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Strategic support of node-consistent cooperative outcomes in dynamic games played over event trees

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Abstract: In this paper, we show that cooperative outcomes in a dynamic game played over an event tree can be supported strategically, that is, to be part of a subgame perfect ε-equilibrium. A numerical example illustrates our results.

Key Words: Stochastic games, S-adapted strategies, cooperative solution, ε-equilibrium, strategic support.

Résumé : Dans cet article, nous montrons que les gains coopératifs dans un jeu dynamique défini sur un arbre d'événements peuvent avoir un support stratégique, c'est-à-dire, faire partie d'un équilibre parfait approximé. Nous illustrons nos résultats avec un exemple numérique.

Mots clés : Jeux stochastiques, stratégies S-adaptées, solution coopérative, support stratégique.

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1 Introduction

Suppose that the players in a dynamic game played over an event tree (DGPET) agree to cooperate and maximize their joint payoff over a given finite planning horizon. The question we ask in this paper is the following: can the resulting cooperative outcomes be sustained as an equilibrium?

In a DGPET, which is a particular stochastic game, the uncertainty is represented by an exogenously given finite event tree, that is, players do not influence the probabilities of transitions between nodes of the tree. As in multistage games (or difference games), a DGPET involves control and state variables. This class of games was initially introduced to examine equilibria in the European natural gas market by Zaccour, 1987 and Haurie et al., 1990. The solution concept proposed in these papers was termed S-adapted equilibrium where S stands for *sample* of realizations of the stochastic process.

Recently, cooperation in DGPET was considered, with the aim of constructing node-consistent solutions, which means that at any node of the event tree, each player's cooperative payoff-to-go in the subgame starting at that node, dominates her noncooperative counterpart. Reddy et al., 2013 proposed a node-consistent Shapley value for this class of games, whereas Parilina and Zaccour, 2015a built a node-consistent core. Node consistency reduces to time consistency in deterministic multistage games, that is, games with an event tree having only one node at each period. The concept of time consistency was first introduced in Petrosjan, 1977; see also Petrosjan and Danilov, 1979. In these references, one idea was to redistribute the players' payoffs along cooperative trajectory by applying an imputation distribution procedure (IDP), which is a decomposition over time of players' payoffs. Since then, a large literature followed on time consistency in dynamic (especially differential) games; see the book by Yeung and Petrosyan, 2006 and the survey by Zaccour, 2008.

To be more precise on our objective here, assume that the players decide to select a particular imputation in the overall cooperative game, say the Shapley value to fix ideas, and next devise a node-consistent IDP. Now, replace for each player her original payoff in each node by its computed IDP gain. Can these modified payoffs be supported strategically, that is, they are part of a (Nash) equilibrium? If at any node there exists a player who benefits if she deviates from the cooperative trajectory, then it implies that the cooperative trajectory is not an equilibrium in the game with payoffs given by IDP. If not, then the conclusion is that the cooperative solution is an equilibrium, and hence is sustainable.

The problem of strategically supporting a cooperative solution is an old one in repeated games. We know from folk theorems that any individually rational outcome (including a cooperative outcome, which is our focus here) can be sustained if the players are sufficiently patient. A folk theorem about the existence of subgame perfect equilibrium in trigger strategies for stochastic games was proved by Dutta, 1995. Recently, Parilina, 2014 provided a condition of strategic support of a cooperative solution in stochastic games. In state-space games, it seems that there is no general folk theorem, but many results for special cases exist. For early contributions in this area, see, e.g., Haurie and Tolwinski, 1985, Tolwinski, 1986, Haurie and Pohjola, 1987 and Haurie et al., 1994, and for more recent ones, see, e.g., Petrosyan, 2008 and Chistyakov and Petrosyan, 2013.[1]

Folk theorems are for infinite-horizon games. It is well-known that cooperation is very hard, if not impossible, to sustain in finite-horizon games. The reason is that players will find it optimal to deviate in the last stage and by a backward induction argument, the result is a noncooperative mode of play throughout the game. Different concepts have been proposed to cope with this difficulty, in particular the idea of subgame-perfect ε-equilibrium; see, e.g., Radner, 1980, Benoit and Krishna, 1985, Solan and Vieille, 2003, Mailath et al., 2005, Flesch et al., 2014 and Flesh and Predtetchinski, 2015. For the class of DGPET, Parilina and Zaccour, 2015b constructed an ε-equilibrium and provided some illustrative examples in environmental economics. This paper takes stock on Parilina and Zaccour, 2015b, but differ in one fundamental way: here, we use the same method for the case where the nodes' payoffs are given by the IDP. Put differently, we wish to check if a node-consistent IDP enjoys the property of being an equilibrium. The short answer is no, but it is in an ε-equilibrium sense.

[1] The books by Dockner et al., 2000 and Haurie et al., 2012 provide a comprehensive introduction to cooperative equilibria in differential games.

The rest of the paper is organized as follows: Section 2 describes the model of the game over event tree, and Section 3 deals with realization of the cooperative solution and problems arise with the realization. Section 4 contains main results. We provide an illustrative example in Section 5, and briefly conclude in Section 6.

2 Game over event tree

We briefly recall the main ingredients of the class of dynamic games played over event trees; see Haurie et al., 2012 for more details.

Let $\mathcal{T} = \{0, 1, \ldots, T\}$ be the set of periods. The exogenous stochastic process is represented by an event tree, which has a root node n^0 in period 0 and a set of nodes $\mathcal{N}^t = \left\{n_1^t, \ldots, n_{N_t}^t\right\}$ in period $t = 1, \ldots, T$. Each node $n_l^t \in \mathcal{N}^t$ represents a possible sample value of the history of the stochastic process up to time t. The tree graph structure represents the nesting of information as one time period succeeds the other. Let $a(n_l^t) \in \mathcal{N}^{t-1}$ be the unique predecessor of node $n_l^t \in \mathcal{N}^t$ on the event-tree graph, $t = 1, \ldots, T$, and $\mathcal{S}(n_l^t) \subset \mathcal{N}^{t+1}$ be the set of all possible direct successors of node $n_l^t \in \mathcal{N}^t$. A path from the root node n^0 to a terminal node n_l^T is called a *scenario*. Each scenario has a probability and the probabilities of all scenarios sum up to 1. We denote by $\pi(n_l^t)$ the probability of passing through node n_l^t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi(n^0) = 1$ and $\pi(n_l^T)$ is equal to the probability of the single scenario that terminates in node n_l^T.

Denote the set of players by $M = \{1, \ldots, m\}$. For each player $j \in M$, we define a set of controls indexed over the set of nodes. Denote by $u_j(n_l^t) \in \mathbb{R}^{m_j}$ the control of player j at node n_l^t, and let $u(n_l^t) = (u_1(n_l^t), \ldots, u_m(n_l^t))$. Let $X \subset \mathbb{R}^p$ be a state set. For each node $n_l^t \in \mathcal{N}^t$, $t = 0, 1, \ldots, T$, let $U_j^{n_l^t} \subset \mathbb{R}^{\mu_j^{n_l^t}}$ be the control set of player j. Denote by $U^{n_l^t} = U_1^{n_l^t} \times \cdots \times U_j^{n_l^t} \times \cdots \times U_m^{n_l^t}$ the product of control sets. A transition function $f^{n_l^t}(\cdot, \cdot) : X \times U^{n_l^t} \mapsto X$ is associated with each node n_l^t. The state equations are given by

$$\begin{aligned} x(n_l^t) &= f^{a(n_l^t)}\left(x\left(a\left(n_l^t\right)\right), u\left(a\left(n_l^t\right)\right)\right), &(1)\\ u\left(a\left(n_l^t\right)\right) &\in U^{a(n_l^t)},\ n_l^t \in \mathcal{N}^t, t = 1, \ldots, T. &(2)\end{aligned}$$

At each node n_l^t, $t = 0, \ldots, T-1$, the reward to player j is a function of the state and the controls of all players, given by $\phi_j^{n_l^t}(x(n_l^t), u(n_l^t))$. At a terminal node n_l^T, the reward to player j is given by function $\Phi_j^{n_l^T}(x(n_l^T))$.

We assume that player $j \in M$ maximizes his expected stream of payoffs discounted at rate λ_j $(0 < \lambda_j < 1)$. The state equations and the reward functions define the following multistage game, where we let

$$\begin{aligned} \mathbf{x} &= \{x(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T\},\\ \mathbf{u} &= \{u(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1\},\end{aligned}$$

and $J_j(\mathbf{x}, \mathbf{u})$ be the payoff to player $j \in M$ that is,

$$J_j(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \lambda_j^t \sum_{n_l^t \in \mathcal{N}^t} \pi(n_l^t)\phi_j^{n_l^t}(x(n_l^t), u(n_l^t)) + \lambda_j^T \sum_{n_l^T \in \mathcal{N}^T} \pi(n_l^T)\Phi_j^{n_l^T}(x(n_l^T)), \qquad (3)$$

$$\begin{aligned} &\text{s.t.}\\ x(n_l^t) &= f^{a(n_l^t)}(x(a(n_l^t)), u(a(n_l^t))), &(4)\\ u(a(n_l^t)) &\in U^{a(n_l^t)}, \quad n_l^t \in \mathcal{N}^t, t = 1, \ldots, T,\\ x(n^0) &= x^0. &(5)\end{aligned}$$

Definition 1 *An admissible* S*-adapted strategy of player j is a vector $\mathbf{u}_j = \{u_j(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1\}$, that is, a plan of actions adapted to the history of the random process represented by the event tree.*

The S-adapted strategy profile of the m players is $\mathbf{u} = (\mathbf{u}_j : j \in M)$. We can thus define a game in normal form $\langle M, (\mathbf{u}_j : j \in M), (W_j(\mathbf{u}, x^0) : j \in M)\rangle$, where $W_j(\mathbf{u}, x^0) = J_j(\mathbf{x}, \mathbf{u})$ is a payoff function of player $j \in M$, and $\mathbf{x}$ is obtained from $\mathbf{u}$ as the unique solution of the state equations that emanate from the initial state x^0.

If the game is played noncooperatively, then the players will seek a Nash equilibrium in S-adapted strategies defined as follows:

Definition 2 *An* S*-adapted Nash equilibrium is an admissible* S*-adapted strategy profile* $\mathbf{u}^N$ *such that for every player* $j \in M$ *the following condition holds:*

$$W_j(\mathbf{u}^N, x^0) \geq W_j\left((\mathbf{u}^N_{-j}, \mathbf{u}_j), x^0\right),$$

where $(\mathbf{u}^N_{-j}, \mathbf{u}_j)$ *is the* S*-adapted strategy profile when all players* $i \neq j$, $i \in M$, *use their Nash equilibrium strategies.*

Remark 1 *Although the* S*-adapted and open-loop equilibria look similar, we note that they differ in the definitions of the state equations and controls. In an open-loop information structure, the controls and the state equations are defined over time. In an* S*-adapted information structure, the controls and the state equations are defined (indexed) over the set of nodes of the event tree.*

If the players agree to cooperate, then they will maximize the sum of their discounted payoffs throughout the entire horizon, that is,

$$\max_{\mathbf{u}_j : j \in M} \sum_{j \in M} W_j\left(\mathbf{u}, x^0\right).$$

Denote the resulting profile of cooperative controls by $\mathbf{u}^*$, that is,

$$\mathbf{u}^* = \arg\max_{\mathbf{u}_j : j \in M} \sum_{j \in M} W_j\left(\mathbf{u}, x^0\right). \tag{6}$$

Further, denote by $\mathbf{x}^* = \{x^*(n^t_l) : n^t_l \in \mathcal{N}^t, t = 0, 1, \ldots, T\}$ the cooperative state trajectory generated by the cooperative control profile $\mathbf{u}^*$.

For later use, we also need to determine the subgame starting at node $n^t_l \in \mathcal{N}^t$ with state $x^*\left(n^t_l\right)$, $t = 1, \ldots, T-1$. This subgame takes place on a tree subgraph $\Gamma(n^t_l)$ of the initial graph. A root of subgraph $\Gamma(n^t_l)$ is node n^t_l. The payoff function of player $j \in M$ in this subgame is given by

$$\begin{aligned} W_j\left(\mathbf{u}(n^t_l), x^*\left(n^t_l\right)\right) &= \sum_{\theta=t}^{T-1} \lambda_j^{\theta-t} \sum_{n^\theta_l \in \mathcal{N}^\theta_\Gamma} \pi(n^\theta_l | n^t_l) \phi_j^{n^\theta_l}(x^*(n^\theta_l), u(n^\theta_l)) \\ &+ \lambda_j^{T-t} \sum_{n^T_l \in \mathcal{N}^T_\Gamma} \pi(n^T_l | n^t_l) \Phi_j^{n^T_l}(x^*(n^T_l)), \end{aligned}$$

where $\mathcal{N}^\theta_\Gamma = \mathcal{N}^\theta \cap \Gamma(n^t_l)$, and $\mathbf{u}(n^t_l) = (\mathbf{u}_j(n^t_l) : j \in M)$ is an S-adapted strategy profile in the subgame, $\mathbf{u}_j(n^t_l) = \{u_j(n^\theta_l) : n^\theta_l \in \Gamma(n^t_l)\}$ is an admissible S-adapted strategy of player j in the subgame starting at node n^t_l, with initial state $x^*(n^t_l)$. The term $\pi(n^\theta_l | n^t_l)$ is the conditional probability[2] that node n^θ_l will be realized if the subgame starts at node n^t_l.

If the players act noncooperatively in the subgame starting at node n^t_l with state $x^*(n^t_l)$ and find the S-adapted Nash equilibrium according to Definition 2, we denote it as $\mathbf{u}^N(n^t_l) = (\mathbf{u}^N_j(n^t_l) : j \in M)$. The S-adapted equilibrium payoff of player j is equal to $W_j\left(\mathbf{u}^N(n^t_l), x^*(n^t_l)\right)$.

If the players cooperate in the subgame starting at node n^t_l with the state $x^*\left(n^t_l\right)$, they maximize the sum of their total discounted payoffs, i.e.,

$$\max_{\mathbf{u}_j(n^t_l) : j \in M} \sum_{j \in M} W_j\left(\mathbf{u}(n^t_l), x^*\left(n^t_l\right)\right),$$

[2] The conditional probability $\pi(n^\theta_l | n^t_l)$ can be calculated by the formula: $\pi(n^\theta_l | n^t_l) = \pi(n^\theta_l)/\pi(n^t_l)$ if $\pi(n^t_l) \neq 0$; otherwise, the subgame starting from node n^t_l cannot materialize.

and the cooperative controls in the subgame are given as follows:

$$\mathbf{u}^*(n_l^t) = \arg\max_{\mathbf{u}_j(n_l^t):j\in M} \sum_{j\in M} W_j\left(\mathbf{u}(n_l^t), x^*\left(n_l^t\right)\right). \tag{7}$$

Therefore, the payoff to player j in the cooperative subgame starting from node n_l^t, with initial state $x^*(n_l^t)$, $n_l^t \in \mathcal{N}^t$, is equal to $W_j\left(\mathbf{u}^*(n_l^t), x^*\left(n_l^t\right)\right)$, $t = 1, \ldots, T$.

Remark 2 *We suppose that the joint-optimization solution and the Nash equilibrium in the whole game and in any subgame are unique. The uniqueness for the joint-optimization solution requires, as usual, strict concavity of the objective function and the control set must be compact and convex. For uniqueness of* S*-adapted Nash equilibrium, we observe that the multistage game has a normal form representation, and therefore the conditions for uniqueness are the same as in classical games with continuous payoffs with constraints as established in Rosen [1965].*

3 Cooperative solution

Suppose that the players decide to cooperate and agree on a particular imputation $y(n^0) = (y_1(n^0), \ldots, y_m(n^0))$ as a solution to whole cooperative game. By being an imputation, $y(n^0)$ satisfies the following equality:

$$\sum_{j\in M} y_j(n^0) = \sum_{j\in M} W_j(\mathbf{u}^*, x^0).$$

Unless the game is trivial, the total before-side-payment payoff $W_j(\mathbf{u}^*, x^0)$ of player $j \in M$, does not coincide with her total after-side-payment payoff given by $y_j(n^0)$. Further, we need to keep in mind that depending on how $y_j(n^0)$ is decomposed over nodes, the player may have more or less interest in continuing cooperation. Now, we proceed with some details.

First, define the characteristic function $V\left(G; n_l^t\right) : 2^m \to \mathbb{R}$, where G is a coalition of players ($G \subset M$), and $V(\varnothing; n_l^t) = 0$. To compute the value $V(G; n_l^t)$ for any coalition $G \subset M$ and any node $n_l^t \in \mathcal{N}^t$, we adopt the γ-characteristic function assumption, which states that when coalition G forms, the left-out-players $(N\backslash G)$ would not join force against G, but only use their individually best-reply strategies. Consequently, the value of γ-characteristic function for coalition G is given by the S-adapted equilibrium outcome of G in the non-cooperative game between members of G maximizing their joint payoff, and non members playing individually, i.e., maximizing their individual payoffs.

Second, we define the imputation set $Y(n_l^t)$ in any subgame starting at node n_l^t in state $x^*(n_l^t)$, that is,

$$Y(n_l^t) = \left\{(y_1(n_l^t), \ldots, y_m(n_l^t)) : \ y_j(n_l^t) \geq V(\{j\}; n_l^t), \forall j \in M, \text{ and } \sum_{j\in M} y_j(n_l^t) = V(M; n_l^t)\right\}.$$

Definition 3 *We call* $(\{\beta_j(n_l^t)\}_{n_l^t\in\mathcal{N}^t, t=0,\ldots,T} : j \in M)$ *an imputation distribution procedure (IDP) of the imputation* $y(n^0) = (y_1(n^0), \ldots, y_m(n^0))$*, where* $\beta_j(n_l^t)$ *is a payment to player* j *at node* n_l^t *in state* $x^*(n_l^t)$*, if for all* $j \in M$ *the following conditions hold:*

$$y_j(n^0) = \sum_{\theta=0}^{T} \lambda_j^\theta \sum_{n_l^\theta \in \mathcal{N}^\theta} \pi(n_l^\theta)\beta_j(n_l^\theta), \tag{8}$$

$$\sum_{j\in M} \beta_j(n_l^t) = \sum_{j\in M} \phi_j^{n_l^t}\left(x^*(n_l^t), u^*(n_l^t)\right), \quad n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1 \tag{9}$$

$$\sum_{j\in M} \beta_j(n_l^T) = \sum_{j\in M} \Phi_j^{n_l^T}(x^*(n_l^T)), \quad n_l^T \in \mathcal{N}^T. \tag{10}$$

Equation (8) means that the expected sum of the discounted payments to player $j \in M$ is equal to her imputation in the whole game. Equations (9), (10) are the conditions for "admissibility" of IDP, i.e., the sum of payments to the players in any node is equal to the sum of payoffs that they can obtain at this node using cooperative controls $u^*(n_l^t)$.

We should notice that only along the cooperative trajectory that players receive payments according to IDP. Off this trajectory, they are paid according to their payoff functions $\phi_j^{n_l^t}(\cdot)$ and $\Phi_j^{n_l^t}(\cdot)$.

Definition 4 *The imputation $y(n^0) \in C(n^0) \subset Y(n^0)$ and corresponding imputation distribution procedure*

$$\left(\left\{\beta_j\left(n_l^t\right)\right\}_{n_l^t \in \mathcal{N}^t, t=1,\ldots,T} : j \in M\right)$$

are called node consistent in the whole game, if for any state $x^*(n_l^t)$, $n_l^t \in \mathcal{N}^t, t = 0, \ldots, T$, there exists $y(n_l^t) = (y_1(n_l^t), \ldots, y_m(n_l^t)) \in C(n_l^t) \subset Y(n_l^t)$ satisfying the following condition:

$$\sum_{\theta=0}^{t-1} \lambda_j^\theta \sum_{n_k^\theta \in \mathcal{N}^\theta} \pi(n_k^\theta)\beta_j(n_k^\theta) + \lambda_j^t \sum_{n_k^t \in \mathcal{N}^t} \pi(n_k^\theta) y_j\left(n_l^t\right) = y_j(x^0). \tag{11}$$

Here $C(n_l^t)$ is a cooperative solution of the cooperative subgame beginning at node n_l^t with state $x^*(n_l^t)$ and it is a subset of the imputation set $Y(n_l^t)$. E.g., it may be the core, the nucleolus, the Shapley value or any other subset.

Any IDP satisfying the conditions in (8)-(10) is admissible. Following Parilina and Zaccour, 2015a, we propose the following node-consistent IDP:

$$\beta_j(n_l^t) = y_j(n_l^t) - \lambda_j \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \pi(n_k^{t+1}|n_l^t) y_j(n_k^{t+1}), \tag{12}$$

and for $t = T$:

$$\beta_j(n_l^T) = \Phi_j^{n_l^T}(x^*(n_l^T)), \tag{13}$$

where $y(n_l^t) = (y_1(n_l^t), \ldots, y_m(n_l^t)) \in C(n_l^t)$ for any $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T-1$. To compute the IDP, the players need to choose an imputation $y(n_l^t)$ belonging to the same cooperative solution[3] in any subgame starting at node n_l^t in state $x^*(n_l^t)$, and follow equations (12) and (13).

If the cooperative solution $C(\cdot)$ is a singleton like the Shapley value, the IDP is uniquely defined by equations (12) and (13), in which $y_j(n_l^t)$ is the jth component of the Shapley value calculated for the subgame beginning at node n_l^t with state $x^*(n_l^t)$.

Remark 3 *Considering the cooperative subgame beginning from node n_l^t, we determine the characteristic function $V\left(G; n_l^t\right)$, which is a function of the state $x^*\left(n_l^t\right)$ as well as the node. To make the notations simpler, we omit the state. Therefore, $V\left(G; n_l^t\right)$ should be read as $V\left(G; n_l^t, x^*\left(n_l^t\right)\right)$. We also omit the state in the notations of the imputation $y(n_l^t)$ and IDP $\beta\left(n_l^t\right)$, which should be read as $y(n_l^t, x^*(n_l^t))$ and $\beta\left(n_l^t, x^*\left(n_l^t\right)\right)$, respectively.*

4 Strategic support of cooperative solution

To recapitulate, in the previous section we stated that if the players agree to cooperate throughout the game, we can implement the agreed upon imputation by changing the initial payoff of player j in node n_l^t, that is, $\phi_j^{n_l^t}(x^*(n_l^t), u^*(n_l^t))$, by $\beta_j(n_l^t)$, $j \in M$ and $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T-1$. Similarly, $\Phi_j^{n_l^T}(x^*(n_l^T))$ was substituted for by $\beta_j(n_l^T)$, $j \in M$ and $n_l^T \in \mathcal{N}^T$. Once this done, the objective is to show that these (new) outcomes can be supported strategically, that is, they correspond to the subgame perfect ε-equilibrium outcomes of a noncooperative DGPET.

[3]Let the cooperative solution be a singleton like the Shapley value. Then in any subgame we need to compute the Shapley value as a cooperative solution. The case where the cooperative solution is the set (e.g., the core) is considered in details by Parilina and Zaccour, 2015a.

We represent the DGPET in extensive form and assume that the players use closed-loop information structure when choosing their strategies. It means that each player knows not only the current node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$ and what she has played on the path leading from the initial node n^0 to $a(n_l^t)$, but also what the other players did in all previous periods. Denote by $P(n_l^t) = (n^0, n_{i_1}^1, \ldots, n_{i_{t-1}}^{t-1}, n_l^t)$ the unique path from initial node n^0 until n_l^t. We call the history of node n_l^t the collection of nodes and corresponding strategy profiles realized on the path $P(n_l^t)$ excepting node n_l^t, that is,

$$H(n_l^t) = \Big(\big(n^0, u(n^0)\big), \big(n_{i_1}^1, u(n_{i_1}^1)\big), \ldots, \big(n_{i_{t-1}}^{t-1}, u(n_{i_{t-1}}^{t-1})\big) \Big).$$

Definition 5 *A behavior strategy of player $j \in M$ in the DGPET is a function associating an action $u(n_l^t)$ with each history $H(n_l^t)$ for each node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T-1$, i.e.,*

$$\sigma_j = \left\{ \sigma_j^{n_l^t} \right\}_{n_l^t \in \mathcal{N}^t, t=0,\ldots,T-1},$$

where

$$\sigma_j^{n_l^t} : H(n_l^t) \longrightarrow U_j^{n_l^t}.$$

Behavior strategy σ_j prescribes to player j the action that should be implemented in each node n_l^t of the event tree $\Gamma(n^0)$. Denote by Σ_j the set of behavior strategies of player j, by $\sigma = (\sigma_1, \ldots, \sigma_m)$ a behavior strategy profile, and by $\Sigma = \Sigma_1 \times \ldots \times \Sigma_m$ the set of possible strategy profiles. For a given behavior strategy profile, we can compute the expected payoff in all subgames, including the whole game, for any given initial state. Denote the payoff of player j in the subgame starting at node n_l^t in state $x(n_l^t)$ as a function of the behavior strategy profile, by

$$\hat{W}_j\big(\sigma, x(n_l^t)\big) = W_j\big(\mathbf{u}(n_l^t), x(n_l^t)\big),$$

where $\mathbf{u}(n_l^t)$ is a trajectory of controls in the subgame starting at node n_l^t with the state $x(n_l^t)$ corresponding to the profile σ.

Below we describe the trigger strategy for the DGPET in detail.

Definition 6 *A strategy profile in behavior strategies $\hat{\sigma}$ is subgame perfect ε-equilibrium if for any player $j \in M$, any node $n_l^t \in \mathcal{N}^t$, any strategy $\sigma_j \in \Sigma_j$ and any history $H(n_l^t)$ the following inequality holds*

$$\hat{W}_j\big(\hat{\sigma}, x(n_l^t)|H(n_l^t)\big) \geqslant \hat{W}_j\big((\hat{\sigma}_{-j}, \sigma_j), x(n_l^t)|H(n_l^t)\big) - \varepsilon,$$

where $\hat{W}_j(\hat{\sigma}, x(n_l^t)|H(n_l^t))$ is the player j's payoff in the subgame starting at node n_l^t with the state $x(n_l^t)$ when players use strategy profile $\hat{\sigma}$ and the history of node n_l^t is $H(n_l^t)$.

To strategically support cooperation in the finite-horizon dynamic game played over an event tree, we shall construct an approximated equilibrium in behavior strategies with a closed-loop information structure. We make the following assumptions about players' behavior:

1. The players want to realize the cooperative trajectory $\mathbf{u}^*$ from (6);
2. The players agree to implement the imputation $y(x^0)$ along the cooperative trajectory $\mathbf{u}^*$;
3. The players adopt the IDP payments as defined by (12) and (13);
4. If a player j deviates from cooperation at node $a\left(n_l^t\right)$, that is, implementing a control $u_j(a\left(n_l^t\right)) \neq u_j^*(a\left(n_l^t\right))$, then cooperation breaks down and all players switch to their Nash equilibrium strategies in the subgame starting at node n_l^t in state $x\left(n_l^t\right) = f^{a(n_l^t)}\left(x^*(a(n_l^t)), (u_{-j}^*(a(n_l^t)), u_j(a(n_l^t)))\right)$.

If player $p \in M$ deviates and cooperation breaks down, her payoff in the subgame starting at n_l^t in state $x\left(n_l^t\right)$ is $W_p(\mathbf{u}^N(n_l^t), x(n_l^t))$, that is, the profit that player p achieves when all players implement Nash equilibrium strategies. Denote by $\hat{\sigma} = (\hat{\sigma}_p : p \in M)$ a behavior strategy profile that prescribes to player $p \neq j$ to implement the cooperative control $u_p^*(n_l^t)$ in node n_l^t if in the history of this node no deviations from cooperative trajectory have been observed, and to implement $u_p^N(n_l^t)$, otherwise. Denote by $\hat{\mathbf{u}}(n_l^t) =$

$\{\hat{u}(n_l^\theta) : n_l^\theta \in \Gamma(n_l^t)\}$ the collection of controls corresponding to strategy profile $\hat{\sigma}$ such that the trigger mode of behavior strategy is implemented in the subgame starting at node n_l^t in state $x(n_l^t)$. If in the history of n_l^θ the individual deviation of player j has been observed, we denote the collection of controls corresponding $\hat{\sigma}$ by $\hat{\mathbf{u}}^j(n_l^\theta)$.

The trigger behavior strategy of a player consists of two behavior types or two modes:

The nominal mode. If the history of node n_l^t coincides with

$$H^*(n_l^t) = \Big(\big(n^0, u^*(n^0)\big), \big(n_{i_1}^1, u^*(n_{i_1}^1)\big), \ldots, \big(n_{i_{t-1}}^{t-1}, u^*(n_{i_{t-1}}^{t-1})\big)\Big), \tag{14}$$

i.e., all players used their cooperative controls on the path $P\left(n_{i_{t-1}}^{t-1}\right)$, that is, from n^0 until $n_{i_{t-1}}^{t-1}$, then player $p, p \in M$ implements $u_p^*(n_l^t)$ in node n_l^t.

The trigger mode. If the history of node n_l^t is such that there exists a node n on the path $P\left(n_{i_{t-1}}^{t-1}\right)$ such that $u(n) \neq u^*(n)$, then player p's strategy is Nash equilibrium strategy calculated for the subgame starting from the successor of n and corresponding state. Here, the history of node n_l^t is such that there exists a node n and at least one deviating player $j \in M$, $j \neq p$, that is, the history $H(n)$ of node n is part of $H^*(n_l^t)$, and $(n, u(n))$ is not part of $H^*(n_l^t)$, but if we replace the control $u_j(n)$ of player j in node n by the cooperative control $u_j^*(n)$, then the pair $(n, (u_{-j}(n), u_j^*(n)))$ will be $(n, u^*(n))$ and part of history $H^*(n_l^t)$.

Formally speaking, the trigger behavior strategy of player $p \in M$ is defined as follows:

$$\hat{\sigma}_p(H(n_l^t)) = \begin{cases} u_p^*(n_l^t), & \text{if } H(n_l^t) = H^*(n_l^t), \\ \hat{u}_p(n_l^t), & \text{if there exists a node } n \text{ on path } P(n_l^t), \\ & \text{such that } u(n) \neq u^*(n), \end{cases} \tag{15}$$

where $\hat{u}_p(n_l^t)$ is player p's control in node n_l^t. The control $\hat{u}_p(n_l^t)$ implements the punishing strategy in the subgame starting in the unique node belonging to the set $\mathcal{S}(n) \cap P(n_l^t)$. The control $\hat{u}_p(n_l^t)$ coincides with $u_p^N(n_l^t)$ calculated as a part of Nash equilibrium for the subgame starting at node $n_1 = \mathcal{S}(n) \cap P(n_l^t)$ in state $x(n_1)$.

To avoid further complicating the notation, we omitted the state argument in the punishing control and the trigger strategy, but we stress that they depend on the state value. Let node n_1 be a direct successor of node n in which player j deviates. The collection of controls $(u_{-j}^*(n), u_j(n))$ is then realized, and the state value in node n_1 can be calculated using the state dynamics $x(n_1) = f^n(x^*(n), (u_{-j}^*(n), u_j(n)))$. The control $\hat{u}_p(n_1)$ is part of the control profile $\hat{u}(n_1) = (u_p^N(n_1) : p \in M)$ where $u_p^N(n_1)$ is a control of player p in node n_1.

Now, in the subgame starting from node $n_l^t \in \mathcal{N}^t$ in state $x(n_l^t)$, the collection of controls punishing player j's individual deviation is given by

$$\hat{\mathbf{u}}^j(n_l^t) = (\hat{\mathbf{u}}_p^j(n_l^t) : p \in M),$$

where $\hat{\mathbf{u}}_p^j(n_l^t) = \left\{\hat{u}_p^j(n_l^\theta) : n_l^\theta \in \Gamma(n_l^t)\right\}$. This collection of controls generates Nash equilibrium trajectory of states in player j's punishment in this subgame, that is,

$$\hat{\mathbf{x}}^j(n_l^t) = \{\hat{x}^j(n_l^\theta) : n_l^\theta \in \Gamma(n_l^t)\}.$$

To construct the trigger strategies, we need to find m punishing strategy profiles for each subgame. Our main result follows.

Theorem 1 *Consider the game played over event tree when players' payoffs on the cooperative trajectory are determined by node-consistent IDP with equations (12) and (13). For any $\varepsilon \geqslant \hat{\varepsilon}$ in the game there exists subgame perfect ε-equilibrium in trigger strategies with players' payoffs $y_1(n^0)$, ..., $y_m(n^0)$, and*

$$\hat{\varepsilon} = \max_{j \in M} \max_{\substack{n_l^t \in \mathcal{N}^t \\ t=1,\ldots,T-1}} \varepsilon_j(n_l^t), \tag{16}$$

where

$$\varepsilon_j(n_l^t) = \max_{u_j(n_l^t)\in U_j^{n_l^t}} \Big\{ \phi_j^{n_l^t}\left(x^*(n_l^t), (u_{-j}^*(n_l^t), u_j(n_l^t))\right) - \beta_j(n_l^t)$$
$$+ \sum_{\theta=t+1}^{T-1} \lambda_j^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \pi(n_l^\theta|n_l^t) \left(\phi_j^{n_l^\theta}(\hat{x}^j(n_l^\theta), \hat{u}^j(n_l^\theta)) - \beta_j(n_l^\theta)\right)$$
$$+\lambda_j^{T-t} \sum_{n_l^T \in \mathcal{N}_\Gamma^T} \pi(n_l^T|n_l^t) \left(\Phi_j^{n_l^T}(\hat{x}(n_l^T)) - \beta_j(n_l^T)\right) \Big\}, \tag{17}$$

where $\hat{u}^j(n_l^\theta)$ is a control profile in node n_l^θ corresponding to a behavior strategy profile $\hat{\sigma}$ determined by (15) and when the trigger mode of the strategy begins in the subgame starting at the node belonging to the set $\mathcal{S}(n_l^t)$ and in state $f^{n_l^t}\left(x\left(n_l^t\right), (u_{-j}^*(n_l^t), u_j(n_l^t))\right)$. Therefore, the differences in the second and third lines also depend on the control $u_j(n_l^t)$. The state $\hat{x}^j(n_l^\theta)$, $n_l^\theta \in \Gamma(n_l^\theta)$ is a state trajectory corresponding to $\hat{\mathbf{u}}^j(n_l^t)$.

Proof. Consider the trigger behavior strategy $\hat{\sigma} = (\hat{\sigma}_p : p \in M)$ defined in (15), and the subgame starting at any node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T-1$. Consider possible histories of any node n_l^t, and compute the benefit of player j deviating at node n_l^t. Her cooperative payoff in this subgame will be given by

$$\hat{W}_j\left(\mathbf{u}^*(n_l^t), x^*(n_l^t)\right) = \beta_j(n_l^t) + \sum_{\theta=t+1}^{T-1} \lambda_j^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \pi(n_l^\theta|n_l^t)\beta_j(n_l^\theta)$$
$$+ \lambda_j^{T-t} \sum_{n_l^T \in \mathcal{N}_\Gamma^T} \pi(n_l^T|n_l^t)\beta_j(n_l^T), \tag{18}$$

where $\mathcal{N}_\Gamma^\theta = \mathcal{N}^\theta \cap \Gamma(n_l^t)$, $\mathbf{u}^*(n_l^t) = (\mathbf{u}_j^*(n_l^t) : j \in M)$ is an S-adapted cooperative strategy profile. In the payoff we use the components of IDP instead of payoffs prescribed initially by payoff functions.

First, consider the case where the history of node n_l^t is $H^*(n_l^t)$. Suppose player j deviates in node n_l^t from the cooperative trajectory. In this case, she may secure the following payoff in the subgame starting at node n_l^t, given the information that the behavior strategy profile $\hat{\sigma} = (\hat{\sigma}_p(\cdot) : p \in M)$ determined by (15) will materialize:

$$\max_{u_j(n_l^t)\in U_j^{n_l^t}} \Big\{ \phi_j^{n_l^t}\left(x^*(n_l^t), (u_{-j}^*(n_l^t), u_j(n_l^t))\right) + \sum_{\theta=t+1}^{T-1} \lambda_j^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \pi(n_l^\theta|n_l^t)\phi_j^{n_l^\theta}\left(\hat{x}^j(n_l^\theta), \hat{u}^j(n_l^\theta)\right)$$
$$+ \lambda_j^{T-t} \sum_{n_l^T \in \mathcal{N}_\Gamma^T} \pi(n_l^T|n_l^t)\Phi_j^{n_l^T}\left(\hat{x}^j(n_l^T)\right) \Big\}, \tag{19}$$

where punishing Nash strategy starts to be implemented in nodes from $\mathcal{S}(n_l^t)$. If the deviation occurs, the players leave the cooperative trajectory and their payoffs are calculated by initially defined functions $\phi_j^{n_l^t}(\cdot)$ and $\Phi_j^{n_l^T}(\cdot)$.

Then, we may compute the benefit from deviation of player j at node n_l^t as a difference between (19) and (18), namely:

$$\varepsilon_j(n_l^t) = \max_{u_j(n_l^t)\in U_j^{n_l^t}} \Big\{ \phi_j^{n_l^t}\left(x^*(n_l^t), (u_{-j}^*(n_l^t), u_j(n_l^t))\right) - \beta_j(n_l^t) \tag{20}$$
$$+ \sum_{\theta=t+1}^{T-1} \lambda_j^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \pi(n_l^\theta|n_l^t) \left(\phi_j^{n_l^\theta}(\hat{x}^j(n_l^\theta), \hat{u}^j(n_l^\theta)) - \beta_j(n_l^\theta)\right)$$
$$+\lambda_j^{T-t} \sum_{n_l^T \in \mathcal{N}_\Gamma^T} \pi(n_l^T|n_l^t) \left(\Phi_j^{n_l^T}(\hat{x}(n_l^T)) - \beta_j(n_l^T)\right) \Big\},$$

Second, suppose that the history of node n_l^t does not coincide with $H^*(n_l^t)$. This means that all players have switched from nominal mode to the trigger one. Player j will have no benefit from deviating in node n_l^t because the players implement their Nash equilibrium strategies regardless of which player (or group of players) has deviated in the previous nodes.

Calculating the maximal benefit from deviation for any subgame and any player given by (20), we obtain the value of $\hat{\varepsilon}$ from the Theorem's statement, that is,

$$\hat{\varepsilon} = \max_{j \in M} \max_{\substack{n_l^t \in \mathcal{N}^t \\ t=1,\dots,T-1}} \varepsilon_j(n_l^t).$$

And for any $\varepsilon \geqslant \hat{\varepsilon}$ the behavior strategy profile determined by (15) is a subgame-perfect ε-equilibrium by construction. □

5 Example

To illustrate the results of the previous section, we consider a three-player stochastic version of the deterministic model of pollution control in Germain et al., 2003. Denote by $M = \{1, 2, 3\}$ the set of players, and by $\mathcal{T} = \{0, 1, \dots, 5\}$ the set of periods. Let $u(n_l^t) = (u_1(n_l^t), u_2(n_l^t), u_3(n_l^t))$ be the vector of countries' emissions of some pollutant and denote by $x(n_l^t)$ the stock of pollution at node n_l^t in time period t. The evolution of this stock is governed by the following difference equation:

$$x(n_l^t) \quad = \quad \big(1 - \delta(a(n_l^t))\big)x(a(n_l^t)) + \sum_{j \in M} u_j(a(n_l^t)), \tag{21}$$

with the initial stock x^0 at root node n^0 being given, and $\delta(n_l^t)$ $(0 < \delta(n_l^t) < 1)$ is the stochastic rate of pollution absorption by nature at node n_l^t. We suppose that $\delta(n_l^t)$ can take two possible values, that is, $\delta(n_l^t) \in \{\underline{\delta}, \overline{\delta}\}$, with $\underline{\delta} < \overline{\delta}$. The event tree is a binary tree, i.e., each node in periods $t = 0, \dots, 5$ has two successors (see Figure 1). The conditional probability of realization of the upward successor of any node is $\frac{1}{4}$ and is $\frac{3}{4}$ for a downward successor. So, for instance, we have probabilities $\pi(n_1^1) = \frac{1}{4}$ and $\pi(n_2^1) = \frac{3}{4}$ in period 1, and probabilities $\pi(n_1^2) = \frac{1}{16}$, $\pi(n_2^2) = \frac{3}{16}$, $\pi(n_3^2) = \frac{3}{16}$, $\pi(n_4^2) = \frac{9}{16}$ for $t = 2$. The root node n^0 and all upward (or left-handed) nodes have the low rate $\underline{\delta}$ of pollution absorption by nature, and all downward (or right-handed) nodes have the high level $\overline{\delta}$ of pollution absorption.

The damage cost is an increasing convex function in the pollution stock having the quadratic form $D_j(x(n_l^t)) = \alpha_j x^2(n_l^t), j \in M$, where α_j is a strictly positive parameter. The cost of emissions is also given by a quadratic function $C_j(u_j(n_l^t)) = \frac{\gamma_j}{2}\left(u_j(n_l^t) - e\right)^2$, where e and γ_j are strictly positive constants.

The total discounted cost $J_j(\mathbf{x}, \mathbf{u})$ to be minimized by player $j \in M$ is given by

$$\sum_{t=0}^{2} \lambda_j^t \sum_{n_l^t \in \mathcal{N}^t} \pi(n_l^t)\left(C_j(u_j(n_l^t)) + D_j(x(n_l^t))\right) + \lambda_j^5 \sum_{n_l^5 \in \mathcal{N}^5} \pi(n_l^5) D_j(x(n_l^5)),$$

where $\mathbf{x} = \{x(n_l^t)\}$ and $\mathbf{u} = \{u(n_l^t)\}$, $n_l^T \in \mathcal{N}^t$, $\lambda_j \in (0, 1)$ is a discount rate of player j, subject to (21), given initial stock $x_0 = 0$ before the game starts and constraints: $u_j(n_l^t) \in [0, e]$ for any player $j \in M$ and any node $n_l^t \in \mathcal{N}^t$, $t = 0, 1, 2$.

We use the following parameters for the numerical simulation:

$$\begin{aligned} &\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3, \\ &\gamma_1 = 0.9, \gamma_2 = 0.8, \gamma_3 = 0.7, \\ &\underline{\delta} = 0.45, \overline{\delta} = 0.8, e = 30, \lambda_1 = \lambda_2 = \lambda_3 = 0.9. \end{aligned}$$

Tables 1 and 2 provide the results regarding the values of maximal benefits from deviation according to (17) and the corresponding values for the case when players do not adopt IDP. (More precisely, we only show the maximal values for each time period.)

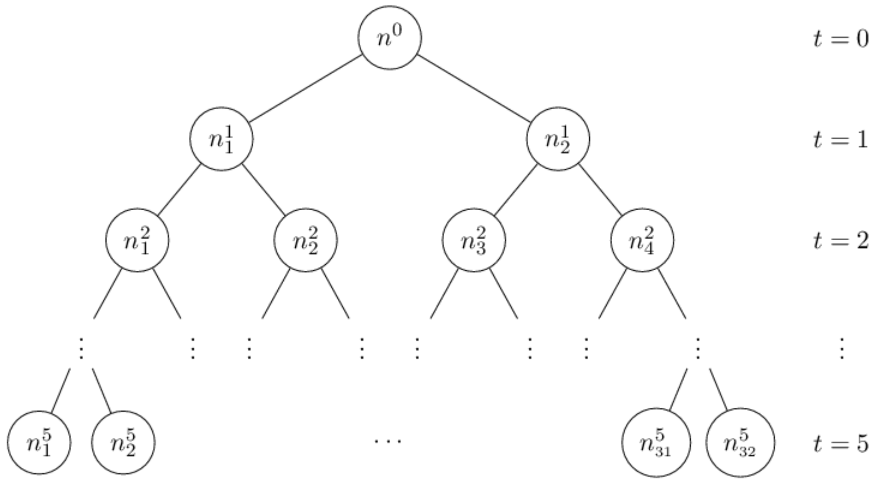

Figure 1: Event tree graph for $T = 5$.

Table 1: The maximal benefits from deviation in time period t calculated for the 6-period game when players adopt IDP.

Time period t	0	1	2	3	4
Player 1	-128.351	-96.4928	-59.492	-23.5305	-0.794881
Player 2	-118.967	-73.6691	-28.0224	12.5117	26.5563
Player 3	-105.177	-33.8832	25.4111	67.5529	61.8055

Table 2: The maximal benefits from deviation in time period t calculated for the 6-period game without IDP.

Time period t	0	1	2	3	4
Player 1	104.714	108.902	96.9909	75.0803	40.6216
Player 2	-103.318	-60.0593	-18.0474	18.2751	28.5478
Player 3	-353.892	-251.621	-141.231	-41.2657	17.8045

The total costs of the three players in the whole game are 2148.7. Tables 1 and 2 show the advantage of adopting the IDP in this game. When players adopt the IDP the maximal benefit of a deviating player or $\bar{\varepsilon}$ is equal to 67.5529 in comparison with 108.902 in the game without using IDP for payoff redistribution. Further, the first time when a player can benefit from deviating is in period $t = 2$ while in the game without IDP the profitable deviation is observed in period $t = 0$. Players 1 and 2 have lower benefits from deviations in the game with IDP than in the game without IDP. But player 3 will have higher incentive to deviate from cooperative trajectory if players adopt the IDP.

6 Concluding remarks

Node consistency, and node consistency for DGPET, can be seen as a necessary condition for sustainability of cooperation, but not a sufficient condition. The reason is that time consistency is not an equilibrium, and consequently the resulting outcomes are not self enforced.

The objective of this paper was to show that node-consistent outcomes are part of a subgame perfect ε-equilibrium. As mentioned earlier, as there is no hope for endowing the cooperative outcomes with an equilibrium property, an approximation was sought here. The simple numerical example showed that deviations can be large and occur in early periods. One conjecture here is that the larger the planning horizon, the later deviations will happen. Further, the results indicate that it is not guaranteed that playing the game with IDP payoffs instead of the original ones, will necessarily reduce the incentive to deviate of all players.

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