. From the table, we see that ignoring survivorship yields an expected drift  $\mu$  of 0.1352, with an associated default probability of 11%. This is an important underestimation of the default probability. This example shows that the survivorship bias on the drift could significantly impact the *perceived* credit quality of a company and eventually tarnish an investor's risk management policy.

Overall, ignoring survivorship can affect the assessment of a company's solvency in a non trivial way, which points out the need to account for this bias in the inference of first-passage structural models.

### 3.1.2 Diffusion coefficient

The NMLE of  $\sigma^2$  corresponds to

$$(\hat{\sigma}^N)^2 = \frac{1}{(n-1)h} \sum_{i=1}^n (r_i - \bar{r})^2 \quad (3)$$

with

$$\bar{r} = \frac{1}{n} \sum_{i=1}^n r_i = \frac{Z_T - z_0}{n}.$$

Directly assessing the bias on this parameter requires the joint distribution of the random variables  $(R_1, R_2, \dots, R_n)$  and the firm's survival over  $[0, T]$ . Since this quantity is difficult to derive analytically, we rely instead on other properties of this estimate to analyze the impact of the survivorship bias. As noted in Merton (1980), increasing the sampling frequency provides more reliable estimates of  $\sigma^2$ . This result hinges on the fact that the quadratic variation of the process converges almost surely, so this estimator is almost surely consistent

$$(\widehat{\sigma^2})^N \rightarrow \sigma^2 \text{ as } h \rightarrow 0$$

for small  $h$ . Thereby, as the observation frequency of the process increases, survivorship considerations become less important for the diffusion coefficient.

To verify this, we perform a simulation exercise in which we compute the NMLE of  $\sigma^2$  for different sample frequencies and sizes. Table 2 shows  $E[(\hat{\sigma}^N)^2 | I_T^Z > 0]$  for  $\sigma^2 = 0.04$  and  $\sigma^2 = 0.16$  at various sampling frequencies (monthly (12), weekly (50) and daily (250)) and over different time horizons (1, 5, 10, and 25 years). All computations are carried out with 10,000 simulations. The initial asset value is set equal to  $A_0 = 110$  to provide high default probabilities and hence a greater potential for survivorship bias.

**Table 2: Conditional expected value of  $(\hat{\sigma}^N)^2$  for  $\sigma^2 = 0.04, 0.16$  at various sampling frequencies (monthly (12), weekly (50) and daily (250)) and over different time horizons (1, 5, 10, 25 years)**

$T$	$\sigma^2 = 0.04$				$\sigma^2 = 0.16$			
	Monthly	Weekly	Daily	Def prob	Monthly	Weekly	Daily	Def prob
1	0.0381	0.0393	0.0399	55.71%	0.1525	0.1566	0.1593	81.65%
5	0.0394	0.0398	0.0400	72.06%	0.1578	0.1594	0.1599	92.05%
10	0.0396	0.0399	0.0400	75.55%	0.1594	0.1595	0.1599	94.54%
25	0.0399	0.0399	0.0400	77.95%	0.1593	0.1598	0.1599	96.75%

Other parameters employed in the simulation are:  $\mu = 0.07, A_0 = 110, L = 100$ . Last column (Def. prob.) gives the probability that the asset crosses the barrier in the interval  $[0, T]$ .

Results in Table 2 show that even if we ignore survivorship when estimating  $\sigma$ , the bias decreases quickly with the sampling frequency. The simulation exercise has been repeated for different values of  $\mu$  (positive or negative) and  $A_0$  yielding similar results. Consequently, our analyses suggest that we can obtain nearly unbiased estimates of  $\sigma^2$  under survivorship by using  $\hat{\sigma}^N$  together with high frequency sampling. Alternatively, the bias can be made very small with weekly or monthly sampling as long as  $T$  increases. For the rest of the paper, we will assume that the survivorship bias on  $\sigma^2$  is small and focus our attention on the MLE of  $\nu$  which has been shown to be much more sensitive to survival.

### 3.2 Conditional MLE

Instead of maximizing the joint *unconditional* density function to obtain the NMLE, we now condition on survival of the company. That is, the conditional MLE (upon survivorship) should be based upon the joint conditional p.d.f.

$$f_{Z_{t_1}, Z_{t_2}, \dots, Z_{t_n} | I_T^Z > 0}(z_1, z_2, \dots, z_n)$$

where  $z_i$  is an observation of the r.v.  $Z_{t_i}$  for  $i = 1, 2, \dots, n$ . The conditional likelihood function is provided in the following proposition.

**Proposition 3** *The conditional likelihood function of the parameter  $\nu$  given that the company survived in the time interval  $[0, T]$  is*

$$\mathcal{L}(\nu) := \frac{1}{\Pr(I_T^Z > 0)} \times \prod_{i=1}^n \frac{1}{\sigma\sqrt{\Delta t}} \phi\left(\frac{z_i - z_{i-1} - \nu\Delta t}{\sigma\sqrt{\Delta t}}\right) \left(1 - \exp\left(\frac{-2z_i z_{i-1}}{\sigma^2 \Delta t}\right)\right) \quad (4)$$

and  $\phi$  is the normal p.d.f. When  $z_n > 0$ , this function converges to 0 when  $\nu \rightarrow \pm\infty$  and therefore  $\nu$  has a finite MLE. (Proofs shown in Appendix C.1 and C.3)  $\square$

We denote by  $\hat{\nu}$  the conditional MLE (CMLE) of  $\nu$  obtained by maximizing the likelihood function given in Equation (4). Since the CMLE of  $\nu$  exists, typical numerical methods can be used to find  $\hat{\nu}$ . Nonetheless, to understand the behavior of this estimate we characterize it with the help of the following proposition.

**Proposition 4** *The CMLE  $\hat{\nu}$  obtained by maximizing Equation (4) is exactly equivalent to solving for  $\nu$  in the expression*

$$z_n = \mathbb{E}[Z_T | I_T^Z > 0]$$

where

$$\mathbb{E}[Z_T | I_T^Z > 0] = z_0 + \nu T + 2z_0 \left( \frac{\Phi\left(\frac{z_0 + \nu T}{\sigma\sqrt{T}}\right)}{\Pr(I_T^Z > 0)} - 1 \right).$$

Moreover,  $Z_T$  is a complete sufficient statistic for the parameter  $\nu$ . (Proof shown in Appendix C.2)  $\square$

Proposition 4 shows that despite having knowledge of the entire path  $\underline{Z}$ , only the initial and final observations  $z_0$  and  $z_n$  matter in determining the CMLE of  $\nu$ , just like in the case of the naive estimator  $\hat{\nu}^N$ . This results in the following undesired properties.

**Proposition 5** *Further properties of the CMLE*

1.  $\mathbb{E}[\hat{\nu} | I_T^Z > 0] - \nu \leq 0$  i.e. the bias of  $\hat{\nu}$  is always negative.
2. When  $z_n \rightarrow 0$ , then  $\hat{\nu} \rightarrow -\infty$ .
3. Moreover,

$$\frac{\mathbb{E}[\hat{\nu} \mathbb{I}_{Z_T < \delta}]}{\nu} \rightarrow \infty$$

and hence

$$\mathbb{E}[\hat{\nu} \mathbb{I}_{Z_T < \delta}] - \nu \rightarrow -\infty.$$

This essentially states that when  $\nu \rightarrow -\infty$ , then the bias in  $\hat{\nu}$  goes to  $-\infty$ .

(Proofs shown in Appendix C.4)  $\square$

A consequence of the first part of Proposition 5 is that correcting for the upward bias in the drift due to survivorship introduces a negative bias in the CMLE. The second and third statement in this proposition provide an explanation about the origin of this negative bias: if a path of a GBM approaches the barrier toward the end of the time interval, then even if  $\nu$  is reasonably high,  $\hat{\nu}$  will tend to  $-\infty$ . Hence, whenever the likelihood of such event is significant, the CMLE will likely introduce a large negative bias.

Table 3 quantifies this negative bias by showing the conditional expected value of  $\hat{\mu}$  for the same combinations of  $\mu$  and  $A_0$  as in Table 1.<sup>4</sup> As it was the case with the NMLE, the default probability is also an important driver of the bias in the CMLE. Over a one-year horizon, we see that the CMLE has an important negative bias that disappears with the solvency of the firm i.e. when we increase  $\mu$  and/or  $A_0$ . When the time horizon is  $T = 10$ , the default probability increases and so does the number of paths approaching the barrier at  $T$  from above. The results of Table 3 illustrate that even in the case of a larger time horizon, the bias still remains significant.

We now assess the economic impact of using the CMLE to correct for the survivorship bias within the context of a fictitious firm. For illustrative purposes, we set again the initial asset value to  $A_0 = 150$  and the drift to  $\mu = 0.05$ . Table 3 shows that the expected CMLE in this case is  $-0.1351$  (recall that the expected NMLE was  $+0.1352$ ). With  $\mu = -0.1351$ , the resulting default probability is 36%, which is more than twice the real default probability of 17%. In this case, using the CMLE grossly overestimates the default probability of the company.

**Table 3: Conditional expected value of  $\hat{\mu}$  for various values of  $\mu$  and  $A_0$**

$\mu - A_0$	1-year horizon					10-year horizon				
	110	150	200	250	300	110	150	200	250	300
-0.1	-0.4668	-0.3767	-0.2300	-0.1479	-0.1164	-0.2883	-0.2862	-0.2815	-0.2758	-0.2697
-0.05	-0.3845	-0.2942	-0.1565	-0.0866	-0.0619	-0.2012	-0.1988	-0.1936	-0.1873	-0.1807
0	-0.3035	-0.2136	-0.0861	-0.0275	-0.0085	-0.1167	-0.1142	-0.1086	-0.1019	-0.0950
0.05	-0.2239	-0.1351	-0.0187	0.0295	0.0440	-0.0360	-0.0334	-0.0276	-0.0209	-0.0142
0.1	-0.1458	-0.0588	0.0460	0.0849	0.0958	0.0400	0.0426	0.0482	0.0544	0.0603
0.15	-0.0693	0.0151	0.1080	0.1390	0.1470	0.1105	0.1129	0.1179	0.1231	0.1277
0.2	0.0055	0.0866	0.1678	0.1921	0.1980	0.1755	0.1775	0.1816	0.1855	0.1887
0.25	0.0786	0.1558	0.2256	0.2444	0.2486	0.2355	0.2370	0.2401	0.2428	0.2448
0.3	0.1498	0.2225	0.2817	0.2960	0.2991	0.2916	0.2928	0.2950	0.2967	0.2978

One-year (ten-year) horizon shown in the left (right) panel ( $T = 1, 10$ ). Other parameters:  $\sigma = 0.3$  and  $L = 100$ .

### 3.3 Debiased conditional MLE

We have shown that while the NMLE is biased upward, the CMLE is biased downward. Typical bias correction methods such as bootstrap or jackknife resampling are not applicable in our context because observations from the conditional stochastic process  $\underline{A}|I_T^A > L$  are not iid. Therefore, we propose a method for debiasing a MLE in non-iid samples based on a series expansion (see Cox and Hinkley (1974), Section 8.4). This technique relies on the following result.

**Proposition 6** *Suppose that  $\hat{\theta}$  is a biased MLE so that we can write*

$$E[\hat{\theta}] = g(\theta) = \theta + \frac{b_1(\theta)}{n} + \frac{b_2(\theta)}{n^2} + \dots$$

*Then the estimator  $\tilde{\theta}$  of  $\theta$  obtained as*

$$\tilde{\theta} = g^{-1}(\hat{\theta})$$

*is first-order unbiased and second-order efficient. (Proof shown in Appendix D)  $\square$*

We can create an unbiased estimate with the following procedure. Let us start by defining

$$\mathcal{E}(\nu) := E[Z_T | I_T^Z > 0]$$

to be some function of  $\nu$ . For some realization  $z_n$  of  $Z_T$ , then from Proposition 4 and 5, the MLE

$$\hat{\nu} = \mathcal{E}^{-1}(z_n)$$

is biased.

<sup>4</sup>Calculations in Table 3 are carried out with numerical integration as explained in computations of Equation (5).

To correct the bias in  $\hat{\nu}$ , we need to compute

$$g(\nu) = \mathbb{E}[\mathcal{E}^{-1}(Z_T)|I_T^Z > 0]. \quad (5)$$

This is commonly done by numerically integrating  $\mathcal{E}^{-1}(Z_T)$  over the entire domain of  $Z_T|I_T^Z > 0$  thus requiring to use a one-variable root-finding algorithm for each element of the integration domain.

From there, we finally find the estimator  $\tilde{\nu}$  by inverting the function  $g$ :

$$\tilde{\nu} = g^{-1}(\hat{\nu}).$$

In practice, one needs to reliably compute  $g(\nu)$  for many values of  $\nu$  before attaining convergence.

We conduct numerical tests to study the effectiveness of the proposed debiasing method. For that purpose, we compare the (conditional) expected NMLE, CMLE and debiased CMLE with the true value of  $\mu$  for various pairs of  $(A_0, \mu)$ . The expected debiased CMLE has been computed by simulation<sup>5</sup> with 20,000 realizations of  $Z_T|I_T^Z > 0$ . The results are shown in Table 4.

**Table 4: Conditional expected value of  $\hat{\mu}^N$  (NMLE),  $\hat{\mu}$  (CMLE) and  $\tilde{\mu}$  (DebCMLE)**

$A_0$	$\mu$	1-year horizon				10-year horizon			
		NMLE	CMLE	DebCMLE	Def prob	NMLE	CMLE	DebCMLE	Def prob
110	-0.1	0.2739	-0.4668	-0.0922	0.8534	0.1089	-0.2883	-0.1004	0.9936
200	-0.1	-0.0642	-0.2300	-0.0574	0.0581	0.0528	-0.2815	-0.1016	0.8989
300	-0.1	-0.0987	-0.1164	-0.0795	0.0013	0.0183	-0.2697	-0.0996	0.7689
110	0	0.3127	-0.3035	0.0209	0.7854	0.1370	-0.1167	0.0097	0.9567
200	0	0.0198	-0.0861	0.0325	0.0292	0.0848	-0.1086	0.0040	0.6271
300	0	0.0005	-0.0085	0.0144	0.0004	0.0565	-0.0950	0.0070	0.4010
110	0.1	0.3574	-0.1458	0.1203	0.7056	0.1812	0.0400	0.1099	0.8571
200	0.1	0.1101	0.0460	0.1321	0.0135	0.1365	0.0482	0.1097	0.2836
300	0.1	0.1001	0.0958	0.1091	0.0001	0.1182	0.0603	0.1095	0.1146
110	0.2	0.4085	0.0055	0.2370	0.6189	0.2454	0.1755	0.2078	0.7165
200	0.2	0.2047	0.1678	0.2218	0.0057	0.2114	0.1816	0.2087	0.0841
300	0.2	0.2000	0.1980	0.2094	0.0000	0.2035	0.1887	0.2079	0.0181
110	0.3	0.4664	0.1498	0.3347	0.5309	0.3265	0.2916	0.3060	0.5825
200	0.3	0.3020	0.2817	0.3213	0.0022	0.3027	0.2950	0.3040	0.0195
300	0.3	0.3000	0.2991	0.3031	0.0000	0.3004	0.2978	0.3031	0.0019

One-year (ten-year) horizon shown in the left (right) panel. Other parameters:  $\sigma = 0.3, L = 100$ . “Def prob” is short for default probability.

In general, the debiasing procedure corrects the upward bias in  $\hat{\mu}^N$  and the downward bias in  $\hat{\mu}$  that we originally obtained in a finite sample. More interestingly, the debiased CMLE works best when the default probability is large which is exactly when the survival bias matters the most. Over large time horizons, the debiasing is also very accurate to the contrary of the NMLE or CMLE. Finally, these results illustrate numerically the statistical properties of the method — namely it is first-order unbiased and second-order efficient.

## 4 Credit risk with survivorship

There are generally two classes of default triggering mechanisms in structural credit risk models. The first one, present in Merton-like models, assumes that default only occurs at maturity if  $A_T$  is insufficient to repay the debt. The second mechanism, associated to first-passage models, supposes that default occurs when  $A_t$  attains a solvency barrier  $L$ . By construction, only the second type of model is subject to the survivorship bias.

<sup>5</sup>Since the debiased CMLE only depends upon  $z_0$  and  $z_n$ , there is no need to sample an entire path. We refer the reader to Appendix A.3 for details about the simulation algorithm.

A structural credit risk model specifies the capital structure of the firm, linking the firm's asset value to its equity. In such model, equity represents an option on the firm's assets. Whereas Merton-like models rely on a European call option to link equity values to the firm's asset prices, first-passage models do so with down-and-out call options.

One difficulty that arises with structural models is the inability of an investor to observe the market value of a firm's assets. Therefore, inference of structural models is often carried out with stock prices rather than directly from the firm's assets.

The first part of this section presents the credit risk model that we use in this paper. The second part explains parameter estimation from stock prices using an extended version of Vassalou and Xing (2004) that accounts for survivorship.

## 4.1 Model specification

Brockman & Turtle (2003) value the outstanding equity of a firm in a framework where  $\underline{A}$  is a GBM and default is triggered as soon as  $A_t$  attains a solvency barrier  $L > A_0$  (as defined in Black & Cox (1976)). They assume that at each point in time, the firm's capital structure is composed of equity  $S$  and liabilities  $D$ . Liabilities consist of not only corporate bonds but also of any other form of payable account. If the firm defaults, all of the asset value is transferred to debt holders. Second, it is assumed that the barrier  $L$  is equal to the nominal value of liabilities, thus requiring surviving firms to be able to fulfill all their obligations during the observation period.

With a constant risk-free rate  $r$  and using standard hedging arguments, the value of a unit of equity corresponds to the value of a down-and-out call option on the firm's assets. This option has a strike price equal to the total liability value  $D$ , a knock-out barrier of  $L$ , and a maturity equal to  $T$ . Thus, within this framework, the value of equity is:

$$\begin{aligned} S_t &:= \mathcal{S}(A_t; \theta) \\ &= A_t \Phi(a_t) - D e^{-r(T-t)} \Phi\left(a_t - \sigma \sqrt{T-t}\right) \\ &\quad - A_t \left(\frac{L}{A_t}\right)^{2\eta} \Phi(b_t) \\ &\quad + D e^{-r(T-t)} \left(\frac{L}{A_t}\right)^{2\eta-2} \Phi\left(b_t - \sigma \sqrt{T-t}\right), \end{aligned} \tag{6}$$

where

$$\begin{aligned} a_t &= \frac{\ln A_t - \ln D + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}}, \\ b_t &= \frac{2 \ln L - \ln A_t - \ln D + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}}, \\ \eta &= \frac{1}{\sigma^2} \left(r + \frac{\sigma^2}{2}\right). \end{aligned}$$

Note that the first two terms of Equation (6) correspond to the equity value in the Merton (1974) model. The third and fourth terms, which depend on the debt ratio  $L/A_t$ , can be viewed as terms that correct Merton's equity value for early default.

## 4.2 Parameter estimation

Assume now that  $\underline{A}$  is not directly observed but as an investor, we only observe  $\underline{S} = \{S_t, t \geq 0\}$  at discrete time points  $0 = t_0 < t_1 < t_2 < \dots < t_n = T$  with  $h = t_i - t_{i-1} = T/n, \forall i = 1, 2, \dots, n$ . Provided that we know all model parameters, then  $S_t = \mathcal{S}(A_t)$  is a monotone increasing function of  $A_t$ . To recover  $A_t$ , it only suffices to invert  $\mathcal{S}$ . As expected,  $\underline{S}$  is affected by the survivorship bias as well due to  $\underline{A}$ .

Suppose that  $D, L, T$  and  $r$  are known so that the goal is to use observed equity prices  $S_0, S_{t_1}, S_{t_2}, \dots, S_{t_n}$  to estimate  $\mu$  and  $\sigma$ . We follow Vassalou and Xing (2004) and Bharath and Shumway (2008) and adopt a two-step approach to estimate these parameters.

First of all, we know from Section 3.1 that  $\sigma$  is practically unaffected by the survivorship bias. Moreover, we know from Duan et al. (2005) that the MLE of Duan (1994) and the iterative procedure of Vassalou and Xing (2004) are equivalent. Therefore, even in the presence of survivorship, we can determine  $\sigma$  using the exact same iterative procedure described by Vassalou and Xing (2004).

For the purpose of the method, let  $\hat{\sigma}^{(k)}$  be the estimate of  $\sigma$  at step  $k$  and let  $\underline{A}^{(k)}$  be the asset value process such that  $A_t = \mathcal{S}^{-1}(S_t; \hat{\sigma}^{(k)})$  i.e. the asset value obtained by inverting the equity price function using  $\hat{\sigma}^{(k)}$ . The algorithm goes as follows:

1. Using an initial estimate  $\hat{\sigma}^{(0)}$  and observed equity prices  $\underline{S}$ , we back out a time series of asset values  $\underline{A}^{(0)}$ .
2. Then,  $\hat{\sigma}^{(1)}$  is computed as the MLE of  $\sigma$  for a GBM having observations  $\underline{A}^{(0)}$  (see Equation 3).
3. Using the estimate  $\hat{\sigma}^{(1)}$  and the same observed equity prices  $\underline{S}$ , we back out another time series of asset values  $\underline{A}^{(1)}$ .
4. Steps 2 and 3 are repeated until convergence.

The procedure stops when  $|\hat{\sigma}^{(k)} - \hat{\sigma}^{(k-1)}| < \epsilon$  for  $\epsilon$  small. Moreover, no estimate of  $\mu$  is necessary at this point because the equity price is independent from that parameter (equity pricing is done under the risk-neutral measure).

Once an estimate of  $\sigma$  is obtained at the  $k$ th iteration, we work with the time series of  $\underline{A}^{(k)}$  to estimate  $\mu$ . Given that  $\mu$  is affected by the survivorship bias, we apply the debiased MLE introduced in Section 3.3.<sup>6</sup>

## 5 Empirical study

The analyses presented in Section 3 have shown the existence of a bias in the estimation of the parameter  $\mu$ . In this section, we assess the economic impact of ignoring or miscorrecting for the survivorship bias when conducting inference of a first-passage structural credit risk model. To this end, we will use the NMLE, the CMLE and the debiased CMLE to assess the credit risk of thousands of companies. We first present the data employed throughout our study and then we investigate several aspects of the empirical differences tied to each of the three estimates of  $\mu$ .

### 5.1 Data, assumptions, and methodology

Our dataset is composed of all firms in the intersection of CRSP<sup>7</sup> and Compustat between January 1979 and December 2016. Sample filters and model inputs are defined following Bharath and Shumway (2008). First, financial companies identified with CUSIP codes 6021, 6022, 6029, 6035, and 6036 are excluded from the sample. Second, model inputs such as face value of debt ( $D$ ) and barrier ( $L$ ) are obtained from Compustat annual data. Debt's face value  $D$  is defined as "Debt in one year" plus half of "Long-term debt" fields in Compustat. We also use this value for the barrier  $L$  because it captures the ability of a firm to service long- and short-term debt, thus indicating the firm's survival capability in the short term. The debt maturity  $T$  is set to 1 year.

The daily equity value is calculated as the price per share from CRSP multiplied by the total number of shares outstanding. Regarding the risk-free rate  $r$ , we employ a monthly time series of 1-year Treasury maturity rates provided by the Board of Governors of the Federal Reserve system.<sup>8</sup>

Our study is conducted as follows. For each firm and month in our sample, we extract the previous 12 months of daily equity values along with the current estimate of the debt's face value, barrier, and risk-free rate to compute estimates of  $\sigma$  and  $\mu$  as discussed in Section 4.2. In the iterative procedure of Section 4.2, we suppose the algorithm has attained convergence when  $\epsilon = 10^{-4}$  with  $\hat{\sigma}^{(0)}$  computed as the sample standard

<sup>6</sup>In the empirical section we also employ the NMLE and CMLE to assess the bias of these estimates.

<sup>7</sup>Center for Research in Security Prices

<sup>8</sup>Available at <http://research.stlouisfed.org/fred/data/irates/gsl>

deviation of daily log equity returns. Moreover, we discard a month if a) there are less than 200 daily equity observations or b) the value of assets was below the debt at any point in the twelve month period. Our final sample consists of 13,794 firms and a total number of 1,231,167 firm-months observations with complete data.

## 5.2 Distribution of $\mu$

We seek to compare the NMLE, CMLE, and debiased CMLE of  $\mu$  across companies. For each given company in the sample, we first compute the median estimate of  $\mu$  over time. Then, we use these estimates to compute percentiles across all firms in the sample for each of the methods under consideration. By comparing different percentiles across companies, we are able to better characterize the bias across companies with different solvency profiles. To conduct statistical analyses, we also report confidence intervals for percentiles.

Table 5 reports percentiles across companies for the estimates of  $\mu$  based upon the NMLE, CMLE and debiased CMLE (DebCMLE). For both the CMLE and NMLE estimates, there is compelling empirical evidence of an important difference between estimates: percentiles are systematically below or above from those computed with DebCMLE. The strongest results are observed for the CMLE, in which all percentiles are statistically below debiased estimates.

Although clearly above the DebCMLE for most of the percentiles, the NMLE is not significantly different from the DebCMLE for large values of  $\mu$ . This result hinges on the fact that for large values of  $\mu$ , the default probability is small and so is the resulting survivorship bias; the correction needed in this case is insignificant. Nonetheless, the CMLE in these cases still exhibits a significant downward bias. Overall, these results suggest that the biases documented in Section 3 are present empirically and can lead to significant differences in the estimates of the drift and, in turn, of the default probability.

**Table 5: Estimates of  $\mu$  (in %) using different estimation procedures**

Percentile	DebCMLE			CMLE			NMLE		
	LB	$\mu$ (%)	UB	LB	$\mu$ (%)	UB	LB	$\mu$ (%)	UB
P10	-6.84	-6.19	-5.35	-19.64	-18.38	-17.17	0.33	0.88	1.30
P25	5.80	6.12	6.53	1.11	1.60	2.11	9.66	9.94	10.23
P50	16.45	16.82	17.12	12.98	13.37	13.76	19.65	19.99	20.35
P75	34.35	35.32	36.46	28.68	29.48	30.18	36.15	37.00	37.77
P90	66.99	69.24	71.44	56.34	57.73	59.47	64.59	66.73	68.84

For each company in the sample, the median estimate of  $\mu$  is computed over time. This table reports percentiles of the latter across companies for the NMLE, CMLE and debiased CMLE methods. Columns DebCMLE, CMLE and NMLE represents estimates of  $\mu$  based upon the debiased conditional MLE, conditional MLE and naive MLE respectively. The lower and upper bounds for the 95% confidence intervals are reported as LB and UB. The sample spans 1979 through 2016 and contains 13,794 firms.

## 5.3 Risk profiles

The previous analysis has shown that for most companies, there can be important differences in estimates of  $\mu$  across the three methods. We now want to study how these differences impact the risk profile of each company. To this end, we group firms according to their default probability (which depends directly on estimates of  $\mu$ ) and compare several characteristics (such as asset volatility, leverage, etc.) across these groups. If all estimation methods produce the same risk profiles, then a group's characteristics will be the same for each of the three methods.

The creation of groups and computation of their characteristics is done as follows. For each estimation method, we compute the 1-year default probability for each month in the sample using Equation 2. Next, we compute the median of this monthly series across time. In a similar fashion, firm characteristics are computed using median values of the following monthly series:

- Asset return  $\mu$ ;
- Asset return volatility  $\sigma$ ;
- Leverage ratio (defined as the ratio between the face value of debt ( $D$ ) and the asset value ( $A_t$ ));<sup>9</sup>
- Risk-adjusted measure of performance defined as the ratio of  $\mu$  over  $\sigma$ ;

<sup>9</sup>We use the last asset value of  $\underline{A}^{(k)}$  in the iterative procedure of Section 4.2.

Companies are then sorted by deciles according to their median default probability (over months). Then, a given characteristic for a decile is calculated as the median of the characteristic across companies within that decile.

Table 6 shows several company characteristics by decile group. In the table, firms in decile 1 have the lowest default probability estimates whereas firms in decile 10 have the largest. Observe that as default probabilities increase (from decile 1 to 10), the leverage and asset volatility increase as well. Given that we observe these monotonic relationships across all three methods, we can conclude that the level and not the ordering of risk profiles is impacted when estimates of  $\mu$  do not properly account for the survivorship bias. When we look at risk-adjusted asset returns (columns Performance), it becomes clear that controlling for the survivorship bias is important in the analysis of companies with high degree of default risk (deciles 6 to 10). For example in the last decile, the risk-adjusted return is still positive when survivorship is ignored whereas it is largely negative when the survivorship bias is miscorrected with the CMLE. Using the debiased CMLE, it is true that these firms underperform but clearly not at the extent suggested by the CMLE.

**Table 6: Parameter characteristics by default probability deciles**

Decile	$\mu$ (%)			$\sigma$ (%)			Leverage(%)			Performance		
	DebCMLE	CMLE	NMLE	DebCMLE	CMLE	NMLE	DebCMLE	CMLE	NMLE	DebCMLE	CMLE	NMLE
1	19.08	18.88	19.59	34.19	34.19	34.19	1.79	1.79	1.79	0.62	0.61	0.62
2	18.58	17.90	19.37	35.80	35.80	35.80	5.84	5.84	5.84	0.59	0.57	0.61
3	19.66	18.82	21.21	37.64	37.64	37.64	8.52	8.52	8.52	0.58	0.55	0.60
4	19.73	18.36	21.40	40.63	40.63	40.63	10.06	10.06	10.06	0.54	0.50	0.59
5	17.73	16.31	21.13	41.50	41.92	38.47	13.11	13.04	14.96	0.48	0.44	0.60
6	17.14	14.81	22.34	45.25	45.20	45.67	15.39	15.30	15.09	0.46	0.40	0.57
7	18.05	13.78	24.02	48.46	48.67	48.17	18.35	18.13	18.48	0.45	0.35	0.59
8	15.19	8.04	21.27	54.42	54.55	54.12	21.71	21.24	21.76	0.36	0.22	0.49
9	11.44	-0.01	16.82	57.76	58.99	59.15	28.07	27.14	27.02	0.25	0.00	0.35
10	-3.47	-33.25	7.41	66.97	63.11	69.79	39.63	41.66	37.61	-0.09	-0.67	0.13

Firms are ordered by default probabilities and grouped by deciles. For each given company and estimation method, the median 1-year default probability is computed over time and the same goes for various firm characteristics such as asset return ( $\mu$ ), asset return volatility ( $\sigma$ ), leverage and the risk-adjusted performance of the firm's assets. Values reported in this table are median risk characteristic computed within each group. Our sample spans 1979 through 2016 and contains 13,794 firms.

## 5.4 Effect on default probabilities

We now explore the survivorship bias effect on the default probability of a firm over time. Specifically, we focus on risky companies across recession periods (NBER business cycle contraction dates). Each month, we compute the 90th percentile of the default probability across firms. We look at the time series of this quantity to determine how the bias shifts the tails of the distribution in periods of high economic uncertainty.

Each panel of Figure 1 displays these percentiles for four recessions: 1981/07-1982/11, 1990/07-1991/03, 2001/03-2001/11 (burst of the dot-com bubble) and 2007/12-2009/06 (Great Recession). Due to the overestimation (underestimation) of  $\mu$  with the NMLE (CMLE), we observe that the default probability is systematically lower (higher) than the debiased estimates.

The overall picture that emerges from these time series is that the solvency of the riskiest companies (as measured by the 90th percentile) is largely affected by ignoring or miscorrecting for the survivorship bias. At the worst of the 1990-1991 recession, the 90th quantile of the default probability computed with the DebCMLE is 60%, which is largely overstated by the CMLE (85%) and significantly understated by the NMLE (45%). Similar differences are observed at the peak of the recent financial crisis — around March 2009. In brief, the gap in these default probabilities varies over time and increases in periods of recession.

## 5.5 Misclassification of companies

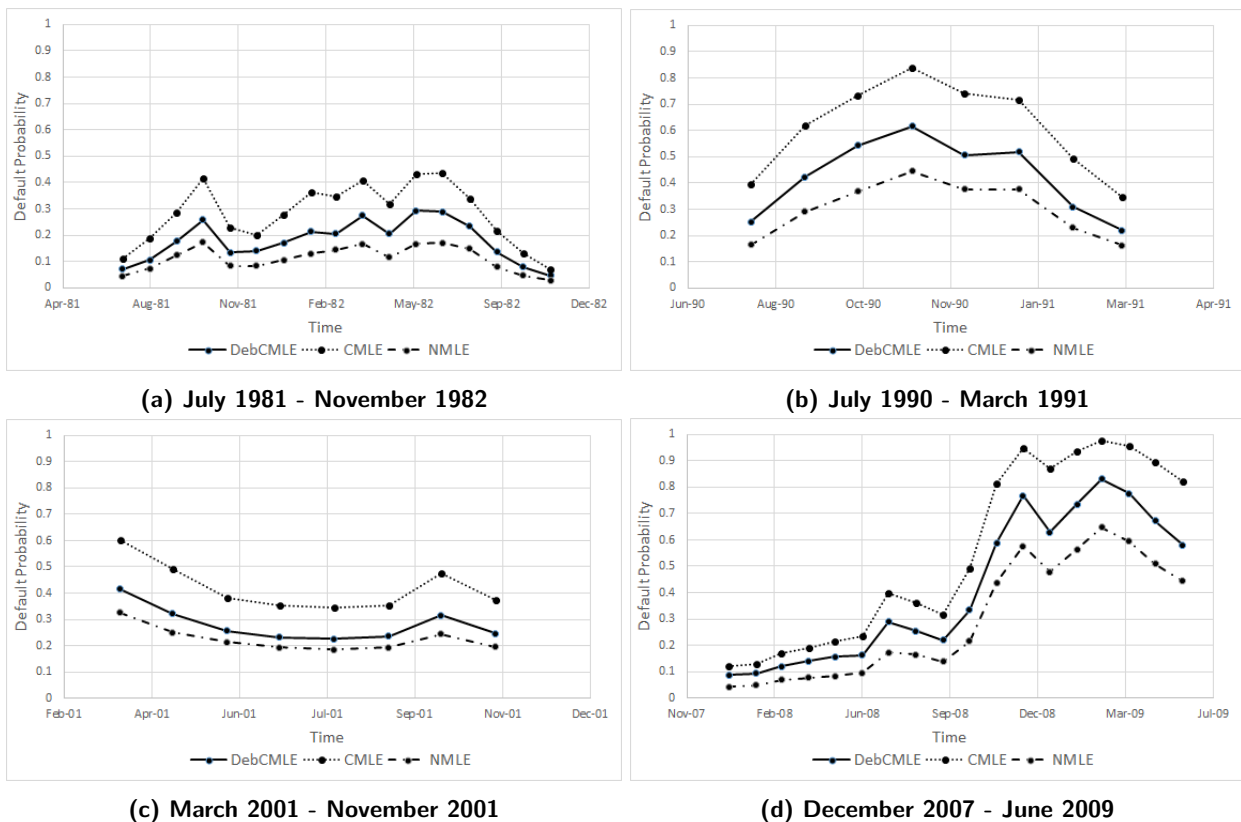
From a risk management perspective, it is important to quantify the proportion of companies over- or under-classified as risky ones. Every year and for each estimation method, we compute the proportion of firms with an average 1-year default probability above a predefined threshold value. Using the debiased CMLE of  $\mu$ ,



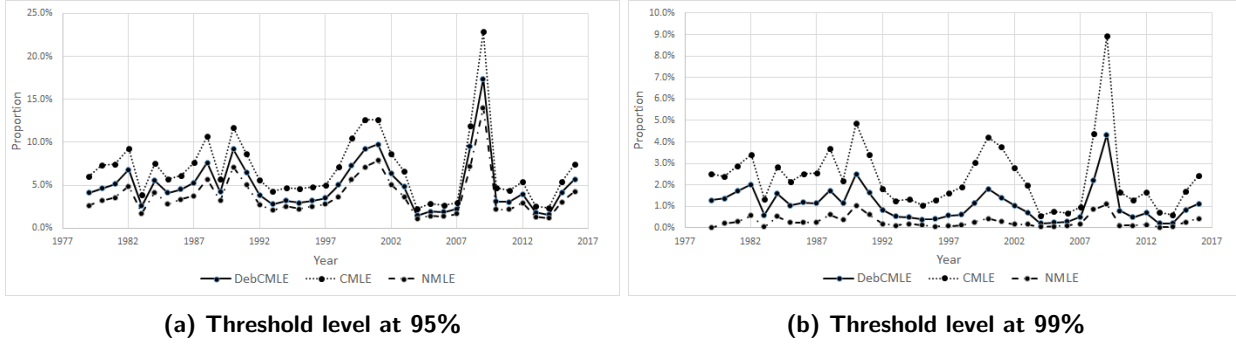
this threshold is set to the 95-th and 99-th percentile of the default probability across all companies and months in the sample. Figure 2 displays the annual proportion of companies having a default probability higher than this threshold.

Panels (a) and (b) show that these proportions have large variability over time with important spikes during economic recessions, in line with our previous results. More importantly, the proportion of misclassified risky firms can be systematically different depending on the estimation method: CMLE provides a larger proportion of risky companies whereas the NMLE a lower number. Given the previous evidence, these differences should not be surprising; however, they show the economic value of accounting for the survivorship bias.

For instance, in Panel A (95% threshold) during the year 2000 the NMLE classifies 7.1% of companies as risky, while the CMLE does so for 12.7% of the sample. In contrast, the proportion that comes from the debiased procedure is 9.2%. These differences become more important when looking at very risky companies (Panel B of in Figure 2 i.e. 99% threshold), in which case the proportions can differ more substantially. At the worst of the Great Recession, the estimate of the true proportion of firms above the 99% threshold is about 5% with the debiasing procedure. In contrast, only 1% of companies are classified as very risky when survivorship is ignored, and 9% receive this classification with the CMLE.



**Figure 1: Evolution of the default probability of the 90-th percentile company across NBER recession periods** This figure shows the monthly 90-th percentile of the default probability across firms using three different estimates of  $\mu$ . DebCMLE represents the estimate of  $\mu$  computed with the debiased conditional MLE. CMLE stands for the conditional MLE of  $\mu$  whereas NMLE is the estimate of  $\mu$  that ignores survivorship. Recession periods, identified below each graph, are defined by NBER as periods of business cycle contractions.



**Figure 2: Proportion of riskiest firms according to the estimate employed for  $\mu$**  This figure shows the proportion of companies with annual default probabilities higher than a given threshold. The threshold is set to the 95-th or 99-th percentile of the default probability across all companies and months computed with the debiased CMLE. With default probabilities computed across firms and methods, each panel shows the proportion of firms having a default probability higher than the threshold for each of the three estimation methods. DebCMLE represents the estimate of  $\mu$  computed with the debiased conditional MLE. CMLE stands for the conditional MLE of  $\mu$  whereas NMLE is the estimate of  $\mu$  that ignores survivorship. The sample spans 1979 through 2016 and contains 13,794 firms.

## 6 Conclusion

In this article, we have studied the maximum likelihood estimation of the GBM conditional on survival. Our results show the existence of an important bias in the drift when (1) we ignore survival of the firm (naive MLE) or (2) when we condition on survival without correcting for the large negative finite sample bias (conditional MLE). Therefore, we proposed a debiasing procedure that is first-order unbiased, second-order efficient, and holds for non-iid samples.

We found that the theoretical biases in the drift translate empirically and economically into important biases in default probabilities. Identifying the proportion of risky companies is an important task for many financial institutions since capital requirements depend on these classifications. Ignoring or miscorrecting for the survivorship bias thus has significant economic consequences for institutional investors and stresses the importance of correctly estimating  $\mu$  for risk management purposes. The fact that these biases expand during periods of financial distress is relevant for internal risk assessment of solvency models.

## Appendices and supplementary material

### Maximum likelihood estimation of first-passage structural credit risk models correcting for the survivorship bias

#### A Conditional distribution upon survival

An important quantity in many derivations is the distribution of  $Z_T$  conditional upon survivorship, that is  $f_{Z_T|I_T^Z > 0}(z)$ . From the joint law of  $(Z_T, I_T^Z)$ , we directly deduce that

$$f_{Z_T|I_T^Z > 0}(z) = \frac{\frac{1}{\sigma\sqrt{T}}\phi\left(\frac{z-z_0-\nu T}{\sigma\sqrt{T}}\right)\left(1 - \exp\left(\frac{-2zz_0}{T\sigma^2}\right)\right)}{\Pr(I_T^Z > 0)} \quad (\text{A1})$$

with  $z_0$  known (see e.g. Musiela & Rutkowski (2005)). One alternative derivation is to rewrite

$$f_{Z_T|I_T^Z > 0}(z) = \frac{\Pr(I_T^Z > 0|Z_T)f_{Z_T}(z)}{\Pr(I_T^Z > 0)} = \frac{\left(1 - \exp\left(\frac{-2zz_0}{T\sigma^2}\right)\right) \times \frac{1}{\sigma\sqrt{T}}\phi\left(\frac{z-z_0-\nu T}{\sigma\sqrt{T}}\right)}{\Pr(I_T^Z > 0)}$$

where the first term is obtained from the distribution of the infimum of a Brownian bridge (see e.g. Karatzas & Shreve (1991)) and the second is simply the p.d.f. of a normal distribution with mean  $\nu T$  and variance  $\sigma^2 T$ .

## A.1 Three-parameter family

Let  $h(z; m, s, \lambda)$  be the p.d.f. of a three-parameter family of distributions determined by

$$h(z; m, s, \lambda) := \frac{1 - e^{-\lambda z}}{s \times \kappa(m, s, \lambda)} \phi\left(\frac{z - m}{s}\right) \quad (\text{A2})$$

where  $\phi$  is the standard normal p.d.f. and

$$\kappa(m, s, \lambda) := \Phi\left(\frac{m}{s}\right) - \exp\left(-\lambda m + \frac{1}{2}\lambda^2 s^2\right) \Phi\left(\frac{m}{s} - \lambda s\right).$$

Then, the p.d.f. of Equation A1 is tied to the p.d.f. of Equation A2. Indeed, we only need to let

$$\begin{aligned} m &= z_0 + \nu T \\ s &= \sigma\sqrt{T} \\ \lambda &= \frac{2z_0}{\sigma^2 T} \end{aligned}$$

to recover the former p.d.f.

Moreover  $\kappa(m, s, \lambda)$  is the survival probability  $\Pr(I_T^Z > 0)$  over the time interval  $[0, T]$  (see e.g. Musiela & Rutkowski (2005)). Indeed it is the probability of survival for a Brownian motion on an interval  $[0, 1]$  having initial value  $z_0 = \frac{\lambda s^2}{2}$ , drift  $\nu = m - \frac{\lambda s^2}{2}$  and diffusion  $\sigma = s$ .

For some proofs, we will also work with an alternative representation of the survival probability, given by

$$k\left(\frac{\nu\sqrt{T}}{\sigma}, \frac{z_0}{\sigma\sqrt{T}}\right) = k\left(m - \frac{\lambda}{2}, \frac{\lambda}{2}\right) = \Phi(m) - e^{2\frac{\lambda}{2}(m - \frac{\lambda}{2})} \Phi(m - \lambda) = \kappa(m, 1, \lambda)$$

where

$$k(x, \delta) = \Phi(x + \delta) - \exp(-2x\delta)\Phi(x - \delta).$$

## A.2 Moments

Let  $Z$  be a random variable whose p.d.f. is given by Equation A2. The first two moments of  $Z$  can be easily obtained from the properties of the score function. The score function with respect to  $m$  is

$$\begin{aligned} \frac{\partial}{\partial m} \ln h(z; m, s, \lambda) &= \frac{1}{s^2} (z - m) - \frac{\partial}{\partial m} \ln(\kappa(m, s, \lambda)) \\ &= \frac{1}{s^2} (z - m) - \frac{1}{\kappa} \left[ \frac{1}{s} \phi\left(\frac{m}{s}\right) - \frac{1}{s} e^{-\lambda m + \lambda^2 s^2 / 2} \phi\left(\frac{m}{s} - \lambda s\right) + \lambda e^{-\lambda m + \lambda^2 s^2 / 2} \Phi\left(\frac{m}{s} - \lambda s\right) \right] \\ &= \frac{1}{s^2} (z - m) - \frac{\lambda}{\kappa} \left[ e^{-\lambda m + \lambda^2 s^2 / 2} \Phi\left(\frac{m}{s} - \lambda s\right) \right] \end{aligned}$$

where  $\kappa$  is short for  $\kappa(m, s, \lambda)$ . Since the expected value of the score function is zero, we obtain

$$\mathbb{E}[Z - m] = \frac{\lambda s^2}{\kappa} \left[ e^{-\lambda m + \lambda^2 s^2 / 2} \Phi\left(\frac{m}{s} - \lambda s\right) \right]$$

and thus

$$\mathbb{E}[Z_T | I_T^Z > 0] = m + \frac{s^2 \lambda}{\kappa(m, s, \lambda)} \exp\left(-\lambda m + \frac{1}{2}\lambda^2 s^2\right) \Phi\left(\frac{m}{s} - \lambda s\right). \quad (\text{A3})$$

The score function with respect to  $s$  is  $\frac{\partial}{\partial s} \ln h(z; m, s, \lambda)$  which is derived as

$$\begin{aligned}
&= \frac{1}{s^3} (z - m)^2 - \frac{\partial}{\partial s} \ln(\kappa(m, s, \lambda)) - \frac{\partial}{\partial s} \ln(s) \\
&= \frac{1}{s^3} (z - m)^2 - \frac{1}{\kappa} \left[ \frac{\partial}{\partial s} \left( \Phi\left(\frac{m}{s}\right) - e^{-\lambda m + \lambda^2 s^2/2} \Phi\left(\frac{m}{s} - \lambda s\right) \right) \right] - \frac{1}{s} \\
&= \frac{1}{s^3} (z - m)^2 - \frac{1}{\kappa} \left[ \left(-\frac{m}{s^2}\right) \phi\left(\frac{m}{s}\right) - p \left(-\frac{m}{s^2} - \lambda\right) \phi\left(\frac{m}{s} - \lambda s\right) - \lambda^2 s p \Phi\left(\frac{m}{s} - \lambda s\right) \right] - \frac{1}{s} \\
&= \frac{1}{s^3} \left[ (z - m)^2 - s^2 + \frac{ms\phi\left(\frac{m}{s}\right) - p(ms + \lambda s^3)\phi\left(\frac{m}{s} - \lambda s\right) + \lambda^2 s^4 p \Phi\left(\frac{m}{s} - \lambda s\right)}{\kappa} \right]
\end{aligned}$$

where

$$p = e^{-\lambda m + \lambda^2 s^2/2}.$$

Again, since the expected value is zero, we obtain

$$\mathbb{E} \left[ \left( \frac{Z - m}{s} \right)^2 \right] = 1 - \frac{\frac{m}{s}\phi\left(\frac{m}{s}\right) - p\left(\frac{m}{s} + \lambda s\right)\phi\left(\frac{m}{s} - \lambda s\right) + \lambda^2 s^2 p \Phi\left(\frac{m}{s} - \lambda s\right)}{\kappa}.$$

Now recall that

$$\begin{aligned}
p\phi\left(\frac{m}{s} - \lambda s\right) &= e^{-\lambda m + \lambda^2 s^2/2} \phi\left(\frac{m}{s} - \lambda s\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\lambda m + \lambda^2 s^2/2 - \frac{1}{2}\left(\frac{m}{s} - \lambda s\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\lambda m - \frac{1}{2}\left(\frac{m}{s}\right)^2 + \lambda m\right) = \phi\left(\frac{m}{s}\right)
\end{aligned}$$

so this becomes

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{Z - m}{s} \right)^2 \right] &= 1 - \frac{\frac{m}{s}\phi\left(\frac{m}{s}\right) - \left(\frac{m}{s} + \lambda s\right)\phi\left(\frac{m}{s}\right) + \lambda^2 s^2 p \Phi\left(\frac{m}{s} - \lambda s\right)}{\kappa} \\
&= 1 + \frac{(\lambda s)\phi\left(\frac{m}{s}\right) - \lambda^2 s^2 p \Phi\left(\frac{m}{s} - \lambda s\right)}{\kappa} \\
&= 1 + \frac{\lambda s}{\kappa} \left( \phi\left(\frac{m}{s}\right) - \lambda p s \Phi\left(\frac{m}{s} - \lambda s\right) \right)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{1}{s^2} \text{Var}[Z - m] &= \left( 1 + \frac{\lambda s}{\kappa} \left( \phi\left(\frac{m}{s}\right) - \lambda p s \Phi\left(\frac{m}{s} - \lambda s\right) \right) \right)^2 - \left[ \frac{\lambda s^2}{\kappa} \left[ p \Phi\left(\frac{m}{s} - \lambda s\right) \right] \right]^2 \\
&= 1 + \frac{\lambda s}{\kappa} \phi\left(\frac{m}{s}\right) - \frac{(\lambda s)^2 p \kappa}{\kappa^2} \Phi\left(\frac{m}{s} - \lambda s\right) - \frac{(\lambda s)^2 p^2}{\kappa^2} \Phi^2\left(\frac{m}{s} - \lambda s\right) \\
&= 1 + \frac{\lambda s}{\kappa} \phi\left(\frac{m}{s}\right) - \frac{(\lambda s)^2 p \left[ \Phi\left(\frac{m}{s}\right) - p \Phi\left(\frac{m}{s} - \lambda s\right) \right] \Phi\left(\frac{m}{s} - \lambda s\right)}{\kappa^2} - \frac{(\lambda s)^2 p^2}{\kappa^2} \Phi^2\left(\frac{m}{s} - \lambda s\right) \\
&= 1 + \frac{\lambda s}{\kappa} \phi\left(\frac{m}{s}\right) - \frac{(\lambda s)^2 p \Phi\left(\frac{m}{s}\right) \Phi\left(\frac{m}{s} - \lambda s\right)}{\kappa^2}.
\end{aligned}$$

Finally,

$$\text{Var}[Z_T | I_T^Z > 0] = s^2 \left( 1 + \frac{\lambda s \times b(m, s, \lambda)}{\kappa(m, s, \lambda)} \right) \quad (\text{A4})$$

where

$$b(m, s, \lambda) = \phi\left(\frac{m}{s}\right) - \frac{\lambda s}{\kappa(m, s, \lambda)} \Phi\left(\frac{m}{s}\right) \Phi\left(\frac{m}{s} - \lambda s\right) \exp\left(-\lambda m + \frac{1}{2}\lambda^2 s^2\right).$$

### A.3 Simulation algorithm

One realization of  $Z_T | I_T^Z > 0$  can be simulated as follows (see e.g. Glasserman & Staum (2001)):

1. Simulate  $Z_T | Z_0 = z_0$  using a normal distribution with mean  $\nu T$  and variance  $\sigma^2 T$ ;
2. Simulate  $U$  a random uniform number. Then a realization of  $I_T^Z | Z_T, Z_0$  can be obtained as

$$I = \exp\left(\frac{1}{2}\left(Z_0 + Z_T + 2 \ln L - \sqrt{(Z_0 - Z_T)^2 - 2\sigma^2 T \ln U}\right)\right); \quad (\text{A5})$$

3. If  $I > 0$ , then keep  $Z_T$ . Otherwise, go back to step 1.

When needed, an entire path of  $\underline{Z}$  can be easily simulated by sampling each increment with the latter algorithm.

A quicker method to simulate a path using a vector-oriented programming language is to sample a GBM without accounting for survivorship. Then, for each sub interval  $[t_{i-1}, t_i]$ , one should simulate  $I_{t_i}^Z | Z_{t_i}, Z_{t_{i-1}}$  using Equation A5 i.e. the infimum given the end points. If any of these infima is below the barrier, then we reject the path entirely and redo this process.

## B Naive MLE

### B.1 Behavior of $\hat{\nu}^N$

When we assume  $\sigma$  is known, then the MLE of  $\nu$  is given by

$$\hat{\nu}^N = \frac{1}{T} \ln(A_T/A_0) = \frac{1}{T}(Z_T - z_0).$$

Using the first moment of  $Z_T | I_T^Z > 0$  (see Equation A3) along with values for  $m, s, \lambda$ , then we get

$$\begin{aligned} \mathbb{E}[\hat{\nu}^N | I_T^Z > 0] &= \nu + \frac{2z_0}{T} \left( \frac{e^{-2z_0\nu/\sigma^2} \Phi\left(\frac{\nu T - z_0}{\sigma\sqrt{T}}\right)}{k\left(\frac{\nu T}{\sigma\sqrt{T}}, \frac{z_0}{\sigma\sqrt{T}}\right)} \right) \text{ or equivalently,} \\ &= \nu + \frac{2z_0}{T} \left( \frac{\Phi\left(\frac{z_0 + \nu T}{\sigma\sqrt{T}}\right)}{k\left(\frac{\nu T}{\sigma\sqrt{T}}, \frac{z_0}{\sigma\sqrt{T}}\right)} - 1 \right) \end{aligned}$$

Since

$$\frac{e^{-2z_0\nu/\sigma^2} \Phi\left(\frac{\nu T - z_0}{\sigma\sqrt{T}}\right)}{k\left(\frac{\nu T}{\sigma\sqrt{T}}, \frac{z_0}{\sigma\sqrt{T}}\right)} > 0$$

the bias is always positive.

### B.2 Asymptotic Behaviour as $T \rightarrow \infty$ .

As  $T \rightarrow \infty$ , the asymptotic behaviour of  $\hat{\nu}^N$  is as follows:

1. If  $\nu > 0$ , then the survival probability converges to a constant

$$\lim_{T \rightarrow \infty} \Pr(I_T^Z > 0) = 1 - \exp\left(\frac{-2z_0\nu}{\sigma^2}\right)$$

and, since the bias  $\frac{2z_0}{T} \left( \frac{\Phi\left(\frac{z_0 + \nu T}{\sigma\sqrt{T}}\right)}{k\left(\frac{\nu T}{\sigma\sqrt{T}}, \frac{z_0}{\sigma\sqrt{T}}\right)} - 1 \right)$  on  $\hat{\nu}^N$  goes to 0 when  $T \rightarrow \infty$ ,  $\hat{\nu}^N$  is strongly consistent and asymptotically unbiased.

2. If  $\nu = 0$ , then  $\hat{\nu}^N$  is asymptotically unbiased and weakly consistent. In this case the survival probability converges to 0, and the conditional distribution of  $\tilde{z}_T = \frac{z_T}{\sigma\sqrt{T}}$  given survival is

$$f_0(z|\tilde{z}_0) = \frac{\phi(z - \tilde{z}_0) [1 - e^{-2\tilde{z}_0 z}]}{\Phi(\tilde{z}_0) - \Phi(-\tilde{z}_0)} \text{ with } \tilde{z}_0 = \frac{z_0}{\sigma\sqrt{T}}.$$

Since  $\tilde{z}_0 \rightarrow 0$  this converges to the half-normal distribution with p.d.f.  $2\phi(z)$ , for  $z > 0$  and with expected value  $\sqrt{\frac{2}{\pi}}$ . Therefore the expected value of  $\hat{\nu}^N$  given survival converges to 0 and the estimator  $\hat{\nu}^N$  is asymptotically unbiased and weakly consistent.

3. If  $\nu < 0$ , then  $\hat{\nu}^N$  has bias conditional on survival approaching  $-\nu$ . Here the conditional p.d.f. of  $\tilde{z}_T = \frac{z_T}{\sigma\sqrt{T}}$  given survival is

$$\begin{aligned} f_{\tilde{\nu}}(z|\tilde{z}_0) &= \frac{\phi(z - \tilde{z}_0 - \tilde{\nu}) [1 - e^{-2\tilde{z}_0 z}]}{k(\tilde{\nu}, \tilde{z}_0)}, \\ &= \text{constant} \times \exp\left(-\frac{z^2}{2} + \tilde{\nu}z\right) (1 - e^{-2\tilde{z}_0 z}) \end{aligned}$$

with  $\tilde{z}_0 = \frac{z_0}{\sigma\sqrt{T}} \rightarrow 0$  and  $\tilde{\nu} = \frac{\nu T}{\sigma\sqrt{T}} \rightarrow -\infty$ . This approaches an exponential distribution with expected value  $\frac{1}{|\tilde{\nu}|}$  so in this case the conditional expected value of  $\tilde{z}_T$  given survival approaches zero,  $E[\hat{\nu}^N] \rightarrow 0$  and its bias approaches  $-\nu$ .

## C Conditional MLE

In this section, we first derive the joint conditional p.d.f. We then show that  $Z_T - z_0$  is a complete and sufficient statistic. Following from this result, we can show that the CMLE exists and is biased downward.

### C.1 Joint conditional pdf

Denote by  $C_{t_i}$  the survival event over the time interval  $[t_{i-1}, t_i]$  i.e.  $C_{t_i}$  is the event  $\{\inf\{Z_t, t_{i-1} \leq t \leq t_i\} > 0\}$ . To simplify the presentation, let us rewrite the joint conditional p.d.f.

$$f_{Z_{t_1}, Z_{t_2}, \dots, Z_{t_n} | I_T^Z > 0}(z_1, z_2, \dots, z_n)$$

as

$$f(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n} | C_{t_1}, C_{t_2}, \dots, C_{t_n}) = \frac{f(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}, C_{t_1}, C_{t_2}, \dots, C_{t_n})}{\Pr(I_T^Z > 0)}.$$

Working on the numerator, we get

$$\begin{aligned} f(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}, C_{t_1}, C_{t_2}, \dots, C_{t_n}) \\ = f(Z_{t_n}, C_{t_n} | Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}) f(Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}) \end{aligned}$$

with

$$\begin{aligned} f(Z_{t_n}, C_{t_n} | Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}) \\ = f(C_{t_n} | Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, Z_{t_n}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}) f(Z_{t_n} | Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}). \end{aligned}$$

From the Markov property of the Brownian motion and the distribution of the infimum of a Brownian bridge, then we can write

$$f(C_{t_n} | Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, Z_{t_n}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}) = f(C_{t_n} | Z_{t_{n-1}}, Z_{t_n}) = 1 - \exp\left(\frac{-2z_{t_n} z_{t_{n-1}}}{\sigma^2 \Delta t}\right).$$

Moreover, again using the Markov property of the Brownian motion, then the conditional density of  $Z_{t_n}$  is

$$f(Z_{t_n} | Z_{t_1}, Z_{t_2}, \dots, Z_{t_{n-1}}, C_{t_1}, C_{t_2}, \dots, C_{t_{n-1}}) = f(Z_{t_n} | Z_{t_{n-1}}) = \frac{1}{\sigma\sqrt{\Delta t}} \phi\left(\frac{z_{t_n} - z_{t_{n-1}} - \nu\Delta t}{\sigma\sqrt{\Delta t}}\right).$$

Therefore, by induction, the joint conditional pdf can be written as

$$f(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n} | C_{t_1}, C_{t_2}, \dots, C_{t_n}) = \frac{\prod_{i=1}^n \frac{1}{\sigma\sqrt{\Delta t}} \phi\left(\frac{z_{t_i} - z_{t_{i-1}} - \nu\Delta t}{\sigma\sqrt{\Delta t}}\right) \left(1 - \exp\left(\frac{-2z_{t_i}z_{t_{i-1}}}{\sigma^2\Delta t}\right)\right)}{\Pr(I_T^Z > 0)}.$$

## C.2 Sufficiency of $Z_T - Z_0$

We begin with a quick review of properties of the *exponential family* of distributions (see e.g. Lehmann (1983), p. 26). An exponential family is a family of distributions indexed by a parameter  $\theta$  with p.d.f. of the form

$$f(x; \theta) = \exp(T(x) - H(\theta))q(x)$$

for functions  $q(x)$ ,  $T(x)$  and  $H(\theta)$ . In general, under the usual regularity conditions, the maximum likelihood estimator in the exponential family is a solution to the score equation

$$\begin{aligned} \frac{\partial}{\partial \theta} \ln(f(X; \theta)) &= 0 \text{ or} \\ T(X) - h(\theta) &= 0 \end{aligned} \tag{C1}$$

where  $h(\theta) = H'(\theta)$ . Since this is an unbiased estimating function,  $E\left[\frac{\partial}{\partial \theta} \ln(f(X; \theta))\right] = 0$  or

$$E[T(X)|\theta] = h(\theta).$$

If the family of distributions is of *full rank*, then the statistic  $T(X)$  is a complete sufficient statistic (Lehmann (1983), page 46).

We now examine the joint distribution of random variables under the joint conditional distribution. Let  $\eta(z_1, z_2, \dots, z_n)$  be some function of observations that does not depend upon  $\nu$ . For the following proof, this function can be modified to  $\dot{\eta}, \ddot{\eta}, \ddot{\eta}$  in different steps of the derivation.

The joint conditional pdf

$$\mathcal{L}(\nu) := \frac{1}{\Pr(I_T^Z > 0)} \times \prod_{i=1}^n \frac{1}{\sigma\sqrt{\Delta t}} \phi\left(\frac{z_{t_i} - z_{t_{i-1}} - \nu\Delta t}{\sigma\sqrt{\Delta t}}\right) \left(1 - \exp\left(\frac{-2z_{t_i}z_{t_{i-1}}}{\sigma^2\Delta t}\right)\right)$$

can be rewritten as

$$\begin{aligned} \mathcal{L}(\nu) &= \frac{\prod_{i=1}^n \frac{1}{\sigma\sqrt{\Delta t}} \phi\left(\frac{z_{t_i} - z_{t_{i-1}} - \nu\Delta t}{\sigma\sqrt{\Delta t}}\right)}{\Pr(I_T^Z > 0)} \times \eta(z_1, z_2, \dots, z_n) \\ &= \frac{\phi\left(\frac{z_{t_n} - z_0 - \nu T}{\sigma\sqrt{T}}\right)}{\Pr(I_T^Z > 0)} \times \dot{\eta}(z_1, z_2, \dots, z_n) \\ &= \frac{\exp\left(-\frac{(z_{t_n} - z_0 - \nu T)^2}{2\sigma^2 T}\right)}{\Pr(I_T^Z > 0)} \times \ddot{\eta}(z_1, z_2, \dots, z_n) \\ &= \frac{\exp\left(\frac{1}{\sigma^2}(z_{t_n} - z_0)\nu\right)}{\Pr(I_T^Z > 0)} \times \ddot{\eta}(z_1, z_2, \dots, z_n). \end{aligned}$$

By the Fisher-Neyman factorization theorem (see e.g. Lehman (1983), p.39), then  $Z_{t_n} - z_0 = Z_T - z_0$  is a sufficient statistic for the family of distributions  $f_{Z_T|I_T^Z > 0}(z)$ . And since  $f_{Z_T|I_T^Z > 0}(z)$  belongs to the linear exponential family of distributions, then  $Z_T$  is a complete sufficient statistic for the parameter  $\nu$ . It follows that the maximum likelihood estimator is a solution to a score equation of the form (see e.g. Lehman (1983), p.46)

$$z_{t_n} - E_\nu[Z_T | I_T^Z > 0] = 0.$$

### C.3 Existence of the CMLE

To show the existence of the CMLE, we rely on the following series expansion of the normal c.d.f. i.e.

$$\Phi(x) \sim \frac{\phi(x)}{|x|} \left(1 - \frac{1}{x^2} + \frac{3}{x^4} - \dots\right)$$

as  $x \rightarrow -\infty$  and

$$1 - \Phi(x) \sim \frac{\phi(x)}{|x|} \left(1 - \frac{1}{x^2} + \frac{3}{x^4} - \dots\right)$$

as  $x \rightarrow \infty$ . Moreover, let us rewrite the survival probability  $\Pr(I_T^Z > 0)$  as a simple function of two parameters i.e.

$$k(x, \delta) = \Phi(x + \delta) - \exp\{-2x\delta\}\Phi(x - \delta)$$

where  $x = \frac{\nu T}{\sigma\sqrt{T}}$  and  $\delta = \frac{z_0}{\sigma\sqrt{T}}$ .

Since  $\Phi(x) \sim \frac{\phi(x)}{-x} \left(1 - \frac{1}{x^2} + \frac{3}{x^4} - \dots\right)$  as  $x \rightarrow -\infty$ , then

$$\begin{aligned} k(x, \delta) &= \Phi(x + \delta) - \exp\{-2x\delta\}\Phi(x - \delta) \\ &\sim \frac{\phi(x + \delta)}{-x - \delta} \left(1 - \frac{1}{(x + \delta)^2}\right) - \exp\{-2x\delta\} \frac{\phi(x - \delta)}{-x + \delta} \left(1 - \frac{1}{(x - \delta)^2}\right) \\ &\sim \frac{\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x + \delta)^2)}{-x - \delta} \left(1 - \frac{1}{(x + \delta)^2}\right) - \frac{\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x - \delta)^2 - 2x\delta)}{-x + \delta} \left(1 - \frac{1}{(x - \delta)^2}\right) \\ &\sim 2\delta\phi(x + \delta) \frac{-x^4 + 2x^2\delta^2 + 3x^2 - \delta^4 + \delta^2}{-x^6 + 3x^4\delta^2 - 3x^2\delta^4 + \delta^6} \\ &\sim 2\delta\phi(x + \delta) \frac{1}{x^2}. \end{aligned}$$

Therefore as  $\nu \rightarrow -\infty$ , we have

$$\begin{aligned} \mathcal{L}(\nu) &= \frac{\exp\left(-\frac{1}{2} \left(\frac{z_{t_n} - z_0 - \nu T}{\sigma\sqrt{T}}\right)^2\right)}{\kappa\left(\frac{\nu T}{\sigma\sqrt{T}}, \frac{z_0}{\sigma\sqrt{T}}\right)} \ddot{\eta}(z_1, z_2, \dots, z_n) \\ &\sim \frac{\phi\left(\frac{z_{t_n} - z_0 - \nu T}{\sigma\sqrt{T}}\right)}{2 \frac{z_0}{\sigma\sqrt{T}} \phi\left(\frac{\nu T + z_0}{\sigma\sqrt{T}}\right) T \nu^2} \ddot{\eta}(z_1, z_2, \dots, z_n) \\ &\sim \nu^2 \exp\left(-\frac{1}{2} \left(\frac{z_{t_n} - z_0 - \nu T}{\sigma\sqrt{T}}\right)^2 + \frac{1}{2} \left(\frac{\nu T + z_0}{\sigma\sqrt{T}}\right)^2\right) \ddot{\eta}(z_1, z_2, \dots, z_n) \\ &\sim \nu^2 \exp\left(\frac{1}{2T\sigma^2} z_{t_n} (2z_0 - Z_T + 2T\nu)\right) \ddot{\eta}(z_1, z_2, \dots, z_n) \\ &\sim \nu^2 \exp\left(\frac{\nu z_{t_n}}{\sigma^2}\right) \ddot{\eta}(z_1, z_2, \dots, z_n) \\ &\rightarrow 0 \text{ as } \nu \rightarrow -\infty \text{ provided } z_{t_n} > 0. \end{aligned}$$

Using the fact that

$$1 - \Phi(x) \sim \frac{\phi(x)}{|x|} \left(1 - \frac{1}{x^2} + \frac{3}{x^4} - \dots\right)$$

as  $x \rightarrow \infty$ , with similar steps we find that

$$k(x, \delta) \sim 1 - e^{-2x\delta}.$$

After a few manipulations, we get that  $\mathcal{L}(\nu) \rightarrow 0$  when  $\nu \rightarrow \infty$  provided that  $Z_T > 0$ .



## C.4 Properties of the CMLE

The CMLE can be written as the solution  $\hat{\nu}$  to the equation

$$\begin{aligned} \frac{Z_T}{\sigma\sqrt{T}} &= h\left(\frac{\nu T}{\sigma\sqrt{T}}, \frac{z_0}{\sigma\sqrt{T}}\right) \\ h(a, b) &:= a + b + 2a \left(\frac{\Phi(a+b)}{k(a, b)} - 1\right). \end{aligned} \quad (\text{C2})$$

Assume that  $T < \infty$  is fixed. We wish to show the following properties of  $\hat{\nu}$ :

1.  $\mathbb{E}[\hat{\nu}|I_T^Z > 0] - \nu \leq 0$  i.e. the bias in  $\hat{\nu}$  is negative.
2. When  $Z_T \rightarrow 0$ , then  $\hat{\nu} \rightarrow -\infty$ .
3. As  $\nu \rightarrow -\infty$ , there exists  $\delta > 0$  such that  $\Pr[Z_T < \delta] \rightarrow 1$  and

$$\mathbb{E}[\hat{\nu}\mathbb{1}_{Z_T < \delta}] - \nu \rightarrow -\infty. \quad (\text{C3})$$

Note that this essentially states that the bias in  $\hat{\nu}$  diverges to  $-\infty$ .

To show these properties we require details concerning the function  $h(a, b)$  for fixed values of  $b > 0$ . We first observe that for fixed  $b > 0$ , the function  $h(\cdot, b)$  is continuous, monotonically increasing, convex, positive, and  $h(a, b) \downarrow 0$  as  $a \downarrow -\infty$ . Moreover, as  $a \uparrow \infty$ , for fixed  $b$ ,

$$\frac{h(a, b)}{a + b} \downarrow 1.$$

For simplicity and without loss of generality, we will show these properties in the case  $\sigma = 1, T = 1$ . Then the maximum likelihood estimator  $\hat{\nu}$  is the unique solution to

$$\begin{aligned} Z_T &= h(\hat{\nu}, z_0) \text{ or} \\ \hat{\nu} &= g(Z_T, z_0) \end{aligned}$$

where  $g(\cdot, z_0)$  is the inverse function of  $h(\cdot, z_0)$ . Since  $h(\cdot, b)$  is differentiable and strictly convex, the inverse is monotone and strictly concave and by Jensen's inequality

$$\begin{aligned} \mathbb{E}_\nu [g(Z_T, z_0)] &< g(\mathbb{E}_\nu [Z_T], z_0) \\ &< g(h(\nu, z_0), z_0) \\ &< \nu \end{aligned}$$

showing that the bias is negative. Since

$$h(a, b) \geq a + b \text{ for all } a,$$

replacing  $a$  by  $g(z, b)$  we have

$$z \geq g(z, b) + b$$

so that

$$Z_T - z_0 \geq \hat{\nu}.$$

The second property follows from the continuity, monotonicity and the property that  $h(a, b) \downarrow 0$  as  $a \downarrow -\infty$ .

To show the third property we wish to show that  $\mathbb{E}[\frac{\hat{\nu}}{\nu}\mathbb{1}_{Z_T < \delta}] \rightarrow \infty$  as  $\nu \rightarrow -\infty$ . To this end we use the asymptotic expression for  $h$ , that is  $h(a, b)(a + b) \rightarrow -b^2$  as  $a \rightarrow -\infty$ . If we replace  $a$  by  $g(z, b)$  in this expression as  $z \rightarrow 0$ , we obtain

$$z(g(z, b) + b) \rightarrow -b^2 \text{ as } z \rightarrow 0.$$

Consider the integral for  $\mathbb{E}[\frac{\hat{\nu}}{\nu}\mathbb{1}_{Z_T < \delta}]$ ,

$$\mathbb{E}\left[\frac{\hat{\nu}}{\nu}\mathbb{1}_{Z_T < \delta}\right] = \int_0^\delta g(z, z_0) \frac{\phi(z - z_0 - \nu) [1 - e^{-2z_0 z}]}{\nu k(\nu, z_0)} dz.$$

Recall that  $z(g(z, z_0) + z_0) \rightarrow -z_0^2$  as  $z \rightarrow 0$ . If we now use the fact that  $k(\nu, z_0) \sim 2z_0\phi(\nu + z_0)\frac{1}{\nu^2}$  as  $\nu \rightarrow -\infty$ , note that as  $z \rightarrow 0$ , the integrand:

$$\begin{aligned}
g(z, z_0) \frac{\phi(z - z_0 - \nu) [1 - e^{-2z_0z}]}{\nu k(\nu, z_0)} &\sim g(z, z_0) \nu^2 \frac{\phi(z_0 + \nu - z) [1 - e^{-2z_0z}]}{2\nu z_0 \phi(\nu + z_0)} \\
&\sim g(z, z_0) \nu \frac{e^{-z^2/2+z(z_0+\nu)} [1 - e^{-2z_0z}]}{2z_0} \\
&\sim -\left(\frac{z_0^2}{z} + z_0\right) \frac{\nu}{2z_0} e^{-z^2/2+z(z_0+\nu)} (1 - e^{-2z_0z}) \\
&\sim -\left(\frac{z_0}{z} + 1\right) \frac{\nu}{2} (1 - e^{-2z_0z}) \\
&\sim -\left(\frac{z_0}{z} + 1\right) \frac{\nu}{2} (2z_0z) \\
&\sim -\nu z_0 (z + z_0) \text{ as } z \rightarrow 0.
\end{aligned}$$

It follows that for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$g(z, z_0) \frac{\phi(z - z_0 - \nu) [1 - e^{-2z_0z}]}{\nu k(\nu, z_0)} > -(1 - \varepsilon)\nu z_0 (z + z_0)$$

for all  $0 < z < \delta$ . Consequently

$$\int_0^\delta g(z, z_0) \frac{\phi(z - z_0 - \nu) [1 - e^{-2z_0z}]}{\nu k(\nu, z_0)} dz > -(1 - \varepsilon)\nu z_0 \int_0^\delta (z + z_0) dz = -\nu(1 - \varepsilon)z_0 \left(\frac{\delta^2}{2} + z_0\delta\right)$$

and this goes to infinity as  $\nu \rightarrow -\infty$ . Moreover since  $Z_T$  converges weakly to 0 as  $\nu \rightarrow -\infty$ , we have  $\Pr[Z_T < \delta] \rightarrow 1$ .

## D Debiased conditional MLE

We now develop the bias correction that is used. Recall that in the exponential family, the maximum likelihood estimator of a parameter  $\theta$  is a solution of the score equation C1. From Section C.2,

$$T(X) - h(\theta) = 0$$

and since this is an unbiased estimating function

$$E[T(X)|\theta] = h(\theta).$$

Thus the maximum likelihood estimator is  $\hat{\theta} = h^{-1}(T)$  assuming invertibility of the function  $h$ . In our application, however, the maximum likelihood estimator can be heavily biased, that is  $E[\hat{\theta}|\theta] - \theta$  can be very large negative (see Equation C3). The idea behind our bias correction is this: having obtained the observed value of the maximum likelihood estimator  $\hat{\theta}_{obs}$  what choice of parameter would result in this as the expected value of the maximum likelihood estimator? In other words, if we denote

$$E[\hat{\theta}|\theta] = g(\theta),$$

we are to solve for the parameter  $\theta$  the equation

$$g(\theta) = \hat{\theta}_{obs}$$

and denoting this ‘‘bias-corrected’’ estimator as  $\tilde{\theta}$ ,

$$\tilde{\theta} = g^{-1}(h^{-1}(T)).$$

While this does not guarantee removal of all bias in the maximum likelihood estimator, it does at least to first order and is therefore similar to methods using the jackknife. The arguments below are similar to those in Cox & Hinkley (1974), Section 8.4.

Suppose  $\hat{\theta}$  is a biased but consistent (ML) estimator satisfying the usual regularity conditions. We assume its expected value has a series expansion

$$\mathbb{E}[h^{-1}(T)|\theta] = g(\theta) = \theta + \frac{b_1(\theta)}{n} + \frac{b_2(\theta)}{n^2} + \dots \quad (\text{D1})$$

where  $b_1(\theta)$  is continuous. Then by standard series inversion,

$$g^{-1}(\theta) = \theta - \frac{b_1(\theta)}{n} + O\left(\frac{1}{n^2}\right).$$

Note that  $\tilde{\theta} = g^{-1}(\hat{\theta})$  satisfies

$$\tilde{\theta} = g^{-1}(\hat{\theta}) = \hat{\theta} - \frac{b_1(\hat{\theta})}{n} + O_p\left(\frac{1}{n^2}\right)$$

and taking expected values

$$\begin{aligned} \mathbb{E}[\tilde{\theta}] &= \mathbb{E}[\hat{\theta}] - \frac{\mathbb{E}[b_1(\hat{\theta})]}{n} + O\left(\frac{1}{n^2}\right) \\ &= \theta + \frac{b_1(\theta) - \mathbb{E}[b_1(\hat{\theta})]}{n} + O\left(\frac{1}{n^2}\right) \\ &= \theta + o\left(\frac{1}{n}\right) \end{aligned}$$

since by the consistency of  $\hat{\theta}$  and the continuity of  $b_1, b_1(\theta) - \mathbb{E}[b_1(\hat{\theta})] \rightarrow 0$  as  $n \rightarrow \infty$ . Recall that the bias-corrected maximum likelihood estimator

$$\hat{\theta} - \frac{b_1(\hat{\theta})}{n}$$

is known to be first order unbiased and second order efficient (see Cox and Hinkley (1974) (Section 8.4)) and our estimator  $\tilde{\theta}$  is equivalent, that is

$$\tilde{\theta} - \left( \hat{\theta} - \frac{b_1(\hat{\theta})}{n} \right) = o_p\left(\frac{1}{n}\right).$$

This argument is often used to support the use of bias correction in cases such as independent and identically distributed random variables, where a jackknife can also be used to remove the bias, since a jackknife allows us to estimate the term  $b_1(\theta)$ .

However in our problem, asset returns conditional upon survival are neither independent nor identically distributed, so methods such as the jackknife are not feasible. We can still continue to use the estimator  $\tilde{\theta}$  and the extent of the debiasing in this problem is very large.

In our application we wish to estimate the drift, the canonical sufficient statistic is

$$T(X) = \frac{Z_T}{\sigma\sqrt{T}},$$

the function  $h$  is given by  $h(a) = h(a, b)$  defined in Equation C2 where  $b = \frac{Z_0}{\sigma\sqrt{T}}$  and the parameter  $\theta$  is  $\theta = \frac{\nu\sqrt{T}}{\sigma}$  from which we get a bias-corrected estimator of the drift:

$$\tilde{\nu} = \frac{\sigma}{\sqrt{T}}\tilde{\theta}.$$

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