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Graph coloring to maximize the number of communicating mobiles in wireless networks

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Abstract: Until recently, graph coloring being a computationally difficult problem, completely dynamic channel allocation was not considered in large scale networks. The combination of virtualization technologies, where powerful centralized allocation algorithms can be implemented, and recent advances in graph coloring algorithms prompts the revisiting of this view. We describe a graph coloring model for maximizing the number of simultaneously communicating mobiles in a wireless network. Since the considered problem is NP-hard, we propose various heuristic algorithms and analyze their performance, in comparison with standard decentralized channel assignment strategies such as fractional frequency reuse. We consider the LTE downlink with the WINNER channel as the reference model. We show that for blocking probabilities below 2%, our scheme typically increases the number of mobile users by 25% for 25 base stations with 120 channels. For small networks, e.g. 10 base stations and 45 mobile users, the algorithms are very close to the optimal channel allocation. The scalable resource allocation scheme has been run on a PC for a thousand users and 25 base stations with 120 channels.

Key Words: Channel assignment, improper vertex coloring.

1 Motivation

Channel allocation in wireless networks has been an active research area in the past years. Resources (frequencies) are increasingly scarce due to the large number of wireless applications, and the amount of users is in constant augmentation [1]. The quality of wireless networks therefore relies more and more on efficient optimization techniques, to reduce the required infrastructure required by increasing the bandwidth or cellular efficiency.

The goal of this paper is to show that efficient algorithms based on graph theory can help maximize the number of simultaneously communicating mobiles in a given network. A similar model was developed in [2] to minimize the number of frequencies needed to ensure simultaneous mobile communication for all users of a network.

Most cellular schemes are based on a fixed geographical resource allocation: within a given region a certain set of frequency bands are reserved to a base station which allocates these on a per needed basis to the local mobile users. The set of resources is typically assigned a priori by the operator and is static. However, current trends point to a virtualization of wireless networks and a much more centralized control of the resources [3]. With such developments, a centralized computing center would control the cells within an acceptable range. This range encompasses a urban area and is determined by transmission latency of the transmission plane or backhaul. If resource management algorithms of acceptable complexity were available, it would thus be physically possible to simultaneously control the channel assignments of hundreds of cell sites. In this paper we not only show that such algorithms are available, but we also demonstrate that it is possible to drastically increase the number of simultaneously communicating users for the standard call-blocking probability of 2%. For example, our scheme increases the number of mobile users by 25% for 25 base stations with 120 channels.

2 Fractional frequency reuse

In their latest incarnations, fourth generation mobile standards can handle superposition of signals up to a certain degree [4]. That is, for a given signal that is denoted as the desired signal, one can tolerate the superposition of all other signal if their projection on the signal space-time is less powerful than the desired signal by a certain factor.

More precisely, let $M = \{m_1, \dots, m_n\}$ be a set of mobiles and $S = \{s_1, \dots, s_t\}$ a set of base stations. Let d_{ip} denote the Euclidian distance between mobile m_i and base station s_p . If we use the standard form of a channel model (eq. 4.23 in [5]) the path loss between the base station and the mobile in dB is expressed as:

$$PL_{ip} = A \log(d_{ip}) + B + C \log\left(\frac{f_c}{5}\right) + X$$

where f_c is the operating frequency in GHz, the parameters A, B, C and X are related to the antenna heights, the medium (urban, sub-urban, walls, floors, etc.) and mostly the presence of a Line Of Sight, LOS. This is an average of the modeled path losses to which we add shadowing, a log normal Gaussian variable with standard deviation ranging from 3 to 10 dB. The received signal power P_{ip} is thus written as:

$$10 \log(P_{ip}) = 10 \log(P_{tx}) - PL_{ip} + 10 \log(S) + 10 \log(R) + G_i + G_p$$

where P_{tx} denotes the transmit power, S is the log normal random variable for shadowing, R is an exponentially distributed short term fading in power (to reflect a Rayleigh distribution of received amplitudes, see also [6]) and G_i, G_p denote the antenna gains in dB of the i th base station and the p th mobile respectively. This model is essentially used to generate received powers for our simulation. We have assumed without much loss of generality that all the base-stations have the same antenna gains and transmit power, all the mobiles also have identical antenna gains and the same propagation scenarios (NLOS, heights, etc.) exist between each base-station and mobile pair. Hence B, C, X and the standard deviation of shadowing are fixed for a given run but the path loss exponent, A , is variable from 30 to 40 dB/decade. The mobile and base-station positions are generated using a uniform distribution from which distances are computed. Then

shadowing and short term fading are randomly synthesized to produce signal powers. In a linear form the received power P_{ip} at s_p of a signal from m_i is thus:

$$P_{ip} = \frac{\alpha r_{ip}}{d_{ip}^\gamma}$$

where

- $\alpha = P_{tx} 10^{\frac{1}{10}(-C \log(\frac{f_c}{f_s}) - B - X + G_i + G_p)}$ is a constant that synthesizes the effects of antenna configuration and position accounting for the type of propagation,
- $r_{ip} = SR$ is the random variable used to generate shadowing and short term fading,
- $\gamma = \frac{A}{10}$ is the attenuation factor which typically varies from 2 in free-space to 4 for Non-Line Of Sight (NLOS) in urban areas.

Each mobile is assumed to be assigned to a given base station. Let $c(i)$ denote the channel, or color, assigned to mobile m_i . Also, for a channel k , let M_k denote the subset of mobiles using channel k , i.e., $m_i \in M_k$ if and only if $c(i) = k$. An admissible coloring scheme would require that the power of all received signals at their assigned base stations are greater, by a given factor $\frac{1}{\theta}$, than the sum of interfering powers received from mobiles which are assigned the same channel. More precisely, for a mobile m_i assigned to a base station s_p , we have the following constraint:

$$\sum_{m_j \in M_{c(i)}, j \neq i} P_{jp} \leq \theta P_{ip} \quad (1)$$

where $\frac{1}{\theta}$ is the maximal admissible Signal-to-Interference Ratio. The current technology can easily deal with an SIR = 6 dB (i.e., $\frac{1}{\theta} = 4$), and sometimes with an SIR < 3 dB (i.e., $\frac{1}{\theta} < 2$).

The optimization problem we are interested in can be formulated as follows: given a set of channels and a set of mobiles assigned to their base stations, which channel should be assigned to each mobile in order to maximize the number of mobiles respecting constraints (1)?

Orthogonal Frequency-Division Multiple Access (OFDMA) is a multi-user version of the popular orthogonal frequency-division multiplexing (OFDM) digital modulation scheme. The latest OFDMA mobile standards such as WiMAX and LTE allocate channels using a technique called Fractional Frequency Reuse (FFR) [3, 7, 8]. As the name suggests, FFR implies using the same frequencies over different geographical areas. Each cell has typically the full system bandwidth, and the problem of co-channel interference at the cell boundaries is resolved by dedicating only part of the available spectrum for the cell edges. More precisely, a cell C_p is associated with every base station s_p , and the assignment of channels to users in C_p is performed without considering how such an assignment is done in other cells. Users at the center of a cell (this notion is detailed below) have access to the whole frequency band F , while only a subset F_p of the frequencies can be used by mobiles located at the border of C_p in order to limit co-channel interference. The subsets F_p are typically chosen so that $|F_p| = \frac{|F|}{3}$ and two mobiles in the borders of two adjacent cells do not have access to the same channels (see Figure 1).

FFR is thus based on a partition of the users into interior and edge users. When the positions of the base stations are not as regular as in Figure 1, Voronoi diagrams are considered [9]. More precisely, a region C_p called Voronoi cell is defined for each base station s_p . It contains all users closer to that base station than to any other. The union of the Voronoi cells defines a Voronoi diagram that divides space into t regions (where t is the number of base stations). In such a case, as pointed out in [7], it can happen that the number of frequencies in F_p (i.e., available for edge users of C_p) is strictly smaller than $\frac{|F|}{3}$. An example is shown in Figure 2 with four base stations so that all pairs of Voronoi cells have adjacent borders. The four color theorem [10] however states that, given any separation of a plane into contiguous regions, no more than four colors are required to color the regions so that no two adjacent regions have the same color. This means that it is always possible to set $|F_p| = \frac{|F|}{4}$ for all cells C_p .

For a mobile m_i , let $a(i)$ be the index of its closest base station - mobile m_i is therefore assigned to $s_{a(i)}$ and belongs to cell $C_{a(i)}$ - and let $b(i)$ denote the index of its second closest base station. We consider the following definition of the center of a cell.

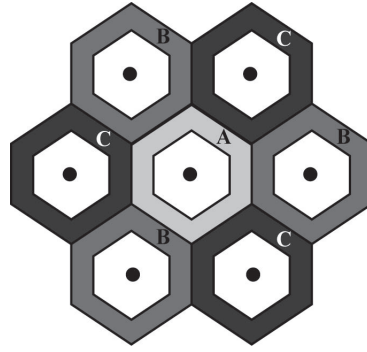


Figure 1: $F = A \cup B \cup C$; every F_p equals A, B or C ; $A \cap B = A \cap C = B \cap C = \emptyset$, $|A| = |B| = |C| = \frac{|F|}{3}$.

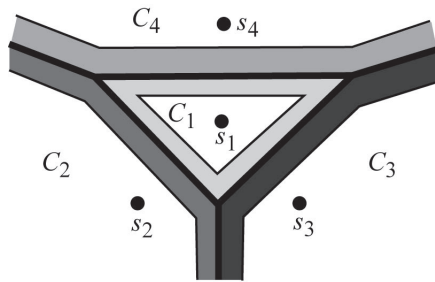


Figure 2: A base station layout where at least one F_p has at most $\frac{|F|}{4}$ frequencies.

Definition 1 A mobile m_i is in the center of $C_{a(i)}$ if and only if $\frac{P_{ib(i)}}{P_{ia(i)}} \leq \frac{\theta}{2}$; otherwise, mobile m_i is at the border of $C_{a(i)}$.

The assignment of a channel ℓ to a mobile m_i must be done without creating too much interference either for mobile m_i or for any other mobile using channel ℓ . More precisely, available channels for m_i are defined as follows.

Definition 2 A channel ℓ is considered as available for m_i if the following inequations are satisfied:

$$\sum_{m_j \in M_\ell} P_{ja(i)} \leq \theta P_{ia(i)}; \tag{2}$$

$$\sum_{m_h \in M_\ell, h \neq j} P_{ha(j)} + P_{ia(j)} \leq \theta P_{ja(j)} \quad \forall m_j \in M_\ell. \tag{3}$$

where M_ℓ denotes the set of mobiles currently using channel ℓ .

If mobile m_i is in the center of its cell $C_{a(i)}$, then any available channel $\ell \in F$ can be assigned to m_i , while only available channels in $F_{a(i)}$ can be chosen for m_i if it is at the border of $C_{a(i)}$. Moreover, every channel can be used at most once in each cell, which can be written as follows:

$$|C_p \cap M_\ell| \leq 1 \quad \forall p = 1, \dots, t, \forall \ell \in F. \tag{4}$$

Taking into account the above constraints, it is not difficult to derive an upper bound on the number of users that can communicate in a given cell when using the FFR technique.

Proposition 1 Let x and y denote the number of mobiles in the center and at the border of a given cell C_p . Then at most

$$\min \{x, |F|\} + \min \{|F_p|, y, \max \{0, |F| - x\}\}$$

of the $x + y$ mobiles can communicate simultaneously.

Proof. Consider any channel assignment that satisfies constraints (2), (3) and (4). If no channel is available for a mobile m_i in the center of C_p while a channel ℓ is assigned to at least one mobile m_j at the border of C_p , then we can keep the number of simultaneously communicating mobiles constant by assigning channel ℓ to mobile m_i instead of m_j . As a consequence, without modifying the number of communicating mobiles, one can reach a channel assignment so that a mobile in the center of C_p has no available channel only if no mobile at the border of C_p is communicating. Hence, for such an assignment, at most $\min\{x, |F|\}$ among the x users in the center of C_p are communicating. Also, at most $|F| - x$ channels are available for the mobiles at the border of C_p , which means that at most $\min\{|F_p|, y, \max\{0, |F| - x\}\}$ among the y users at the border of C_p are communicating. \square

Corollary 1 Consider a network with t base stations s_1, \dots, s_t and n mobiles m_1, \dots, m_n . Let x_p and y_p denote the number of users in the center and at the border of C_p , respectively. If channels are assigned using the FFR technique, then the following value is an upper bound on the ratio of the number of simultaneously communicating mobiles over the total number of potential users:

$$\frac{\sum_{p=1}^t \{\min\{x_p, |F|\} + \min\{|F_p|, y_p, \max\{0, |F| - x_p\}\}}}{n} \quad (5)$$

Notice that there is no guarantee that the upper bound of the above Corollary can be reached. For example, it may happen that some channels in the center of a cell are not available because of interferences with mobiles in other cells. Although FFR has proven to be an effective resource allocation technique, it lacks global vision of the network. Since this scheme is of fixed configuration, the spectrum allocation cannot adjust according to the dynamic cell load. For example, it may happen that a mobile m_i at the border of its cell $C_{a(i)}$ has no available channel in $F_{a(i)}$, while a channel ℓ that does not belong to $F_{a(i)}$ and is not assigned to any mobile in $C_{a(i)}$ is available for m_i . The assignment of channel ℓ to mobile m_i is only made possible with a centralized channel assignment strategy. The following section presents such a centralized strategy, based on graph theory concepts.

3 A graph coloring problem

Improper coloring is studied in [11] to model carrier-based inter-cell interference, and in [2] for the minimization of the number of channels needed to satisfy the demand of all users in a wireless network. Improper coloring is defined as follows. Let $G = (V, W, \omega)$ be a complete directed graph with vertex set V , and with two functions W and ω that associate a weight $W(v)$ to every vertex $v \in V$ and a weight $\omega(u, v)$ to every arc in G . For a real number θ and an integer number k , we say that a function $c : V \rightarrow \{1, \dots, k\}$ is a θ -improper k -coloring of G if, for every vertex $v \in V$, the following constraint is satisfied:

$$\sum_{u \neq v \mid c(v)=c(u)} \omega(u, v) \leq \theta W(v). \quad (6)$$

For a given directed graph $G = (V, W, \omega)$ and a real number θ , the *Directed Weighted Improper Coloring Problem* (DWICP) is to determine the minimum integer k such that G has a θ -improper k -coloring. The decision version of this problem, denoted k -DWICP, is to determine whether G admits a θ -improper k -coloring, where k and θ are given.

The link between the channel assignment problem and the DWICP is the following. For a set $M = \{m_1, \dots, m_n\}$ of n mobiles and a set $S = \{s_1, \dots, s_t\}$ of t base stations, we can construct a complete directed graph $G = (M, W, \omega)$ with vertex set $V = M$, with an arc in both directions between each pair of mobiles, and with $W(m_i) = P_{ia(i)}$ and $\omega(m_j, m_i) = P_{ja(i)}$. The construction of G is illustrated in Figure 3.

It follows from these weight definitions that equations (1) and (6) are equivalent, which means that finding a channel assignment satisfying (1) is equivalent to determining a θ -improper k -coloring of G .

We study here a slightly different problem, called *Partial Directed Weighted Improper Coloring Problem* (PDWICP). It is defined on the same directed complete graph $G = (M, W, \omega)$. The number k of available

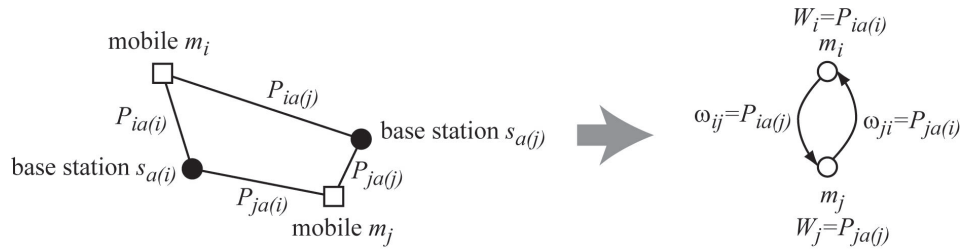


Figure 3: Construction of a directed complete graph associated with a channel assignment problem.

colors is given as well as θ , and the objective is to color as many vertices as possible while satisfying constraints (6). The example in Figure 4 has vertices that correspond to three mobiles. The numbers on the vertices and on the edges are their weights. If only $k = 1$ color is available and $\theta = \frac{1}{2}$, then only two vertices (a and b or a and c) can be colored. Indeed, if b and c receive the same color, then the sum of the weights of the arcs entering b is at least 3 which is larger than $\theta W(b) = \frac{5}{2}$.

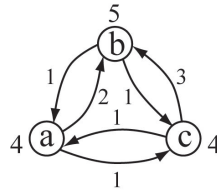


Figure 4: A graph G where at most two of the three vertices can be colored in a $\frac{1}{2}$ -improper 1-coloring.

Note that the k -DWICP is equivalent to determining whether the optimal value of the PDWICP is equal to the number $|M|$ of vertices in G . Since the k -DWICP is \mathcal{NP} -complete ([2]), we conclude that the PDWICP is \mathcal{NP} -hard.

4 A Boolean linear programming model

The PDWICP can be formulated as a Boolean linear programming problem. Indeed, let $G = (V, W, \omega)$ be a complete directed graph with vertex set V , and with weights $W(v)$ on the vertices and weights $\omega(u, v)$ on the arcs. Consider also a real number θ and an integer k . For a vertex $v \in V$ and a integer (color) $i \in \{1, \dots, k\}$, let x_{vi} be a Boolean variable that equals 1 if and only if vertex v takes color i . Also, let z_v be a Boolean variable that equals 1 if and only if a color is assigned to v . The PDWICP can be modeled as follows:

$$\max \sum_{v \in V} z_v \tag{7}$$

$$\text{s.t. } \sum_{i=1}^k x_{vi} \leq 1 \quad \forall v \in V \tag{8}$$

$$\sum_{i=1}^k x_{vi} \geq z_v \quad \forall v \in V \tag{9}$$

$$\sum_{u \neq v} \omega(u, v) x_{ui} \leq \theta W(v) + \alpha (1 - x_{vi}) \quad \forall v \in V, \quad \forall i \in \{1, \dots, k\} \tag{10}$$

$$x_{vi}, z_v \in \{0, 1\} \quad \forall v \in V, \quad \forall i \in \{1, \dots, k\} \tag{11}$$

The objective (7) is to maximize the number of colored vertices. Constraints (8) impose that every vertex can receive at most one color, while constraints (9) link the x with the z variables, imposing a zero value to z_v when vertex v has none of the k colors. Constraints (10) ensure that the colored vertices satisfy constraints

(6), where α is a large integer. One can set for example $\alpha = n\Omega$, where Ω is the largest edge weight in G . In such a case, constraints (10) do not impose any restriction if v does not have color i (i.e., when $x_{vi} = 0$).

The optimal solution of this Boolean linear programming model can only be obtained for relatively small problems (see Section 6 for more details). The model will however be useful to validate the heuristics proposed in the next section.

5 Heuristic algorithms

The standard graph coloring problem is to color the vertices of an undirected graph so that no two adjacent vertices receive the same color. Many heuristics have been proposed for this problem and we show in this section how to adapt the most popular ones to the PDWICP. These algorithms are sequential and thus efficient execution-time wise, which is an important criteria for real time wireless networks.

Assume that some vertices of G have already been colored, and let V_ℓ ($\ell = 1, \dots, k$) denote the set of vertices having color ℓ . Suppose we want to assign color i to an uncolored vertex v . In standard graph coloring, such an assignment is possible only if V_i does not contain any vertex adjacent to v . For the PDWICP, the coloring has to remain θ -improper when assigning color i to v . In order not to violate constraints (6), we have to impose constraints similar to (2) and (3). This justifies the following definition of an *available* color for v .

Definition 3 A color i is considered as available for v if the following constraints are satisfied:

$$\sum_{u \in V_i} \omega(u, v) \leq \theta W(v) \quad (12)$$

$$\omega(v, u) + \sum_{x \in V_i, x \neq u} \omega(x, u) \leq \theta W(u) \quad \forall u \in V_i. \quad (13)$$

Consider now a vertex v and a set X of vertices (possibly including v). Most sequential coloring algorithms use decision rules based on the number $deg_X(v)$ of vertices in X adjacent to v . For example, with $X = V$, $deg_V(v)$ is the degree of v which corresponds to the number of edges incident to v . For the PDWICP, we use the following measure $\mu_X(v)$ instead of $deg_X(v)$:

$$\mu_X(v) = \frac{\sum_{u \in X \setminus \{v\}} \omega(u, v)}{W(v)}.$$

5.1 Adaptations of the Welsh-Powell algorithm

The Welsh-Powell algorithm ([12]) is a greedy heuristic based on a static order of the vertices. It can be described as follows: the vertices are first ordered by non-increasing values $deg_V(v)$; the algorithm then considers the vertices in that order and gives to each one the smallest available color (integer) not already used by at least one of its neighbors. We propose a similar algorithm for the PDWICP. More precisely, the vertices are first ordered by non increasing values $\mu_V(v)$. Each vertex v then receives the smallest available color $i \in \{1, \dots, k\}$ for v . If no color i is available for v , then v is not colored. This algorithm, called WP1, is summarized in Table 1. It can be shown that it runs in $\mathcal{O}(n^2)$, provided appropriate data structures are used.

Experiments have shown that the WP1 algorithm may create very small color classes. This is due to the fact that it may happen that two vertices u and v are put in the same color class while $\omega(u, v)$ represents a large portion of the interference tolerance $\theta W(v)$ at v . To avoid such situations, we have implemented a slightly different adaptation of the Welsh-Powell algorithm, called WP2. When going through the ordered list L , we first consider that a color i is available for v if, in addition to constraints (12) and (13), color i satisfies the following constraints, where δ is a given real number in $[0, 1[$:

$$\omega(u, v) \leq \delta \theta W(v) \quad \text{and} \quad \omega(v, u) \leq \delta \theta W(u) \quad \forall u \in V_i. \quad (14)$$

Table 1: First adaptation of the Welsh-Powell algorithm.

Algorithm 1: WP1

INITIALIZATION:
Determine an ordered list $L = (v_1, \dots, v_n)$ of the vertices of G so that v_i appears earlier than v_j in L whenever $\mu_V(v_i) > \mu_V(v_j)$;
Set $V_\ell = \emptyset$ for $\ell = 1, \dots, k$;

COLORING:
for $v = v_1$ **to** v_n **do**
| If there exists at least one available color for v in $\{1, \dots, k\}$, then choose the smallest one ℓ and
| add v to V_ℓ ;
end

We then go through list L a second time, only considering the uncolored vertices. These are colored using constraints (12) and (13) (but not constraints (14)). The algorithm is summarized in Table 2. It is easy to observe that the proposed modification does not affect the algorithm's complexity, which means that it also runs in $\mathcal{O}(n^2)$, provided appropriate data structures are used. In our experiments, we use $\delta = 0.1, 0.2, \dots, 0.9$ and take the best result over the 9 runs.

Table 2: Second adaptation of the Welsh-Powell algorithm.

Algorithm 2: WP2

INITIALIZATION:
Determine an ordered list $L = (v_1, \dots, v_n)$ of the vertices of G so that v_i appears earlier than v_j in L whenever $\mu_V(v_i) > \mu_V(v_j)$;
Set $V_\ell = \emptyset$ for $\ell = 1, \dots, k$;

COLORING:
for $\ell = 1$ **to** k **do**
| **for** $v = v_1$ **to** v_n **do**
| | If v is not colored and constraints (12), (13) and (14) are satisfied for v with color ℓ , then add v
| | to V_ℓ ;
| **end**
| **for** $v = v_1$ **to** v_n **do**
| | If v is not colored and constraints (12) and (13) are satisfied for v with color ℓ (i.e., color ℓ is
| | available for v), then add v to V_ℓ ;
| **end**
end

5.2 Adaptation of the DSATUR algorithm

Given a partial standard coloring of the vertices of an undirected graph, the *saturation degree* of a vertex v is defined as the number of different colors appearing on vertices adjacent to v . The DSATUR algorithm, proposed by Brélaz [13], colors the vertices sequentially, choosing at each step the uncolored vertex with highest saturation degree. If several vertices maximize this value, the algorithm chooses one with a maximum number of uncolored adjacent vertices. The color assigned to the selected vertex is then the smallest color not appearing on an adjacent vertex. An adaptation of Brélaz's algorithm was used in [14, 15] and also in [3] where it proved to be very efficient as it achieved a 33% gain in cell throughput over conventional FFR.

The DSATUR algorithm can easily be adapted to the PDWICP. For this purpose, we consider the set U of uncolored vertices which have at least one available color. At the beginning of the algorithm, U is therefore set equal to the vertex set V . Then, at each step of the algorithm, we consider a vertex $v \in U$ with maximum *saturation degree* $dsat(v)$ defined as the number of non available colors for v , i.e., the number of

colors $i \in \{1, \dots, k\}$ that do not satisfy constraints (12) and/or (13). In case of ties, we choose one with maximum value $\mu_U(v)$. We then remove v from U and assign to v the smallest available color in $\{1, \dots, k\}$. Also, we update $dsat(u)$ for all vertices $u \in U$ and remove those for which $dsat(u) = k$ (since none of the k colors is available for them). We finally update $\mu_U(v)$ for all vertices still belonging to U . The adaptation of the DSATUR algorithm to the PDWICP, called DSAT, is summarized in Table 3. It can be shown that this algorithm has a $\mathcal{O}(n^3)$ worst case complexity.

Table 3: Adaptation of the DSATUR algorithm.

Algorithm 3: DSAT

INITIALIZATION:
Set $U = V$ and $V_\ell = \emptyset$ for $\ell = 1, \dots, k$;
Set $dsat(v) = 0$ and $\mu_U(v) = \mu(v)$ for all vertices v ;

COLORING:
while $U \neq \emptyset$ **do**
 Determine a vertex $v \in U$ with maximum saturation degree $dsat(v)$. In case of ties, choose one with maximum value $\mu_U(v)$;
 Choose the smallest available color $i \in \{1, \dots, k\}$ for v and add v to V_i ;
 Remove v from U and update $dsat(u)$ for all $u \in U$;
 Remove from U all vertices u for which $dsat(u) = k$;
 Update $\mu_U(u)$ for all vertices in U ;
end

5.3 Adaptation of the RLF algorithm

The *Recursive Largest First* (RLF), proposed by Leighton [16], builds a sequence V_1, \dots, V_k of color classes. Consider the construction of one color class V_ℓ , let U denote the set of uncolored vertices, and let W be the set (initially empty) of uncolored vertices with at least one neighbor in V_i . Every time a vertex in U is chosen to be moved to V_i , all its neighbors in U are moved from U to W . The first vertex $v \in U$ to be included in V_i is the one with the largest number of adjacent vertices in U . The rest of V_i is built as follows : while U is not empty, the next vertex to be moved from U to V_i is one having the largest number of neighbors in W . Ties are, if possible, broken by choosing a vertex with the smallest number of neighbors in U .

The adaptation of the RLF algorithm to the DWICP is done as follows. When constructing V_i , W is defined as the set of uncolored vertices v for which color i is not available (i.e., at least one of constraints (12) and (13) is violated if color i is assigned to v). Moreover, instead of counting how many vertices in $X = U$ or W are adjacent to a given vertex v , we use the $\mu_X(v)$ measure. The proposed adaptation of the RLF algorithm to the PDWICP, called RLF, is summarized in Table 4. It can be shown that it runs in $\mathcal{O}(n^3)$ with suitable data structures.

6 Numerical results

To test the performance of the proposed heuristics, random instances were generated in the same way as in [2]. More precisely, positions of mobiles and base stations were randomly generated according to a uniform distribution in the Euclidian plane, forbidding situations where base stations are too close to one another. Signals' powers were computed according to the formulas given in Section 2, with a propagation loss factor γ equal to 4.

We first compare the optimal value, denoted OPT, provided by the Boolean linear programming model, with the values produced by the four proposed heuristics WP1, WP2, DSAT and RLF. Such a comparison is only possible for instances of relatively small size since the time needed to compute OPT grows exponentially with the number n of users. Average results obtained for 50 instances with $t = 5$ base stations, $n = 25$ mobiles, $k = 15$ channels and $\theta = 0.25$ (i.e., an SIR = 6 dB) are summarized in Figure 5. In abscissa, we

Table 4: Adaptation of the RLF algorithm.

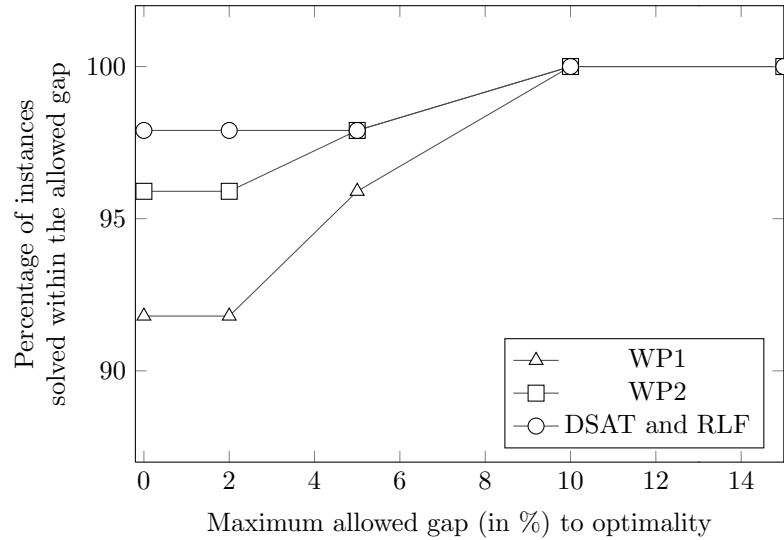
Algorithm 4: RLF**INITIALIZATION:**Set $V_\ell = \emptyset$ for $\ell = 1, \dots, k$;**COLORING:****for** $i = 1$ *to* k **do** Let U be the set of uncolored vertices; Choose a vertex $v \in U$ with largest value $\mu_U(v)$; Add v to V_i and move to W all vertices in U for which color i is no more available; **while** $U \neq \emptyset$ **do** Select a vertex $u \in U$ with largest value $\mu_W(u)$ and move it to V_i . In case of ties, choose one with smallest value $\mu_U(u)$; Move to W all vertices in U for which color i is no more available; **end****end**

Figure 5: Comparisons between exact and heuristic algorithms.

consider various maximum allowed gaps (expressed in percentage) between OPT and the value produced by the heuristics, while the ordinate indicates the percentage of instances for which the given gap was not exceeded. For example, considering WP1, we can read that 92% of the instances (i.e., 46 out of 50) were solved to optimality, while the maximum gap between OPT and the obtained value has never exceeded 10%. WP2 performs better than WP1 since it produces 96% optimal solutions (i.e. 48 out of 50 instances). RLF and DSAT, which have a complexity of $O(n^3)$, give better results than WP1 and WP2, whose running time is in $O(n^2)$.

Next, we compare the performances of the four heuristics with two upper bounds related to the Fractional Frequency Reuse (FFR) technique. As stated in Corollary 5 in Section 2, the following value is an upper bound on the ratio of the number of simultaneously communication mobiles over the total number of potential users :

$$\frac{\sum_{p=1}^t \{\min \{x_p, |F|\} + \min \{|F_p|, y_p, \max \{0, |F| - x_p\}\}\}}{n}.$$

We denote FFR3 the value of this bound obtained by considering $|F_p| = \frac{|F|}{3}$. As already mentioned, FFR3 is a very optimistic bound which can hardly be reached, not only because some channels in the center of a cell possibly create interferences with mobiles in other cells, but also because real base station layouts do not permit to fix $|F_p| = \frac{|F|}{3}$ (see Figure 2). A more realistic upper bound, denoted FFR4, is therefore obtained by fixing $|F_p| = \frac{|F|}{4}$.

Computational results for different values of n (number of mobiles), t (number of base stations), and k (number of channels) are reported in Tables 5, 6 and 7. Each entry in these tables corresponds to the average coverage ratio (i.e., the percentage of mobiles to which a channel could be assigned) taken over 20 instances. A coverage ratio of 98% corresponds to the typical accepted call-blocking probability of 2%. The optimal value OPT is given for the small instances having $t = 10$ base stations and $k = 12$ channels. The numbers in columns 'FFR3' and 'FFR4' are the values of the two upper bounds multiplied by 100. Figures 12, 13, and 8 show the statistical distribution of these results for the triplets $(n, t, k) = (20, 10, 12)$, $(200, 15, 48)$ and $(800, 25, 120)$.

For networks having $t = 10$ base stations and $k = 12$ available channels, we observe that WP1, WP2, DSAT and RLF not only produce a better coverage ratio than the optimistic upper bound FFR3, but are also close to optimality. Indeed, it takes less than 20 mobiles to get a coverage ratio below 98% (i.e., a call-blocking probability of 2%) with the FFR technique, while this ratio is reachable for 35 mobiles when using WP1, WP2, DSAT, or RLF, and for 40 mobiles when using the Boolean linear programming model. Moreover, the statistical distributions in Figure 12 show that these results are stable since the boxplots of WP1, WP2, DSAT, RLF consist of a unique point (if we except outliers).

The results are similar for networks having $t = 15$ base stations and $k = 48$ channels : while 190 mobiles are sufficient to go below a coverage ratio of 98% with the decentralized FFR technique, this coverage limit is reached by WP1, WP2, DSAT and RLF with about 220 mobiles.

The results obtained for larger and more realistic wireless networks with up to $t = 25$ base stations and $k = 120$ channels lead to interesting observations. We first note that the coverage ratio of 98% is reached by the FFR technique with about 750 mobiles while 200 additional mobiles can communicate using the WP2 heuristic. The performances of WP1, DSAT and RLF are however not as good as those of WP2. For example, while the coverage ratio of WP2 for 1050 users is 95.5%, it lies between 84.3% and 86.4% for the other heuristics. Such a decrease of about 10% represents more than 100 users. This bad behavior of WP1, DSAT and RLF can be explained as follows. When the optimal coverage ratio is strictly smaller than 100%, it seems reasonable to remove from the network those users having large interferences with other mobiles. By keeping these users in the network and assigning them a channel, one takes the risk of preventing many other users from communicating. WP1, DSAT and RLF do not take this risk into account. The situation is different for WP2 since the first visit of the ordered list L is done so that two users that interfere much with one another do not get the same channel. Note however that the observed bad behavior of WP1, DSAT and RLF only occurs for a low coverage ratio, much below the standard acceptable call-blocking probability of 2%.

Since WP2 can be considered as the winner of the above comparisons, we finally compare it in Figures 9, 10, and 11 with the optimistic upper bound FFR3. For $(t, k) = (10, 12)$, we observe that the call-blocking probability (CBP) of 2% (shown with an horizontal line) is reached with 20 users with the FFR technique, while 15 additional users can communicate with WP2. For $(t, k) = (15, 48)$, the number of communicating users with the CBP of 2% increases from 190 to 220, while the increase is from 750 to 950 users for $(t, k) = (25, 120)$.

We have performed similar tests with $\gamma = 3$. The results are shown in Figures 12 and 13. A decrease of γ increases the interferences, which implies that less users can communicate simultaneously. As expected, we see by comparing Figures 11 and 12 that the number of communicating users with $\theta = \frac{1}{4}$ and a CBP of 2% is reduced from 950 to 780 when decreasing γ from 4 to 3, while the upper bound FFR3 is decreased from 750 to 650.

The case $(\theta, \gamma) = (\frac{1}{2}, 3)$ is less constrained than $(\frac{1}{4}, 3)$ since the maximal admissible SIR $\frac{1}{\theta}$ is decreased from 4 to 2. We can see in Figure 13 that the coverage ratio of 98% is reached by the FFR technique with

Table 5: Coverage ratio (in %) with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 10$ base stations, $k = 12$ channels.

n	OPT	FFR3	FFR4	WP1	WP2	DSAT	RLF
10	100.0	100.0	100.0	100.0	100.0	100.0	100.0
15	100.0	99.7	97.7	100.0	100.0	100.0	100.0
20	100.0	97.5	93.5	100.0	100.0	100.0	100.0
25	99.4	97.0	92.6	99.4	99.4	99.4	99.4
30	99.8	95.7	90.7	99.0	99.3	99.5	98.7
35	99.4	93.7	88.6	98.3	98.7	98.3	98.4
40	97.9	91.0	84.9	94.6	95.5	96.3	95.3
45	97.0	90.4	84.0	89.9	93.1	92.9	91.6

Table 6: Coverage ratio (in %) with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 15$ base stations, $k = 48$ channels.

n	FFR3	FFR4	WP1	WP2	DSAT	RLF
75	100.0	99.9	100.0	100.0	100.0	100.0
100	100.0	99.7	100.0	100.0	100.0	100.0
125	99.2	97.8	100.0	100.0	100.0	100.0
150	99.4	98.2	100.0	100.0	100.0	100.0
175	98.8	95.5	100.0	100.0	100.0	100.0
200	97.0	93.4	99.2	99.4	99.7	99.1
225	94.4	89.4	94.6	96.9	97.2	96.3
250	94.0	88.2	93.6	95.6	94.9	94.2

Table 7: Coverage ratio (in %) with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 25$ base stations, $k = 120$ channels.

n	FFR3	FFR4	WP1	WP2	DSAT	RLF
500	99.6	98.8	100.0	100.0	100.0	100.0
550	99.4	98.0	100.0	100.0	100.0	100.0
600	99.2	97.9	100.0	100.0	100.0	100.0
650	98.7	97.1	99.4	99.8	99.4	99.4
700	98.8	96.1	99.4	99.9	99.6	99.7
750	97.8	94.7	99.4	99.8	98.6	99.5
800	97.4	93.7	97.8	99.3	97.8	97.8
850	97.5	93.5	96.4	98.7	96.5	96.1
900	97.0	93.1	97.1	99.2	94.8	95.6
950	96.9	92.5	93.5	97.8	93.0	93.0
1000	95.4	90.7	87.3	96.4	88.0	86.4
1050	94.9	89.4	85.0	95.5	86.4	84.3

about 830 mobiles while 80 additional mobiles can communicate using the WP2 heuristic. In summary, it appears that the advantage of the proposed heuristic over the FFR technique does not depend on the values of θ and γ , i.e., on the threshold SIR or the propagation characteristics. The minimal gain of the coloring model is about 10% in all situations.

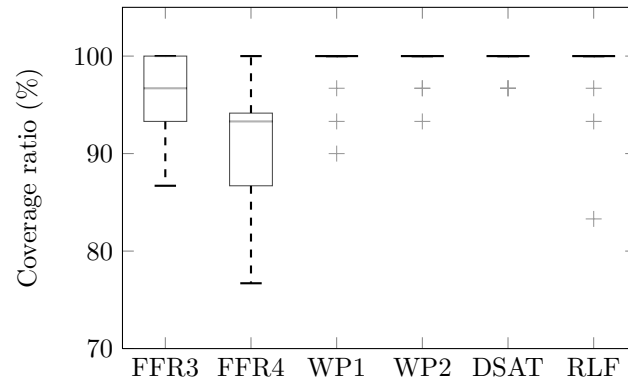


Figure 6: Boxplot of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 10$ base stations, $k = 12$ channels, $m = 30$ mobiles.

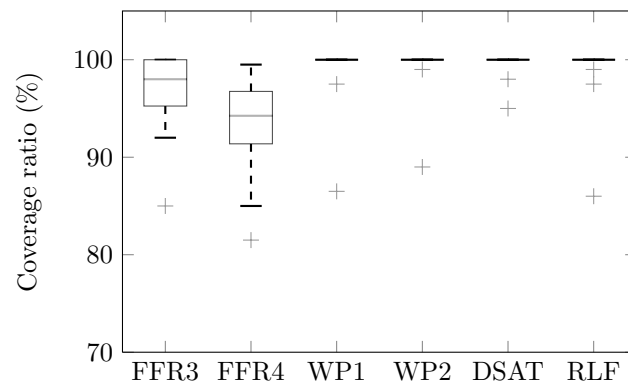


Figure 7: Boxplot of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 15$ base stations, $k = 48$ channels, $m = 200$ mobiles.

fig9

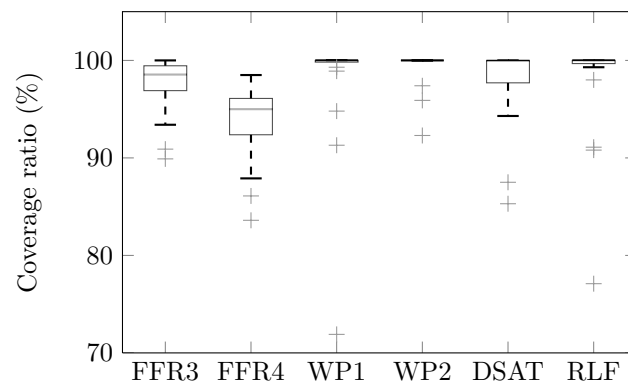


Figure 8: Boxplot of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 25$ base stations, $k = 120$ channels, $m = 800$ mobiles.

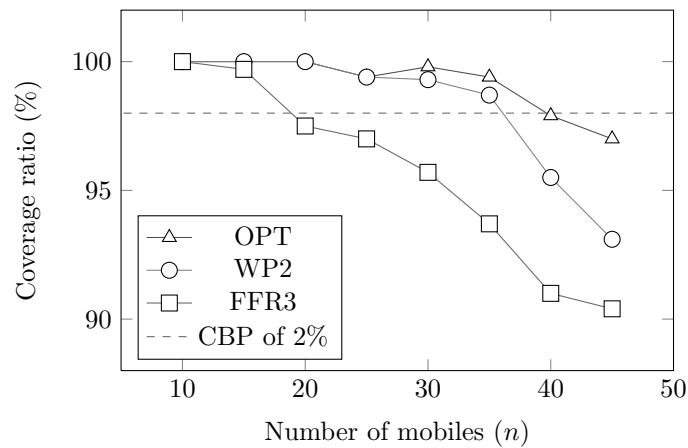


Figure 9: Evolution of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 10$ base stations, $k = 12$ channels.

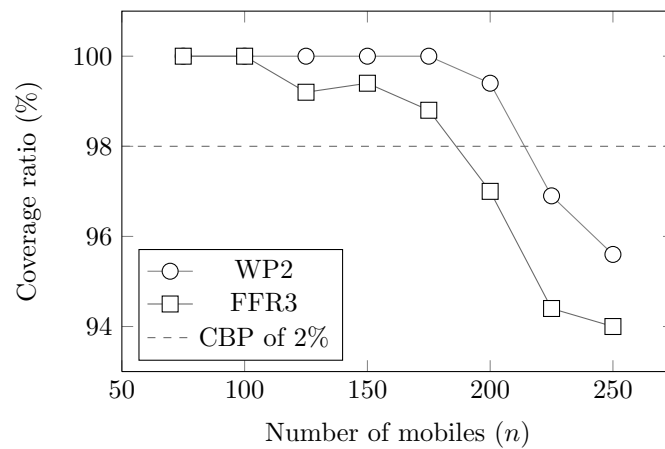


Figure 10: Evolution of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 15$ base stations, $k = 48$ channels.

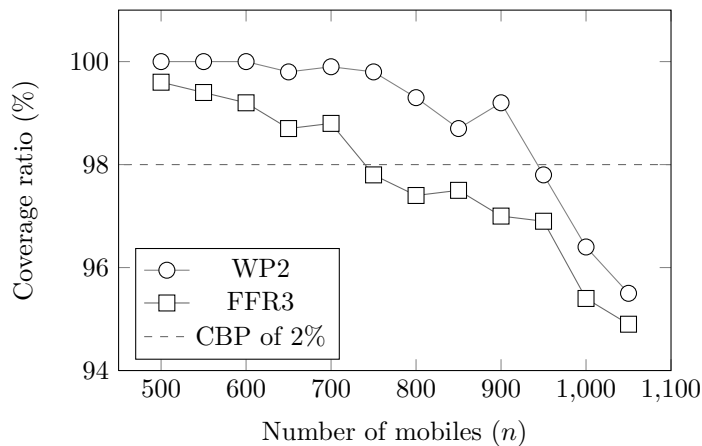


Figure 11: Evolution of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 4)$, $t = 25$ base stations, $k = 120$ channels.

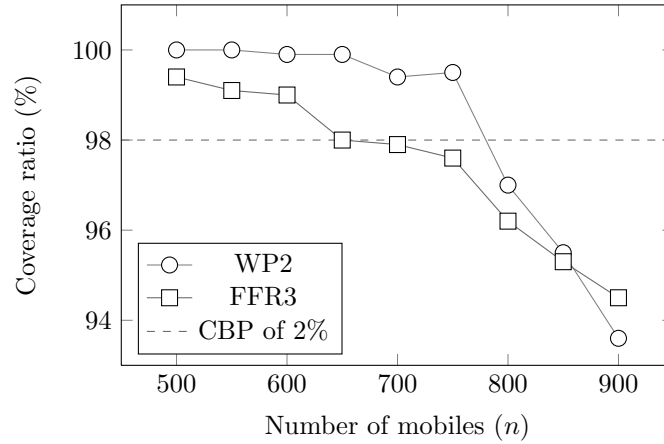


Figure 12: Evolution of the coverage ratio with $(\theta, \gamma) = (\frac{1}{4}, 3)$, $t = 25$ base stations, $k = 120$ channels.

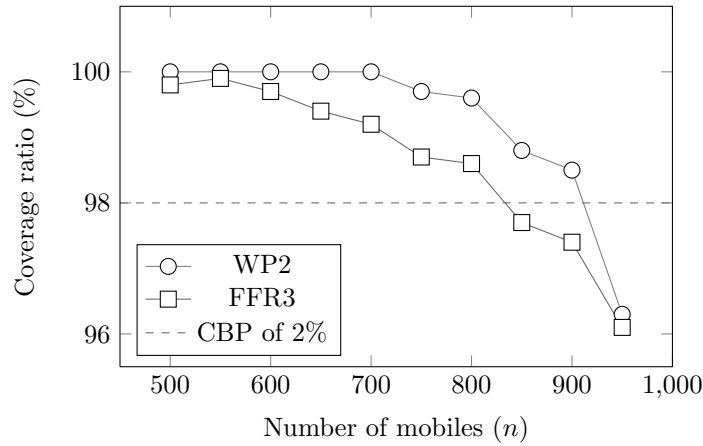


Figure 13: Evolution of the coverage ratio with $(\theta, \gamma) = (\frac{1}{2}, 3)$, $t = 25$ base stations, $k = 120$ channels.

7 Conclusion

Today's wireless networks should allow as many users as possible to communicate simultaneously, without compromising the quality of the signals. A detour into graph theory has shown how to achieve such a goal in an efficient way, and how to perform better than today's phone operator systems that mostly use decentralized channel assignment strategies such as fractional frequency reuse.

The new proposed strategy relies on sequential graph coloring algorithms. In particular, the WP2 heuristic runs in $\mathcal{O}(n^2)$ (where n is the number of users), which means that the algorithm is suitable for real-time channel assignments. Experiments have clearly shown that the proposed centralized strategy, when compared to the FFR technique, drastically increases the number of communicating users with the standard call-blocking probability of 2%. The next step will consist in developing on-line algorithms that can deal with users who appear, disappear and move in real time in a network.

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