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Abstract: We introduce an efficient approach to evaluate counterparty risk and we compute the Credit Valuation Adjustement for derivatives having early exercise features. The approach is flexible and can account for wrong-way risk and various models for the underlying risk factor's dynamics. Numerical experiments are presented to illustrate the efficiency of the method.

Key Words: Credit risk, Credit valuation adjustment, Dynamic programming.

Résumé :   Nous proposons une nouvelle approche efficace pour l'évaluation du risque de contrepartie et le calcul de l'ajustement correspondant pour des produits financiers avec possibilité d'exercice anticipé. Cette approche est flexible, peut tenir compte de la corrélation entre facteurs de risque, et peut s'appliquer à une large gamme de modèles d'évolution de ces facteurs. Nous présentons des essais numériques qui illustrent l'efficacité et la versatilité de la méthode.

Mots clés :   Risque de crédit, CVA, programmation dynamique.

1 Introduction

The financial crisis of 2007–2008 highlighted a number of shortcomings in the regulation of financial institutions. One of these was the underestimation of *counterparty risk*, defined as the risk of incurring losses in mark-to-market derivative portfolio values in the event of a counterparty default. As a response to this shortcoming, the third installment of the Basel Accords (Basel III) advocated a strengthening of risk coverage measures. This included, more specifically, adding the credit valuation adjustment to the capital requirements of financial institutions in order to deal with possible counterparty default, and upgrading counterparty credit risk management standards by considering wrong-way risk.

The *credit valuation adjustment* (CVA) is a risk capital charge that measures the expected potential loss in derivative portfolio values from counterparty default. The CVA charge can increase significantly when considering *wrong-way risk* (WWR), which is the additional risk faced when the underlying risk factors and the default of the counterparty are correlated. The CVA caught the attention of researchers in quantitative finance due to the complexity underlying its evaluation. Brigo and Masetti (2005) give a general pricing formula for the CVA, which can be seen as a call option on the derivative portfolio value with a random maturity corresponding to the counterparty default date. While the computation of the CVA is straightforward for European-style derivatives (Klein, 1996; Gregory, 2010), this is not the case for derivatives with an early exercise opportunity, because the CVA is then path-dependent (exposure falls to zero after exercise). Complexity is again increased when considering WWR, that is, when allowing a correlation between the default process and the derivatives' underlying risk factors to exist. In most practical cases, the CVA is presently evaluated using computationally intensive simulation-based methods.

A number of approaches have been developed to incorporate the impact of credit risk on the value of derivatives. These approaches can be divided into two major categories, according to the way the default event is modeled, that is, either using structural or intensity models. In structural models, the default event for a given firm is related to the evolution of some of its structural variables, while in intensity models, default is governed by an exogenous Poisson (or Cox) process. Although a structural framework is intuitive, its calibration for pricing needs is challenging. On the other hand, an intensity-based approach is more direct and allows for straightforward calibration since default hazard rates can be easily extracted from the observed credit default swap premiums (see for instance Brigo and Masetti (2005) and Gregory (2010)).

CVA evaluation has been addressed under both frameworks, leading to analytical expressions for European options (e.g. Klein (1996) in the structural model case and Gregory (2010) in the intensity model case). However, the issue of CVA evaluation is much more complicated for options with early-exercise opportunities. The least-square Monte Carlo (LSMC) method, which originated with Longstaff and Schwartz (2001), is extensively used in the financial industry to approximate the exposure of an American derivative. While LSMC can be useful to approximate the optimal early-exercise strategy, it introduces statistical errors and is generally recognized not to be very accurate for the estimation of the continuation value, which defines the exposure of the derivative contract. Moreover, simulation-based approaches that are presently used involve two separate steps: the default-free derivative value is first evaluated independently from any counterparty default concern, and then used in a Monte Carlo setting involving the simulation of the default process along with the market risk factors to estimate the exposure at default (Cesari et al., 2010; Brigo and Pallavicini, 2007). Such an approach can only work under the assumption that counterparty risk does not alter the exercise mechanism, which is an unrealistic simplification since the default driver is generally an observable market process (either a set of structural variables or a hazard rate process).

CVA evaluation is also complex for derivative products where the exercise leads to a physical contract rather than direct cash flows (the typical example is an interest rate swaption). In that case, one needs to account for the CVA resulting from entering the contract at some future date in an unknown state of the world. Since simulation always starts from one point in space and time, evaluating the CVA for different states of the world and possibly different valuation dates requires nested simulations that rapidly exceed computational capacity. The common practice is presently to ignore the possibility of default after exercise, which creates inconsistencies in the valuation of the CVA, and may lead to incorrect assessments of counterparty risk.

In this paper, we introduce a new approach to price counterparty risk, possibly under WWR, based on a recursive formulation for the CVA of a derivative security with path-dependent features. We consider a general recovery function that may incorporate many counterparty risk features, such as recovery assumptions, collateral posting and netting agreements, and we account for the relation between the CVA and the exercise mechanism when this mechanism is at the discretion of one of the parties in the contract. Our formulation gives rise to a dynamic programming (DP) algorithm that may be used to evaluate counterparty risk corresponding to any exercise strategy or stochastic stopping time. This algorithm is much more efficient than the currently available methods, thereby providing an accurate evaluation without the need for costly simulation. Moreover, the algorithm provides more than a point estimate: it yields the value of a vulnerable derivative and its CVA for all possible values of the underlying asset and of time to maturity in a single execution. The CVA pricing model is implemented using an intensity model for counterparty default, which can be calibrated to market data. Numerical implementations are based on efficient DP interpolation techniques, as described in Breton and de Frutos (2012).

The paper is organized as follows. Section 2 proposes a general model for the computation of the CVA in a default intensity framework. Section 3 illustrates the application of the CVA model to various types of contracts and recovery functions. Section 4 reports on numerical experiments. Section 5 is a conclusion.

2 Credit valuation adjustment model

In this section, we develop a general model and a recursive characterization of the CVA that can be used for defaultable derivative contracts, with or without early exercise opportunities.

2.1 Notation

Consider a defaultable contract with inception date $t = 0$ and maturity T. Denote by $(Y_t)_{0 \leq t \leq T}$ the (possibly multidimensional) process of the underlying factors, including the risk-free interest rate, denoted by $(r_t)_{0 \leq t \leq T}$. Let τ be the default time of the counterparty. τ is assumed to represent the first jump time of a Cox process with intensity process $(\lambda_t)_{0 \leq t \leq T}$, also called the hazard rate process. We assume that the process of all market quantities $X_t = (Y_t, \lambda_t)_{0 \leq t \leq T}$ is Markovian.

We consider contracts with stopping features that are introduced through a stopping time κ with respect to the filtration generated by the process $(X_t)_{0 \leq t \leq T}$. A stopping mechanism $\mathcal{H}$ is characterized by a collection of sets $(H_t)_{0 \leq t \leq T}$, such that the stopping event happens at t if $X_t \in H_t$. The stopping event may lead for instance to immediate cash flows (e.g. the payoff of an option) or to a physical contract (e.g. an interest rate swap). Conditional on no prior default at a given date $t \in [0, T]$ where $X_t = x$, we denote by $V_t^D(x; \mathcal{H})$ the expected sum of discounted cash flows of a defaultable claim, and by $V_t(x; \mathcal{H})$ the expected sum of discounted cash flows of a counterparty-risk-free claim with the same characteristics, when the stopping mechanism $\mathcal{H}$ is used. Finally, we denote by $V_t^D(x)$ and $V_t(x)$ respectively the value of the defaultable claim and of the corresponding risk-free claim, that is,

$$V_t^D(x) \equiv V_t^D(x; \mathcal{H}^*) = \max_{\mathcal{H}} \left\{ V_t^D(x; \mathcal{H}) \right\} \tag{1}$$

$$V_t(x) \equiv V_t(x; \mathcal{N}) = \max_{\mathcal{H}} \left\{ V_t(x; \mathcal{H}) \right\}, \tag{2}$$

where $\mathcal{H}^*$ (resp. $\mathcal{N}$) is the stopping mechanism that is optimal for the defaultable (resp. default-free) claim.

In our general formulation, the stopping time κ can be exogenous (deterministic or stochastic), or it can be at the discretion of one of the parties. When the stopping time is exogenous (including the case where $\kappa = T$), the set of possible stopping mechanisms is a singleton and the argument $\mathcal{H}$ may be dropped from $V_t^D(x; \mathcal{H}) = V_t^D(x)$ and $V_t(x; \mathcal{H}) = V_t(x)$.

The amount that is recovered (or paid) by the investor in case of default is often expressed as a fixed proportion of the claim; for the moment, however, we do not assume any particular form and simply define two general recovery processes, denoted $R_t^a(x)$ and $R_t^b(x)$ where $R_t^a(x)$ is the (possibly negative) amount

recovered if the stopping event occurs *after* the default event (i.e., $\kappa > \tau$), and $R_t^b(x)$ is the (possibly negative) amount recovered if the stopping event occurs *before* the default event (i.e., $\kappa < \tau$), when $X_t = x$.

For a given $u \geq t$, denote by

$$\begin{aligned} \Lambda_t(u) &\equiv \exp\left(-\int_t^u \lambda_s ds\right), \\ \Gamma_t(u) &\equiv \exp\left(-\int_t^u r_s ds\right). \end{aligned}$$

In what follows, the notation $\mathbb{E}_t[\cdot]$ represents the expectation under the risk-neutral measure, conditional on no prior default and on the information available up to t. In this framework, the conditional risk-neutral default probability in $(t, u]$ is given by

$$D_t(u) = 1 - \mathbb{E}_t\left[\Lambda_t(u)\right].$$

It will be useful to recall that

$$\int_t^u \lambda_s \Lambda_t(s) ds = 1 - \Lambda_t(u)$$

and that, for a given function f,

$$\mathbb{E}_t[1_{(t,u]}(\tau) f(\tau)] = \mathbb{E}_t\left[\int_t^u f(s) \lambda_s \Lambda_t(s) ds\right]$$

where the function $1_A(\cdot)$ is defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for a given function V, we define

$$\begin{aligned} V^+(x) &= \max\left\{V(x); 0\right\} \\ V^-(x) &= \min\left\{V(x); 0\right\}. \end{aligned}$$

2.2 General pricing formula

The CVA capital charge required by Basel III is equal to the expected potential loss from a derivative contract due to a possible default of the counterparty at some future time. Conditional on no prior default at a given date $t \in [0, T]$ where $X_t = x$, we then have

$$\text{CVA}_t(x) = V_t(x) - V_t^D(x).$$

On the other hand, for a given stopping mechanism $\mathcal{H}$, we define the expected potential loss from a derivative contract at $(t, X_t = x)$ as the difference between the expected discounted cash flows for the default-free and the defaultable claims when the stopping mechanism $\mathcal{H}$ is used for both contracts:

$$C_t^{\mathcal{H}}(x) \equiv V_t(x; \mathcal{H}) - V_t^D(x; \mathcal{H}).$$

We then have

$$\begin{aligned} \text{CVA}_t(x) &= V_t(x) - V_t^D(x) \\ &= V_t(x; \mathcal{N}) - V_t^D(x; \mathcal{H}^*) \\ &= V_t(x; \mathcal{N}) - V_t(x; \mathcal{H}^*) + C_t^{\mathcal{H}^*}(x). \end{aligned}$$

When the stopping time is exogenous, the stopping mechanism is unique, $\mathcal{N} = \mathcal{H}^*$, and $\text{CVA}_t(x) = C_t^{\mathcal{H}}(x)$. However, when $\mathcal{N} \neq \mathcal{H}^*$, the CVA can be decomposed into two parts: the first part is due to the change in

the optimal stopping mechanism when a contract is subject to counterparty risk, while the second part is the expected loss under the stopping mechanism that is optimal for the defaultable claim.

Under a given stopping mechanism $\mathcal{H}$, we denote by $F^{\mathcal{H}}_{[s,u]}$ (respectively $F^{\mathcal{H}}_{[s,u)}$) a random variable representing the sum of the cash flows promised by the contract during the time interval $[s,u]$ (respectively $[s,u)$) discounted back at s. According to the risk-neutral pricing principle, conditional on no prior default,

$$
\begin{aligned}
V_t(x;\mathcal{H}) &= \mathbb{E}_t\left[F^{\mathcal{H}}_{(t,T]}\right]; \\
V_t^D(x;\mathcal{H}) &= \mathbb{E}_t\left[1_{(T,\infty]}(\tau)F^{\mathcal{H}}_{[t,T]}\right] + \mathbb{E}_t\left[1_{(t,T]}(\tau)F^{\mathcal{H}}_{[t,\tau)}\right] \\
&\quad + \mathbb{E}_t\left[1_{(t,\kappa]}(\tau)\Gamma_t(\tau)R^a_\tau(X_\tau) + 1_{(\kappa,T]}(\tau)\Gamma_t(\tau)R^b_\tau(X_\tau)\right].
\end{aligned}
$$

For the defaultable claim, if default happens after maturity T, then all the promised cash flows are earned (first term); but in case of early default, only cash flows between t and τ are received (second term). The third term corresponds to the amount recovered at the default event, depending on the relative position of the stopping event (after or before default). The expected potential loss under the stopping mechanism $\mathcal{H}$ is then given by

$$
\begin{aligned}
C_t^{\mathcal{H}}(x) &= V_t(x;\mathcal{H}) - V_t^D(x;\mathcal{H}) \\
&= \mathbb{E}_t[1_{(t,\kappa]}(\tau)\Gamma_t(\tau)\left(V_\tau(X_\tau;\mathcal{H}) - R^a_\tau(X_\tau)\right)] \\
&\quad + \mathbb{E}_t[1_{(\kappa,T]}(\tau)\Gamma_t(\tau)\left(V_\tau(X_\tau;\mathcal{H}) - R^b_\tau(X_\tau)\right)]. \tag{3}
\end{aligned}
$$

For contracts with no stopping features, ($\kappa = T$), $\mathrm{CVA}_t(x) = C_t^{\mathcal{H}}(x)$ and the CVA pricing formula reduces to

$$
\mathrm{CVA}_t(x) = \mathbb{E}_t[1_{(t,T]}(\tau)\Gamma_t(\tau)\left(V_\tau(X_\tau) - R^a_\tau(X_\tau)\right)]. \tag{4}
$$

When $R^a_t(x) = RV_t^+(x) + V_t^-(x)$, where $R \in [0,1]$ is a constant recovery factor, we recover the CVA pricing formula of Brigo and Masetti (2005):

$$
\mathrm{CVA}_t(x) = (1-R)\,\mathbb{E}_t[1_{(t,T]}(\tau)\Gamma_t(\tau)V_\tau^+(X_\tau)].
$$

For some derivative contracts, for example with early exercise opportunities or with desactivating barriers, the stopping time κ is stochastic, which makes the direct valuation used in (4) infeasible. In the following section, we propose a recursive characterization of the CVA that can be used for any stopping mechanism, as long as there is a finite number of possible stopping dates.

2.3   Recursive pricing formula

To this end, we define the set $\mathcal{T} = \{t_m, m = 0, ..., M\}$ of discrete *evaluation dates*, where $t_M \equiv T$. The set $\mathcal{T}$ includes all the dates where a cash flow is promised in the contract, all possible stopping dates, which we assume to be finite in number, and any other date where the CVA needs to be evaluated. In this setting, the characterization of a stopping mechanism $\mathcal{H}$ is reduced to a discrete collection of sets H_m, $m = 1, ..., M$, such that the stopping event happens at t_m if $X_{t_m} \in H_m$. The set $\mathcal{T}$ of evaluation dates coincides with the set of possible stopping dates without loss of generality, since it suffices to set $H_m = \varnothing$ when stopping is not possible at t_m. Define

$$
\begin{aligned}
\delta_m &\equiv \exp\left(-\int_{t_m}^{t_{m+1}} \lambda_s ds\right) = \Lambda_{t_m}(t_{m+1}), m = 0, ..., M-1 \\
\beta_m &\equiv \exp\left(-\int_{t_m}^{t_{m+1}} r_s ds\right) = \Gamma_{t_m}(t_{m+1}), m = 0, ..., M-1.
\end{aligned}
$$

Conditional to $\kappa \geq t_m$ and to no prior default at date t_m, we now compute the expected loss corresponding to a given stopping mechanism $\mathcal{H}$.

For $x \notin H_m$ (i.e., $\kappa \geq t_{m+1}$), we have, using (3)

$$
\begin{aligned}
C_{t_m}^{\mathcal{H}}(x) &= \mathbb{E}_{t_m}[1_{(t_m,\kappa]}(\tau)\Gamma_{t_m}(\tau)\left(V_\tau(X_\tau;\mathcal{H}) - R_\tau^a(X_\tau)\right)] \\
&\quad + \mathbb{E}_{t_m}[1_{(\kappa,T]}(\tau)\Gamma_{t_m}(\tau)\left(V_\tau(X_\tau;\mathcal{H}) - R_\tau^b(X_\tau)\right)] \\
&= \mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}} \Gamma_{t_m}(s)\left(V_s(X_s;\mathcal{H}) - R_s^a(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right] \\
&\quad + \mathbb{E}_{t_m}\left[\int_{t_{m+1}}^{\kappa} \Gamma_{t_m}(s)\left(V_s(X_s;\mathcal{H}) - R_s^b(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right] \\
&\quad + \mathbb{E}_{t_m}\left[\int_{\kappa}^{T} \Gamma_{t_m}(s)\left(V_s(X_s;\mathcal{H}) - R_s^a(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right] \\
&= L_{m,m+1}^{a;\mathcal{H}}(x) + \mathbb{E}_{t_m}\left[\beta_m\delta_m C_{t_{m+1}}^{\mathcal{H}}\left(X_{t_{m+1}}\right)\right].
\end{aligned}
$$

The first term

$$
L_{m,m+1}^{a;\mathcal{H}}(x) = \mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}} \Gamma_{t_m}(s)\left(V_s(X_s;\mathcal{H}) - R_s^a(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right]
$$

corresponds to the expected loss upon default at $(t_m, X_{t_m} = x)$ for a contract promising a payoff equal to $V_t(x;\mathcal{H})$ and maturing at the deterministic date t_{m+1} when the recovery is $R_t^a(x)$.

The second term is obtained by

$$
\begin{aligned}
&\mathbb{E}_{t_m}\left[\int_{t_{m+1}}^{\kappa} \Gamma_{t_m}(s)\left(V_s(X_s;\mathcal{H}) - R_s^a(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right] \\
&\quad + \mathbb{E}_{t_m}\left[\int_{\kappa}^{T} \Gamma_{t_m}(s)\left(V_s(X_s;\mathcal{H}) - R_s^b(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right] \\
&= \mathbb{E}_{t_m}\left[\beta_m\delta_m \int_{t_{m+1}}^{\kappa} \Gamma_{t_{m+1}}(s)\left(V_s(X_s;\mathcal{H}) - R_s^a(X_s)\right)\lambda_s\Lambda_{t_{m+1}}(s)ds\right. \\
&\quad \left. + \beta_m\delta_m \int_{\kappa}^{T} \Gamma_{t_{m+1}}(s)\left(V_s(X_s;\mathcal{H}) - R_s^b(X_s)\right)\lambda_s\Lambda_{t_{m+1}}(s)\right] \\
&= \mathbb{E}_{t_m}\left[\beta_m\delta_m C_{t_{m+1}}^{\mathcal{H}}\left(X_{t_{m+1}}\right)\right].
\end{aligned}
$$

On the other hand, for $x \in H_m$ (i.e., $\kappa = t_m$), we have

$$
\begin{aligned}
C_{t_m}^{\mathcal{H}}(x) &= \mathbb{E}_{t_m}[1_{(t_m,T]}(\tau)\Gamma_{t_m}(\tau)\left(V_\tau(X_\tau;\cdot) - R_{t_m}^b(X_\tau)\right)] \\
&= \mathbb{E}_{t_m}\left[\int_{t_m}^{T} \Gamma_{t_m}(s)\left(V_s(X_s) - R_{t_m}^b(X_s)\right)\lambda_s\Lambda_{t_m}(s)ds\right] \\
&\equiv L_{m,M}^b(x).
\end{aligned}
$$

The term $L_{m,M}^b(x)$ corresponds to the expected loss at $(t_m, X_{t_m} = x)$ for a contract with no stopping feature and maturing at the deterministic date $t_M = T$, when the recovery is $R_t^b(x)$.

We therefore obtain a recursive definition of the expected loss corresponding to a given stopping mechanism $\mathcal{H}$, conditional on no prior default and no prior stopping at date t_m:

$$
\begin{aligned}
C_{t_m}^{\mathcal{H}}(x) &= \left(L_{m,m+1}^{a;\mathcal{H}}(x) + \mathbb{E}_{t_m}\left[\beta_m\delta_m C_{t_{m+1}}^{\mathcal{H}}\left(X_{t_{m+1}}\right)\right]\right)1_{\overline{H_m}}(x) \\
&\quad + L_{m,M}^b(x)1_{H_m}(x), \quad m = 0,...,M-1 && (5) \\
C_T^{\mathcal{H}}(x) &= 0. && (6)
\end{aligned}
$$

Equations (5)–(6) apply to any stopping mechanism, as long as there is a finite number of possible stopping dates. The limiting case where stopping can happen at any time is obtained by letting $M \to \infty$. In practice, the number of possible stopping dates is constrained by the observation frequency. In particular, Equation (5) applies both to exogenous stopping mechanisms (e.g. barrier deactivation or deterministic expiry) and to stopping mechanisms that are at the discretion of one of the parties to the contract (e.g. early exercise).

As mentioned above, when the stopping mechanism is exogenous, the argument $\mathcal{H}$ can be dropped and $\mathrm{CVA}_{t_m}(x) = C_{t_m}(x)$. We then obtain a recursive characterization of the CVA that can be very interesting in many cases, for instance, when the CVA of a contract maturing at a near deterministic date is relatively easy to evaluate.

2.4   Optimal exercise

We now characterize the optimal exercise strategy of a defaultable claim with a finite number of exercise opportunities. Denote by $V_{t_m}^e(x)$ the exercise payoff at t_m when $X_{t_m} = x$, $m = 0, ..., M$. Without loss of generality, we can assume that the set of exercise opportunities coincides with $\mathcal{T}$ by setting $V_{t_m}^e(x) = -\infty$ when exercise is not allowed at t_m.

Recall that $\mathcal{H}^*$ is the stopping mechanism corresponding to the optimal exercise strategy of the defaultable claim. Accordingly, $V_{t_m}^D(x) = V_{t_m}^D(x; \mathcal{H}^*)$ denotes the value of the defaultable claim at t_m when $X_{t_m} = x$, obtained by using the optimal exercise strategy for this defaultable claim, and $V_{t_m}(x; \mathcal{H}^*)$ denotes the expected payoff of the corresponding risk-free claim under the same exercise strategy.

Define an auxiliary stopping mechanism $\mathcal{F}$ that consists, at each exercise date $t_m \in \mathcal{T}$, of holding the claim until the next exercise date, and of using the stopping mechanism $\mathcal{H}^*$ from then on. From (5), the expected loss due to counterparty risk resulting from using this strategy is given by

$$
\begin{aligned}
C_{t_m}^{\mathcal{F}}(x) &= L_{m,m+1}^{a,\mathcal{F}}(x) + \mathbb{E}_{t_m}\left[\beta_m \delta_m C_{t_{m+1}}^{\mathcal{H}^*}\left(X_{t_{m+1}}\right)\right] \\
&= V_{t_m}(x; \mathcal{F}) - V_{t_m}^D(x; \mathcal{F}), \quad m = 0, ..., M-1.
\end{aligned}
$$

Because there are no intermediate cash flows, i.e.,

$$
V_u\left(X_u; \mathcal{F}\right) = \mathbb{E}_u\left[\Gamma_u(t_{m+1}) V_{t_{m+1}}\left(X_{t_{m+1}}; \mathcal{H}^*\right)\right] \text{ for } u \in [t_m, t_{m+1}],
$$

we have, for $m = 0, ..., M-1$,

$$
\begin{aligned}
L_{m,m+1}^{a,\mathcal{F}}(x) &= \mathbb{E}_{t_m}\left[(1 - \delta_m)\beta_m V_{t_{m+1}}\left(X_{t_{m+1}}; \mathcal{H}^*\right)\right] \\
&\quad + \mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}} \Gamma_{t_m}(s) R_s^a(X_s)\lambda_s \Lambda_{t_{m+1}}(s)ds\right]; \\
V_{t_m}(x; \mathcal{F}) &= \mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}\left(X_{t_{m+1}}; \mathcal{H}^*\right)\right].
\end{aligned}
$$

Consequently, with the fact that $C_{t_{m+1}}^{\mathcal{H}^*}(x) = V_{t_{m+1}}(x; \mathcal{H}^*) - V_{t_{m+1}}^D(x)$, we obtain

$$
\begin{aligned}
V_{t_m}^D(x; \mathcal{F}) &= \mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}\left(X_{t_{m+1}}; \mathcal{H}^*\right)\right] \\
&\quad - L_{m,m+1}^{a,\mathcal{H}^*}(x) - \mathbb{E}_{t_m}\left[\beta_m \delta_m C_{t_{m+1}}^{\mathcal{H}^*}\left(X_{t_{m+1}}\right)\right] \\
&= \mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}} \Gamma_{t_m}(s) R_s^a(X_s)\lambda_s \Lambda_{t_{m+1}}(s)ds\right] \\
&\quad + \mathbb{E}_{t_m}\left[\beta_m \delta_m V_{t_{m+1}}^D(X_{t_{m+1}})\right].
\end{aligned}
$$

Now consider the alternative strategy $\mathcal{E}$, which consists of exercising at t_m. The expected loss resulting from using this strategy and the default-free value of the claim under $\mathcal{E}$ are

$$
\begin{aligned}
C_{t_m}^{\mathcal{E}}(x) &= L_{m,M}^b(x) \\
V_{t_m}(x; \mathcal{E}) &= V_{t_m}^e(x),
\end{aligned}
$$

and therefore

$$V_{t_m}^D(x;\mathcal{E}) = V_{t_m}^e(x) - L_{m,M}^b(x).$$

We then have, for $m = 0, ..., M - 1$, conditional on no prior default or exercise at t_m

$$
\begin{aligned}
V_{t_m}^D(x) &= \max\left\{V_{t_m}^D(x;\mathcal{E}); V_{t_m}^D(x;\mathcal{F})\right\} & (7)\\
&= \max\left\{V_{t_m}^e(x) - L_{m,M}^b(x);\right. \\
&\qquad \mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}} \Gamma_{t_m}(s)R_s^a(X_s)\lambda_s\Lambda_{t_{m+1}}(s)ds\right] \\
&\qquad \left. +\mathbb{E}_{t_m}\left[\beta_m\delta_m V_{t_{m+1}}^D(X_{t_{m+1}})\right]\right\} & (8)\\
H_m^* &= \{x : V_{t_m}^D(x) = V_{t_m}^e(x) - L_{m,M}^b(x)\}; & (9)\\
V_{t_m}(x;\mathcal{H}^*) &= V_{t_m}^e(x)1_{H_m^*}(x) + \mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}(X_{t_{m+1}};\mathcal{H}^*)\right]1_{\overline{H_m^*}}(x); & (10)\\
C_{t_m}^{\mathcal{H}^*}(x) &= V_{t_m}(x;\mathcal{H}^*) - V_{t_m}^D(x), & (11)
\end{aligned}
$$

with the terminal condition

$$V_T^D(x) = V_T(x;\mathcal{H}^*) = V_T^e(x). \tag{12}$$

Equation (7) characterizes the optimal exercise strategy of a vulnerable claim. Equation (8) yields a recursive expression for the (optimal) value of a defaultable claim. It is important to notice that the function $V_{t_m}^D(x)$ is continuous in x, which is not the case for the expected payoff under a stopping strategy that is suboptimal at t_m.

2.5 Naive strategy

Recall that $\mathcal{N}$ is the stopping strategy that maximizes the default-free value of the contract, so that $V_{t_m}(x;\mathcal{N}) = V_{t_m}(x)$. This strategy, which we call the *naive strategy,* is characterized by the following dynamic program:

$$
\begin{aligned}
V_T(x) &= V_T^e(x) & (13)\\
V_{t_m}(x) &= \max\{V_{t_m}^e(x); \mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}(X_{t_{m+1}})\right]\}, \quad m = 0, ..., M - 1; & (14)\\
N_m &= \{x : V_{t_m}(x) = V_{t_m}^e(x)\}. & (15)
\end{aligned}
$$

Using (5), the expected payoff of a defaultable claim under the naive strategy is then, for $m = 0, ..., M - 1$,

$$
\begin{aligned}
V_{t_m}^D(x;\mathcal{N}) &= \left(\mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}(X_{t_{m+1}})\right] - L_{m,m+1}^{a,\mathcal{N}}(x)\right. \\
&\qquad -\mathbb{E}_{t_m}\left[\beta_m\delta_m C_{t_{m+1}}(X_{t_{m+1}};\mathcal{N})\right]\right) 1_{\overline{N_m}}(x) \\
&\qquad + \left(V_{t_m}^e(x) - L_{m,M}^b(x)\right)1_{N_m}(x) \\
&= \left(\mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}} \Gamma_{t_m}(s)R_s^a(X_s)\lambda_s\Lambda_{t_{m+1}}(s)ds\right]\right. \\
&\qquad +\mathbb{E}_{t_m}\left[\beta_m\delta_m V_{t_{m+1}}^D(X_{t_{m+1}};\mathcal{N})\right]\right) 1_{\overline{N_m}}(x) \\
&\qquad + \left(V_{t_m}^e(x) - L_{m,M}^b(x)\right)1_{N_m}(x)
\end{aligned}
$$

where it is apparent that the naive strategy is not optimal, that is, $V_{t_m}^D(x;\mathcal{N}) < V_{t_m}^D(x)$.

3 Application examples

In this section, we illustrate the application of Equations (4) and (5)–(6) for the computation of the CVA and the evaluation of vulnerable contracts, for various types of contracts and loss functions, and we show how to account for WWR in this general framework.

3.1   European options

Consider a European option paying $V^e(X_T)$ at maturity T, so that $V_t(x) = \mathbb{E}_t\left[\Gamma_t(T)V^e(X_T)\right]$ and suppose that $R_t^a(x) = RV_t(x)$ where $R \in [0,1]$ is a constant recovery factor. Because there is no stopping feature in this case, we have $\mathrm{CVA}_t(x) = C_t(x)$ and $R_t^b(x) = V_t(x)$. From (4), we then have, at $(t, X_t = x)$,

$$
\begin{aligned}
\mathrm{CVA}_t(x) &= \int_t^T \mathbb{E}_t\left[\Gamma_t(s)\left(1-R\right)\left(V_s(X_s)\right)\lambda_s\Lambda_t(s)\right]ds \\
&= \int_t^T \mathbb{E}_t\left[\Gamma_t(s)\left(1-R\right)\mathbb{E}_s\left[\Gamma_s(T)V^e(X_T)\right]\lambda_s\Lambda_t(s)\right]ds \\
&= (1-R)\mathbb{E}_t\left[\Gamma_t(T)V^e(X_T)\int_t^T \lambda_s\Lambda_t(s)ds\right] \\
&= (1-R)\mathbb{E}_t\left[\Gamma_t(T)V^e(X_T)\left(1-\Lambda_t(T)\right)\right]
\end{aligned}
\tag{16}
$$

and

$$
V_t^D(x) = \mathbb{E}_t\left[\Gamma_t(T)V^e(X_T)\left(1-\left(1-R\right)\left(1-\Lambda_t(T)\right)\right)\right].
\tag{17}
$$

Notice that in order to compute the expectation $\mathbb{E}_t[\cdot]$ in (16) and (17), we need to specify the correlation between the hazard rate and the market factors. If such a correlation exists, we are in the presence of the so-called wrong-way risk (or it can be right-way risk). If, however, we assume that the default intensity is independent from the other market factors, we obtain

$$
\begin{aligned}
\mathrm{CVA}_t(x) &= (1-R)\mathbb{E}_t\left[\Gamma_t(T)V^e(X_T)\right]\mathbb{E}_t\left[\left(1-\Lambda_t(T)\right)\right] \\
&= (1-R)D_t(T)V_t(x).
\end{aligned}
$$

If default probabilities $D_t(T)$ can be expressed in closed form (for instance if we adopt an affine term structure model for the hazard rate), then the CVA of a defaultable European option is easily obtained as a fraction of the value of an equivalent risk-free option.

For the particular case where $R = 0$, Equation (17) becomes

$$
\begin{aligned}
V_t^D(x) &= \mathbb{E}_t\left[\Gamma_t(T)\Lambda_t(T)V^e(X_T)\right] \\
&= \mathbb{E}_t\left[\exp\left(-\int_t^T\left(r_s+\lambda_s\right)ds\right)V^e(X_T)\right].
\end{aligned}
$$

Notice that the default rate λ_s appears as an additional spread in the discount rate. By making the discount rate higher, this spread incorporates counterparty risk into the pricing formula. If default intensity is independent from the other market factors, this reduces to

$$
V_t^D(x) = V_t(x)\left(1-D_t(T)\right).
\tag{18}
$$

Formula (18) is simple and quite intuitive: the risky value is the default-free value multiplied by the probability of surviving until maturity T.

3.2   Bermudan options

Consider a Bermudan option that can be exercised at any date $t_m \in \mathcal{T}$ where the exercise payoff is $V_{t_m}^e(X_{t_m})$. As in the European option case, we assume that $R_t^a(x) = RV_t(x)$, where R is a constant recovery factor and where $V_t(x)$ is the default-free value of the option at $(t, X_t = x)$ under the optimal exercise strategy for the default-free option (the naive strategy). Because exercise leads to immediate cash flows, $R_t^b(x) = V_t(x)$.

We then have

$$
\mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}}\Gamma_{t_m}(s)R_s^a(X_s)\lambda_s\Lambda_{t_{m+1}}(s)ds\right] = \mathbb{E}_{t_m}\left[\beta_m\left(1-\delta_m\right)RV_{t_{m+1}}(X_{t_{m+1}})\right],
$$

and therefore,

$$V_{t_m}^D(x) = \max\left\{ V_{t_m}^e(x); \right.$$
$$\left. \mathbb{E}_{t_m}\left[\beta_m\left((1-\delta_m)\, RV_{t_{m+1}}(X_{t_{m+1}}) + \delta_m V_{t_{m+1}}^D(X_{t_{m+1}}) \right) \right] \right\},$$

where $V_{t_m}^D(x)$ is the value of the defaultable option (i.e., under the exercise strategy $\mathcal{H}^*$). If the option is not exercised, the first term involves the amount recovered if default happens before t_{m+1}, while the second term involves the value of the defaultable option conditional on no prior default or exercise at t_{m+1}. If the default intensity is independent from the other market factors, this reduces to

$$V_{t_m}^D(x) = \max\left\{ V_{t_m}^e(x); D_{t_m}(t_{m+1})RV_{t_m}(x) \right.$$
$$\left. + (1 - D_{t_m}(t_{m+1}))\,\mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}^D(X_{t_{m+1}}) \right] \right\}.$$

Finally, if $R = 0$, we obtain, in the general case,

$$V_{t_m}^D(x) = \max\left\{ V_{t_m}^e(x); \mathbb{E}_{t_m}\left[\beta_m \delta_m V_{t_{m+1}}^D(X_{t_{m+1}}) \right] \right\},$$

which reduces to

$$V_{t_m}^D(x) = \max\left\{ V_{t_m}^e(x); (1 - D_{t_m}(t_{m+1}))\,\mathbb{E}_{t_m}\left[\beta_m V_{t_{m+1}}^D(X_{t_{m+1}}) \right] \right\}$$

when default intensity is independent from the other market factors.

3.3   Interest rate swaps

Consider an interest-rate-payer swap, where the principal is normalized to 1 and the swap rate is γ. Assume that the fixed and floating payments are exchanged on the same dates, denoted by t_m, $m = 1, ..., M$, where $\Delta_m = t_m - t_{m-1}$ is the length of period m. In that case, $X_t = (r_t, \lambda_t)$ and $V_t(x)$ denotes the default-free market value of the swap at $(t, r_t = x)$. Assume that $R_t^a(x) = RV_t^+(x) + V_t^-(x)$, where $R \in [0, 1]$ is a constant recovery factor, and $R_t^b(x) = V_t(x)$ (no stopping feature).

If the floating rate first resets at t_i, then the value of the swap at some date $t \leq t_i$ is

$$V_t(x) = \mathbb{E}_t\left[\Gamma_t(t_i)\right] - \gamma \sum_{m=i+1}^{M} \Delta_m \mathbb{E}_t\left[\Gamma_t(t_m)\right] - \mathbb{E}_t\left[\Gamma_t(T)\right].$$

If zero-coupon bond prices $P_t(r, T) = \mathbb{E}_t\left[\Gamma_t(T)\right]$ at $(t, r_t = r)$ can be obtained in closed form, which is the case for most popular interest-rate models, then the swap value V_t can also be expressed in closed form:

$$V_t(x) = P_t(x, t_i) - \gamma \sum_{m=i+1}^{M} \Delta_m P_t(x, t_m) - P_t(x, T). \tag{19}$$

The CVA of an interest-rate swap is defined by

$$\mathrm{CVA}_t(x) = (1-R)\int_t^T \mathbb{E}_t\left[\Gamma_t(s)\max\left\{V_s(x); 0\right\}\lambda_s\Lambda_t(s)\right]ds.$$

Equivalently, since there is no stopping feature in this case, $\mathrm{CVA}_t(x) = C_t(x)$, and using the recursive expression (5), we get

$$L_{m,m+1}^a(x) = (1-R)\mathbb{E}_{t_m}\left[\int_{t_m}^{t_{m+1}}\Gamma_{t_m}(s)\max\left\{V_s(x); 0\right\}\lambda_s\Lambda_{t_m}(s)ds\right] \tag{20}$$

$$\mathrm{CVA}_{t_m}(x) = L_{m,m+1}^a(x)$$
$$+ \mathbb{E}_{t_m}\left[\beta_m\delta_m\mathrm{CVA}_{t_{m+1}}(X_{t_{m+1}})\right],\ m = 0, ..., M-1 \tag{21}$$

$$\mathrm{CVA}_T(x) = 0. \tag{22}$$

Clearly, the recursive formulation is much easier to evaluate. If the Δ_i are sufficiently small, one can make either one of the commonly used approximations (see, for instance, Brigo and Pallavicini (2007)):

$$
\begin{aligned}
L_{m,m+1}^a(x) &\simeq (1-R)\max\{V_{t_m}(x);0\} D_{t_m}(t_{m+1}) \text{ (anticipated default)};\\
L_{m,m+1}^a(x) &\simeq (1-R)\mathbb{E}_{t_m}\left[\beta_m\delta_m\max\{V_{t_{m+1}}(X_{t_{m+1}});0\}\right] \text{ (postponed default)}.
\end{aligned}
$$

The value of a defaultable swap is then given by

$$V_{t_m}^D(x) = V_{t_m}(x) - \text{CVA}_{t_m}(x). \tag{23}$$

3.4   Interest-rate Bermudan swaptions

A swaptions is an option to enter into a swap contract. We consider a Bermudan swaption with a set $\mathcal{T} = \{t_m, m = 0, ..., M\}$ of discrete dates at which the option holder has the right to enter a payer swap locked at the same date, and with the remaining subsequent dates as payment dates. Denote by $V_{t_m}(x)$ the (default-free) value of this swap at $(t_m, x = X_{t_m})$. The default-free value of the swaption at $(t_m, x = X_{t_m})$, denoted by $W_{t_m}(x)$, is given by the following dynamic program:

$$
\begin{aligned}
W_{t_m}(x) &= \max\left(V_{t_m}(x); \mathbb{E}_{t_m}\left[\beta_m W_{t_{m+1}}(X_{t_{m+1}})\right]\right), \quad m = 0, ..., M-1 \tag{24}\\
W_T(x) &= 0. \tag{25}
\end{aligned}
$$

To compute the swaption value under counterparty risk, the CVA of the swap itself should be taken into account because, when the swaption is exercised, the investor does not get immediate cash flows but rather enters into a new contract that promises future cash flows. Accordingly, assume that the recovery value for the swaption if default occurs before exercise is $R_t^a(x) = R_w W_t(x)$, while the recovery value for the underlying swap is $R_t^b(x) = R V_t^+(x) + V_t^-(x)$, where the recovery rate R_w and R are the recovery rates of the swaption and the swap respectively.

Using (8), the value of the defaultable swaption is then given by

$$
\begin{aligned}
W_{t_m}^D(x) &= \max\Big\{V_{t_m}^D(x); \mathbb{E}_{t_m}\Big[\big(\beta_m(1-\delta_m)R_w W_{t_{m+1}}(X_{t_{m+1}})\\
&\quad + \delta_m W_{t_{m+1}}^D(X_{t_{m+1}})\big)\Big]\Big\} \tag{26}\\
W_T^D(x) &= 0. \tag{27}
\end{aligned}
$$

To compute the CVA of a swaption, the DP recursion (20)–(23) is first used, yielding the value of the defaultable swap for all dates as a function of $x = (r, \lambda)$. The recursion (24)–(27) is then used to compute the default-free and defaultable values of the swaption. The CVA of the swaption is the difference $\text{CVA}_{t_m}^W(x) = W_{t_m}(x) - W_{t_m}^D(x)$. Notice that simulation-based methods usually do not account for the CVA of the swap when evaluating the CVA of a swaption, because the computational burden of obtaining this CVA for each exercise date as a function of the state vector is too great.

3.5   Wrong-way risk

Wrong-way risk (WWR) is the additional risk implied by a dependence between counterparty credit quality and market factors. The general formulations (4) and (5)–(6) in the present paper allow for the existence of a correlation between market risk factors and default intensity.

One simple way to model WWR is to specify a relationship between default-free risk variables and the hazard rate. For instance, one may assume that

$$\lambda_t = f(t, V_t),$$

where f is some deterministic positive function of time and risk factors, and where the dependence upon time is helpful for calibration purposes. Hull and White (2012) propose the following model:

$$\lambda_t = \exp(g(t) + h Y_t), \tag{28}$$

where h is a constant and g is a deterministic function of time that can be calibrated to the observed term structure of credit spreads. Actually, one can also include some random noise in the above relationship; however, unless the standard deviation of the noise is relatively large, including noise complicates the computations without significantly affecting the results (Hull and White 2012).

Under (28), the hazard rate is fully specified by the process Y_t so that the computation of the CVA does not require an additional state variable, and the computational burden of evaluating the CVA is the same as that of evaluating a default-free contract. This makes the choice of the above model interesting in a DP context; moreover, it allows for the incorporation of WWR into the pricing procedure without increasing its numerical complexity.

4   Numerical experiments

In this section, we report on various numerical experiments that illustrate the efficiency of the DP approach to CVA valuation. A first set of experiments compares the results obtained using the recursive pricing formula to those obtained using the standard Monte Carlo approach. A second set of experiments illustrates features of the CVA for various application examples and market models. All experiments were done using an AMD A6-6310 APU processor with 1.8 GHz of power and 8 GB of RAM. Implementation details of our numerical experiments for both the simulation and the DP methods are provided in the appendix.

4.1   Bermudan options under GBM: Comparative results

This first experiment is used to assess the efficiency of the DP approach with respect to existing CVA evaluation methods. We assume a geometric Brownian motion model for the asset price dynamics and a constant hazard rate λ, $X_t = S_t$, where the price dynamics under the risk-neutral measure are described by

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z_t\right), \tag{29}$$

where Z_t is a standard Gaussian random variable, r is the risk-free rate and σ is the volatility.

We compute the CVA at date t_0 of a Bermudan put option with a strike K and a maturity T of one year offering $M = 100$ exercise opportunities, with a zero recovery rate ($R = 0$), under both the optimal strategy $\mathcal{H}^*$ and the naive strategy $\mathcal{N}$, as defined in (1)–(2).

We first compare the results obtained for the naive strategy in terms of values and computational burden, using DP and Monte Carlo simulation. Simulation is performed using a sample of 1,000,000 scenarios, requiring around 60 CPU seconds. Table 1 reports on the required CPU times and on the precision reached according to the number of grid points, while using the DP procedure.

Table 1: Precision and CPU time according to the grid size, DP approach, log-normal model.

n	50	100	150
CPU time (seconds)	0.187	0.343	0.578
Precision	10^{-5}	10^{-8}	10^{-9}

Table 2 compares adjusted prices $V_0^D(S_0)$ obtained using DP to the 95% confidence intervals obtained by simulation for various parameter values. The length of these confidence intervals is of the order of 10^{-3}; all DP prices are inside the intervals. One can observe the efficiency of our proposed approach in precision, computation time and memory requirements: while 60 seconds are required to reach a precision of 10^{-3} by simulation using 10^6 samples, the DP approach reaches a precision of 10^{-5} in less than 0.2 seconds using 50 grid points.

Figure 1 compares the expected losses in value and in percentage of the default-free value, according to the exercise strategy and to the moneyness of the option. One can observe that the expected losses are

Table 2: Adjusted price of a Bermudan put option in the log-normal model without correlation. Parameters are $K = 50$, $r = 0.05$, $T = 1$, $M = 100$.

λ	σ	DP	Simulation
0.1	0.2	2.8792	[2.8777, 2.8807]
0.1	0.15	2.0091	[2.0083, 2.0102]
0.1	0.25	3.7595	[3.7576, 3.7615]
0.05	0.2	2.9594	[2.9587, 2.9608]
0.05	0.15	2.0605	[2.0599, 2.0613]
0.05	0.25	3.8699	[3.8682, 3.8711]
0.15	0.2	2.8017	[2.7996, 2.8031]
0.15	0.15	1.9592	[1.9576, 1.9599]
0.15	0.25	3.6528	[3.6501, 3.6547]
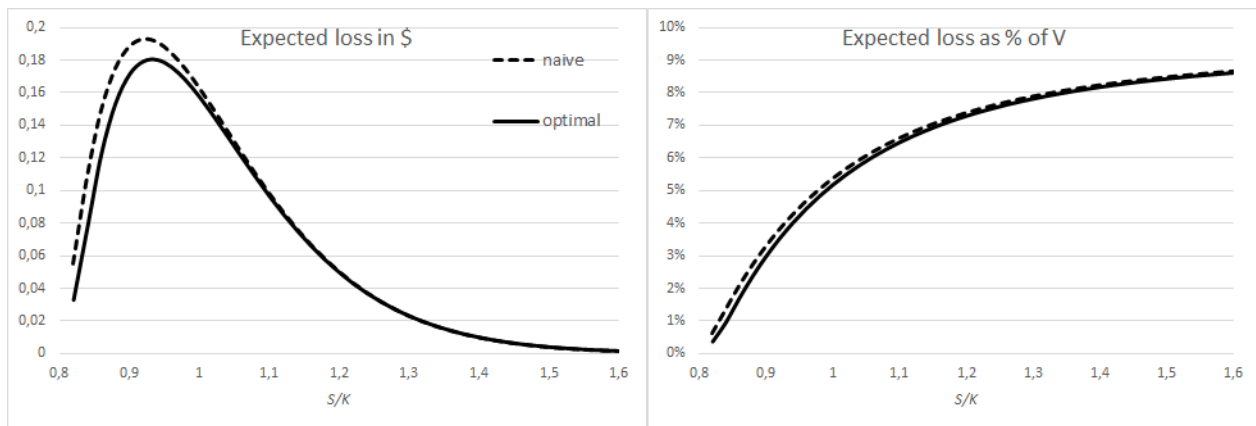
Figure 1: CVA of a Bermudan option at inception for the optimal and naive strategies, as a function of moneyness. Lognormal model with constant hazard rate; parameters are $r = 0.05$, $\sigma = 0.2$, $T = 1$, $M = 100$, $\lambda = 0.1$.

highest when the option is at the money. When the option is in the money, counterparty risk is not very significant since the investor will generally exercise early. On the other hand, when the option is deep out of the money, the option value becomes so small that the adjustment eventually vanishes – even if it still represents a relatively sizeable percentage of the option value.

One can observe that using the optimal exercise strategy results in a smaller expected loss than the one resulting from the naive strategy, and that the difference is larger near the exercise barrier. The CVA of the contract is equal to the expected losses associated with the optimal strategy. It is interesting to note that approximating the CVA using the expected losses associated with the naive strategy is common practice. This is due to the fact that the computation of the optimal strategy of a defaultable contract is not possible using Monte Carlo simulation. On the other hand, the DP recursion (7)–(12) directly provides both strategies and the CVA corresponding to the optimal strategy. Finally, notice that the value of a vulnerable claim under the naive strategy is discontinuous at the exercise barrier, resulting in arbitrage opportunities.

4.2   Impact of WWR

Our second experiment introduces WWR for the Bermudan put option in the GBM model (29) considered in the previous section. We assume that the dependence between the hazard rate and the asset value is described by two parameters denoted $\overline{\lambda}$ and h, such that, during the time interval $[t_m, t_{m+1}]$,

$$\lambda_{t_m} = \overline{\lambda} \exp(hS_{t_m}), \tag{30}$$

where h measures the amount of right-way or wrong-way risk ($h < 0$ indicates WWR[1]). Clearly, the introduction of such a dependence does not modify the numerical complexity of either the DP or simulation approaches.

In order to assess the impact of WWR, we compare the CVA at date 0, obtained using the default model (30), with the one obtained when assuming a constant hazard rate equal to λ_0. The left panel of Figure 2 shows, as a function of S_0, the hazard rate $\lambda_0 = \overline{\lambda} \exp(hS_0)$ and the additional CVA when $\overline{\lambda} = 2$ and $h = -0.06$. The right panel of Figure 2 shows, as a function of S_0, the factor h satisfying $\lambda_0 = 0.1$ and the additional CVA when $\overline{\lambda} = 2$. Finally, Figure 3 shows, as a function of $\overline{\lambda}$, the value of h such that $\lambda_0 = 0.1$ at $S_0 = K$, and the additional CVA.
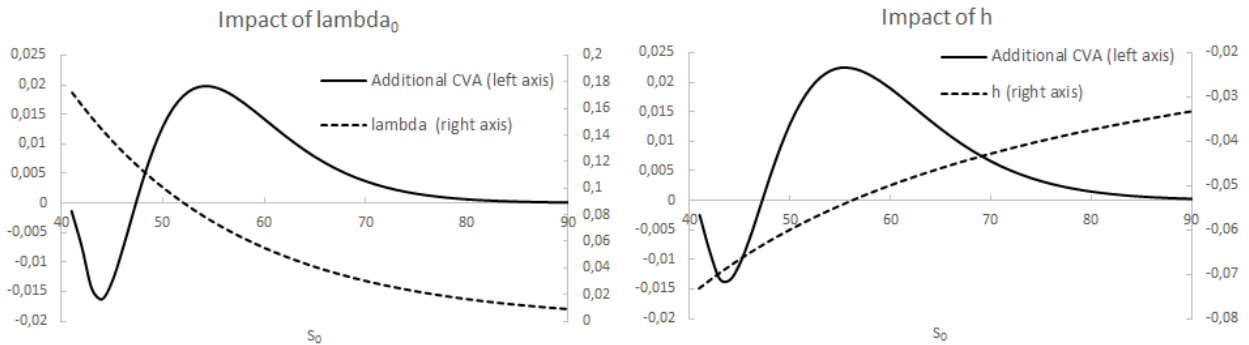

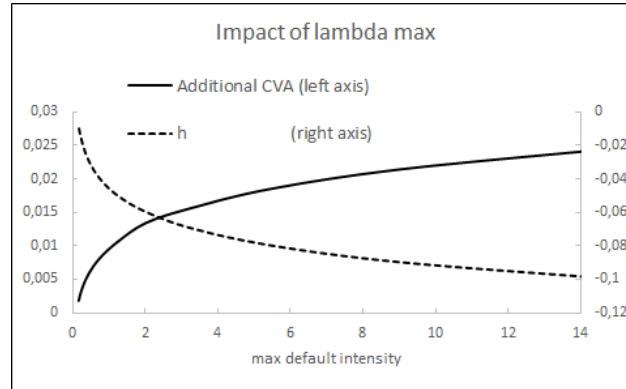Figure 2: Impact of WWR, Bermudan put option. Parameters are $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$.

Figure 3: Impact of WWR, Bermudan put option. Parameters are $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$.

It is interesting to note that the presence of WWR may imply a decrease in the CVA for Bermudan options. Indeed, when the price of the underlying asset is low, a negative correlation between the hazard rate and the price of the underlying asset increases the probability of early exercise, and consequently decreases counterparty risk. This is not the case for European options, as illustrated in Figure 4.

4.3 Jump-diffusion model

Our third set of results is obtained by specifying a different market model, namely, Merton's (1976) jump-diffusion model. Accordingly, the price dynamics under the risk-neutral measure are described by

$$S_t = S_0 \exp\left((r - e^{\eta + \frac{\phi^2}{2}}\alpha - \frac{1}{2}\sigma^2)t + \sigma Z_t \right) \prod_{i=1}^{N_t} (J_i + 1) \tag{31}$$

---

[1]For a put option, a decrease in the price of the underlying asset increases both the exposure and the probability of default.
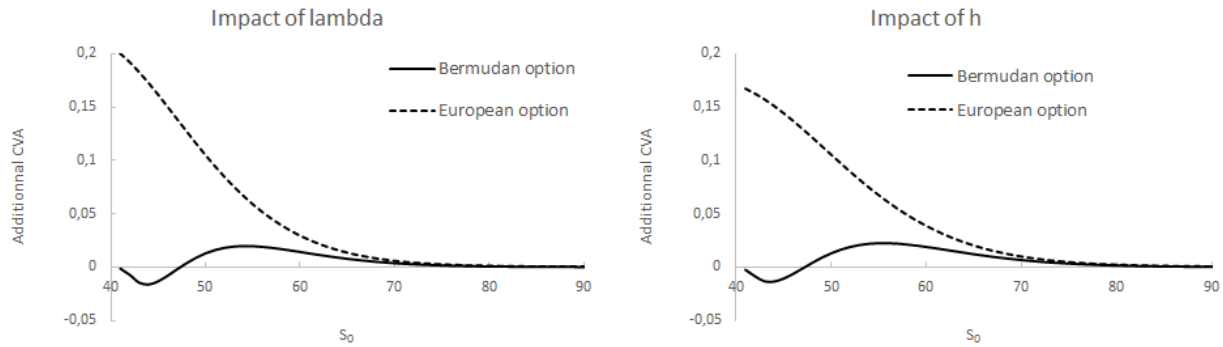
Figure 4: Impact of WWR on European and Bermudan put options. Parameters are $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$.

where the jump size $\log(J_i + 1)$ is a Gaussian random variable of mean η and standard deviation ϕ, and where N_t is a Poisson process with intensity α. In turbulent times, when counterparty risk is present, jumps in the asset price may well be a better assumption than the constant volatility of the lognormal model. As pointed out previously, the DP algorithm can accommodate any market model. However, a more complex model and added volatility may require additional processing time and number of grid points to attain a given level of precision. To illustrate the accuracy of the DP procedure, Figure 5 shows the convergence of the DP price to the analytical Merton formula for a European risk-free option, as the number of grid points increases.
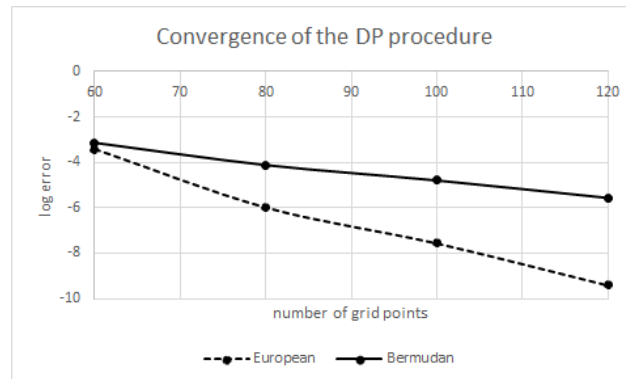

Figure 5: Log error as a function of the number of grid points in the DP procedure for Merton's jump-diffusion model. Parameters are $S_0 = K = 50$, $r = 0.05$, $\sigma = 0.2$, $\alpha = 0.25$, $\eta = 0$, $\phi = 0.1$, $T = 1$. Benchmark for the European case is the Merton analytical formula. Benchmark for the Bermudan case ($M = 100$) is computed with 600 grid points.

Figure 5 and Table 3 also illustrate the results obtained for the corresponding vulnerable Bermudan option with a zero recovery rate (optimal exercise strategy) for various grid sizes, using the DP approach (the benchmark is the value at $n = 600$). It shows that a precision of 10^{-4} is attained with 60 grid points in around 3 seconds.

Finally, Table 4 presents the results obtained using simulation and DP for various parameter values. Simulation results are presented for both the naive and optimal strategies for comparison purposes; however, it is important to note that exercise strategies cannot be obtained by simulation, and must be previously computed using DP. Simulations were performed using 10^6 samples and required 78 CPU seconds, with a precision of the order of 10^{-2}.

Table 3: Precision and computing time as a function of the number of grid points, vulnerable Bermudan option under the jump-diffusion model. Parameters are $K = 50$, $r = 0.05$, $\sigma = 0.2$, $\alpha = 0.25$, $\eta = 0$, $\phi = 0.1$, $T = 1$, $M = 100$.

n	60	80	100	120	Benchmark
Value	2.98608573	2.985426525	2.985333179	2.9853466	2.985349358
Error	0.000736372	7.71675E-05	1.61791E-05	2.7581E-06	
CPU seconds	3.125	4	4.8438	5.8281	

Table 4: Adjusted price of a Bermudan put option in the jump-diffusion model. Parameters are $S_0 = K = 50$, $r = 0.05$, $\alpha = 0.25$, $\eta = 0$, $\phi = 0.1$, $T = 1$, $M = 100$.

| λ	σ	DP(optimal)	Simulation(optimal)	DP(naive)	Simulation(naive)
0.1	0.2	2.9853	[2.9832, 2.9860]	2.9796	[2.9776, 2.9806]
0.1	0.15	2.1373	[2.1361, 2.1381]	2.1341	[2.1336, 2.1356]
0.1	0.25	3.8513	[3.8484, 3.8522]	3.8427	[3.8419, 3.8459]
0.05	0.2	3.0646	[3.0634, 3.0655]	3.0631	[3.0619, 3.0641]
0.05	0.15	2.1902	[2.1893, 2.1907]	2.1894	[2.1886, 2.1901]
0.05	0.25	3.9583	[3.9575, 3.9602]	3.9561	[3.9559, 3.9587]
0.15	0.2	2.9109	[2.9097, 2.9130]	2.8989	[2.8959, 2.8995]
0.15	0.15	2.0873	[2.0857, 2.0880]	2.0804	[2.0799, 2.0823]
0.15	0.25	3.7512	[3.7504, 3.7548]	3.7333	[3.7295, 3.7343]

4.4 Stochastic hazard rate

We now consider a stochastic hazard rate following a CIR process (Cox et al., 1985; Schönbucher, 2003):

$$d\lambda_t = \xi(\theta - \lambda_t)dt + \nu\sqrt{\lambda_t}dB_t,$$

where B_t is a Brownian motion. We assume that S_t follows a geometric Brownian motion, independent from λ_t, and that $R = 0$. Here, the state vector $X_t = (S_t, \lambda_t)$ is bidimensional, requiring a two-dimensional discretization grid, where n_S and n_λ are the number of discretization points for the underlying asset price and the hazard rate respectively.

In the European case, the value of a vulnerable option can be obtained analytically, and Figure 6 illustrates the convergence of the DP price to the analytical value as the grid size and configuration change. A precision of 10^{-4} can be reached in 3 seconds with 60 grid points for the underlying asset and 80 for the hazard rate.
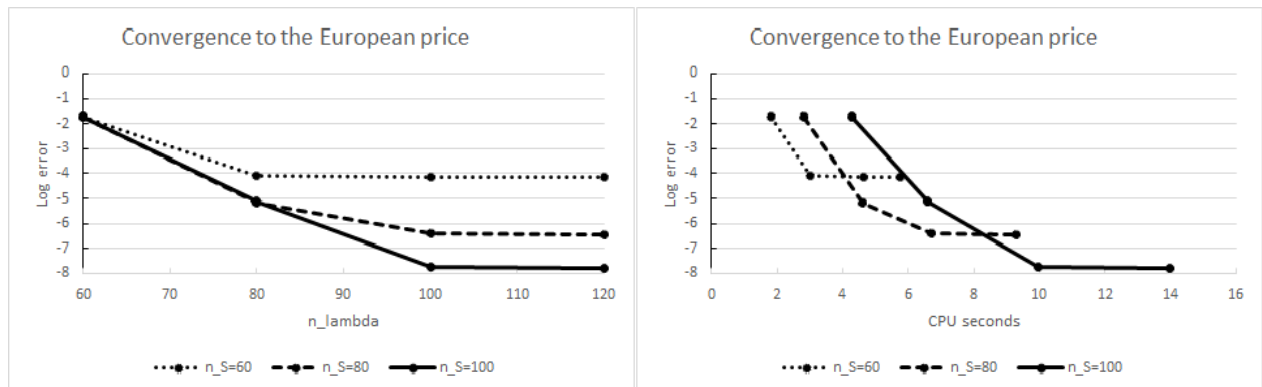
Figure 6: Precision (log error) as a function of grid size and computation time (CPU seconds). Parameter values are $S_0 = K = 50$, $\lambda_0 = 0.1$, $r = 0.05$, $\sigma = 0.2$, $\theta = 0.1$, $\xi = 0.5$, $\nu = 0.2$, $M = 12$, $T = 1$.

Figure 7 provides the convergence information for a corresponding Bermudan option with 12 exercise opportunities. In this case, the benchmark is approximated by the value obtained by DP with $n_S = 300$ and $n_\lambda = 300$. A precision of 10^{-5} is reached in 10 seconds with a 100×100 grid.
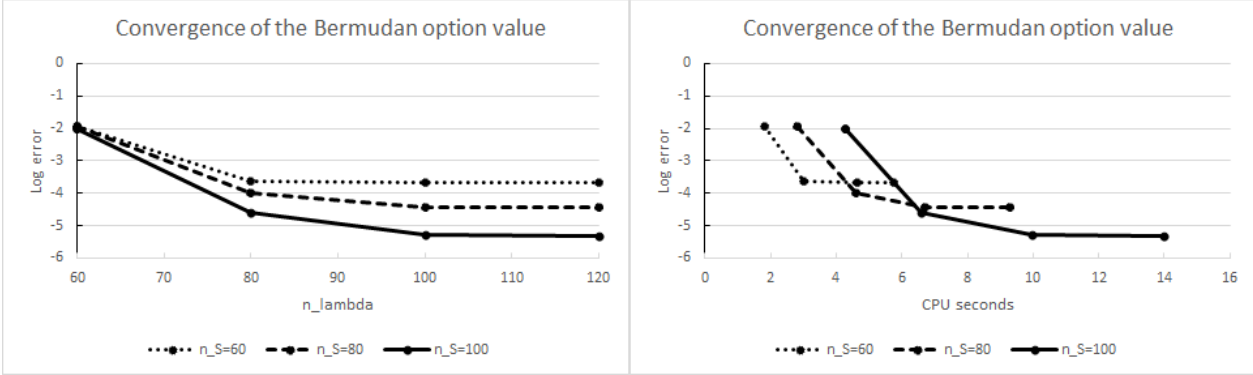

Figure 7: Precision (log error) as a function of grid size and computation time (CPU seconds). Parameter values are $S_0 = K = 50$, $\lambda_0 = 0.1$, $r = 0.05$, $\sigma = 0.2$, $\theta = 0.1$, $\xi = 0.5$, $\nu = 0.2$, $M = 12, T = 1$.

4.5 Swaps and swaptions with stochastic hazard rate and WWR

In our last set of results, we consider swaps and swaptions, assuming that the evolution of the interest and hazard rates is described by CIR processes, that is

$$dr_t = \xi_r(\theta_r - r_t)dt + \nu_r\sqrt{r_t}dB_t^1, \tag{32}$$
$$d\lambda_t = \xi_\lambda(\theta_\lambda - \lambda_t)dt + \nu_\lambda\sqrt{\lambda_t}dB_t^2. \tag{33}$$

WWR is present when the two processes are correlated, where ρ denotes the correlation coefficient between the Brownian motions B_t^1 and B_t^2:

$$dB_t^1 dB_t^2 = \rho dt.$$

In practice, Equations (32) and (33) could also be shifted by deterministic functions in order to match the observed term structure of interest rates and credit spreads.

Under (32)–(33), when $\rho = 0$, the joint density of the state variable $\left(r_{t_{m+1}}, \lambda_{t_{m+1}}\right)$ can be expressed analytically. When the correlation is not null, no closed-form densities are available. In this case, the CIR processes can be approximated by Vasicek processes:

$$dr_t = \xi_r(\theta_r - r_t)dt + \nu_r\sqrt{r_{t_m}}dB_t^1 \text{ for } t \in [t_m, t_{m+1}]$$
$$d\lambda_t = \xi_\lambda(\theta_\lambda - \lambda_t)dt + \nu_\lambda\sqrt{\lambda_{t_m}}dB_t^2 \text{ for } t \in [t_m, t_{m+1}].$$

Table 5 compares the CVA (in basis points) of an interest rate swap, obtained by Monte Carlo simulation and by the DP recursion (20)–(22) for various values of ρ, at $(t = 0, r_0, \lambda_0)$. The DP procedure uses $128 \times 128 = 16,384$ grid points, while the simulation uses $500,000$ samples and 180 time-discretization nodes. All DP results are within the simulation confidence intervals. The DP procedure requires less memory and computation time to produce the CVA at all payment dates and for all possible values of the state vector than simulation requires to produce a single estimate. The impact of correlation is illustrated in Figure 8.

We now evaluate a Bermudan swaption to enter the swap evaluated in Table 5. Table 6 presents the default-free value and the CVA of the swaption at $(t = 0, r_0, \lambda_0)$ in basis points, when the optimal exercise strategy is used. The first CVA evaluation (CVA1) is computed by assuming that the swap is default-free. The second evaluation (CVA2) accounts for the possibility of counterparty default in the swap contract, where we assume that the hazard rate process is the same for both the swaption and the swap, and that recovery upon default is null in both cases ($R = 0$).
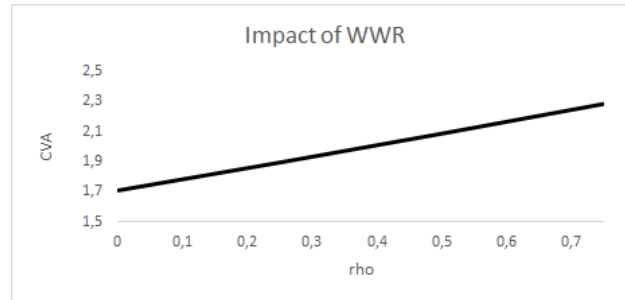
Figure 8: Impact of ρ on the CVA of a swap. Parameters are as in Table 5.

Table 5: CVA of a swap, CIR model with correlation. Parameters are $\gamma = 0.05$, $T = 1$, $M = 12$, $R = 0$, $r_0 = 0.05$, $\theta_r = 0.05$, $\xi_r = 0.5$, $\nu_r = 0.1$, $\lambda_0 = 0.1$, $\theta_\lambda = 0.1$, $\xi_\lambda = 0.5$, $\nu_\lambda = 0.2$.

ρ	DP	Simulation
0	1.7104	$[1.6693, 1.7312]$
0.25	1.8958	$[1.8447, 1.9102]$
0.50	2.0873	$[2.0296, 2.1001]$
0.75	2.2847	$[2.2769, 2.3513]$
CPU seconds	48.48	92.83

Table 6: CVA of a swaption, CIR model with correlation. Parameters are as in Table 5.

ρ	Default-free	CVA1	CVA2
0	36.7740	6.3108	10.0919
0.25	36.7740	7.3799	10.7681
0.50	36.7740	8.4463	11.2375
0.75	36.7740	9.5137	12.0284
CPU seconds	0.219	53	80

A first observation is that neglecting the counterparty risk of the underlying swap may lead to significant undervaluation of the CVA, reaching 37% in this example for $\rho = 0$. A second observation is that the time required to compute the CVA of the swaption by DP is of the same order as the time required to compute a single instance of the CVA of the underlying swap by simulation. This means that, even with a very coarse grid, accounting for the CVA of the underlying swap using a simulation approach is computationally prohibitive.

5   Conclusion

This paper proposes a recursive formulation of the CVA that allows its evaluation using a dynamic programming approach. The DP algorithm can be used to evaluate the CVA of contracts with optional or stochastic stopping times, and can accommodate a wide range of market and default models. The DP approach is computationally more efficient than Monte Carlo simulation, providing a complete characterization of the CVA at all possible stopping times and for all possible states of the world, and doing so in less time and using less memory than simulation requires for a single evaluation. Moreover, for contracts with optional exercise features, the DP approach allows for the computation of the optimal exercise strategy and provides the corresponding CVA, which is not possible using Monte Carlo simulation.

Illustrative examples are provided to show the flexibility of the proposed approach, and numerical experiments for various contracts and models illustrate its precision and efficiency.

6 Appendix: Implementation details

6.1 Dynamic program

The recursive approach to CVA valuation can be equated to solving the following general dynamic program:

$$v_m(x) = f_m\left(\mathbb{E}\left[G_m v_{m+1}\left(X_{t_{m+1}}\right) | X_{t_m} = x\right]\right), \ m = 0, ..., M-1 \tag{34}$$
$$v_M(x) = 0 \tag{35}$$

where f_m is a known function and G_m is a random variable, possibly multidimensional, and where we assume that the joint density of $\left(G_m, X_{t_{m+1}}\right)$ under the risk-neutral measure, conditional on $X_{t_m} = x$, is known. To simplify the exposition, we describe the implementation when the state space is unidimensional, where $x \in [0, \infty)$. Suppose that the function v_{m+1} is known analytically on $[0, \infty)$. At a given x, since both the joint density and the function v_{m+1} are analytical, computation of $\mathbb{E}\left[G_m v_{m+1}\left(X_{t_{m+1}}\right) | X_{t_m} = x\right]$ amounts to evaluating the integral of an analytically known function.

In order to solve the dynamic program (34)–(35), we compute v_m on a finite grid and use a spectral interpolation scheme to obtain an analytical interpolation function $\hat{v}_m$ approximating v_m. Starting from the known function v_M, this process yields, by backward induction, analytical interpolation functions $\hat{v}_m(x)$ for all evaluation dates t_m.

More precisely, define a set $\mathcal{G} = \{x_j, j = 1, ..., n\}$ of n grid points, such that

$$0 < x_1 < x_2 < ... < x_n < \infty$$

and a family of n basis functions, denoted by $(\psi_j)_{j=1,...,n}$. An interpolation function $\hat{v}_m(x)$ is defined by

$$\hat{v}_m(x) = \left\{ \begin{array}{c} \sum_{j=1}^n c_j^m \psi_j(x) \text{ if } x \in [x_1, x_n] \\ o(x) \text{ if } x \notin [x_1, x_n], \end{array} \right.$$

where o is an extrapolation function characterizing the behavior of v outside the localization interval, and where the coefficients c_j satisfy the linear system

$$v_m(x_i) = \sum_{j=1}^n c_j^m \psi_j(x_i), \ i = 1, ..., n.$$

We use a spectral interpolation scheme with Chebyshev polynomials as basis functions. The use of these interpolating functions is known to be efficient when combined with Chebyshev interpolation nodes (Breton and de Frutos (2012)), and is often characterized by an exponential convergence. The computation of the interpolating coefficients c_j can be performed using a fast Fourier transform (FFT) algorithm.

Moreover, we evaluate the integrand for the computation of the expected value

$$\mathbb{E}\left[G_m v_{m+1}\left(X_{t_{m+1}}\right) | X_{t_m} = x\right]$$

on $\mathcal{G}$ and interpolate it using the same spectral interpolation scheme. The integration over the interval $[x_1, x_n]$ of the resulting interpolation function is analytic, and corresponds to the Clenshaw-Curtis quadrature.

The extension of this approach to cases where the state space is multidimensional (corresponding to, e.g., asset prices, stochastic volatilities, stochastic interest rates) is straightforward and involves a multidimensional grid and multidimensional Chebyshev interpolation. However, the computational burden of the recursive approach increases significantly with the dimension of the state space.

The recursive approach to CVA valuation yields M analytical functions, which can be used to evaluate the CVA at any date $t_m \in \mathcal{T}$, and for any possible value of the market factors and hazard rate. Each function is completely characterized by the n coefficients $c_j^m, j = 1, ..., n$.

To conclude, it is worth mentioning that, for contracts with early exercise opportunities, an exercise barrier divides the state space into two regions (H_m and $\overline{H_m}$) at t_m. Along this exercise barrier, the value

function may present discontinuities (if the exercise strategy is not optimal) or changes in its curvature. As a consequence, the interpolation of the value function by a polynomial may be less precise near the exercise barrier. In our implementation, we adjusted the localization interval, at each evaluation date, so that its boundary would coincide with the exercise barrier. This adjustment requires the numerical solution of the equation $\hat{V}_{t_m}^D(x) = V_{t_m}^e(x)$, where $\hat{V}_{t_m}^D$ is a polynomial of degree n. We found that setting a boundary of the localization interval to coincide with the exercise barrier can significantly improve the accuracy and the convergence of the algorithm.

6.2 Simulation

For options with early exercise opportunities, our simulation experiment is performed assuming that the value and exercise strategy of the risk-free option have already been computed and are available for all exercise dates as a function of the asset price. Actually, we obtained these by solving the default-free dynamic program (13)–(15) using spectral interpolation, as described in the preceding section.

We then simulate the default time τ and the underlying asset price trajectory using antithetic variates to reduce simulation variance. For each sample path, we record the default time τ, the corresponding time index j such that $\tau \in (t_{j-1}, t_j]$, and the first date t_k at which the price of the underlying asset is below the exercise barrier. On a given sample path, if $j > k$, default occurs after the exercise of the option and the exposure is 0. If, however, $j \leq k$, default occurs during the time interval $(t_{j-1}, t_j]$ while the option is still alive, and the exposure is set to the non-recovered fraction of the discounted default-free value. The CVA of the vulnerable option is obtained by averaging the exposures on all sample paths.

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