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Abstract: In this paper, we propose a diffusion model for a subscription service. The evolution over time of the number of subscribers is governed by a differential equation combining two processes, namely, a customer acquisition process and a customer attrition process. Assuming profit-maximization behavior of the firm, we use dynamic programming to optimize the customer equity and determine optimal customer relationship marketing expenditures. We implement an augmented Kalman filter with continuous state and discrete observations to estimate the model's parameters using market data of two well-known companies in the telecommunications sector.

To the best of our knowledge, this is the first paper to model acquisition and retention efforts in the context of a diffusion model. By doing so, we extend the literature on product diffusion to services, that is, beyond its traditional area of durable (and occasionally non-durable) products. By the same token, we contribute to the literature on customer relationship marketing (CRM) where social interactions have been overlooked. Our analytical and numerical results provide a better understanding of the relationship between the optimal customer equity, the customer lifetime value, the prospect lifetime value and the optimal acquisition and retention spending. Our model and estimation approach give the tools for assessing empirically the role of CRM spending, social interactions and other factors in the service subscription dynamics. Our empirical results show indeed that CRM spending and external incentives have a significant effect on acquisition and retention processes, and that this effect is market specific.

Key Words: Diffusion models, subscription service, customer retention, customer acquisition, optimal spending, dynamic programming.

Résumé :   Dans cet article, nous proposons un nouveau modèle de diffusion dans le contexte de services d'abonnement. L'évolution dans le temps du nombre d'abonnés est régi par une équation différentielle qui incorpore deux processus, à savoir, un processus d'acquisition de clientèle et un autre d'attrition de clientèle. Nous utilisons la programmation dynamique pour maximiser le capital client et déterminer les dépenses optimales en rétention et acquisition de clientèle. Une analyse de sensibilité est effectuée pour évaluer l'impact des principaux paramètres du modèle sur les résultats. Enfin, nous illustrons nos résultats avec trois cas d'entreprises opérant dans le secteur de la télévision payante.

Mots clés : Modèles de diffusion, service d'abonnement, rétention de clientèle, acquisition de clientèle, dépenses optimales, programmation dynamique.

1 Introduction

With the sustained improvement in Information and Communication Technologies (ICT), subscription-based services have experienced rapid growth in recent years. In 2013, the International Telecommunications Union estimates the number of mobile-cellular subscriptions to be 6.8 billion, worldwide, corresponding to a global penetration of 96%. Beyond their traditional use (home phones, cellular phones), subscription services have revolutionized the film, TV and digital media industries by introducing many new services such as YouView, iTunes, Apple TV, Netflix and other Internet streaming media services. Despite the growing role of subscription services in the modern economy, the amount of research published on the diffusion of these services remains modest compared to the significant literature on diffusion of new products. According to Peres et al. (2010), the few studies focusing on service diffusion have overlooked one fundamental aspect: the relationship between customers and service providers, particularly the marketing-expenditure management to initiate and to maintain this relationship. Indeed, the growth of these services is not restricted to the number of adopters in each period, but also depends on the number of subscribers who stay with the service. The development and maintenance of long-term relationships with customers is at the core of customer relationship marketing (CRM). This task requires an efficient marketing-expenditure management, which takes into account the costs and benefits of marketing, sales and customer interactions. In the same review, Peres et al. (2010) highlighted the need to incorporate CRM concepts into the diffusion framework for the service industry. They consider that modeling should be more directed toward tying diffusion and CRM concepts in describing the influence of relationship metrics on the growth and profitability of customers and firms.

This paper attempts to fill the gap by developing a new framework for modeling the diffusion of subscription services by considering the role of CRM expenditures over time. We propose a model where service growth is described by two processes: a customer acquisition process and a customer retention process. Each is influenced by internal incentives provided by the firm (marketing efforts) and by external incentives that include all other factors not related to marketing expenditures. Using a dynamic-programming approach, we determine the optimal acquisition and retention outlays to maximize customer equity. We start with a no-contagion effect scenario for which analytical results can be obtained. Our results shed an interesting light on the relationship between the marginal customer equity, giving by the difference between the customer lifetime value and the prospect lifetime value, and the optimal CRM strategies. We conduct a sensitivity analysis to assess how marketing effectiveness, external incentives, margin and discount rate influence the optimal acquisition and retention strategies. Next, we conduct numerical simulations for the contagion effect scenario. The results show that the optimal customer equity is almost linear in the number of subscribers. The optimal retention rate is constant in both scenarios, while the optimal acquisition rate is decreasing when a contagion effect is retained and is constant when the contagion effect is neglected. Finally, we propose an augmented Kalman filter with continuous state and discrete observations to estimate our model parameters using data from two-known service providers, namely, Sky Deutschland AG and DIRECTV. Here, the objectives are: (i) to assess the importance and magnitude of CRM spending on the diffusion process, which to the best of our knowledge has not been done before; and (ii) to determine and analyze the optimal CRM strategies in a real context. Our results reveal that CRM spending and external incentives have a significant impact on acquisition and retention processes. While Sky CRM expenditures are below the optimal level, DIRECTV over-spends in acquisition. By applying optimal strategies, DIRECTV could have saved 13.4% of total CRM expenditures and increased its total customer equity by 13.3%.

The paper is organized as follows: In Section 2, we give an overview of the literature on the diffusion of services and on optimal CRM spending. In Section 3, we develop a diffusion model that links CRM expenditures to customer acquisition and retention. In Section 4, we determine the optimal acquisition and retention expenditure allocation with and without a contagion effect, and in Section 6, we proceed to an empirical illustration. Section 7 briefly concludes.

2   Literature background

Our approach draws on two branches of the literature, namely, on the diffusion of new products (or services) in marketing, and on customer-relationship marketing. In the next two subsections, we briefly report on the papers most relevant to ours.

2.1   Diffusion of services

The new-product-diffusion literature is one of the most active in marketing science. Since the seminal paper by Bass (1969), hundreds of papers have been published, addressing a large variety of issues and contexts pertaining to the diffusion of durable products (see the surveys in Mahajan et al. 1990, and Peres et al. 2010). Surprisingly, little effort has been dedicated to modeling and forecasting the growth of service markets, despite its considerable evolution during the past few decades.

The sparse literature addressing services and using the Bass-type (or S-shaped) model includes empirical papers aimed at forecasting service growth in certain sectors such as telecommunications (see, e.g., Botelho and Pinto 2004, Jongsu and Minkyu 2009, Michalakelis et al. 2008). Other papers have added features to the basic Bass model. For instance, Krishnan et al. (2000) studied brand-level diffusion model in the cellular telephone industry, and analyzed the impact of new players on the diffusion dynamics for the category and for existing brands. Jain et al. (1991) and Islam and Fiebig (2001) extended the modeling framework to incorporate supply restrictions, and evaluated their impact on the growth of a new service in the telecommunications markets.

In the above references, the models did not account for the role of marketing variables in the diffusion process. Mesak and Darrat (2002) examined the impact of interdependence between consumers' and retailers' adoption processes on the optimal pricing policy for new subscriber services. Fruchter and Rao (2001) studied two components of the pricing decision: service access pricing and usage pricing. They show that the adoption rate can be boosted by keeping the membership fee low. These studies share a shortcoming, namely, they model the diffusion of services as though they were durable goods. However, service diffusion differs from durable goods diffusion because it involves two processes: adoption and retention. The influence of customer retention on service growth has generally been neglected. However, disregarding customer attrition will likely distort forecasts and lead to underestimation of price sensitivity (Danaher 2002). Libai et al. (2009) were probably the first to incorporate customer attrition into the Bass diffusion model. They showed that customer attrition considerably affects the market growth of a new service as well as the stock market value of service firms. However, their modeling framework did not incorporate marketing variables into either the acquisition or the retention process. Therefore, the adoption and attrition rates were assumed to be constant over time. All these papers have used the aggregate approach to model service diffusion. In many marketing applications, especially for subscription services, it is important to anticipate the timing of adoption and defection at the individual level. Therefore, alternative models have been proposed to disaggregate the diffusion process to the customer level. These models introduce customer heterogeneity into the diffusion process by incorporating explanatory variables. Numerous studies have attempted to examine service adoption drivers (see, e.g., Wareham et al. 2004, Prins and Verhoef 2007, Nam et al. 2010, Landsman and Givon 2010, Katona et al. 2011, Nitzan and Libai 2011) and service defection drivers (see, e.g., Li 1995, Bolton 1998, Bolton and Lemon 1999, Reinartz and Kumar 2003, Verhoef 2003). The two main influences on adoption and retention decisions are those under firm's control (price, advertising, direct mailing, service quality, loyalty programs, etc.) and those related to the individual (demographic profile, economic profile, satisfaction, usage patterns, personal social network/word-of-mouth, etc.).

2.2   Optimal CRM spending

It is well established that the duration of the relationship with customers has important financial implications for firms (Bolton 1998, Reinartz and Kumar 2003, Nagar and Rajan 2005, Rust and Chung 2006). Consequently, how to manage CRM expenditures optimally has received considerable interest from both practitioners and researchers in recent years. Blattberg and Deighton (1996), along with others, used a decision calculus approach to separately determine the optimal acquisition spending and the optimal retention

spending that maximize individual customer equity. Several extensions have been proposed to their model. For instance, Berger and Bechwati (2001) dealt with allocating budgets between acquisition and retention under several different market situations. Pfeifer (2005) examined the context in which the cost to acquire a new customer is five times the cost of retaining an existing one. He demonstrated that the optimal allocation depends on costs (average costs vs. marginal costs). Calciu (2008) focused on comparing the Blattberg and Deighton model to its extensions. He also proposed an alternative to the "lost-for-good" assumption, by stating that a lost customer may leave a service for a limited time but then return.

Reinartz et al. (2005) presented a conceptual framework that established the link between customer acquisition, relationship duration and customer profitability. They studied the case in which the firm can invest in customer acquisition and retention through several communication channels (telephone, face-to-face and email). Using parameter estimates, the authors simulate the allocation of resources between customer acquisition and customer retention under several scenarios. Reinartz et al. (2005) find that a suboptimal allocation to retention spending generates a greater impact on long-term customer profitability than suboptimal acquisition spending. Furthermore, they show that the optimal choice of communication channels depends on the maximizing criteria.

In a competitive context, Musalem and Joshi (2009) consider two firms competing for customer relationship in two periods, by investing in acquisition and retention efforts. Their model is based on utility functions that depend on the intrinsic customer preference and the on effectiveness of CRM efforts. They suggest that retention efforts should be focused on the moderately responsive customers, while acquisition efforts should be most aggressively targeted towards the competition's moderately profitable customers.

Surprisingly, none of these papers addressed the question of optimal CRM spending in a dynamic diffusion context. This omission is particularly striking for subscription services.

3   The model

Denote by m the market potential of a service, and by $N(t)$ the number of subscribers at time $t \in [0, \infty)$. The remaining market potential at time t is then given by $m - N(t)$. Let $a(t)$ be the conditional probability that a potential customer will subscribe to this service at time t, given that he is not an actual user. This conditional probability represents the customer acquisition rate at time t. Denote by $r(t)$ the customer retention rate, which measures the conditional probability that a current client will remain with the firm at time t. The evolution of the number of subscribers can then be represented by the following differential equation:

$$\dot{N}(t) = \frac{dN(t)}{dt} = a(t)(m - N(t)) - (1 - r(t))N(t), \quad N(0) = N_0. \tag{1}$$

Equation (1) presents the balance between new and lost customers at each time t. Observe that contrary to the case of durable products, which has been extensively studied in the literature, the number of "adopters" $N(t)$ is not necessarily monotonically increasing over time. (Clearly, knowing up front that a variable is necessarily increasing or decreasing over time simplifies considerably the analysis.) Further, we highlight from the outset that our diffusion model relies on the following hypothesis: a customer who stops his subscription is not lost forever, but joins the untapped market $m - N(t)$. An alternative to this hypothesis is the "lost-for-good" one made in, e.g., Berger and Nasr (1998). Our hypothesis implicitly means that the consumer's cost of switching back and force to another provider is relatively low, which is not unrealistic is some service industries, e.g., the telecommunications industry.[1] Here, a customer may leave a company for a while and come back later on because of a special offer or an improvement in service quality (Libai et al. 2009).

Denote by $A(t)$ the expenditure to acquire a new subscriber and by $R(t)$ the expenditure to retain an existing one. The acquisition spending includes the cost of different marketing actions to attract new subscribers, e.g., incentives (premiums, special savings), advertising. The retention spending refers to marketing expenditures in terms of, e.g., loyalty programs, direct-mail campaigns. Following Berger and Nasr-Bechwati (2001),

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[1] Or at least an envisioned one now that long-term contracts are not anymore the business model in this sector.

Pfeifer (2005) and Calciu (2008), we assume that acquisition and retention rates are related to marketing expenditures through the following exponential functions:

$$a(t) = \gamma_a \left(1 - e^{-f_0 - f_1 A(t) - q\frac{N(t)}{m}} \right), \tag{2}$$

$$r(t) = \gamma_r \left(1 - e^{-h_0 - h_1 R(t)} \right), \tag{3}$$

where q is a nonnegative imitation coefficient; γ_a and γ_r are ceiling parameters belonging to $(0, 1]$; and f_0, h_0, f_1 and h_1 are positive parameters. Our formulation of the acquisition and retention rates deserves the following comments:

1. The acquisition (resp. retention) ceiling rate γ_a (resp. γ_r) is the maximum proportion of targeted prospects who would be acquired (resp. retained) if there were no limit to spending, that is,

$$\lim_{A(t) \to \infty} a(t) = \gamma_a; \quad \lim_{R(t) \to \infty} r(t) = \gamma_r.$$

   The actual values of these ceiling parameters, as in fact all others, may vary across industries and firms within an industry.

2. The positive parameters f_1 and h_1 measure the effectiveness of marketing efforts. They could also be interpreted as the customer's sensitivity toward CRM spending. For simplicity, we assume that this effectiveness is independent on the number of subscribers $N(t)$, an assumption that is clearly worth relaxing in future work.

3. The floor rates are determined by the positive constants f_0 and h_0, and are equal to $\gamma_a \left(1 - e^{-f_0 - q\frac{N(t)}{m}} \right)$ for acquisition and $\gamma_r \left(1 - e^{-h_0} \right)$ for retention. One can think of the parameters f_0 and h_0 as measures of the external incentives to join or leave the service. An example of an external incentive is the switching cost, which has a significant role in customer retention (Jones et al., 2000) since it means it is not free for a consumer to swap. Switching costs include learning cost, transaction cost and all efforts resulting from changing service providers. Social barriers may also reduce customer defection (Woisetschläger et al. 2011). On the acquisition side, some customers come to a firm as a result of their own initiative rather than as a result of marketing efforts. These self-determined customers perceive their act of service adoption as self-instigated (Dholakia 2006). In contrast, firm-determined customers adopt a new product/service in response to firm's incentives.

4. The acquisition rate benefits from word-of-mouth (imitation or contagion), measured by the imitation coefficient times the percentage of subscribers, that is, $q\frac{N(t)}{m}$. Note that we only retain positive word-of-mouth, which is a simplifying assumption as unsatisfied lost customers can damage the reputation of the company by reporting bad experiences in their circles. Still, we believe that the introduction of contagion in the subscription dynamics is an important contribution to the literature. Indeed, in the studies that investigated telecommunications services diffusion (e.g., Meade and Islam (2006)), and in almost all the CRM literature (e.g., Berger and Nasr-Bechwati (2002), Pfeifer (2005) and Calciu (2008)), contagion effects were typically ignored.

5. The functional forms in (2) and (3) have two important properties. First, for all nonnegative $A(t)$ and $R(t)$, the resulting values of $a(t)$ and $r(t)$ are in the interval $[0, 1]$. This is consistent with our definition of $a(t)$ and $r(t)$ as conditional probabilities. In the new-product-diffusion literature, this property has often been neglected. Second, they exhibit strictly diminishing returns to acquisition (retention) spending. Indeed, we clearly have

$$\frac{da(t)}{dA(t)} = \gamma_a f_1 e^{-f_0 - f_1 A(t) - q\frac{N(t)}{m}} > 0, \qquad \frac{d^2 a(t)}{d(A(t))^2} = -\gamma_a f_1^2 e^{-f_0 - f_1 A(t) - q\frac{N(t)}{m}} < 0,$$

$$\frac{dr(t)}{dR(t)} = \gamma_r h_1 e^{-h_0 - h(t)R(t)} > 0, \qquad \frac{d^2 r(t)}{d(R(t))^2} = -\gamma_r h_1^2 e^{-h_0 - h(t)R(t)} < 0.$$

The specifications in (2) and (3) assume that marketing activities can be separated into those that affect acquisition and those that affect retention. Clearly, this may be true for certain companies but may be

difficult to argue for others. For example, advertising efforts are designed to build awareness (necessary for acquisition) and to remind consumers to buy (necessary for retention). As firms offering subscription services tend to separate the two types of marketing expenditures in their financial reports, we distinguish between acquisition and retention spending and between acquisition and retention effects. If this were not the case, we would then simply define one marketing expenditure variable that affects both acquisition and retention.

Following the literature (see, e.g., Gupta et al. 2004, Libai et al. 2009), we suppose that the service provider selects the acquisition and retention strategies that maximize customer equity (CE). According to Wiesel et al. (2008), the calculation of this value combines three customer metrics, namely, (i) the net present value of customer revenue, (ii) the net present value of customer acquisition expenditures and (iii) the net present value of customer retention expenditures. Formally, the optimization problem is stated as follows:

$$J = \max_{R,A} CE = \int_0^\infty \left[N(t) g - N(t) R(t) - (m - N(t)) A(t) \right] e^{-\rho t} dt, \tag{4}$$

$$\dot{N}(t) = a(t)(m - N(t)) - (1 - r(t)) N(t), \quad N(0) = N_0 > 0, \tag{5}$$

$$a(t) = \gamma_a \left(1 - e^{-f_0 - f_1 A(t) - q \frac{N(t)}{m}} \right), \tag{6}$$

$$r(t) = \gamma_r \left(1 - e^{-h_0 - h_1 R(t)} \right), \tag{7}$$

where g is the net revenue per subscriber, and ρ is the discount rate. $A(t)$ and $R(t)$ are the control variables and $N(t)$ is the state variable. The last two equations in the above problem are equivalent to

$$A(a) = -\frac{1}{f_1} \left(\ln \left(1 - \frac{a}{\gamma_a} \right) + f_0 + q \frac{N}{m} \right), \tag{8}$$

$$R(r) = -\frac{1}{h_1} \left(\ln(1 - \frac{r}{\gamma_r}) + h_0 \right). \tag{9}$$

Substituting for $A(a)$ and $R(r)$ in the objective function (4), the dynamic optimization problem of the service provider becomes

$$J = \max_{a, r \in [0,1]} CE = \int_0^\infty \left(N(t) g + \frac{N(t)}{h_1} \left(\ln(1 - \frac{r(t)}{\gamma_r}) + h_0 \right) \right.$$
$$\left. + \frac{(m - N(t))}{f_1} \left(\ln \left(1 - \frac{a}{\gamma_a} \right) + f_0 + q \frac{N(t)}{m} \right) \right) e^{-\rho t} dt, \tag{10}$$

$$\dot{N}(t) = a(t)(m - N(t)) - (1 - r(t)) N(t), \quad N(0) = N_0 > 0, \tag{11}$$

where the acquisition rate $a(t)$ and the retention rate $r(t)$ are the new control variables and $N(t)$ is the state variable. Optimal CRM expenditures $A(t)$ and $R(t)$ are easily calculated based on equations (8) and (9).

As we shall solve numerically the optimization problem, it is easier to have a state variable that has natural lower and upper bounds. This is achievable by defining the new state variable $x(t) = \frac{N(t)}{m}$, which is the penetration rate of the service and assumes its values in the interval $[0, 1]$. Implementing this change of state variable, the above optimization problem can be written equivalently as

$$J = \max_{a, r \in [0,1]} CE = m \int_0^\infty \left(x(t) g + \frac{x(t)}{h_1} \left(\ln(1 - \frac{r(t)}{\gamma_r}) + h_0 \right) \right.$$
$$\left. + \frac{(1 - x(t))}{f_1} \left(\ln(1 - \frac{a(t)}{\gamma_a}) + f_0 + q x(t) \right) \right) e^{-\rho t} dt, \tag{12}$$

$$\dot{x}(t) = a(t)(1 - x(t)) - (1 - r(t)) x(t), \quad x(0) = x_0 > 0. \tag{13}$$

We shall omit from now on the time argument when no ambiguity may arise.

4 Optimal customer acquisition and retention policies

To determine the optimal acquisition and retention policies, we need to solve the standard dynamic programming problem in (12)–(13). Denote by $V(x)$ the value function, that is, the maximal customer equity value that can be achieved when the penetration rate is equal to x, assuming that the service provider implements the optimal acquisition and retention policies. The Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\rho V\left(x\right) \;\; = \;\; \max_{a,r} \left(m\left(xg + \frac{x}{h_1}\left(\ln(1 - \frac{r}{\gamma_r}) + h_0 \right) + \frac{(1-x)}{f_1}\left(\ln(1 - \frac{a}{\gamma_a}) + f_0 + qx \right) \right) \right. \tag{14}$$
$$\left. + V'\left(x\right)\left(a\left(1 - x\right) - \left(1 - r\right)x \right) \right)$$

Assuming an interior solution, differentiating the right-hand side with respect to a and r, and equating to zero we get

$$a^{*}\left(x\right) \;\; = \;\; \gamma_a - \frac{m}{f_1 V'\left(x\right)}, \tag{15}$$

$$r^{*}\left(x\right) \;\; = \;\; \gamma_r - \frac{m}{h_1 V'\left(x\right)}. \tag{16}$$

Inserting these values in (14), to obtain

$$\rho V\left(x\right) \;\; = \;\; m\left(xg + \left(\frac{x}{h_1} + \frac{(1-x)}{f_1} \right)\left(\ln m - \ln V'\left(x\right) \right) \right.$$
$$+ \frac{x}{h_1}\left(-\ln h_1 \gamma_r + h_0 \right) + \frac{(1-x)}{f_1}\left(-\ln f_1 \gamma_a + f_0 + qx \right) \right) \tag{17}$$
$$+ V'\left(x\right)\left(\gamma_a\left(1 - x\right) - \left(1 - \gamma_r\right)x \right) - \frac{mx}{h_1} - \frac{m}{f_1}\left(1 - x\right),$$

and in the dynamics (13), to get

$$\dot{x} = \left(\gamma_a - \frac{m}{f_1 V'\left(x\right)} \right)\left(1 - x\right) - \left(1 - \gamma_r + \frac{m}{h_1 V'\left(x\right)} \right)x.$$

The HJB equation (17) does not admit any apparent analytical solution and has to be solved numerically, which we will do in the next section. In the following subsection, we analyze the special case where $q = 0$.

4.1 No contagion effects

As mentioned earlier, the literature in CRM has ignored contagion effects in the dynamics of subscription to a service. Wu and Chu (2010) provided an empirical support to this assumption and showed that Gompertz and logistic models outperform the Bass model in forecasting the diffusion of mobile telephony in Taiwan. Ignoring contagion, imitation or word-of-mouth effects can be justified on the ground that they are really negligible or because at the end of the day positive and negative effects cancel out and do not affect the actual outcome. One can also invoke mathematical tractability and the desire to have an analytical solution that gives some hints regarding optimal policies. We treat in some details this special case for a reason that will become apparent in the next section. The following proposition shows that if the assumption of absence of contagion effect is suitable, then the solution is in closed form and easy to interpret.

Proposition 1 *In the absence of contagion effects, i.e., $q = 0$, the value function is given by*

$$V\left(x\right) = \Gamma x + \Omega,$$

and the acquisition and retention policies by

$$a^* (x) \quad = \quad a^* = \gamma_a - \frac{m}{f_1 \Gamma}, \tag{18}$$

$$r^* (x) \quad = \quad r^* = \gamma_r - \frac{m}{h_1 \Gamma}, \tag{19}$$

where Γ is the solution of the following equation:[2]

$$H (\Gamma) = B \Gamma + C \ln(\Gamma) + D = 0, \tag{20}$$

with

$$
\begin{aligned}
B \quad &= \quad \frac{(\rho + 1 + \gamma_a - \gamma_r)}{m} \\
C \quad &= \quad \frac{1}{h_1} - \frac{1}{f_1} \\
D \quad &= \quad \left(\frac{1}{f_1} - \frac{1}{h_1} \right) \ln m + \frac{1}{h_1} \left(\ln h_1 \gamma_r - h_0 + 1 \right) - \frac{1}{f_1} \left(\ln f_1 \gamma_a - f_0 + 1 \right) - g
\end{aligned}
$$

and

$$\Omega = \frac{m}{\rho} \left(\gamma_a \frac{\Gamma}{m} + \frac{1}{f_1} \left(\ln \left(\frac{m}{\Gamma f_1 \gamma_a} \right) + f_0 - 1 \right) \right). \tag{21}$$

Proof. We set $q = 0$ in the HJB equation (17) and assume that the value function is linear and given by

$$V (x) = \Gamma x + \Omega, \tag{22}$$

and hence $V' (x) = \Gamma$. Substituting for $V (x)$ and $V' (x)$ in (17), we get

$$
\begin{aligned}
\rho \left(\Gamma x + \Omega \right) \quad = \quad &m \Bigg(xg + \ln(m) \left(\frac{x}{h_1} + \frac{(1 - x)}{f_1} \right) - \ln (\Gamma) \left(\frac{x}{h_1} + \frac{(1 - x)}{f_1} \right) \\
&+ \frac{x}{h_1} \left(- \ln (h_1 \gamma_r) + h_0 \right) + \frac{(1 - x)}{f_1} \left(- \ln (f_1 \gamma_a) + f_0 \right) \Bigg) \\
&+ \Gamma \left(\gamma_a (1 - x) - (1 - \gamma_r) x \right) - \frac{mx}{h_1} - \frac{m}{f_1} (1 - x).
\end{aligned}
$$

By identification, we obtain Γ and Ω. Next, it suffices to substitute in (15) and (16) to get the result. $\qquad \square$

A first observation is that the optimal acquisition and retention policies are independent of the state variable x, and the value function is linear. To interpret this value function in (22), let us rewrite it as a function of the state variable N

$$V (N) = \frac{\Gamma}{m} N + m \frac{\Omega}{m} = N\theta + m\mu = N (\theta + \mu) + (m - N) \mu,$$

where

$$\theta = \frac{\Gamma}{m}, \quad \mu = \frac{\Omega}{m}.$$

Then, we see that the value function is decomposed into two parts, namely, the value of actual customers, given by $N (\theta + \mu)$, and the value of untapped market potential, given by $(m - N) \mu$. In this sense, the sum of coefficients $(\theta + \mu)$ can be interpreted as the customer life value (CLV), that is, the present value of all future profits generated by an existing customer (Kamakura et al. 2005). Similarly, we interpret μ as the prospect lifetime value (PLV), that is, the value of a potential customer. This last measure provides valuable information to current and potential investors, creditors, managers and other possible stakeholders, in terms of assessing the financial performance of future prospects or potential customers. The few papers (Pfeifer and Farris 2004, Calciu 2008) that focused on the PLV concept argue that the firm should invest to convince prospects with a high PLV to become customers. Other studies (Gupta and Lehmann 2003, Gupta

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[2]Note that $H (\Gamma)$ has a Lambert function form.

and Zeithaml 2006) claim that customer-acquisition decisions should be based on CLV. Here, we show that the service provider's CRM policy is based, among other elements, on the difference between CLV and PLV. This difference is given by the coefficient θ and represents the marginal customer equity.

A second observation is that, although we have a closed-form solution to the HJB equation, the parameter Γ is only given implicitly by (20). To get additional insight into the solution, let us study the function (20):

$$H(\Gamma) = B\Gamma + C\ln(\Gamma) + D, \tag{23}$$

Deriving (23), we obtain

$$H'(\Gamma) = B + \frac{C}{\Gamma}, \quad H''(\Gamma) = -\frac{C}{\Gamma^2}.$$

Therefore, $H(\Gamma)$ can be characterized as follows:

1. If $C > 0$, that is, the acquisition effectiveness is greater than the retention effectiveness ($f_1 > h_1$), then $H(\Gamma)$ is strictly concave and increasing from $-\infty$. There exists one Γ satisfying (20).
2. If $C = 0$, that is, $f_1 = h_1$, then we would have $\Gamma = -\frac{D}{B}$. As B is strictly positive for all parameter values, we must have $D < 0$ for Γ to be positive, which is intuitively and conceptually appealing; otherwise the acquisition and retention rates (18 and 19) become greater than the ceiling rates.
3. If $C < 0$, which occurs when $f_1 < h_1$, then $H(\Gamma)$ is strictly convex and reaches its minimum at $-\dfrac{B}{C} > 0$.

   To obtain at least one solution for (20), we then have to assume that $H\left(-\dfrac{B}{C}\right) \le 0$.

Inserting (18)–(19) in (8)–(9) yields the following optimal CRM expenditures:

$$A^* = \frac{1}{f_1}\left(\ln\left(\gamma_a f_1 \theta\right) - f_0\right), \tag{24}$$

$$R^* = \frac{1}{h_1}\left(\ln\left(\gamma_r h_1 \theta\right) - h_0\right). \tag{25}$$

The above equations show that the optimal acquisition and retention expenditures are independent of the number of subscribers. Equations (24)–(25) also show that the optimal acquisition and retention expenditures depend, through the marginal customer equity θ, on all of the model's parameters. Furthermore, the optimal expenditures are increasing with θ. Similarly, optimal rates, given by (18) and (19), become closer to the ceiling rates for high values of θ and closer to the floor rates for low values of θ. This means that, when the marginal customer equity is high, the provider should invest more on the acquisition and retention of customers.

For the above expenditures to be non negative, θ must satisfy the following inequality:

$$\theta \ge \theta_m = \min\left(\frac{e^{f_0}}{\gamma_a f_1}, \frac{e^{h_0}}{\gamma_r h_1}\right), \tag{26}$$

otherwise, the firm will not invest in CRM and the acquisition and retention rates would only depend on ceiling and external parameters, that is,

$$a = \gamma_a\left(1 - e^{-f_0}\right), \quad r = \gamma_r\left(1 - e^{-h_0}\right).$$

Note that in the absence of external factors leading to subscription and retention ($f_0 = h_0 = 0$), the service provider will be then out of business. Interestingly, as we will see in the next section, we observe empirically that the parameters f_0 and h_0 are significantly different from zero. If $\theta < \frac{e^{f_0}}{\gamma_a f_1}$, then the service provider should not invest in acquisition. Similarly, if $\theta < \frac{e^{h_0}}{\gamma_r h_1}$, then it is optimal not to invest in retention.

A recurrent question in CRM is whether the firm should invest more in acquisition or retention. In our case, the answer depends on the parameter values. Indeed, the difference

$$
\begin{aligned}
A^* - R^* &= \frac{1}{f_1} \ln \left(\gamma_a f_1 \theta \right) - \frac{1}{h_1} \ln \left(\gamma_r h_1 \theta \right) - \frac{f_0}{f_1} + \frac{h_0}{h_1}, \\
&= \left(\rho + 1 + \gamma_a - \gamma_r \right) \theta - \frac{1}{f_1} + \frac{1}{h_1} - g
\end{aligned}
$$

Let Γ^* be the value of Γ when A^* is equal to R^*, that is,

$$
\theta^* = \frac{g + \dfrac{1}{f_1} - \dfrac{1}{h_1}}{\left(\rho + 1 + \gamma_a - \gamma_r \right)}.
$$

When $\theta_m \leq \theta \leq \theta^*$, the service provider should invest more in acquisition. When the marginal customer equity θ exceeds θ^*, retention expenditures should be higher.

Finally, the steady state of the dynamics is given by

$$
x_{ss} = \frac{a^*}{1 + a^* - r^*}, \tag{27}
$$

a level at which the number of new subscribers is equal to the number of lost subscribers. As $a(t)$ and $r(t)$ are constant over time, it is easy to verify that the penetration rate at time t is given by

$$
x(t) = \left(x_0 - x_{ss} \right) e^{-(1 + a^* - r^*)t} + x_{ss}, \tag{28}
$$

where a^* and r^* are given by (18)–(19).

4.1.1 Sensitivity analysis

Table 1 summarizes the effect of varying the model parameters on acquisition and retention policies ($\nearrow$ increasing, $\searrow$ decreasing). The computations of derivatives are provided in Appendix 1.

Table 1: Sensitivity Analysis.

	a^*	A^*	r^*	R^*
g	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$
ρ	$\searrow$	$\searrow$	$\searrow$	$\searrow$
γ_a	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$
γ_r	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$
h_0	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$
f_0	$\searrow$	$\searrow$	$\searrow$	$\searrow$
h_1	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow\searrow$
f_1	$\nearrow$	$\nearrow\searrow$	$\searrow$	$\searrow$

Impact of f_1: A higher acquisition effectiveness leads to lower retention expenditures, and consequently to lower retention rate. The intuition behind this result is that when acquisition effectiveness increases, retention becomes less an issue and retention expenditures can be safely reduced. Put differently, for a firm that is highly effective at attracting customers, it would be less costly to acquire (including previously lost) customers than to invest in retaining them. It is interesting also to note that the best response to an increase in the acquisition effectiveness depends on the actual level of spending in acquisition; see Appendix.

Impact of h_1: When retention effectiveness increases, acquisition expenditures (and consequently the acquisition rate) and the retention rate increase. When a firm that is efficient at having long-term relationships with its customers, which, indeed, is what retaining customer is all about, then it is worth investing heavily to acquire these customers. Taking into account the above result, we see that increasing retention effectiveness and acquisition effectiveness do not yield the same impacts. In term of impact on the retention spending, when the retention effectiveness increases, the service provider has interest to invest more in customer retention until reaching a certain level. After this level, retention expenditures should go down.

Impact of g and ρ: A higher margin per subscriber leads to an increase in customer relationship marketing expenditures and rates. This result is intuitive, as an increase in the margin means a more profitable customer relationship, and therefore, the service provider has an incentive to increase its investment in acquisition and retention. A higher discount rate leads to the opposite effect because future cash flows are less valued, and consequently, the firm is better off reducing its current investment in acquisition and retention.

Impact of f_0 and h_0:   When the external incentives f_0 to subscribe to the service go up, then the optimal acquisition and retention expenditures decline, and consequently, the corresponding rates. This result is intuitive: if the service provider benefits from free positive marketing circumstances, e.g., lower switching cost, word-of-mouth, then there is less need to invest in marketing to reach the same outcome. When the external incentives to remain with the service h_0 increases, the service provider can retain more subscribers with less retention expenditures and should invest more in acquisition to improve its acquisition rate.

5   Numerical solution

To determine the acquisition and retention policies, we need to solve the HJB equation in (17), which we rewrite in a more compact form as follows:

$$\rho V(x) = m \left[xg + (\ln m - \ln V'(x)) C(x) + \frac{Ax}{h_1} + \frac{(1-x)}{f_1} B(x) - C(x) \right] + V'(x) D(x), \qquad (29)$$

where

$$A = -\ln(h_1 \gamma_r) + h_0, \quad B(x) = -\ln(f_1 \gamma_a) + f_0 + qx,$$
$$C(x) = \frac{x}{h_1} + \frac{1-x}{f_1}, \quad D(x) = \gamma_a(1-x) - (1-\gamma_r)x.$$

As stated earlier, this equation does not have an apparent closed-form solution and we therefore need to solve it numerically. This is done following a two-step procedure:

1. From equation (29), we determine the derivative function V' by using a non-linear least-square method. Here, we use the Levenberg-Marquardt algorithm, which is a standard technique and available in Matlab.[3]
2. Once V' is obtained as a function of x and V, we use the Runge-Kutta-Fehlberg algorithm to solve the resulting ordinary differential equation $V'(x) = f(x, V)$.[4] This algorithm is also implemented in Matlab.

Our model has 10 parameters, namely,

$$
\begin{array}{rcl}
\text{Market size} & : & m, \\
\text{Diffusion parameter} & : & q, \\
\text{Acquisition rate parameters} & : & \gamma_a, f_0, f_1, \\
\text{Retention rate parameters} & : & \gamma_r, h_0, h_1, \\
\text{Revenue per subscriber} & : & g, \\
\text{Discount rate} & : & \rho.
\end{array}
$$

We ran a large number of simulations varying the different parameter values around the following base case:

---

[3]The unfamiliar and interested reader can learn on this technique by reading, as a start, the article in Wikipedia (see https://en.wikipedia.org/wiki/Levenberg%E2%80%93Marquardt_algorithm). Some tutorial references are provided at the end of the article.

[4]Again, the unfamiliar and interested reader can start by visiting the Wikipedia page on numerical solutions of ordinary differential equations at https://en.wikipedia.org/wiki/Numerical_methods_for_ordinary_differential_equations.

$$m \;\; = \;\; 100 \times 10^6, \;\; q = 0.2, \;\; \gamma_a = 0.05, \;\; f_0 = 0.01, \;\; f_1 = 0.5,$$
$$\gamma_r \;\; = \;\; 0.9, \;\; h_0 = 0.5, \;\; h_1 = 0.3, \;\; g = 20, \;\; \rho = 2.2\%.$$

The results are presented in a series of figures. In each one of them, we draw the value function varying one parameter, while the others are kept at their base case levels.

Figure 1 exhibits the value function for different values of the market size m when the diffusion parameter q is equal zero. This result comes at no surprise as we have shown analytically that the value function is linear for $q = 0$. In Figures 2a-2d we plot the value function for different values of q and m. The values of q are inspired from what has been found empirically in the large literature on diffusion of new products, and those of m cover a wide range of market sizes, i.e., from 20 to 100 millions. Clearly, the value function is almost linear and increasing in the penetration rate x. This result has been obtained in **all** simulations. To save on space, we only print few examples in Figures 3a–3h.[5]
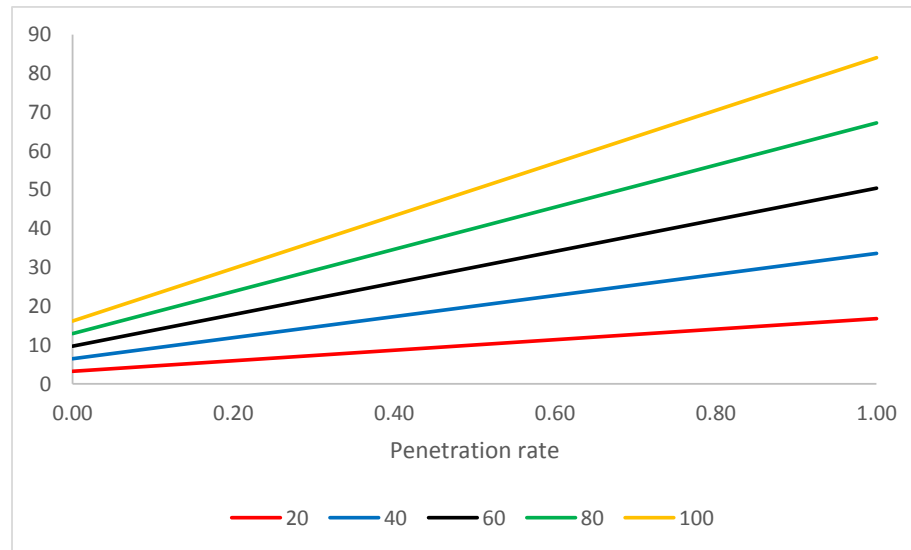

Figure 1: Value function (in 10^8) for different values of m and $q = 0$.

To see the intuition behind this robust result, namely, that the value function is (almost) linear increasing in x, first observe that $B(x), C(x)$ and $D(x)$ in (29) are all affine functions in x. Now, inspecting (29), it is easy to see that the only term that prevents this equation to hold true for a linear $V(x)$ is $-\frac{mxB(x)}{f_1}$. What the numerical results are telling us is that this term is in fact negligible (or very small) when compared to the rest of the terms in equation (29), even when m is taken very large, i.e., 100 millions. Based on these numerical results, we state the following claim.

Claim 1 *In the presence of a contagion effect, the optimal customer equity is an almost linear increasing function in the penetration rate of the service.*

Under the above claim, we have the following result, which shows that accounting for contagion has a qualitative important impact on the acquisition spending policy.

Corollary 1 *In the presence of a contagion effect, the optimal acquisition spending decreases with the number of subscribers, while the optimal retention rate is independent of that number.*

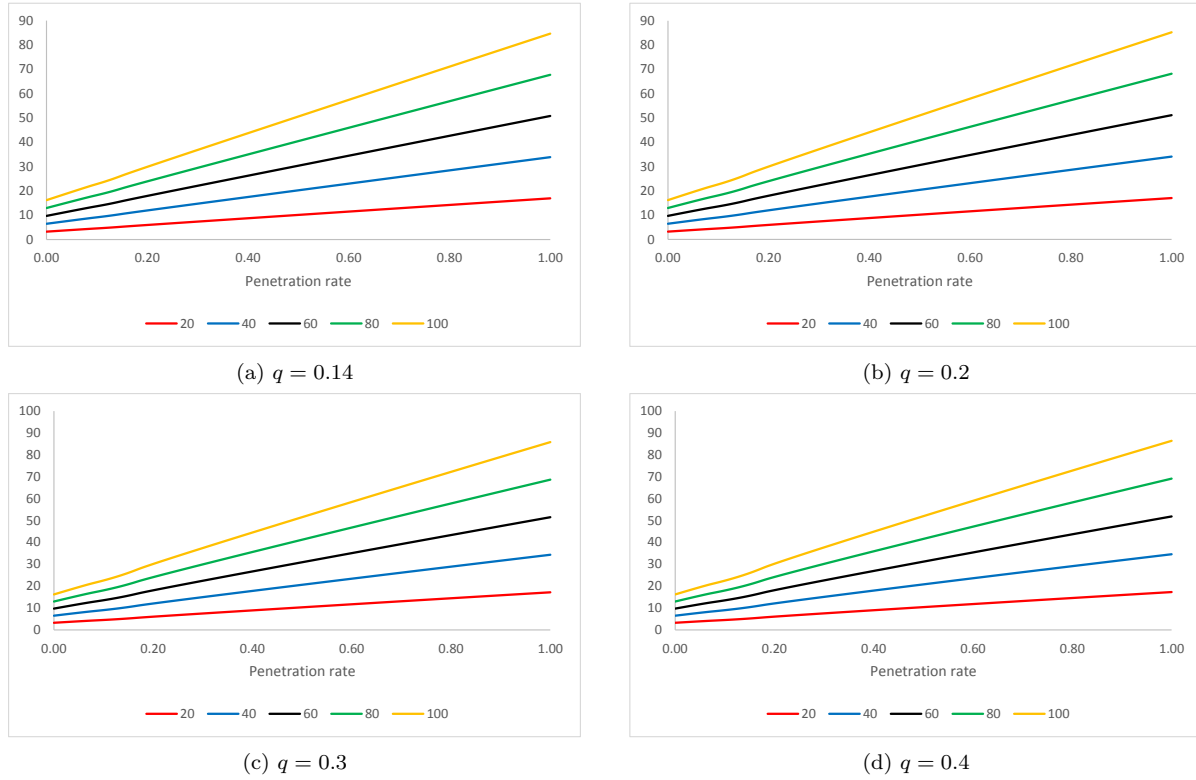[5] Other results can be provided by the authors upon request.

(a) $q = 0.14$

(b) $q = 0.2$

(c) $q = 0.3$

(d) $q = 0.4$

Figure 2: Value function (in 10^8) for different values of m and q.

Proof. If $V(x)$ is linear, then $V'(x)$ is constant and independent of x and hence of N. Substituting for the optimal acquisition and retention policies from (15) and (16) into (8) and 9, we obtain the following optimal acquisition and retention spending:

$$A = -\frac{1}{f_1}\left(\ln\left(\frac{m}{\gamma_a f_1 V'(x)}\right) + f_0 + q\frac{N}{m}\right), \tag{30}$$

$$R = -\frac{1}{h_1}\left(\ln\left(\frac{m}{\gamma_r h_1 V'(x)}\right) + h_0\right). \tag{31}$$

Differentiating with respect to N, we get

$$\frac{\partial A}{\partial N} = -\frac{q}{mf_1} < 0, \qquad \frac{\partial R}{\partial N} = 0,$$

and hence the result. $\qquad\square$

6 Empirical study

Our model stipulates that the evolution of the number of subscribers to a service depends on CRM expenditures. In this section, we propose to empirically test the validity of such assertion and eventually to assess the importance of such spending on the diffusion process (acquisition and retention). To the best of our knowledge, this is the first attempt to investigate the role of CRM in the diffusion process in a real business context. In particular, we wish to shed an empirical light on the significance and magnitude of customer's sensitivity toward CRM spending and external influence terms of our model, namely, $(f_1; h_1)$ and $(f_0; h_0)$, respectively. Next, we use estimated parameters to calibrate our optimization model and compute optimal acquisition and retention spending.

Data for the study pertain to two well-known service providers, namely, Sky Deutschland AG and DIRECTV. Sky Deutschland AG is a leading pay-TV company that operates in Germany and Austria, and

(a) Various values of f_0

(b) Various values of f_1

(c) Various values of h_0

(d) Various values of h_1

(e) Various values of γ_a

(f) Values of γ_r

(g) Various values of g
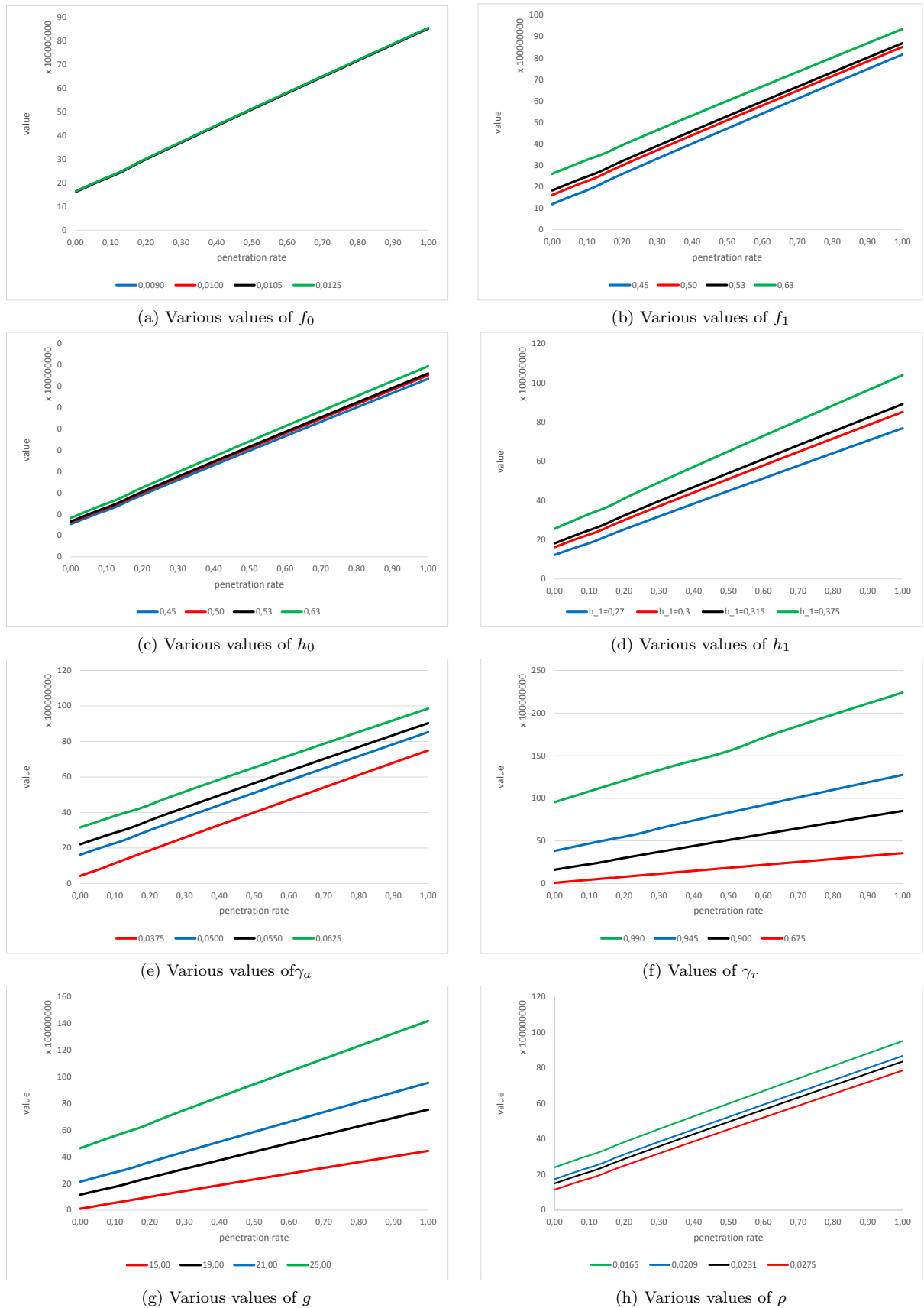
(h) Various values of ρ

Figure 3: Value function for different parameter values.

offers different categories of basic and premium digital subscription television channels via satellite and cable television. DIRECTV is also a leading provider of digital television entertainment services. This company operates in the United States, Brazil, Mexico and other countries in Latin America. To avoid problems of data comparability and the use of different monetary units, we only retain the U.S. market. We use quarterly customer and financial data that are available in annual reports, namely, the number of subscribers, the number of new subscribers, the number of lost subscribers, revenues and operating expenses, and total acquisition and retention spending. The following transformations were performed to fit our needs in terms of estimating the model's parameters (see Tables 2 and 3 for some descriptive statistics):

Margin: The margin per subscriber is given by the difference between total revenues and operating costs (excluding acquisition and retention expenditures) during one quarter, divided by the total number of subscribers. To smooth the computation, we follow Gupta et al. (2004) and Libai (2009) and take the average over the preceding four quarters.

Acquisition spending per prospect: The acquisition spending per prospect is the ratio of total acquisition expenditures to the size of the remaining (untapped) market at the beginning of a quarter.

Retention spending per customer: This is obtained by dividing total retention expenditures by the current number of subscribers at the beginning of each quarter.

Acquisition rate: This is the number of newly acquired customers during a given quarter divided by the size of the remaining market.

Retention rate: This is given by the ratio of the number of retained subscribers to the total number of subscribers at the beginning of each time period.

Table 2: Descriptive statistics for DIRECTV.

Data period: From March 2006 to March 2015				
Quarterly Statistics	Mean	St. Deviation	Min	Max
Number of new subscribers ('000)	990.16	96.98	839	1,280
Acquisition rate (a)	1.03%	0.10%	0.87%	1.34%
Acquisition spending per prospect ($\$CA$)	6.37	1.16	4.03	8.3
Number of lost subscribers ('000)	845.57	95.59	664	1,051
Retention rate (r)	95.41%	0.35%	94.58%	95.94%
Retention spending per subscriber ($\$CR$)	15.18	1.92	9.36	19.37
Revenue per subscriber ($\$g$)	108.36			
Total number of subscribers at the first quarter ('000)	15,133			
Total number of subscribers at the last of quarter ('000)	20,352			
Number of households (millions)	115			

Table 3: Descriptive statistics for Sky Deutschland AG.

Data period: From September 2007 to June 2015				
Quarterly Statistics	Mean	St. Deviation	Min	Max
Number of new subscribers ('000)	159.00	51.40	58	284
Acquisition rate (a)	0.35%	0.12%	0.13%	0.64%
Acquisition spending per prospect (€CA)	1.14	0.37	0.42	2.01
Number of lost subscribers ('000)	103.75	26.832	65	171
Retention rate (r)	96.29%	1.46%	93.16%	98.20%
Retention spending per subscriber (€CR)	6.41	0.79	4.48	7.59
Revenue per subscriber (€g)	33.60			
Total number of subscribers at the first quarter ('000)	2,495			
Total number of subscribers at the last quarter ('000)	4,225			
Number of households (millions)*	48			

Finally, we proceeded as follows to estimate the market size and the discount rate:

Market size: Since subscriptions to TV services are household based, we set the market size equal to the total number of households.

Discount rate: Researchers and practitioners suggest a range of 8% to 16% for the annual discount rate (Gupta et al. 2004). Following Gupta et al. (2004), we estimated the annual discount rate using the weighted average cost of capital. For both Sky and DIRECTV, this rate is equal to 9%,[6] which translates into a quarter discount rate of 2.2%.

6.1 Estimation and results

To estimate the different parameters of our diffusion model (1), we apply the Kalman filter method (i.e., designed to estimate state variables of a dynamic system). The basic form of the standard Kalman filter consists of two sets of equations: system equations, called also transition equations, which describe the evolution of the state vector α_t , and measurement equations, which describe the relation between the state of the system and the vector of observations y_t. More specifically, we have

$$\begin{aligned}
\text{Transition equations} &: & \alpha_{t+1} &= F\left(\alpha_t, u_t, \beta, t\right) + w_t, \\
\text{Measurement equations} &: & y_t &= G\left(\alpha_t, u_t, \beta, t\right) + v_t,
\end{aligned}$$

where $F\left(.\right)$ and $G\left(.\right)$ are vectorial functions of state α_t, control vector $\left(u_t\right)$, parameter vector $\left(\beta\right)$ and time t. The error terms $w_t \sim N\left(0, \sigma_w^2 \mathrm{I}\right)$ and $v_t \sim N\left(0, \sigma_v^2 \mathrm{I}\right)$,where I denotes the identity matrix. These errors capture the uncertainties in the model and the measurement noise, respectively. Unfortunately, several difficulties prevent a direct application of the standard Kalman filter to the estimation of diffusion models. First, the functions $F\left(.\right)$ and $G\left(.\right)$ can be nonlinear. Second, the system equations and the measurement equations are not of the same type: whereas the first is a continuous differential equation, the second is observed at discrete time intervals. Third, the standard Kalman filter treats the parameter vector β as given, while it is often unknown. To overcome the preceding limitations, we followed the approach in Naik et al (2008) where an Extended Kalman Filter method were applied to determine the vector β maximizing the log-likelihood function. Unfortunately, our results turned out to be unreliable due to the complicated (exponential) form of the retention and acquisition functions, and to the over parametrization (8 means and 12 covariances) of our diffusion model. Next, we applied the approach introduced by Xie et al (1997), that is, the Augmented Kalman Filter with Continuous state and Discrete observations (AKFCD). The AKFCD combines the Extended Kalman Filter for nonlinear system with continuous state and discrete observations and the Augmented Filter for parameter estimation by incorporating the parameter vector β in the state vector α_t. To represent our diffusion model in a state form, we let $\beta = \left(\gamma_a, f_0, f_1, q, \gamma_r, h_0, h_1\right), \alpha_t = \left(N_t, \beta\right)'$, $u_t = \left(A_t, R_t\right)$ and $y_t = \left(\tilde{N}_t, \tilde{New}_t, \tilde{Lost}_t\right)$, where $\tilde{N}_t, \tilde{New}_t$ and $\tilde{Lost}_t$ are the three main measurements published by the two companies in their quarterly reports, that is, the number of subscribers, the number of new subscribers and the number of churners. Then, the nonlinear multivariate state-space is

$$\begin{aligned}
\alpha_{t+1} &= \begin{pmatrix} N_{t+1} \\ \beta_{t+1} \end{pmatrix} = \begin{pmatrix} N_t + a_t(m - N_t) - (1 - r_t)\,N_t \\ \beta_t \end{pmatrix} + w_t, \\
y_t &= \begin{pmatrix} \tilde{N}_t \\ \tilde{New}_t \\ \tilde{Lost}_t \end{pmatrix} = \begin{pmatrix} 1 \\ a_t(m - N_t) \\ (1 - r_t)\,N_t \end{pmatrix} + v_t.
\end{aligned}$$

The AKFCD algorithm is a Bayesian updating procedure. The unknown parameters (means of α_0 and covariance matrix) are initialized based on prior distributions. The parameters are estimated recursively through two processes: a time updating process and a measurement updating process; see Xie et al. (1997). Unlike Xie et al, who constructed the prior estimates based on the research results reported in literature, we selected initial parameters as follows: (i) means and covariances of β_0 are based on the results of a linear

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[6]http://www.wikiwealth.com/

regression of equations (3) and (2) (see Appendix 3); (ii) means and covariances of N_0, w_0 and v_0 are based on maximum log-likelihood criteria as in Naik et al. (2008).

Parameter estimates and fit statistics are provided in Table 4. These results call for the following comments:

1. The marketing efficiency parameters (f_1, h_1) all have the expected (positive) sign, which confirms the positive impact of CRM expenditures on service growth. The customer sensitivity toward CRM spending is higher in the Sky market than in the DIRECTV market.
2. The imitation effect is significant only for DIRECTV. At first quarter of 2006, the imitation allowed to acquire 0.05% of prospects (more than 52K new subscribers).
3. Due to external incentives (f_0), DIRECTV can acquire 0.21% of potential customers in each period. This percentage represents 6.7% of the maximum rate that DIRECTV can reach (0.21/3.15). For Sky, less than 0.1% of prospects can be acquired without acquisition effort, which represents only around 4% of the ceiling rate. The external factors influencing a subscriber to stay with the service are (again) stronger in the case of DIRECTV than for Sky. Indeed, whereas 91.81% of DIRECTV subscribers keep the service for the next period without any retention spending, Sky would lose almost 15% of their subscribers if they make no retention effort.

A first general conclusion is that CRM expenditures indeed influence the service growth. A second conclusion is that customer sensitivity towards CRM spending as well as the external incentives are market specific, that is, their impact varies greatly across companies/markets.

To compute the optimal acquisition and retention expenditures, we used the estimated parameter values in (24) and (25). Table 5 reports the optimal spending strategies and customer metrics for each company.

A comparison of observed and optimal results calls for the following comments:

1. Figure 4 shows the observed and predicted number of subscribers for the two service providers. The two curves are very close in the case of DIRECTV, whereas our model surestimates the numbers for Sky. There many reasons that could explain the difference in the case of Sky, among them the data to which we had access and the possibility that the firm is not really optimizing.
2. Contrasting the average observed spending in Tables 2 and 3 to the optimal CRM policies in Table 5, we conclude that the two companies should invest more in retention, specially DIRECTV which must double its retention spending per subscriber. In fact, during the last years, customer retention has been a big challenge for this firm. To illustrate, in the third quarter of 2014, DIRECTV attracted $1,023,000$ new subscribers but lost $1,051,000$ existing ones. Clearly, DIRECTV is mainly focusing on

Table 4: Parameter estimates.

	Sky Deutschland AG		DIRECTV	
	Parameters	s.e	Parameters	s.e
Acquisition Equation				
Ceiling rate (γ_a)	2.50%***	0.50%	3.15%***	0.31%
Floor rate at the first quarter	0.10%		0.26%	
Spending effectiveness (f_1)	0.0577***	0.0095	0.0245***	0.0036
External incentives (f_0)	0.0410***	0.0075	0.0696***	0.0114
Contagion effect (q)	0.0172	0.2201	0.1282***	0.0492
Retention Equation				
Ceiling rate (γ_r)	99.63%***	0.0062	100%***	0.0793
Floor rate	86.51%		91.81%	
Spending effectiveness (h_1)	0.4153***	0.0637	0.0603***	0.0074
External incentives (h_0)	2.0247***	0.3699	2.503***	0.0776

(***) Significant at 1%

Table 5: Optimal spending strategies and customer metrics.

	Sky Deutschland AG	DIRECTV
Optimal CLV	€863.00	$1633.14
Optimal PLV	€41.17	$151.09
Optimal acquisition spending at first quarter	€1.68	$1.94
Optimal acquisition rate	0.31%	0.40%
Optimal retention spending	€9.15	$32.99
Optimal retention rate	99.34%	98.88%

(a) Sky   (b) DIRECTV
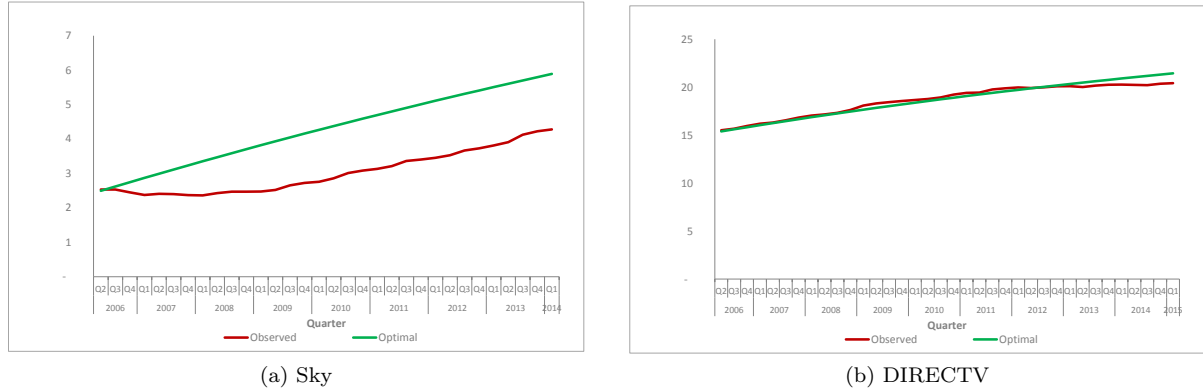
Figure 4: Evolution of the number of subscribers (10^6).

customer acquisition. On average, the observed acquisition spending per prospect is more than three times than the computed optimal level. By cutting its acquisition spending and reinvesting in retention, DIRECTV could have saved 13.4% of total CRM expenditures, increased its total customer equity by 13.3% and reached a higher number of subscribers in the first quarter of 2015.

3. Contrary to DIRECTV, Sky acquisition spending has been under the optimal level for all periods but in the last quarter of 2014; see Figure 5. Applying the optimal CRM strategies would have increased Sky customer equity by 26%, despite an increase of total CRM spending by 54%.

7   Concluding remarks

In this paper, we proposed a model linking the concepts and processes of customer relationship marketing to the diffusion of new services. Although CRM has become a central issue for many firms and research is burgeoning, little work has addressed the problem of customer acquisition and retention expenditures in the context of subscription services. Clearly, customer relationship duration in this context affects hugely the service growth and the customer profitability. Our dynamic model incorporates these (acquisition and retention) expenditures, and the service growth explicitly accounts for the two processes of customer acquisition and customer attrition. By using dynamic programming, we introduced an innovative approach to determine the optimal acquisition and retention spending that maximize the customer equity. In the no-contagion case, we showed that the optimal customer equity is linear in the number of subscribers and represents the sum of the value of existing customers and the value of the remaining market. Moreover, we found that optimal acquisition and retention policies are constant throughout the service growth and does not depend on the penetration rate. The marginal customer equity, given by the difference between the customer lifetime-value and the prospect lifetime-value, plays a crucial role in the determination of the optimal CRM expenditures. When a contagion effect is retained, our numerical simulations showed that the optimal customer equity can be well approximated by a linear function. In this case, we obtained that the optimal retention spending is constant, as in the no-contagion case, whereas the optimal acquisition spending becomes a decreasing function in the penetration rate.

(a) Acquisition spending Sky

(b)Acquisition spending DIRECTV

(c) Retention spending Sky

(d) Retention spending DIRECTV

Figure 5: Evolution of CRM spending (10^6).

For empirical evidence regarding the impact of acquisition and retention efforts on the services growth, we analyzed real data of two pay-TV companies. A Kalman filter technique was used to estimate the model parameters. The results confirmed the positive impact of CRM expenditures on the service growth and the significant presence of external factors that influence acquisition and retention processes. However, the two impacts are market specific. We obtained also that one could cut overall CRM expenditures (by reducing the acquisition spending and increasing the retention spending) and still increase the customer equity.

Finally, we mention three possible extensions to our work. First, we assumed that individual acquisition and retention costs do not vary with the penetration rate. This could explain why optimal spending policies are constant in the no-contagion effect scenario. Several researchers have shown that the innovators are less price sensitive than other consumers (Goldsmith, 1996, Goldsmith and Newell, 1997, Goldsmith et al. 2005, Park et al. 2010). The acquisition cost of early adaptors should be low compared to laggards who are more resistant to change. Likewise, the retention behavior could change with the penetration rate (Fader and Schmittlein 1992). Based on these remarks, an interesting extension to this work would to be to assume that marketing effectiveness varies with the penetration rate. Second, our model excludes the "lost-for-good"scenario and assumes that all lost customers join the untapped market. In this sense, the model might be extended by assuming that only a part of lost customers become again prospects and studying the sensitivity of the optimal policies toward this new parameter. Finally, our study is developed in a non-competitive market. Though the external incentives parameters could capture the effect of marketing efforts of other firms, a competitive framework with strategic interactions might yield additional insights on the role of CRM expenditures in the diffusion of subscription services.

Appendices

Appendix 1   Sensitivity analysis

The sign of the derivatives in Table 1 are based on the following analysis. First, recall that we have

$$H\left(\rho, \gamma_a, \gamma_r, g, h_0, f_0, h_1, f_1, m, \Gamma\right) = \left(\rho + 1 + \gamma_a - \gamma_r\right)\frac{\Gamma}{m} + \left(\frac{1}{h_1} - \frac{1}{f_1}\right)\ln\left(\frac{\Gamma}{m}\right)$$

$$+\frac{1}{f_1}\left(f_0 - \ln\left(\gamma_a\right) - \ln\left(f_1\right) - 1\right) - \frac{1}{h_1}\left(h_0 - \ln\left(\gamma_r\right) - \ln\left(h_1\right) - 1\right) - g = 0,$$

We use the implicit function theorem to determine the effect of a model parameter on Γ. Let y_i be a parameter, then we have

$$\frac{\partial \Gamma}{\partial y_i} = -\frac{\frac{\partial H}{\partial x_i}}{\frac{\partial H}{\partial \Gamma}}, \ y_i.$$

We first determine the value of $\frac{\partial H}{\partial \Gamma}$ and its sign, that is,

$$\frac{\partial H}{\partial \Gamma} = \frac{\left(\rho + 1 + \gamma_a - \gamma_r\right)}{m} + \left(\frac{1}{h_1} - \frac{1}{f_1}\right)\frac{1}{\Gamma},$$

$$= \frac{\rho + 1 + a^* - r^*}{m} > 0.$$

We can show also that

$$\begin{aligned}
H\left(\rho, \gamma_a, \gamma_r, g, h_0, f_0, h_1, f_1, m, \Gamma\right) &= \left(\rho + 1 + \gamma_a - \gamma_r\right)\frac{\Gamma}{m} + \frac{1}{h_1} - \frac{1}{f_1} - +CR^* - CA^* - g = 0, \\
&= \Gamma\frac{\partial H}{\partial \Gamma} + CR^* - CA^* - g = 0, \\
\Gamma\frac{\partial H}{\partial \Gamma} &= CA^* + g - CR^*.
\end{aligned}$$

The derivatives of H with respect to the different parameters are given by

$$\frac{\partial H}{\partial f_0} = \frac{1}{f_1}, \quad \frac{\partial H}{\partial h_0} = -\frac{1}{h_1}, \quad \frac{\partial H}{\partial \gamma_a} = \frac{\Gamma}{m} - \frac{1}{\gamma_a f_1}, \quad \frac{\partial H}{\partial \gamma_r} = -\frac{\Gamma}{m} + \frac{1}{\gamma_r h_1}, \quad \frac{\partial H}{\partial g} = -1,$$

$$\frac{\partial H}{\partial \rho} = \frac{\Gamma}{m}, \quad \frac{\partial H}{\partial f_1} = \left(\frac{1}{f_1}\right)^2\left(\ln\left(\frac{\Gamma \gamma_a f_1}{m}\right) - f_0\right), \quad \frac{\partial H}{\partial h_1} = -\left(\frac{1}{h_1}\right)^2\left(\ln\left(\frac{\Gamma \gamma_r h_1}{m}\right) - h_0\right),$$

$$\frac{\partial H}{\partial m} = -\frac{\Gamma}{m}\frac{\partial H}{\partial \Gamma}.$$

The derivatives of Γ with respect to the different parameters are given by

$$\frac{\partial \Gamma}{\partial f_0} = -\frac{1}{f_1\frac{\partial H}{\partial \Gamma}} < 0, \quad \frac{\partial \Gamma}{\partial h_0} = \frac{1}{h_1\frac{\partial H}{\partial \Gamma}} > 0,$$

$$\frac{\partial \Gamma}{\partial \gamma_a} = -\frac{\left(\frac{\Gamma}{m} - \frac{1}{\gamma_a f_1}\right)}{\frac{\partial H}{\partial \Gamma}} = -\frac{\Gamma}{m}\frac{a^*}{\gamma_a\frac{\partial H}{\partial \Gamma}} < 0, \quad \frac{\partial \Gamma}{\partial \gamma_r} = -\frac{-\frac{\Gamma}{m} + \frac{1}{\gamma_r h_1}}{\frac{\partial H}{\partial \Gamma}} = \frac{\Gamma}{m}\frac{r^*}{\gamma_r\frac{\partial H}{\partial \Gamma}} > 0,$$

$$\frac{\partial \Gamma}{\partial g} = \frac{1}{\frac{\partial H}{\partial \Gamma}} > 0, \quad \frac{\partial \Gamma}{\partial \rho} = -\frac{\Gamma}{m\frac{\partial H}{\partial \Gamma}} < 0, \quad \frac{\partial \Gamma}{\partial m} = \frac{\Gamma}{m} > 0,$$

$$\frac{\partial \Gamma}{\partial f_1} = -\frac{1}{\frac{\partial H}{\partial \Gamma}}\left(\frac{1}{f_1}\right)^2\left(\ln\left(\frac{\Gamma}{m}\gamma_a f_1\right) - f_0\right) = -\frac{CA^*}{f_1\frac{\partial H}{\partial \Gamma}} < 0,$$

$$\frac{\partial \Gamma}{\partial h_1} = \frac{1}{\frac{\partial H}{\partial \Gamma}}\left(\frac{1}{h_1}\right)^2\left(\ln\left(\frac{\Gamma}{m}\gamma_r h_1\right) - h_0\right) = \frac{CR^*}{h_1\frac{\partial H}{\partial \Gamma}} > 0,$$

$$\ln\left(\frac{\Gamma}{m}\gamma_a f_1\right) - f_0 \geq 0 \text{ and } \ln\left(\frac{\Gamma}{m}\gamma_r h_1\right) - h_0 \geq 0 \text{ (floor rates constraints).}$$

The derivatives of CA^* and CR^* are given by

$$CA^* = \frac{1}{f_1}\ln\left(\frac{\gamma_a f_1 \Gamma}{m}\right) - \frac{f_0}{f_1},$$

$$\frac{\partial CA^*}{\partial f_0} = \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial f_0}}{\Gamma} - \frac{1}{f_1} = -\frac{1}{f_1}\left(1 + \frac{1}{f_1\Gamma\frac{\partial H}{\partial \Gamma}}\right) < 0, \quad \frac{\partial CA^*}{\partial h_0} = \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial h_0}}{\Gamma} = \frac{1}{h_1 f_1 \Gamma\frac{\partial H}{\partial \Gamma}} > 0,$$

$$\frac{\partial CA^*}{\partial \gamma_a} = \frac{1}{f_1}\left(\gamma_a + \frac{\frac{\partial \Gamma}{\partial \gamma_a}}{\Gamma}\right) = \frac{1}{\gamma_a f_1}\left(1 - \frac{a^*}{m\frac{\partial H}{\partial \Gamma}}\right) = \frac{1}{\gamma_a f_1}\left(1 - \frac{a^*}{\rho + 1 + a^* - r^*}\right) > 0 \text{ since } r^* \leq 1,$$

$$\frac{\partial CA^*}{\partial \gamma_r} = \frac{\partial CA^*}{\partial \gamma_r} = \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial \gamma_r}}{\Gamma} = \frac{1}{f_1 m}\frac{r^*}{\gamma_r \frac{\partial H}{\partial \Gamma}} > 0,$$

$$\frac{\partial CA^*}{\partial g} = \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial g}}{\Gamma} = \frac{1}{f_1\Gamma\frac{\partial H}{\partial \Gamma}} > 0, \quad \frac{\partial CA^*}{\partial \rho} = \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial \rho}}{\Gamma} = -\frac{1}{f_1 m\frac{\partial H}{\partial \Gamma}} < 0, \quad \frac{\partial CA^*}{\partial m} = -\frac{1}{f_1}\frac{1}{m} + \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial m}}{\Gamma} = 0,$$

$$\frac{\partial CA^*}{\partial f_1} = -\left(\frac{1}{f_1}\right)^2\ln\left(\gamma_a f_1 \frac{\Gamma}{m}\right) + \frac{1}{f_1}\left(\frac{1}{f_1} + \frac{\frac{\partial \Gamma}{\partial f_1}}{\Gamma}\right) + f_0\left(\frac{1}{f_1}\right)^2,$$

$$= \frac{1}{f_1}\left(\frac{1}{f_1} + \frac{\frac{\partial \Gamma}{\partial f_1}}{\Gamma} - CA^*\right) = \frac{1}{f_1}\left(\frac{1}{f_1} - \frac{1}{\Gamma}\frac{CA^*}{f_1\frac{\partial H}{\partial \Gamma}} - CA^*\right),$$

$$= \frac{1}{f_1}\left(\frac{1}{f_1} - CA^*\left(1 + \frac{1}{f_1\Gamma\frac{\partial H}{\partial \Gamma}}\right)\right),$$

$$= \frac{1}{f_1}\left(\frac{1}{f_1} - CA^*\left(1 + \frac{\gamma_a - a^*}{\rho + 1 + a^* - r^*}\right)\right) = \frac{1}{f_1}\left(\frac{1}{f_1} - CA^*\left(\frac{\rho + 1 + \gamma_a - r^*}{\rho + 1 + a^* - r^*}\right)\right),$$

$$\frac{\partial CA^*}{\partial f_1} > 0 \text{ if } CA^* < \frac{1}{f_1}\left(\frac{\rho + 1 + a^* - r^*}{\rho + 1 + \gamma_a - r^*}\right); \quad < 0 \text{ else,}$$

$$\frac{\partial CA^*}{\partial h_1} = \frac{1}{f_1}\frac{\frac{\partial \Gamma}{\partial h_1}}{\Gamma} = \frac{1}{f_1\Gamma\frac{\partial H}{\partial \Gamma}}\left(\frac{1}{h_1}\right)^2[\ln(\Gamma\gamma_r h_1) - h_0] = \frac{CR^*}{h_1 f_1 \Gamma\frac{\partial H}{\partial \Gamma}} \geq 0,$$

$$CR^* = \frac{1}{h_1}\ln\left(\frac{\gamma_r h_1 \Gamma}{m}\right) - \frac{h_0}{h_1},$$

$$\frac{\partial CR^*}{\partial f_0} = \frac{1}{h_1}\frac{\frac{\partial \Gamma}{\partial f_0}}{\Gamma} = -\frac{1}{f_1 h_1 \Gamma\frac{\partial H}{\partial \Gamma}} < 0,$$

$$\frac{\partial CR^*}{\partial h_0} = \frac{1}{h_1}\frac{\frac{\partial \Gamma}{\partial h_0}}{\Gamma} - \frac{1}{h_1} = \frac{1}{h_1}\left(\frac{1}{h_1\Gamma\frac{\partial H}{\partial \Gamma}} - 1\right),$$

$$= \frac{1}{h_1}\left(\frac{\gamma_r - r^*}{\rho + 1 + a^* - r^*} - 1\right) = \frac{1}{h_1}\frac{\gamma_r - \rho - 1 - a^*}{\rho + 1 + a^* - r^*} < 0,$$

$$\frac{\partial CR^*}{\partial \gamma_a} = \frac{1}{h_1}\frac{\frac{\partial \Gamma}{\partial \gamma_a}}{\Gamma} = -\frac{1}{h_1\gamma_a m\frac{\partial H}{\partial \Gamma}} < 0,$$

$$\frac{\partial CR^*}{\partial \gamma_r} = \frac{1}{h_1}\left(\frac{1}{\gamma_r} + \frac{\frac{\partial \Gamma}{\partial \gamma_r}}{\Gamma}\right) = \frac{1}{\gamma_r h_1}\left(1 + \frac{r^*}{m\frac{\partial H}{\partial \Gamma}}\right) = \frac{1}{\gamma_r h_1}\frac{1 + \rho + a^*}{1 + \rho + a^* - r^*} > 0,$$

$$\frac{\partial CR^*}{\partial g} = \frac{1}{h_1}\frac{\frac{\partial \Gamma}{\partial g}}{\Gamma} = \frac{1}{h_1\Gamma\frac{\partial H}{\partial \Gamma}} > 0, \quad \frac{\partial CR^*}{\partial \rho} = \frac{1}{h_1}\frac{\frac{\partial \Gamma}{\partial \rho}}{\Gamma} = -\frac{1}{h_1\frac{\partial H}{\partial \Gamma}} < 0, \quad \frac{\partial CR^*}{\partial m} = -\frac{1}{h_1}\frac{1}{m} + \frac{1}{h_1}\frac{\frac{\partial \Gamma}{\partial m}}{\Gamma} = 0,$$

$$\frac{\partial CR^*}{\partial f_1} = -\frac{1}{h_1\Gamma\frac{\partial H}{\partial \Gamma}}\left(\frac{1}{f_1}\right)^2\left(\ln\left(\frac{\Gamma}{m}\gamma_a f_1\right) - f_0\right) = -\frac{CA^*}{h_1 f_1 \Gamma\frac{\partial H}{\partial \Gamma}} \leq 0,$$

$$\frac{\partial CR^*}{\partial h_1} = -\left(\frac{1}{h_1}\right)^2 \ln\left(\gamma_r h_1 \frac{\Gamma}{m}\right) + \frac{1}{h_1}\left(\frac{1}{h_1} + \frac{\frac{\partial \Gamma}{\partial h_1}}{\Gamma}\right) + h_0\left(\frac{1}{h_1}\right)^2,$$

$$= \frac{1}{h_1}\left(\frac{1}{h_1} + \frac{\frac{\partial \Gamma}{\partial h_1}}{\Gamma} - CR^*\right) = \frac{1}{h_1}\left(\frac{1}{h_1} + \frac{1}{\Gamma}\frac{CR^*}{h_1\frac{\partial H}{\partial \Gamma}} - CR^*\right),$$

$$= \frac{1}{h_1}\left(\frac{1}{h_1} + CR^*(\frac{\gamma_r - r^*}{\rho + 1 + a^* - r^*} - 1)\right) = \frac{1}{h_1}\left(\frac{1}{h_1} - CR^*(\frac{1 + \rho + a^* - \gamma_r}{\rho + 1 + a^* - r^*})\right),$$

$$\frac{\partial CR^*}{\partial h_1} > 0 \text{ if } CR^* < \frac{\rho + 1 + a^* - r^*}{h_1\left(1 + \rho + a^* - \gamma_r\right)}, < 0 \text{ else.}$$

Derivatives of a with respect to each parameter are given by

$$a^* = \gamma_a - \frac{m}{f_1\Gamma},$$

$$\frac{\partial a^*}{\partial f_0} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial f_0} = -\frac{m}{\left(f_1\Gamma\right)^2\frac{\partial H}{\partial \Gamma}} < 0, \frac{\partial a^*}{\partial h_0} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial h_0} = \frac{m}{f_1 h_1\left(\Gamma\right)^2\frac{\partial H}{\partial \Gamma}} > 0$$

$$\frac{\partial a^*}{\partial \gamma_a} = 1 + \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial \gamma_a} = 1 + \frac{m}{f_1\left(\Gamma\right)^2}\left(-\frac{\Gamma}{m}\frac{a^*}{\gamma_a\frac{\partial H}{\partial \Gamma}}\right) = 1 - \frac{1}{\gamma_a f_1\Gamma}\left(\frac{a^*}{\frac{\partial H}{\partial \Gamma}}\right)$$

$$= 1 - \frac{\gamma_a - a^*}{\gamma_a}\left(\frac{a^*}{\rho + 1 + a^* - r^*}\right) > 0$$

$$\frac{\partial a^*}{\partial \gamma_r} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial \gamma_r} = \frac{r^*}{f_1\gamma_r\Gamma\frac{\partial H}{\partial \Gamma}} > 0$$

$$\frac{\partial a^*}{\partial g} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial g} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{1}{\frac{\partial H}{\partial \Gamma}} > 0, \frac{\partial a^*}{\partial \rho} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial \rho} = -\frac{1}{f_1\Gamma\frac{\partial H}{\partial \Gamma}} < 0, \frac{\partial a^*}{\partial m} = 0$$

$$\frac{\partial a^*}{\partial f_1} = m\left(\frac{\Gamma + f_1\frac{\partial \Gamma}{\partial f_1}}{\left(f_1\Gamma\right)^2}\right) = m\left(\frac{\Gamma - \frac{CA^*}{\frac{\partial H}{\partial \Gamma}}}{\left(f_1\Gamma\right)^2}\right)$$

$$= \frac{m}{\Gamma\left(f_1\right)^2}\left(1 - \frac{CA^*}{\Gamma\frac{\partial H}{\partial \Gamma}}\right) = \frac{m}{\Gamma\left(f_1\right)^2}\left(\frac{g - CR^*}{CA^* + g - CR^*}\right) > 0$$

$$\frac{\partial a^*}{\partial h_1} = \frac{m}{f_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial h_1} = \frac{mCR^*}{h_1 f_1\left(\Gamma\right)^2} > 0$$

Derivatives of r with respect to each parameter are given by

$$r^* = \gamma_r - \frac{m}{h_1\Gamma},$$

$$\frac{\partial r^*}{\partial f_0} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial f_0} = -\frac{m}{f_1 h_1\left(\Gamma\right)^2\frac{\partial H}{\partial \Gamma}} < 0, \frac{\partial r^*}{\partial h_0} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial h_0} = \frac{m}{\left(h_1\Gamma\right)^2\frac{\partial H}{\partial \Gamma}} > 0,$$

$$\frac{\partial r^*}{\partial \gamma_a} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial \gamma_a} = \frac{m}{h_1\left(\Gamma\right)^2}\left(-\frac{\Gamma}{m}\frac{a^*}{\gamma_a\frac{\partial H}{\partial \Gamma}}\right) = -\frac{a^*}{\gamma_a h_1\Gamma\frac{\partial H}{\partial \Gamma}} < 0,$$

$$\frac{\partial r^*}{\partial \gamma_r} = 1 + \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial \gamma_r} = 1 + \frac{m}{h_1\left(\Gamma\right)^2}\frac{\Gamma}{m}\frac{r^*}{\gamma_r\frac{\partial H}{\partial \Gamma}} = 1 + \frac{r^*}{h_1\Gamma\gamma_r\frac{\partial H}{\partial \Gamma}} > 0,$$

$$\frac{\partial r^*}{\partial g} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial g} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{1}{\frac{\partial H}{\partial \Gamma}} > 0, \frac{\partial r^*}{\partial \rho} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial \rho} = -\frac{1}{h_1\Gamma\frac{\partial H}{\partial \Gamma}} < 0, \frac{\partial r^*}{\partial m} = 0,$$

$$\frac{\partial r^*}{\partial f_1} = \frac{m}{h_1\left(\Gamma\right)^2}\frac{\partial \Gamma}{\partial f_1} = -\frac{m}{h_1 f_1\left(\Gamma\right)^2}\frac{CA^*}{\frac{\partial H}{\partial \Gamma}} < 0,$$

$$\frac{\partial r^*}{\partial h_1} = \frac{\Gamma + h_1\frac{\partial \Gamma}{\partial h_1}}{\left(h_1\Gamma\right)^2} = \frac{m}{\left(h_1\right)^2\Gamma}\left(1 + \frac{CR^*}{\Gamma\frac{\partial H}{\partial \Gamma}}\right) = \frac{m}{\left(h_1\right)^2\Gamma}\frac{CA^* + g}{CA^* + g - CR^*} > 0.$$

Appendix 2   Numerical results

We provide a series of figures illustrating that indeed the value function is almost linear for different parameter values. In each figure, we let one parameter takes different values, while others remain at their benchmark's levels.

Appendix 3   Initialization of Kalman filter parameters

To initialize our Kalman filter algorithm, we estimated the seven parameters, that is, the ceiling rate (γ_a or γ_r), the spending effectiveness (f_1 or h_1) and the external incentives (f_0 or h_0) and the word-of-mouth effect (q) by using a linear regression method. We adopted the linearization approach proposed by Horsky and Simon (1983), also in a context of new-product diffusion model, i.e.,

$$\ln\left(1 - \frac{a(t)}{\gamma_a^k}\right) = -f_1 A(t) - f_0,$$

$$\ln\left(1 - \frac{r(t)}{\gamma_r^k}\right) = -h_1 R(t) - h_0,$$

where γ_a^k and γ_r^k are the values at iteration k, with $\gamma_a^k \in [\gamma_a^{\max}, 1]$ and $\gamma_r^k \in [\gamma_r^{\max}, 1]$. This procedure deserves some explanations. The lower bound in each of the intervals, that is, $\gamma_a^{\max}$ and $\gamma_r^{\max}$, is the highest value observed in the data set. The reason for this choice is twofold. First, the ceiling rate is the maximum rate that can be reached when there is no limit to spending. Therefore, this rate cannot logically be less than an already empirically observed one. Second, a ceiling rate below this maximum leads to a technical problem, namely, a logarithm of a negative number for all values exceeding this rate. Adopting a mesh size of 0.01%, we run regressions for all admissible values of γ_a^k and γ_r^k, and select the result exhibiting the highest goodness of fit $\left(R^2\right)$ for each equation.

References

Bass, F.M. 1969. A new product growth for model consumer durables. Management Science 15(5) 215–227.

Berger, P.D., N. Nasr-Bechwati. 2001. The allocation of promotion budget to maximize customer equity. OMEGA 29(1) 49–61.

Berger, P. D., N. Nasr-Bechwati. 1998. Customer lifetime value: Marketing models and applications. Journal of Interactive Marketing 12(1) 17–29.

Bolton, R.N. 1998. A dynamic model of the duration of the customer's relationship with a continuous service provider: The role of satisfaction. Marketing Science 17(1) 45–65.

Bolton, R.N., K.N. Lemon. 1999. A Dynamic model of customers' usage of services: Usage as an antecedent and consequence of satisfaction. Journal of Marketing Research 36(2) 171–186.

Botelho, A., L.C. Pinto. 2004. The diffusion of cellular phones in Portugal. Telecommunications Policy 28(5–6) 427–437.

Blattberg, R.C., J. Deighton. 1996. Manage marketing by the customer equity test. Harvard Business Review 74(4) 136–144.

Calciu, M. 2008. Numeric decision support to find optimal balance between customer acquisition and retention spending. Journal of Targeting, Measurement and Analysis for Marketing 16(3) 214–227.

Danaher, P.J. 2002. Optimal pricing of new subscription services: Analysis of a market experiment. Marketing Science 21(2) 119–138

Dholakia, U.M. 2006. How customer self-determination influences relational marketing outcomes: Evidence from longitudinal field studies. Journal of Marketing Research 43(1) 109–120.

Fader, P.S., D.C. Schmittlein. 1993. Excess behavioral loyalty for high-share brands: Deviation from the Dirichlet model for repeat purchasing. Journal of Marketing Research 30(4) 478–493.

Fruchter, G.E., R.C. Rao. 2001. Optimal membership fee and usage price over time for a network service. Journal of Service Research 4(1) 3–14.

Goldsmith, R.E. 1996. Service innovativeness and price sensitivity: An exploratory study. Association of Marketing Theory and Practice Proceedings 5 85–91.

Goldsmith, R.E., S.J. Newell. 1997. Innovativeness and price sensitivity: managerial, theoretical, and methodological issues. Journal of Product & Brand Management 6(3) 163–74.

Goldsmith, R.E., D. Kim, L.R. Flynn, W.-M. Kim. 2005. Price sensitivity and innovativeness for fashion among Korean consumers. The Journal of Social Psychology 145(5) 501–508.

Gupta, S., D.R. Lehmann. 2003. Customers as assets. Journal of Interactive Marketing 17(1) 9–24.

Gupta, S., D.R. Lehmann, J. A. Stuart. 2004. Valuing customers. Journal of Marketing Research 41(1) 7–18.

Gupta, S., V. Zeithaml. 2006., Customer, metrics and their impact on financial performance. Marketing Science 25(6) 718–739.

Horsky, D., L.S. Simon. 1983. Advertising and the diffusion of new products. Marketing Science 2(1) 1–17.

Islam, T., D.G. Fiebig. 2001. Modelling the development of supply-restricted telecommunications markets. Journal of Forecasting 20(4) 249–264.

Jain, D., V. Mahajan, E. Muller. 1991. Innovation diffusion in the presence of supply restrictions. Marketing Science 10(1) 83–90.

Jones, M.A., D.L. Mothersbaugh, S.E. Beatty. 2000. Switching barriers and repurchase intentions in services. Journal of Retailing 76(2) 259–274.

Jongsu, L., L. Minkyu. 2009. Analysis on the growth of telecommunication services: A global comparison of diffusion patterns. Applied Economics 41(24) 3143–3150.

Katona, Z., P.P. Zubcsek, M. Sarvary. 2011. Network effects and personal influences: The diffusion of an online social network. Journal of Marketing Research 48(3) 425–443.

Krishnan, T.V., F.M. Bass, V. Kumar. 2000. Impact of a late entrant on the diffusion of a new product/service. Journal of Marketing Research 37(2) 269–278.

Landsman, V., M. Givon. 2010. The diffusion of a new service: Combining service consideration and brand choice. Quantitative Marketing and Economics 8(1) 91–121.

Libai, B., E. Muller, R. Peres. 2009. The Diffusion of services. Journal of Marketing Research 46(2) 163–175.

Li, S. 1995. Survival analysis. Marketing Research 7 17–23.

Mahajan, V., E. Muller, F.M. Bass. 1990. New product diffusion models in marketing: A review and directions for research. The Journal of Marketing 54(1) 1–26.

Meade, N., T. Islam. 2006. Modelling and forecasting the diffusion of innovation- A 25-year review. International Journal of Forecasting 22(3) 519–545.

Mesak, H.I., A.F. Darrat. 2002. Optimal pricing of new subscriber services under interdependent adoption processes. Journal of Service Research 5(2) 140–153.

Michalakelis, C., D. Varoutas, T. Sphicopoulos. 2008. Diffusion models of mobile telephony in Greece. Telecommunications Policy 32(3–4) 234–245.

Musalem, A., Y.V. Joshi. 2009. How much should you invest in each customer relationship? A competitive strategic approach. Management Science 28(3) 555–565.

Nagar, V., M.V. Rajan. 2005. Measuring customer relationships: The case of the retail banking industry. Management Science 51(6) 904–919.

Nam, S., P. Manchanda, P.K. Chintagunta. 2010. The effect of signal quality and contiguous word of mouth on customer acquisition for a video-on-demand service. Marketing Science 29(4) 690–700.

Nitzan, I., B. Libai. 2011. Social effects on customer retention. Journal of Marketing 75(6) 24–38.

Prasad, A.N., A. Prasad, and S.P. Sethi 2008. Building brand awareness in dynamic oligopoly markets. Management Science 54(1) 129–138.

Park, J.E., J. Yu, J.X. Zhou. 2010. Consumer innovativeness and shopping styles. Journal of Consumer Marketing 27(5) 437–446.

Peres, R., E. Muller, V. Mahajan. 2010. Innovation diffusion and new product growth models: A critical review and research directions. International Journal of Research in Marketing 27(2) 91–106.

Pfeifer, P.E. 2005. The optimal ratio of acquisition and retention costs. Journal of Targeting, Measurement and Analysis for Marketing 13(2) 179–188.

Pfeifer, P.E., P.W. Farris. 2004. The elasticity of customer value to retention: The duration of a customer relationship. Journal of Interactive Marketing 18(2) 20–31.

Prins, R., P.C. Verhoef. 2007. Marketing communication drivers of adoption timing of a new e-Service among existing customers. Journal of Marketing 71(2) 169–183.

Reinartz, W.J., V. Kumar. 2003. The impact of customer relationship characteristics on profitable lifetime duration. Journal of Marketing 67(1) 77–99.

Reinartz, W. J., J. S. Thomas, V. Kumar. 2005. Balancing acquisition and retention resources to maximize customer profitability. Journal of Marketing 69(1) 63–79.

Rust, R.T., T.S. Chung. 2006. Marketing models of service and relationships. Marketing Science 26(6) 560–580.

Verhoef, P.C. 2003. Understanding the effect of customer relationship management efforts on customer retention and customer share development. Journal of Marketing 67(4) 30–45.

Wareham, J., A. Levy, W. Shi. 2004. Wireless diffusion and mobile computing: implications for the digital divide. Telecommunications Policy 28(5–6) 439–457.

Wiesel, T., B. Skiera, G.J. Villanueva. 2008. Customer equity: An integral part of financial reporting. Journal of Marketing 72(2) 1–14.

Woisetschläger, D.M., P. Lentz, H. Evanschitzky. 2011. How habits, social ties, and economic switching barriers affect customer loyalty in contractual service settings. Journal of Business Research 64(8) 800–808.

Wu, F.-S., W.-L. Chu. 2010. Diffusion models of mobile telephony. Journal of Business Research 63(5) 497–501.

Xie, J., X.M. Song, M. Sirbu, and Q. Wang. 1997. Kalman filter estimation of new product diffusion models. Journal of Marketing Research 34(3) 378–393.