

**Twin Graphs for OTN Physical Topology
Design**

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Abstract: In this paper we propose the use of twin graphs for optical transport network (OTN) physical topology design. Some properties inherent to twin graphs are fault tolerance, performance, cost, scalability, and planarity. Twin graphs can be easily generated by a recursive method. Moreover, the additional capacity required to implement path dedicated protection against single link or node failures is smaller in twin graphs than in existing OTN topologies.

Key Words: Network physical topology design, optical transport networks, twin graphs.

Résumé : Dans cet article, nous proposons l'utilisation d'une classe de graphes appelés graphes jumeaux (twin graphs) pour la conception de topologies physiques pour des réseaux optiques de transport (OTN). Ce type de graphe a plusieurs propriétés intéressantes, telles que la tolérance aux pannes, la performance et le coût. Dans le cas d'une panne individuelle, la performance d'un réseau optique modélisé par un graphe jumeau ne change pas substantiellement, parce qu'il y a des chemins de service et de protection de même longueur (en nombre d'arêtes) pour toutes les paires de nœuds non-adjacents. Des graphes jumeaux à $n + 1$ nœuds peuvent être facilement obtenus à partir de graphes jumeaux à n nœuds, et deux graphes jumeaux peuvent toujours être réunis dans le but d'obtenir un autre graphe jumeau, ce qui assure la capacité des réseaux à grandir (*scalability*). En outre, les ressources supplémentaires requises pour permettre la protection avec des chemins dédiés contre des pannes individuelles dans un nœud ou une arête sont moins importantes dans des graphes jumeaux que dans des réseaux optiques de transport existants.

Mots clés : Conception de topologies physiques, réseaux optiques de transport, graphes jumeaux.

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1 Introduction

Physical topology design plays an important role in the performance of Optical Transport Networks (OTN). Therefore, several fundamental factors must be taken into account, such as capital expenditure (CAPEX), performance, fault tolerance (survivability), and scalability. Simultaneously, some constraints also have to be considered, such as diameter and node degree.

The OTN physical topology design can be modelled using a broad class of graphs called 2-connected graphs, that ensure network survivability against any single node or link failure. Given n nodes, the 2-connected graph minimizing the number of links is the ring. So, rings meet the cost and the survivability factors. However, in the case of a link failure, all traffic passing through this link must be rerouted in the opposite way, and will pass through all other links, overloading the network. This also leads to a large resource requirement for protection schemes. Thus, rings do not meet the performance and (indirectly) the scalability factors.

In this paper we propose the use of a subclass of 2-connected graphs called twin graphs [1] for the design of OTN physical topologies. Besides the survivability, twin graphs have attractive properties in load distribution under normal and faulty operating conditions, and in resource requirement for path dedicated protection. In the case of a single failure the performance of a network modelled by a twin graph does not change substantially, because there are equal length working and backup paths for all non-adjacent pairs of nodes. Another properties of twin graphs are discussed throughout this paper.

This paper is organized as follows: Section 2 presents the definition of twin graphs. In Section 3 we present constructive methods taken from [2] to generate twin graphs. In Section 4 we explore some interesting features of twin graphs relevant to OTN physical topology design. Finally, in Section 5 we point out one interesting subject that will be further investigated.

2 What is a twin graph?

Let $G = G(V, E)$ denote a graph consisting of a set $V(G)$ of vertices or nodes and a set $E(G)$ of edges or links, each link uv connecting a pair of nodes $(u, v) \in G$. The order of G is $n = |V(G)|$ and its size is $m = |E(G)|$. All graphs in this work are connected, undirected, unweighted, without loops or multiple links. A graph is connected if there is a path between every pair of nodes $(u, v) \in G$.

Two nodes $u, v \in V(G)$ are said adjacent if the link uv belongs to $E(G)$. The neighborhood $\Gamma(v)$ of a node $v \in V(G)$ is the set of nodes adjacent to v in G , and its cardinality defines the degree of v . If two nodes $u, v \in V(G)$ have the same neighborhood, we say that (u, v) is a twin pair in G .

The distance $dist_G(u, v)$ between nodes $u, v \in V$ in G is the number of links in any of the shortest paths between u and v . A shortest path between u and v is called a u - v geodesic. The diameter $diam(G)$ is the length of the greatest geodesic in G . The transmission of G is the sum of all distances in G , denoted by $H(G) = \sum_{\forall u, v \in V(G)} dist_G(u, v)$ [3].

A graph G is 2-connected if and only if there are at least 2 node-disjoint paths between every two nodes $u, v \in G$. Since node-disjoint paths are also link-disjoint, a network topology modelled by a 2-connected graph survives either against any single node or link failure, because in this case each pair of nodes is still connected by a path avoiding the failure. However, there is no constraint about the length of this path, with respect to the length of a geodesic in the original graph. In this way, the class of 2-geodetically-connected graphs has been defined as a special subset of 2-connected graphs [4]. A graph G is 2-geodetically-connected (2-GC for short) if and only if there are at least 2 node-disjoint geodesics between every two non-adjacent nodes $u, v \in G$. So, in case of a single node or link failure, every pair of non-adjacent nodes is still connected by an alternative geodesic avoiding the failure.

If a graph G of given order has the minimum size to satisfy a certain property, we say that G is minimum with respect to this property. For instance, given n , the minimum connected graphs, which are called trees, have size $m = n - 1$.

Now we are able to define twin graphs as follows: every twin graph is a minimum 2-GC graph. Each twin graph has order $n \geq 4$ and size $m = 2n - 4$ [1]. As an example, Figure 1 illustrates a twin graph of order 10 where one can verify that it has only 16 links and there are at least two node-disjoint geodesics connecting any pair of non-adjacent nodes.

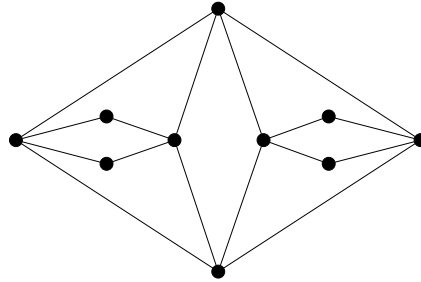


Figure 1: A twin graph of order 10.

The class of twin graphs is situated with respect to other graph classes in the set diagram shown in Figure 2. Twin graphs can be recursively defined from the cycle of order 4 [1] using the concept of twin pairs defined above. Moreover, two twin graphs can be merged in order to build another twin graph. Both processes are explained in the next section. For further information on twin graphs we refer the reader to [1, 2, 4, 5, 6, 7].

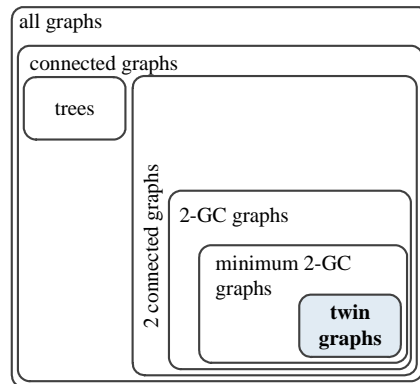


Figure 2: Twin graphs placement, with respect to other graph classes.

3 How to build a twin graph?

We can always add a node to a twin graph to obtain a new one, and in general it can be done in several ways, since twin pairs are not unique. Moreover, we can always merge two twin graphs in order to build another twin graph, and it requires only 4 additional links.

3.1 Growing a twin graph

Twin graphs can be recursively defined as follows [1]: the cycle of order 4 is a twin graph; if G is a twin graph of order n and (u, v) is a twin pair in G , then the addition of a new node v' by using links uv' and vv' produces a twin graph G' of order $n + 1$. So, in order to grow a twin graph G , we need only to:

1. Identify a twin pair (u, v) in G ;
2. Build G' , where $V(G') = V(G) \cup \{v'\}$ and $E(G') = E(G) \cup \{uv', vv'\}$.

Figure 3 shows the four possible ways (up to isomorphism) for growing the twin graph of order 10 shown in Figure 1 in order to build a twin graph of order 11.

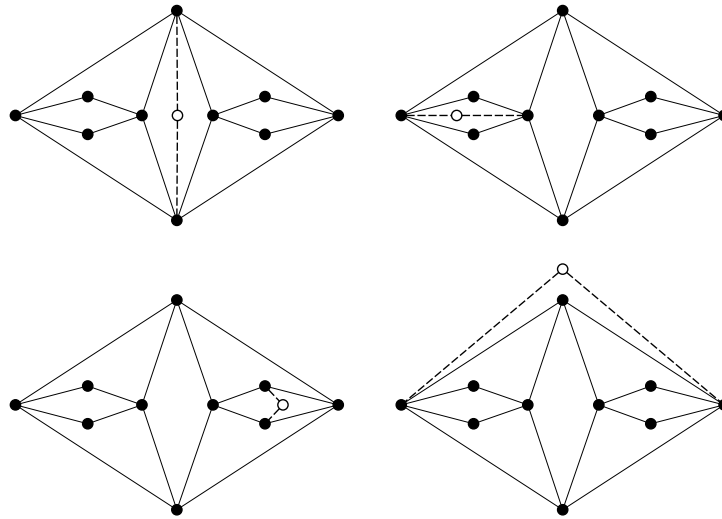


Figure 3: Growing a twin graph. Dotted lines represent the 4 possible ways (up to isomorphism) for growing the twin graph of order 10 shown in Figure 1 in order to build a twin graph of order 11.

3.2 Merging two twin graphs

In order to merge two twin graphs G_1 and G_2 , we need only to:

1. Identify a twin pair (u_1, v_1) in G_1 and a twin pair (u_2, v_2) in G_2 ;
2. Build $G = G(V, E)$, where $V = V(G_1) \cup V(G_2)$ and $E = E(G_1) \cup E(G_2) \cup \{u_1u_2, u_1v_2, u_2v_1, v_1v_2\}$.

Figure 4 illustrates the merging process described above. Figure 4(a), (b) shows two twin graphs of order 5 and 8, respectively, and Figure 4(c) identifies the twin pair we have selected for illustrating the process, and how to connect the 4 links. Then, Figure 4(d) shows the resulting twin graph of order 13, after redrawing.

4 Why twin graphs are interesting for OTN physical topology design?

4.1 Survivability, performance, and cost

Twin graphs relate survivability to performance, preserving distances in the remaining network after a single node or link failure, and it is done by minimizing the number of links.

A simple way to evaluate the cost of a network is the mean degree, which is a key variable to model OTN topologies [8].

The mean degree \bar{d} of twin graphs is calculated as follows:

$$\bar{d} = \frac{2m}{n} = \frac{4n - 8}{n} = 4 - \frac{8}{n}. \quad (1)$$

From (1) we obtain $2 \leq \bar{d} < 4$, independently of n . It is important to observe that it is in agreement with the mean degree of existing OTN topologies [8].

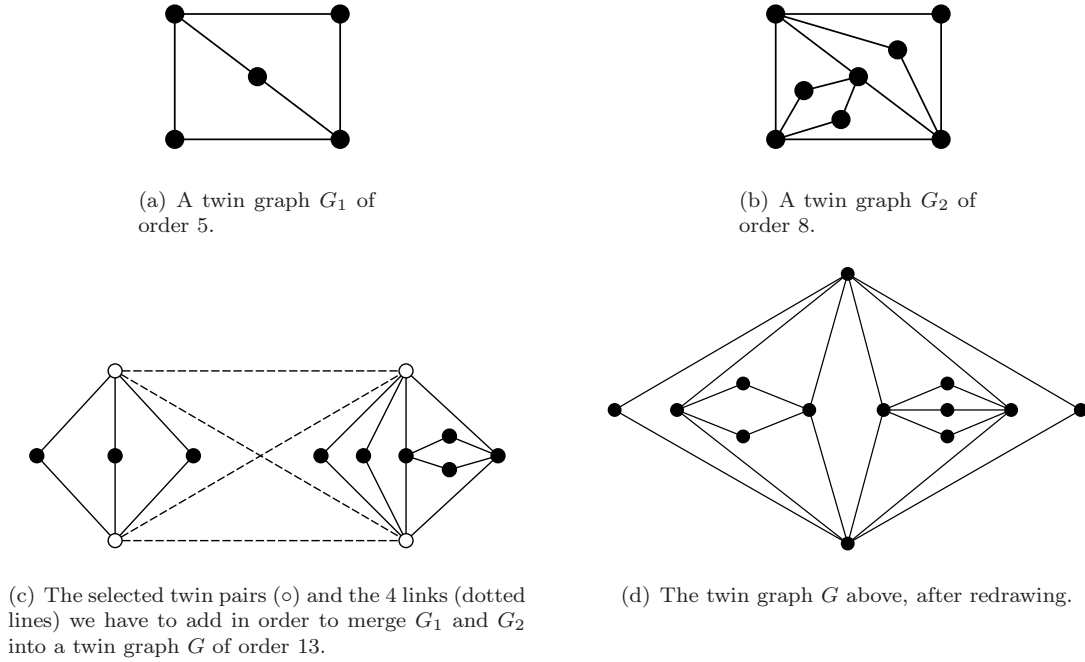


Figure 4: Merging two twin graphs of different orders.

4.2 Planarity

Since all twin graphs are planar [6], it is easy to embed a twin graph in a network covering a particular geographic area.

4.3 Scalability, and computational cost to identify a twin graph

Network topologies modelled by twin graphs are scalable, due to the growing and the merging processes shown in the previous section. Both processes must start with twin graphs. Fortunately, it is computationally easy to identify twin graphs. Using the recognition algorithm provided by [5], the problem of deciding if a graph of order n and size m is a twin graph or not can be solved in $O(mn) = O(n^2)$.

4.4 Topology diversity

An important feature of twin graphs is that, given the network order, the number of possible topologies is restricted. That makes it easy to solve the physical topology design problem. In spite of its cardinality, it is possible to find a good trade-off between diameter and maximum degree.

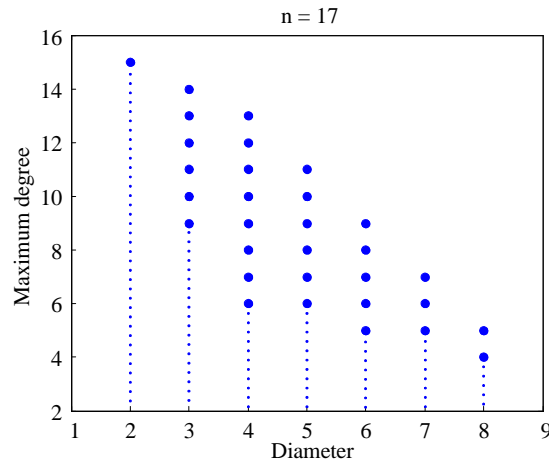
To demonstrate these characteristics, we have built all twin graphs of order n , for $n \leq 17$, using the methods explained in Section 3. For each n , the cardinality of this set is shown in the second column of Table 1. As expected, the number of twin graphs does not decrease as n increases, since all twin graphs of order $n + 1$ can be generated from a twin graph of order n .

For each n there is a twin graph of diameter 2, namely, the complete bipartite graph $K_{2,n-2}$ [1]. Besides this, for each n , $n \leq 17$, we have found at least one twin graph with diameter ranging from 2 to $\lfloor n/2 \rfloor$. This is summarized in the third column of Table 1.

Finally, the fourth column of Table 1 summarizes the maximum degree range for twin graphs of order n , $n \leq 17$. Figure 5 shows in more detail the relationship between diameter and maximum degree for twin graphs of order 17. For instance, for a particular application where maximum degree is critical, there is a

Table 1: Cardinality, diameter range and maximum degree range of twin graphs with n nodes, for $n \leq 17$.

n	Cardinality	Diameter	Maximum degree
4	1	2 – 2	2 – 2
5	1	2 – 2	3 – 3
6	2	2 – 3	3 – 4
7	2	2 – 3	4 – 5
8	4	2 – 4	4 – 6
9	5	2 – 4	4 – 7
10	9	2 – 5	4 – 8
11	13	2 – 5	4 – 9
12	23	2 – 6	4 – 10
13	35	2 – 6	4 – 11
14	63	2 – 7	4 – 12
15	102	2 – 7	4 – 13
16	182	2 – 8	4 – 14
17	310	2 – 8	4 – 15

Figure 5: Maximum degree *versus* diameter of twin graphs of order 17. A symbol (●) represents the existence of at least one twin graph with this configuration.

twin graph of diameter 8 and maximum degree 4. On the other extreme, if the diameter is the most critical parameter, we have the twin graph of diameter 2 and maximum degree 15. For intermediate situations there are many possible configurations, such as a twin graph of diameter 4 and maximum degree 6.

4.5 Link and node protection coefficients

The link protection coefficient is defined as the additional capacity required to implement path dedicated protection against single link failures [9, 10]. We show that twin graphs need a protection coefficient smaller than the protection coefficient of existing OTN topologies [9]. The node protection coefficient is analogously defined here, and then we show that twin graphs have the best possible node protection coefficient. Throughout this subsection we assume uniform all-to-all traffic and minimum hop routing.

4.5.1 Link protection coefficient

The link protection coefficient K_p^l measures the additional capacity required to implement path dedicated protection against single link failures [9, 10]. It is given by:

$$K_p^l = \frac{h^l}{h}, \quad (2)$$

where h^l is the mean distance for backup paths used in case of a link failure, and h is the mean distance for working paths. The mean distance for working paths, considering unit demands is

$$h = \frac{1}{D} \frac{H}{2} = \frac{H}{n(n-1)}, \quad (3)$$

where $D = n(n-1)/2$ is the number of bidirectional demands, considering all-to-all communication; and the total distance for working paths in a graph G is the transmission H .

The mean distance for backup paths used in case of a link failure is given by:

$$h^l = \frac{H^l}{n(n-1)}, \quad (4)$$

where H^l is the total distance for backup paths used in case of a link failure (second link-disjoint shortest paths).

Note that H can be decomposed as $H = H_1 + H_{2+}$, where the term H_1 is due to pairs at unit distance, i.e., to adjacent pairs, and H_{2+} is due to pairs at distance 2 or more. Since $H_1 = 2m$, we can write:

$$H = 2m + H_{2+}. \quad (5)$$

Then, the link protection coefficient is:

$$K_p^l = \frac{h^l}{h} = \frac{\frac{H^l}{n(n-1)}}{\frac{H}{n(n-1)}} = \frac{H^l}{H}. \quad (6)$$

In a twin graph, the distance between a node pair (u, v) can be changed only if the link connecting u and v fails, and in this case a distance 1 becomes 3, since each node belongs to (at least) one cycle of order 4 [6]. Then:

$$H_1^l = 3H_1 \quad (7)$$

$$H_{2+}^l = H_{2+} \quad (8)$$

$$K_p^l = \frac{H^l}{H} = \frac{H_1^l + H_{2+}^l}{H_1 + H_{2+}} = \frac{6m + H - 2m}{2m + H - 2m} = \frac{4m}{H} + 1. \quad (9)$$

From [3], if a graph G has order n and size m , then:

$$H \geq 2n(n-1) - m. \quad (10)$$

Using $m = 2n - 4$, we have for twin graphs:

$$H \geq 2n(n-3) + 8. \quad (11)$$

Then,

$$K_p^l \leq \frac{n^2 + n - 4}{n^2 - 3n + 4}. \quad (12)$$

Existing OTN topologies of order between 9 and 100 have K_p^l between 1.39 and 1.99 [9]. From (12) we notice that for $n \geq 11$ twin graphs are a better choice for modelling OTN topologies, from the viewpoint of link protection coefficient. They are particularly interesting for modelling large networks, since $K_p^l \leq 1.20$ for $n \geq 21$, and $K_p^l \leq 1.10$ for $n \geq 40$.

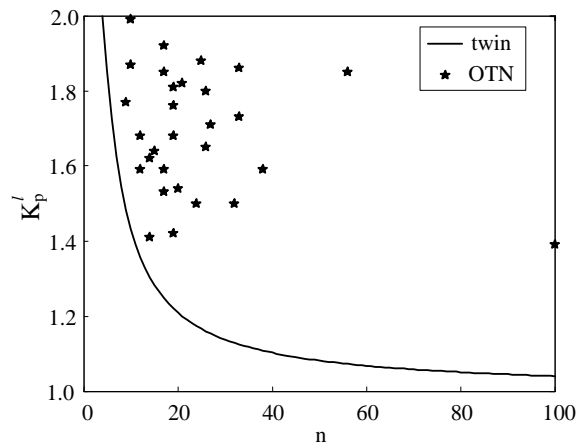


Figure 6: Maximum link protection coefficient K_p^l of twin graphs with $n \leq 100$ nodes, and K_p^l of the 29 existing OTN topologies reported on [9].

4.5.2 Node protection coefficient

Analogously to the link protection coefficient, the node protection coefficient K_p^n measures the additional capacity required to implement path dedicated protection against single node failures. It is given by:

$$K_p^n = \frac{h^n}{h}, \quad (13)$$

where h^n is the mean distance for backup paths used in case of a node failure, and h is again the mean distance for working paths. The mean distance for backup paths used in case of a node failure is $h^n = H^n/n(n-1)$, where H^n is the total distance for backup paths used in case of a node failure.

Since in a twin graph the distance between any pair of nodes does not change in case of a node failure:

$$H_1^n = H_1 \quad (14)$$

$$H_{2+}^n = H_{2+} \quad (15)$$

$$K_p^n = \frac{h^n}{h} = \frac{H^n}{H} = \frac{H_1^n + H_{2+}^n}{H_1 + H_{2+}} = 1. \quad (16)$$

Then, twin graphs always have the best possible node protection coefficient.

5 Conclusion

In this paper we proposed the use of twin graphs for OTN physical topology design, showing several interesting features of these graphs useful to model such networks.

Regarding the additional capacity measured by the protection coefficients K_p^n and K_p^l , one could expect that a node failure is worse than a link one, since a node removal implies the removal of all its adjacent links, and in this case we would have $K_p^n > K_p^l$. We observe however that there is an infinite class of graphs reaching $K_p^n = 1$, but there are no graphs (without multiple links) reaching $K_p^l = 1$. This will be explored in a future work.

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