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Abstract

This paper presents a framework where many existing structural credit risk models can be made hybrid by using a transformation of leverage to define the default intensity. The approach supports stochastic interest rates as well. As the default trigger is not purely specified by an exogenous barrier, the model produces endogenous random recovery rates that are negatively correlated to the default probabilities, which is consistent with the empirical findings. The key contributions of the paper are as follows. First, default intensity is defined as an increasing and convex transformation of leverage. Second, the recovery rate model uses leverage at default in a cascade manner to account for different debtholders seniority. Third, the approach is implemented on a firm-by-firm basis using maximum likelihood and the unscented Kalman filter (UKF). This accounts for trading noises which are deviations from theoretical prices. Finally, the non-linearity between default intensity and leverage is investigated empirically on each company of the CDX NA IG and HY indices using monthly CDS data from January 2004 to May 2008.

Key Words: risk management, credit risk, structural, reduced-form, hybrid, default intensity, recovery rate, unscented Kalman filter (UKF).

Résumé

Cet article présente une approche permettant d'adapter plusieurs modèles de risque de crédit de type structurel afin de les rendre hybrides en construisant un processus d'intensité qui est fonction du ratio d'endettement. Cette approche s'adapte même en présence de taux d'intérêt stochastique. Puisque le défaut n'est pas entièrement déterminé par une barrière spécifiée de façon exogène, le modèle produit des taux de recouvrement stochastique corrélés négativement avec la probabilité de défaut, ce qui est en accord avec les faits empiriques. Les principales contributions de cette recherche sont les suivantes : premièrement, le processus d'intensité est une fonction croissante et convexe du ratio d'endettement. Deuxièmement, le taux de recouvrement utilise le ratio d'endettement au moment du défaut tout en incorporant un effet de cascade afin de tenir compte de la séniorité de la dette. Troisièmement, la méthode proposée peut être appliquée firme par firme en utilisant le maximum de vraisemblance et le filtre de Kalman "unscented" (UKF). Cela permet de prendre en considération les bruits de marché qui créent des déviations par rapport aux prix théoriques. Finalement, la relation non-linéaire entre le processus d'intensité et le ratio d'endettement est étudiée empiriquement sur chacune des compagnies des indices CDX NA IG et HY en utilisant des données mensuelles de CDS de janvier 2004 à mai 2008.

1 Introduction and review of the literature

Credit risk, which is the potential loss arising from a default of payment by an obligor, mainly comes from two sources (Madan & Unal (1998)): the uncertainty regarding the timing of default and the amount lost by the creditor at the moment of default. The literature on the former is very rich and encompasses the structural, reduced-form (intensity), and more recently, hybrid models. Although many authors have investigated the empirical behavior of recovery rates, very few of them have integrated observed features of recovery rates in credit risk models. In this paper, a hybrid credit risk framework that incorporates recovery rate risk is presented.

Hybrid credit risk models combine features of structural and reduced-form models. One important class of hybrid models is based upon the fact that there is an important informational gap between managers of the firm and investors. Only managers can observe the true market value of its assets and liabilities whereas investors receive imperfect information. For example, in Duffie & Lando (2001), investors receive periodic financial statements that are a noisy estimate of market values. Jarrow & Protter (2004) argues that the gap between reduced-form and structural models is given by the amount of information observed by investors. The latter model is appropriate when the information set of investors and managers is the same so that default is fully predictable. In the opposite case, default takes investors by surprise. Other models also relax the complete information assumption, notably, Çetin, Jarrow, Protter & Yildirim (2004), Giesecke (2004) and Giesecke and Goldberg (2004a, b).

Other contributions integrate both reduced-form and structural models. In Madan & Unal (2000), a model for the value of assets and liabilities is presented and default occurs when a single and random loss, occurring at a random time, is larger than the value of the equity. The resulting model is built with an intensity-based approach. The Chen et al. (2004, 2005) papers introduce a model where the credit state of the firm (interpreted as either the rating of the company or its distance to default) is a Cox-Ingersoll-Ross (CIR)-type of affine process with gamma-distributed jumps. A second process, dependent on interest rates and on the credit state of the firm, is required. Default either occurs as soon as the first process hits a barrier or when the second process jumps. Thus, default comes from either a predictable or an unpredictable process. Bakshi, Madan & Zhang's (2006) model is a reduced-form model based on Vasicek-type state variables. One of the latter is the leverage of the firm. It should be noted that all models presented in these papers provide for non-zero short-term credit spreads, something that cannot be found in most structural models.[1]

Recovery rates have long been considered a constant fraction or a random exogenous random variable in the literature. However, recovery rates are inversely related to default probabilities and this has been mainly documented by Edward Altman. For example, in Altman & Kishore (1998), the recovery rate of a AAA company is 68% whereas a creditor of a CCC company should expect a recovery of 38%. Moreover, Altman et al. (2004) and Altman (2006) survey the current literature of credit risk models and emphasize the importance of relating the recovery rate to the probability of default. The very recent literature is slowly integrating stochastic recovery rates that are inversely related to default probabilities. In Bakshi, Madan & Zhang (2006b), the recovery rate is obtained as the exponential of minus the default intensity (corrected by constants). In Das & Hanouna (2009), the authors use several mathematical functions (logit, probit, arctan) that transform a default intensity (defined on the real line) to a [0,1] value. They obtain decreasing term structures of recovery rates. Pan & Singleton (2008) discuss the issue of econometric identification between recovery rate and default intensity. The authors argue that using the recovery of face value assumption is sufficient to solve this issue. Other significant contributions to stochastic recovery rates are Andersen & Sidenius (2004), Gaspar & Slinko (2008) and Hocht & Zagst (2009). These authors use a stochastic recovery rate framework but do not further discuss of this specific issue in empirical studies.

The fact that the market value of assets and liabilities are not observed by investors is mentioned in the literature at least since Jarrow & Turnbull (2000) and Jarrow & Protter (2004). One way to solve this issue is to use an estimation technique that uses observed data to infer the true value of assets/liabilities. This is in line with the maximum likelihood estimation (MLE) approach of Duan (1994, 2000) and Duan et al. (2003) in which the equity price is treated as a one-to-one transformation of the assets. This method

[1] Zhou (2001) is a notable exception since the latter is a structural model where the assets have a jump component.

applied to structural models provides unbiased estimates of the asset volatility, given that the equity model is appropriate. Because observed equity prices may contain randomness unrelated to their true theoretical value, it has been shown by Duan & Fulop (2009) that asset volatility estimates obtained by the MLE technique of Duan (1994) may overstate the true asset volatility. To account for possible trading noises in equity prices, Duan & Fulop (2009) propose a clever adaptation of the Auxiliary Particle Filter of Pitt & Shephard (1999) to the context of Merton (1974). The result is a simulated-MLE extension of Duan (1994) which only works with equity prices and the Merton (1974) model.

This paper presents a framework where many structural credit risk models can be made hybrid by using a transformation of leverage to define default intensity. This transformation should obey two simple properties. First, default probability should increase with leverage. Second, a shock on leverage should have a greater impact on the default probability when the leverage of the company is high than if it were low. As a result, the greater (smaller) the convexity of the transformation, the closer the model is to a structural (reduced-form) credit risk model. Once dynamics for assets and liabilities are determined, the framework can be used to model the two sources of credit risk, i.e. the moment of default and the amount of loss at default. The consequences of the approach are that short-term credit spreads are significantly different from zero and more importantly, time-varying stochastic recovery rates can be directly derived from the value of assets and liabilities at the moment of default. These recovery rates are consistent with the empirical findings that the default probability and loss given default are proportional. Notably, the stochastic recovery rate model presented gives rise to a term structure of recovery rates that is either increasing, decreasing or hump-shaped. The rationale behind the term structure of expected recovery rates is simple and can be interpreted with similar arguments as credit spread curves. For highly-rated firms, the major risk is a downgrade to which is associated a smaller recovery rate. Consequently, the term structure of expected recovery rates is usually downward sloping. For a risky firm, if it survives, its rating should improve, leading to a larger recovery rate. For such a firm, the term structure is usually upward sloping.

The key contributions of the paper are as follows. First, default intensity is defined as an increasing and convex transformation of leverage. This is different from Bakshi, Madan & Zhang (2006a) since their default intensity is a constant times leverage, which is a linear transformation. Moreover, Bharath and Shumway (2006) and Duffie, Saita, Wang (2007) use a Cox proportional hazards model to define a default intensity. Although this is a non-linear function of leverage, they use this approach to find possible determinants of default, bankruptcies, merger, etc. This is a regression-based analysis estimated on the whole set of companies at once, whereas the model in this paper is estimated on a firm-by-firm basis. The second important contribution is that the value of assets and liabilities at default can be used in a cascade structure to account for different debtholders seniority. Thus, the proposed model replicates recovery rates inversely proportional to default probabilities without using an arbitrary mathematical function that do not have any economic or financial foundation. Third, the approach is implemented on a firm-by-firm basis using maximum likelihood and the unscented Kalman filter (UKF).[2] This technique allows estimation of the parameters of the model (structural and reduced-form component) using multiple sources of information and accounting for possible trading noise. The aforementioned papers use a single source of observations (a single time series of equity price, bond price or CDS premium) whereas the technique is capable of using the whole term structure of CDS premiums for example. Finally, possible uses of the model range from pricing credit sensitive instruments to risk management applications (credit VaR, risk-based capital, etc.).

The non-linearity between default intensity and leverage is investigated empirically. To do so, the term structure of CDS premiums is used to estimate the parameters of each company in the sample of 225 firms of the CDX NA IG and CDX NA HY indices. The dataset comprises the evolution of monthly CDS premiums between January 2004 and May 2008 for various companies in many sectors and credit ratings. It was found that for non-investment grade companies, default intensity is much more sensitive to changes in leverage than for investment-grade firms. The critical level of leverage over which default becomes more likely is lower for the former firms than the latter. This means that investors tolerate a lower level of leverage for risky companies and that changes in their leverage have a greater impact on default probabilities. As a result of

[2] Although the use of these filters in this setting is original (to the best of the authors' knowledge), their application in finance is not new and they have been applied successfully with equity options (see Carr & Wu (2007) and Bakshi, Carr & Wu (2008)) and affine term structure models (see Christoffersen, Jacobs, Karoui, Mimouni (2009)).

the credit crisis, default intensities became less sensitive to leverage in 2006–2008 than in 2004–2006. Finally, a decreasing term structure of expected recovery rates is observed for most companies but the structure is increasing for CCC companies. Between 2004–2006 and 2006–2008, recovery rates dropped severely and the curves became steeper.

The paper is structured as follows. In the next section, the general modeling approach is presented. In Section 3, the pricing of defaultable zero-coupon bonds and CDS in the context of the framework is discussed. A family of transformations is also introduced. Section 4 presents different capital structures that can be used with the model. It is important to note that the framework is not limited to these dynamics of assets and liabilities. Section 5 shows how to use non-linear filtering techniques in conjunction with the model. Section 6 presents the empirical study. Section 7 presents the conclusions, while the most important proofs are left in the appendix. Other proofs and a detailed description of the numerical techniques used are included in a technical report available upon request to the authors.

2 Model

It is widely agreed and supported empirically that the main determinants of default are the leverage, its volatility and the interest rate (see for example Ericsson, Jacobs & Oviedo (2009) and references therein). This is also supported theoretically by structural models of default. However, a large portion of the spreads remains unexplained even when accounting for these three variables combined. In addition, structural models in general have failed to appropriately represent the short-term level of credit spreads, mainly because default is a predictable process. To solve these issues, the literature has turned to structural models with incomplete information or reduced-form models. The common feature of these classes of models is the presence of a surprise element that adds randomness to the default trigger.

The approach that is proposed in this paper falls into the hybrid credit risk model category. The main idea is that the sensitivity of the credit risk of the firm to its debt ratio determines how a default occurs. The result is a hybrid between a pure reduced-form and a structural model. The model is then used to build a recovery rate distribution that is tied to the time-varying solvency of the firm.

2.1 Structural framework

As a starting point of the model, assume that the total value of the assets of the firm are represented by the continuous-time stochastic process $\{A_t : t \geq 0\}$. The obligations of the company toward their creditors are defined by the liabilities process, $\{L_t : t \geq 0\}$, which may also be interpreted as the default threshold. The risk-free spot interest rate is denoted by $\{r_t : t \geq 0\}$. Formally, the filtration $\{\mathcal{G}_t : t \geq 0\}$ is generated by r, A and L (with the usual regularity conditions). Different dynamics for the assets and liabilities of the firm can be considered. Examples are presented in Section 4 but the model is not limited to those capital structures.

Structural models usually define the moment of default τ as the first moment that the assets cross the value of the liabilities, i.e. $\tau \equiv \inf\{t > 0 : A_t < L_t\}$. This results that τ is $\mathcal{G}$-predictable. Default in the proposed model is rather defined using a reduced-form default trigger, that is highly correlated with the debt ratio of the firm. This is discussed next.

2.2 Reduced-form framework

The default trigger of reduced-form models is mostly based upon Lando (1998), that is, it represents the first jump of a Cox process. In that case, the default time satisfies

$$\tau \equiv \inf\left\{t > 0 : \int_0^t H_u du > E_1\right\} \tag{1}$$

where E_1 is an exponential random variable with mean 1, which is independent of $\{\mathcal{G}_t : t \geq 0\}$ and $\{H_t : t \geq 0\}$ is the default intensity process.

Define the debt ratio $X_t \equiv L_t/A_t$ so that $\{X_t, t \geq 0\}$ results in a $\mathcal{G}-$adapted continuous-time stochastic process. It is assumed that H_u is a function of the debt ratio X_u i.e. $H_u = h(X_u)$ where $h : \mathbb{R} \to [0, \infty[$ is a firm-specific deterministic transformation, known as the sensitivity function.

It is required that h is an increasing function since default probabilities should increase with the debt ratio. As a consequence, the firm will not necessarily default as soon as its debt ratio approaches some critical threshold but it simply means that its default likelihood increases. The firm may very well survive and improve its financial status. The inverse is also true, that is, a company with a low debt ratio may have a significant default probability, depending on the importance of the capital structure on the ultimate default probability.

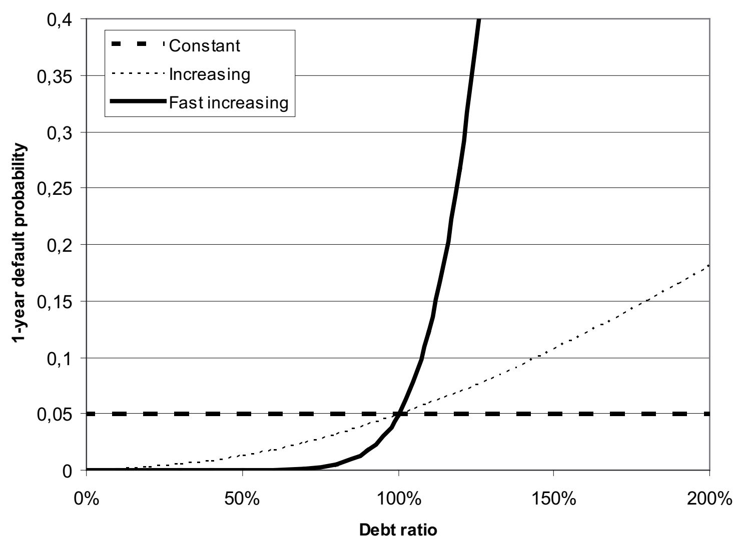
A typical structural model (with a default threshold at 100%) can be found from the proposed model when

$$h(x) = \begin{cases} 0, & x < 1 \\ \infty, & x \geq 1. \end{cases} \tag{2}$$

2.3 Intensity process

The firm-specific sensitivity function h plays a major role in the model: it gauges how the debt ratio affects the default probability. Figure 1 shows the impact of the function h on the one-year default probability, for different h functions, assuming the debt ratio remains constant throughout the year.

Figure 1: One-year default probability as a function of the debt ratio. The functions used are 0.05 (constant), $0.05x^2$ (increasing) and $0.05x^{10}$ (fast increasing).



For example, if $h(x) = 0.05$ (a constant), then default takes everyone by surprise and occurs independently of the debt ratio. When it increases very rapidly, say $h(x) = 0.05x^{10}$, then the one-year default probability is very low when the debt ratio is below 100% or even close to 100%, but increases rapidly afterwards. Thus, when the debt ratio approaches 100%, it does not mean the company defaults but the likelihood of such an event is very high. When the function h increases more slowly, say $h(x) = 0.05x^2$, it is much more likely for the firm to default with debt ratios lower than 100%, thus increasing the default probability. It illustrates that the model is capable of generating a default even if the company has a solid financial status, in terms of debt ratio.

The most simple interpretation of h is that it represents the sensitivity of the credit risk of the firm to its debt ratio. Two companies with similar capital structures may have different default probabilities since they may operate in different industries or that one firm may be more exposed to liquidity crises or simply because investors do not necessarily trust the financial statements of one firm. Since an increase of Δx of the debt ratio should have more impact on the default probability if the debt ratio is already important, h should be a convex function,[3] that is

$$h(x_1 + \Delta x) - h(x_1) < h(x_2 + \Delta x) - h(x_2)$$

for $x_1 < x_2$. The steeper (in the sense of the second derivative of h) the sensitivity function is, the more important is leverage in the determination of the default probability.

2.4 Recovery rate model

The vast majority of credit risk models require an exogenous specification of the recovery rate R, i.e. independent of the capital structure of the firm upon default. Most of the time, it is set as a constant often estimated from the seniority of the bond or chosen exogenously based on empirical researches such as Carty & Lieberman (1996) and Altman & Kishore (1996). In CreditMetrics (1997), a beta distributed random recovery rate, independent of the default process, is used. However, Altman et al. (2004) and Altman (2006) both argue the importance of having a recovery rate structure that is inversely related to the default probability.

Instead of an exogenous random recovery rate, one can use a structural model and build a proxy of R from the value of the assets and liabilities at the moment of default. However, most of these models will fail to provide interesting recovery rates. For example, in Black & Cox (1976), default occurs as soon as the assets cross the liabilities. Thus, the ratio of assets over liabilities at default is 100% and there is no room for lower recovery rates. In Longstaff & Schwartz (1995) and other similar models (see Collin-Dufresne and Goldstein (2001)), default occurs at the first passage of assets to some arbitrary threshold so that recovery is deterministic or determined arbitrarily by the threshold. Merton (1974) has random recovery, but default can only occur at the maturity of the zero-coupon debt. In summary, taking the value of assets and liabilities at the moment of default is rarely sufficient to obtain interesting recovery rates with pure structural models. An exogenous random variable, function or not of the capital structure, has to be defined.

A significant contribution of the proposed model is that the state variable may be used to construct the random recovery rate. Since default comprises an element of surprise, the assets at default will very likely be lower than the value of the liabilities and taking the ratio of the two at default is indeed a way to obtain realistic recovery rates that depend on the capital structure of the firm. Moreover, legal fees, liquidation costs and different seniority can be accounted for to obtain a recovery rate distribution that is closer to reality.

Importantly, this approach is not designed to be an exact valuation of the amount of money to be received by the specific debtholders (junior and senior) and equityholders upon default. The following model should be seen as an approach where the time-varying solvency of the firm will have an impact on the recovery rate distribution. The financial situation of a firm may deteriorate (improve) so that its mean recovery rate (given default) may decrease (increase) accordingly. Morever, surprise elements, through the sensitivity function, will also determine the distribution of the recovery rate given default. The recovery rate model presented is described as being endogenous since it comes from the value of the assets and liabilities at the moment of default.

Assume that the liquidation and legal fees represent a fraction κ of the market value of assets at default. The approximated value of the assets available to debtholders at default time is $A_\tau^* = \min((1-\kappa) A_\tau; L_\tau)$. Suppose there are two classes of bondholders: junior and senior. The senior bondholders represent $100\omega\%$ of the liabilities and the junior debtholders, $100(1-\omega)\%$. Thus, in case of default, senior investors have a right on the first $100\omega\%$ of the assets and the juniors take what remains. One can represent the recovery rate (with respect to the market value) for both senior and junior debtholders as

$$R_\tau^{(S)} = \frac{\min(A_\tau^*; \omega L_\tau)}{\omega L_\tau} \text{ and } R_\tau^{(J)} = \frac{A_\tau^* - \min(A_\tau^*; \omega L_\tau)}{(1-\omega) L_\tau}. \tag{3}$$

[3] Other authors have proposed transformations of leverage for their intensity functions. For example, Bakshi et al. (2006) consider the case $h(x) = \theta x$, which is linear.

Note that these recovery rates do not represent a fraction of the face value but a fraction of the market value of the liabilities at default. With one class of investors, R_τ becomes

$$R_\tau = \frac{A^*_\tau}{L_\tau} = \min\left((1-\kappa)\frac{A_\tau}{L_\tau};1\right). \tag{4}$$

Because the recovery rate is a random variable endogenously derived from the assets and liabilities, it serves both risk management and pricing purposes. When it is used for pricing credit derivatives, it should be used with the risk-neutral measure $\mathbb{Q}$. Therefore, comparison of average recovery rates with the existing empirical literature is only appropriate under the objective probability measure $\mathbb{P}$.

3 Pricing credit-sensitive assets

This section is devoted to the computation of the survival probability, the pricing of defaultable zero-coupon bonds and CDS. The framework can be used to price other credit-sensitive securities, such as coupon bonds.

3.1 Survival probability

Throughout this paper, it is assumed that $\mathcal{H}_t = \sigma\left\{\mathbb{I}_{\{\tau\le s\}}, s\le t\right\}$ is the σ-algebra that contains the information regarding the survival of the firm and $\mathcal{F}_t = \sigma\left(\mathcal{G}_t \cup \mathcal{H}_t\right)$ is the σ-algebra that contains all information. The indicator of default $\mathbb{I}_{\{\tau\le s\}}$ takes the value 1 if default occurred before s and 0 otherwise (survival). Thus, the conditional survival probability at time T, knowing the information available at time t is[4]

$$S_t(T) = \Pr\left(\tau > T \middle| \mathcal{F}_t\right) = \mathbb{I}_{\{\tau>t\}}\mathrm{E}\left[\exp\left(-\int_t^T H_u du\right)\middle| \mathcal{G}_t\right]. \tag{5}$$

This survival probability may be computed under the objective measure $\mathbb{P}$ as well as under the risk neutral measure $\mathbb{Q}$ or the $T-$forward measure $\mathbb{Q}_T$. To emphasize the dependence upon the chosen probability measure, the notation $S_t^{\mathbb{P}}(T)$, $S_t^{\mathbb{Q}}(T)$ and $S_t^{\mathbb{Q}_T}(T)$ will be used.

Assume that the dynamics of the debt ratio are

$$dX_t = \mu_t^{(X)} X_t dt + \sigma_t^{(X)} X_t dB_t$$

where $\{B_t : t\ge 0\}$ is a Brownian motion, $\sigma_t^{(X)}$ is a deterministic function of time and $\left\{\mu_t^{(X)} : t\ge 0\right\}$ is a predictable process. In Section 4, different capital structures are presented under which this assumption holds.

Throughout this paper, the intensity process is built upon the transformation h given by

$$h(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}, \alpha>0, \theta>0 \tag{6}$$

which is the hazard rate function of a Weibull distribution with shape parameter α and scale parameter θ. A value of $\alpha > 1$ is required for the transformation to be increasing with x. When $\alpha > 2$, the function h is convex, meaning that a small increase in the debt ratio has a greater impact when the firm already has a large debt ratio. The parameters α and θ are responsible for the sensitivity of the survival of the firm toward its debt ratio. θ is known as the critical level of leverage or the default threshold because the default intensity is large when leverage is greater than θ and small otherwise. The rate at which default intensity will converge to either 0 or infinity is guided by α.

Applying Itô's lemma to the particular shape of the intensity function implies that the dynamics of the default intensity $\{H_t : t\ge 0\}$ are

$$dH_t = \mu_t^{(H)} H_t dt + \sigma_t^{(H)} H_t dB_t \tag{7}$$

[4] See Lando (2004), p.115.

where $H_0 = h(X_0)$,

$$\mu_t^{(H)} = (\alpha-1)\,\mu_t^{(X)} + \frac{1}{2}(\alpha-1)(\alpha-2)\left(\sigma_t^{(X)}\right)^2 \text{ and } \sigma_t^{(H)} = (\alpha-1)\,\sigma_t^{(X)}.$$

Although θ does not appear in equation (7), it affects the level of the initial default intensity H_0.

It is shown in Appendix A.1 that if $\mu_t^{(H)}$ and $\sigma_t^{(H)}$ are not stochastic, then the survival probability $S_t(T) = S^*\left(T-t^*, H^*_{T-t^*}; T\right)$ where $t^* = T-t$ is the time-to-maturity, $H^*_{t^*} = H_{T-t^*}$ is the intensity process expressed as a function of time-to-maturity, and the function $S^*(t^*, H^*_{t^*}; T)$ satisfies the partial differential equation (PDE)

$$-\frac{\partial S^*}{\partial t^*} + \mu^{(H)}_{T-t^*} H^*_{t^*} \frac{\partial S^*}{\partial H^*} + \frac{1}{2}\left(\sigma^{(H)}_{T-t^*} H^*_{t^*}\right)^2 \frac{\partial^2 S^*}{\partial (H^*)^2} = H^*_{t^*} S^* \tag{8}$$

with the boundary conditions

$$\begin{aligned} S^*(0, H^*_{t^*}; T) &= 1 \\ \lim_{H^*_{t^*}\to\infty} S^*(t^*, H^*_{t^*}; T) &= 0 \\ \lim_{H^*_{t^*}\to 0} S^*(t^*, H^*_{t^*}; T) &= 1. \end{aligned}$$

Whenever $\mu_t^{(H)} = \mu_H$ and $\sigma_t^{(H)} = \sigma_H$ are constant functions of time, Dothan (1978)'s quasi-closed form solution of (8) can be used directly. Thus, the survival probability is $S_0(t^*) = S^*(0, H_0^*; t^*)$ where

$$S^*(0, H_0^*; t^*) = \frac{1}{\pi^2} w^p \int_0^\infty \int_0^\infty g(x,y)\,dxdy + \frac{2}{\Gamma(2p)} w^p K_{2p}\left(2\sqrt{w}\right), \tag{9}$$

$w = \frac{2H_0^*}{\sigma_H^2}$ and $p = \frac{1}{2} - \frac{\mu_H}{\sigma_H^2}$. Moreover,

$$g(x,y) = \sin\left(2\sqrt{w}\sinh y\right)\exp\left(-\sigma_H^2 \frac{4p^2+x^2}{8} t^*\right) x\cosh\left(\frac{\pi x}{2}\right) \mathbb{R}\left(\Gamma\left(-p + i\frac{x}{2}\right)\right)^2,$$

where i is the imaginary unit, $\mathbb{R}$ is the real part of a complex number, $\Gamma(\bullet)$ is the gamma function and $K_n(x)$ is the modified Bessel function of the second kind of order n.

Under stochastic interest rates, Dothan's solution no longer works, so that numerical methods should be used to solve the PDE in (8). One can use an explicit finite difference method to compute the survival probability. For more complex cases, where the value of the assets and liabilities need to be tracked in order to compute an endogenous recovery rate, a tree method such as Schönbucher (2002) is recommended.

3.2 Bond prices

3.2.1 Without recovery assumptions

The unrealistic zero-recovery zero-coupon bond is first considered since it is used as a building block for other credit-sensitive instruments. Given that the discount bond has a maturity of T years, the payoff of the bond is given by $\mathbb{I}_{\{\tau>T\}}$ and the price is

$$V_0(t,T) = \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right)\mathbb{I}_{\{\tau>T\}}\middle|\mathcal{F}_t\right]\mathbb{I}_{\{\tau>t\}} = \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T (r_u + H_u)\,du\right)\middle|\mathcal{G}_t\right].$$

In the particular case of constant interest rate, the last expression becomes $V_0(t,T) = e^{-r(T-t)} S_t^{\mathbb{Q}}(T)$. With stochastic interest rate, the usual change of numéraire allows the expression of the bond price as the product of the riskless zero-coupon bond value $p_t(T)$ paying one dollar at time T and the survival probability under the $T-$forward martingale measure $\mathbb{Q}_T$:

$$V_0(t,T) = p_t(T)\, S_t^{\mathbb{Q}_T}(T). \tag{10}$$

3.2.2 With endogenous recovery assumptions

To price credit derivatives or coupon bonds on a single company, recovery specifications are crucial. As a first step, the amount of money that will be paid to the security holder upon default needs to be determined as well as the moment when it will be paid. Most recovery assumptions are defined around the following three hypotheses (for an overview of the recovery specification and the valuation of bonds, see Duffie & Singleton (1999)): (1) a fraction R of the face value of the bond payable at its maturity (RFV) (ex.: Lando (1998)); (2) a fraction R of an equivalent risk-free zero-coupon bond payable upon default (RT) (ex.: Jarrow & Turnbull (1995)); (3) a fraction R of the market value of an equivalent defaultable bond (RMV) (ex.: Duffie & Singleton (1999), Bakshi, Madan & Zhang (2006));

Although the amount payable and its exact timing are defined in the clauses of the contract, the fraction R to be used in the valuation must still be appropriately estimated. Ideally, it should be consistent with the default generating process used to value the assets and liabilities. Thus, when pricing a credit-sensitive asset, an assumption must be made on R and on the type of payment upon default.

Following the discussion of Section 2.4, it is assumed throughout the paper that the recovery rate used is defined as in equations (3) and (4). With the RMV assumption, Duffie & Singleton (1999) note that it is necessary to define the loss process for their assumption to work. Obviously, $1-R_\tau$ (or $1-R_\tau^{(S)}$ or $1-R_\tau^{(J)}$) defined in equations (3) and (4) is a natural choice for their approach.

Assume that the investor recovers a fraction R_τ of an equivalent Treasury bond at default time τ. Then, the time t value of a risky zero-coupon bond[5] is

$$\begin{aligned} V_R(t,T)\,\hat{E} &= \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right)\mathbb{I}_{\{\tau>T\}} + \exp\left(-\int_t^\tau r_u du\right) R_\tau p_\tau(T)\,\mathbb{I}_{\{\tau\le T\}}\middle|\,\mathcal{F}_t\right]\mathbb{I}_{\{\tau>t\}} \\ &= \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T (r_u+H_u)\,du\right)\middle|\,\mathcal{G}_t\right] \\ &\quad +\mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right)\int_t^T R_s H_s \exp\left(-\int_t^s H_u du\right) ds\,\middle|\,\mathcal{G}_t\right]. \end{aligned}$$

Since r_u, R_s and H_s are dependent random variables, it is difficult to obtain a neat closed-form expression as with (10). Numerical methods such as Schönbucher (2002) or simulations are required to evaluate such an expression.

3.3 Credit default swap

A credit default swap (CDS) is a credit derivative intended to provide protection against a default, within a predetermined period of time. In the most basic type of CDS (settled in cash), the protection seller provides for a payment of par minus recovery upon default, which covers the loss in case of default[6]. In exchange, the protection buyer pays a periodic premium, usually four times a year, that ceases if there is a default. This spread is usually fixed such that the expected present value (PV) of losses equals the expected PV of premiums.

Given that the CDS matures at T, the expected PV of losses is

$$\mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^\tau r_u du\right) LGD(\tau)\,\mathbb{I}_{\{\tau\le T\}}\middle|\,\mathcal{F}_t\right] \tag{11}$$

where $LGD(\tau)$ is the loss given default random variable. With the RT assumption, the loss given default can be computed as $LGD(\tau)=1-R_\tau p_\tau(T)$. The expected PV of losses becomes

$$\mathrm{E}^{\mathbb{Q}}\left[\int_t^T LGD(s)\,H_s\exp\left(-\int_t^s (r_u+H_u)\,du\right) ds\,\middle|\,\mathcal{G}_t\right].$$

[5] See Lando (2004), equation (5.6), p.117.

[6] We refer to Hull (2009) for more details on the cash flows and dynamics of CDS.

To simplify the presentation, assume that a premium of 1 is paid at times $t_i < T$. In this case, the expected PV of premiums is given by

$$\mathrm{E}^{\mathbb{Q}}\left[\sum_{t_i}\exp\left(-\int_t^{t_i} r_u du\right)\mathbb{I}_{\{t\le t_i<\tau\}}\middle|\mathcal{F}_t\right]\mathbb{I}_{\{\tau t\}}. \tag{12}$$

Then, the periodic premium is the ratio of (11) over (12).

As with defaultable bonds, r_u, H_u and $LGD\,(u)$ are dependent random variables, so it is not possible to obtain a closed-form expression for the expected default payoff and premiums. Again, the application of numerical methods such as Schönbucher (2002) are required.

4 Capital structures

In this section, examples of classical capital structures illustrate how it can be used along with the intensity approach presented herein. However, the framework is not restricted to those models.

Unless stated otherwise, it is assumed throughout this section that under the real-world measure $\mathbb{P}$, the dynamics of the assets are characterized by the SDE (stochastic differential equation)

$$dA_t = \mu_A\,(t)\,A_t dt + \sigma_A A_t dB_t^{\mathbb{P}}$$

where $\left\{B_t^{\mathbb{P}} : t \ge 0\right\}$ is a standard $\mathbb{P}-$Brownian motion, $\mu_A\,(t)$ is a predictable process and σ_A is a diffusion coefficient. To proceed with the risk neutralization, it is supposed that the assets of the firm are traded, that is, the market is composed of at least a risk-free asset and the assets of the firm themselves. Finally, the function h is given by equation (6).

Three different debt structures are presented: under constant risk-free rate, the debt either grows at a (1) constant rate or is (2) stochastic and correlated with the assets; under stochastic interest rates, the debt is a (3) risk-free zero coupon bond.

4.1 Constant risk-free rate

Throughout this section, the risk-free asset is assumed to grow at a constant rate r.

4.1.1 Deterministic debt growth

The first model to be considered is a very basic one in which the capital structure of the firm is similar to what has been proposed by Merton (1974) with $\mu_A\,(t) = \mu_A$. The debt of the firm grows at rate $\beta \ge r$ since risky firms do not finance themselves at the risk free rate. The ordinary differential equation of the debt value is $dL_t = \beta L_t dt$. The debt ratio is

$$X_t = \frac{L_t}{A_t} = \frac{L_0 e^{\beta t}}{A_t}.$$

Applying Itô's lemma, one can show that

$$dX_t = X_t\left(\beta - \mu_A + \sigma_A^2\right)dt - X_t\sigma_A dB_t^{\mathbb{P}},$$

which is a geometric Brownian motion (GBM). Applying the h transformation onto the debt ratio and using Itô's lemma, one can show that

$$dH_t = \mu_H^{\mathbb{P}} H_t dt - \sigma_H^{\mathbb{P}} H_t dB_t^{\mathbb{P}} \tag{13}$$

where $H_0 = h\,(X_0)$,

$$\mu_H^{\mathbb{P}} = (\alpha-1)\left(\beta-\mu_A+\sigma_A^2\right) + \frac{1}{2}\sigma_A^2\,(\alpha-1)\,(\alpha-2) \text{ and } \sigma_H^{\mathbb{P}} = (\alpha-1)\,\sigma_A. \tag{14}$$

Applying the risk neutralization to the asset process $\{A_t : t \geq 0\}$, the default intensity, under the martingale measure $\mathbb{Q}$ is a GBM, with drift

$$\mu_H^{\mathbb{Q}} = (\alpha - 1)\left(\beta - r + \sigma_A^2\right) + \frac{1}{2}\sigma_A^2 (\alpha - 1)(\alpha - 2)$$

and diffusion $\sigma_H^{\mathbb{Q}} = (\alpha - 1)\sigma_A$. Note that, letting $\beta = r$ and $L_0 = Fe^{-r\overline{T}}$, the debt becomes a risk-free zero-coupon bond with face value F and maturity $\overline{T}$. This basic model can be seen as an extension of Merton's (1974) model to accommodate for default prior to the debt maturity.

Since the drift of the default intensity decreases when r increases, one can deduce that such a capital structure will provide credit spreads negatively related to the interest rates. The parameter β needs to be exogenously specified or estimated.

4.1.2 Stochastic growth rate

Instead of holding a single zero-coupon bond debt, a company may have a set of financial commitments that each have a specific behavior. Because modeling the behavior of each single debt issue would be difficult (and hard to infer from balance sheet data), it is possible to aggregate the value of each issue and assume, as an approximation, that the total amount of the debt follows a GBM. This is similar to assuming that the total value of all the assets held on the balance sheet follows a GBM. Consequently, the solvency of the company would rely on the volatility of the liabilities and the extent to which the total assets hedge the total liabilities.

Assume the debt behaves as a GBM correlated with the assets, that is,

$$dL_t = \mu_L L_t dt + \sigma_L L_t \left(\rho dB_t^{\mathbb{P}} + \sqrt{1-\rho^2} d\widetilde{B}_t^{\mathbb{P}}\right)$$

where the standard $\mathbb{P}-$Brownian motions $\left\{B_t^{\mathbb{P}} : t \geq 0\right\}$ and $\left\{\widetilde{B}_t^{\mathbb{P}} : t \geq 0\right\}$ are independent. The correlation parameter ρ can be interpreted as an (*ex post*) measure of hedging. Indeed, a company that has an appropriate risk management policy will invest in assets that behave closely to its liabilities. In addition to having similar asset and liability drifts and diffusions, a high level of dependence should also be seen between variations in assets and liabilities. This is approximated by the correlation parameter ρ. Thus, a poor hedge would be observed if the assets move almost independently from the liabilities, increasing the default risk of the firm. The converse also applies since a better hedge, measured by a very high correlation between the assets and the debt, would lower the range of possible outcomes of the debt ratio.

The dynamics of the debt ratio X_t under this capital structure becomes

$$dX_t = \mu_X^{\mathbb{P}} X_t dt + \sigma_X X_t dB_t^{\mathbb{P}} + \widetilde{\sigma}_X X_t d\widetilde{B}_t^{\mathbb{P}} \tag{15}$$

where

$$\mu_X^{\mathbb{P}} = \mu_L - \mu_A + \sigma_A^2 - \rho\sigma_A\sigma_L, \ \sigma_X = \rho\sigma_L - \sigma_A, \text{ and } \widetilde{\sigma}_X = \sqrt{1-\rho^2}\sigma_L.$$

Applying Itô's lemma, one obtains that $\{H_t : t \geq 0\}$ is a GBM with drift and diffusion parameters satisfying

$$\begin{aligned} \mu_H^{\mathbb{P}} &= (\alpha - 1)\left(\mu_L - \mu_A + \sigma_A^2\right) + \frac{1}{2}(\alpha - 1)(\alpha - 2)\left(\sigma_A^2 + \sigma_L^2\right) - \rho\sigma_A\sigma_L(\alpha - 1)^2 \\ \text{and } \sigma_H^{\mathbb{P}} &= (\alpha - 1)\sqrt{\sigma_A^2 + \sigma_L^2 - 2\rho\sigma_L\sigma_A}. \end{aligned}$$

Note that as ρ increases, the drift and diffusion coefficients of the intensity process decrease.

For the risk neutralization, it is assumed that the liabilities of the firm, in addition to its assets and the risk-free asset, are traded. The tradeability of L_t can be justified when it is interpreted as an aggregate financial product, that is, one that approximates the total value of a large sum of tradeable assets. In this case, under the martingale measure $\mathbb{Q}$, the drift and diffusion parameters of the default intensity are

$$\mu_H^{\mathbb{Q}} = \sigma_A^2 (\alpha - 1) + \frac{1}{2}(\alpha - 1)(\alpha - 2)\left(\sigma_A^2 + \sigma_L^2\right) - (\rho\sigma_A\sigma_L)(\alpha - 1)^2$$

and $\sigma_H^{\mathbb{Q}} = \sigma_H^{\mathbb{P}}$.

At this point, it is important to insist that the dynamics of the previous and upcoming capital structures are very different. Consequently, the effect of the transformation of the debt ratio might be different and the behavior of the default intensity may change as well. This further complicates empirical comparisons between capital structures.

4.2 Stochastic risk-free rate

Consider a more realistic framework where the risk-free rate can evolve stochastically, possibly in conjunction with the assets of the firm. In the finance literature, incorporation of random interest rates in credit risk models mainly depend on the type of model. With many reduced-form models (see for example Duffee (1999) and Bakshi et al. (2006)), the default intensity and the random interest rates are part of a multi-factor interest rate model. With structural models, many authors (see for example Longstaff & Schwartz (1995), Collin-Dufresne & Goldstein (2001), etc.) have modeled the assets and the short rate process as two dependent processes.

The approach presented here is somewhat similar to the one used in many structural models. The main difference resides in the definition of the interest rate process: the dynamics of the forward rate curve is modeled rather than the short rate, in a way consistent with Heath, Jarrow and Morton (1992). This approach will encompass the Hull & White (1990b) extended Vasicek model and the Ho & Lee (1986) model.

Assume that the market is composed of K risk-free zero-coupon bonds with maturities $T_1, \ldots, T_K$ and a risky asset A_t. Thus, the instantaneous forward rate process under the measure $\mathbb{P}$ is defined by

$$df_t(T) = \left(\sigma_t^{(f)}(T)\right)^\top (\lambda_t - \mathbf{\Sigma}_t(T))\,dt + \left(\sigma_t^{(f)}(T)\right)^\top d\mathbf{B}_t^{\mathbb{P}}$$

where $f_t(T)$ is the instantaneous forward rate determined at t for maturity T, $\sigma_t^{(f)}(T)$ is a $K+1$ vector of deterministic functions of time[7], $\mathbf{\Sigma}_t(T) = -\int_t^T \sigma_t^{(f)}(s)\,ds$, $\left\{\mathbf{B}_t^{\mathbb{P}} : t \geq 0\right\}$ is a vector of $K+1$ independent $\mathbb{P}-$Brownian motions, and λ_t is a vector that contains the risk premium associated to each Brownian motion, that will be determined according to the risk-free bond prices, as in Heath, Jarrow and Morton (HJM) (1992). If λ is not stochastic, the forward rates have a Gaussian distribution, leading to potentially negative rates.

The risky asset $\{A_t : t \geq 0\}$ dynamic is

$$dA_t = \left(r_t + \lambda_t^\top \sigma_t^{(A)}\right) A_t dt + \left(\sigma_t^{(A)}\right)^\top A_t d\mathbf{B}_t^{\mathbb{P}} \tag{16}$$

where $\sigma_t^{(A)}$ is a $K+1$ vector of deterministic functions of time. Furthermore, the price $p_t(T)$ at time t of a risk-free zero-coupon bond with maturity T satisfies[8]

$$dp_t(T) = m_t(T)\,p_t(T)\,dt + \mathbf{\Sigma}_t^\top(T)\,p_t(T)\,d\mathbf{B}_t^{\mathbb{P}},\ 0 \leq t \leq T$$

where $m_t(T) = r_t + \lambda_t^\top \mathbf{\Sigma}_t(T)$.

If the company has a risk-free zero-coupon debt with maturity $\overline{T}$, then the value of its debt L_t is equal to $Fp_t\left(\overline{T}\right)$. Thus, the debt ratio X_t is obtained from $X_t = Fp_t\left(\overline{T}\right)A_t^{-1}$ where F is the face value of the debt. The process followed by the debt ratio is derived using Itô's lemma:

$$dX_t = X_t\left(m_t\left(\overline{T}\right) - r_t - \lambda_t^\top \sigma_t^{(A)} + \sigma_t^{\top(A)}\sigma_t^{(A)} - \mathbf{\Sigma}_t^\top\left(\overline{T}\right)\sigma_t^{(A)}\right)dt + X_t\left(\mathbf{\Sigma}_t\left(\overline{T}\right) - \sigma_t^{(A)}\right)^\top d\mathbf{B}_t^{\mathbb{P}}$$

with $X_0 = Fp_0\left(\overline{T}\right)/A_0$. The details are presented in Appendix A.2. It is no longer the case that the debt ratio is a $\mathbb{P}-$GBM since it evolves with the short rate, which is random. However, the debt ratio has a lognormal distribution under the martingale measure $\mathbb{Q}$.

[7]The last element of $\sigma_t^{(f)}(T)$ should be zero so that the asset process will depend on an extra source of noise.
[8]See for example in Björk (1998) for a detailed proof.

In order to price bonds and any other credit-sensitive assets, it was discussed in Section 3.2.1 that the behavior of the debt ratio under the T-year forward risk neutral measure $\mathbb{Q}_T$ has to be determined. It is shown in Appendix A.2 that

$$d\left(\frac{A_t}{p_t(T)}\right) = \frac{A_t}{p_t(T)}\left(\sigma_t^{(A)} - \mathbf{\Sigma}_t(T)\right)^\top d\mathbf{B}_t^{\mathbb{Q}_T}.$$

However, the maturity of the debt is $\overline{T}$, not T, so one needs to determine the SDE followed by $p_t\left(\overline{T}\right)/A_t$ under $\mathbb{Q}^T$. It is shown in Appendix A.2 that the drift and diffusion of the debt ratio under $\mathbb{Q}^T$ are

$$\begin{aligned}\mu_t^{(X)}\left(T,\overline{T}\right) &= \mathbf{\Sigma}_t^\top\left(\overline{T}\right)\mathbf{\Sigma}_t(T) - \mathbf{\Sigma}_t^\top(T)\,\sigma_t^{(A)} + \left(\sigma_t^{(A)}\right)^\top\sigma_t^{(A)} - \mathbf{\Sigma}_t^\top\left(\overline{T}\right)\sigma_t^{(A)}\\ \text{and } \sigma_t^{(X)}\left(T,\overline{T}\right) &= \mathbf{\Sigma}_t\left(\overline{T}\right) - \sigma_t^{(A)}.\end{aligned}$$

Finally, the drift and diffusion of the default intensity, under the $\mathbb{Q}^T$ measure, are

$$\begin{aligned}\mu_t^{(H)}\left(T,\overline{T}\right) &= \mu_t^{(X)}\left(T,\overline{T}\right)(\alpha-1) + \frac{1}{2}(\alpha-1)(\alpha-2)\left(\sigma_t^{(X)}\left(T,\overline{T}\right)\right)^\top\sigma_t^{(X)}\left(T,\overline{T}\right)\\ \text{and } \sigma_t^{(H)}\left(T,\overline{T}\right) &= (\alpha-1)\,\sigma_t^{(X)}\left(T,\overline{T}\right).\end{aligned}$$

Since the drift and diffusion of the default intensity process are time-varying and deterministic, Dothan (1978)'s result no longer applies in this case. Thus, numerical methods should be used, as it was noted in Section 3.1.

5 Estimation with filtering techniques

5.1 Introduction

A filtering technique is a statistical and recursive algorithm that filters noisy observations (measurements) over time to obtain the best estimate of the evolution of some unobserved phenomenon (state). Filtering algorithms are very popular for signal processing applications in electronics, music and picture processing. The most widely known filtering algorithm is the Kalman filter which is used when the relation between the observations and the unobserved variable is linear. When the state variable is a Gaussian time series and the noise between the state variables and observations is also Gaussian, the Kalman filter provides the best estimate (in the sense of minimum mean square error (MSE)) of the state. In the setting of this paper, the state (unobserved) variable is the market value of assets and the measurements are the derivatives prices which are obtained from noisy observations (trading noise) of their theoretical prices.

In many applications, the relation between observations and the state is not necessarily linear so that the Kalman filter is suboptimal. For example, the equity price is a non-linear transformation of the asset value in the Black-Scholes' setting, so that estimation of the Merton (1974) model with trading noise cannot be accomplished with a Kalman filter. Adjustments have to be added to account for such non-linearities.

Non-linear filtering techniques are numerous and can be classified into two classes: non-linear Kalman filters and particle filters. Non-linear Kalman filters are algorithms that adapt the Kalman filter for non-linearities. Those are the extended Kalman filter (EKF) and the unscented Kalman filter (UKF).

All filters are designed to provide best estimates of the state variable, given that the parameters are known. In all cases, the likelihood function (or quasi-likelihood function) can be built and optimized to recover the parameters of the model. This is usually done in two steps repeatedly until convergence: filtering and likelihood maximization. The advantage of using these techniques in the estimation of credit risk models is that they account for trading noise without having to numerically invert the pricing function, as in Duan (1994) (without trading noise) or Duan & Fulop (2009) (with trading noise).

5.2 State-space representation

In the credit risk model to be estimated, the unobserved variable[9], which is the market debt ratio (see Section 4.1) evolves in continuous-time as a geometric Brownian motion i.e.

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

with strong solution

$$X_{t+\Delta t} = X_t \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\left(W_{t+\Delta t} - W_t\right)\right).$$

The filtering techniques presented only work in discrete-time so a discretization scheme is applied. Assume the time interval $[0, T]$ is split into n smaller time intervals so that $\Delta t = T/n$ and $t_k = k\Delta t$, $k = 0, 1, \ldots, n$. Defining $x_k \equiv \ln X_{t_k}$, one obtains

$$x_k = x_{k-1} + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_k, \quad k = 1, 2, \ldots, n \tag{17}$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are i.i.d. (independent and identically distributed) standard normal random variables (r.v.). equation (17) represents the state equation in a state-space model.

Define the random variable $Y_k^{(i)}$ as the price at time t_k of some derivative i ($i = 1, 2, \ldots, N$) dependent on the credit risk of a given company (equity, CDS, corporate bonds) and the column vector $\mathbf{Y}_k^\top \equiv \left[Y_k^{(1)}, \ldots, Y_k^{(N)}\right]$ as the set of these prices. For example, with $N = 10$, one can use the whole term structure of CDS prices (maturities of 1–10 years) to infer the dynamics of the debt ratio (or assets in other structural models). One can also combine equity with CDS and bonds prices, as long as they rely on the same state variable. The relation between the derivative price and the state variable is given by the observation equation

$$Y_k^{(i)} = g^{(i)}\left(X_k\right) \ \exp\left(\nu_k^{(i)}\right)$$

where the trading noise or pricing error is multiplicative. Moreover, $g^{(i)}(.)$ is the i-th derivative price function (which is non-linear), $\nu_k^{(i)}$ is the trading noise on the i-th derivative with variance $\left(\delta^{(i)}\right)^2$ i.e. $\nu_k^{(i)} \sim N\left(0, \delta^{(i)}\right)$. It is important to note that each $\nu_k^{(i)}$ is i.i.d. over time, independent from ε_k but may be dependent across derivatives. Thus, $\nu_k^\top \equiv \left[\nu_k^{(1)}, \ldots, \nu_k^{(N)}\right]$ has a multivariate normal distribution (i.i.d. over time), with mean $\mathbf{0}$ and variance $\mathbf{R}$ where

$$\begin{aligned} \mathbf{R} &= \operatorname{diag}(\delta) \ \rho \ \operatorname{diag}(\delta) \\ \delta^\top &\equiv \left[\delta^{(1)}, \ldots, \delta^{(N)}\right]. \end{aligned}$$

Note that ρ is some correlation matrix that determines the dependence in the pricing errors across derivatives. Furthermore, diag(δ) is the operator that creates a square matrix with diagonal elements corresponding to δ. In summary, $\mathbf{R}$ is the covariance matrix of the trading noise.

Defining $y_k^{(i)} \equiv \ln\left(Y_k^{(i)}\right)$, $\mathbf{y}_k^\top \equiv \left[y_k^{(1)}, \ldots, y_k^{(N)}\right]$ and $f^{(i)}(x_k) \equiv \ln\left(g^{(i)}\left(\exp\left(x_k\right)\right)\right)$, one gets that the observation equation can be written as

$$y_k^{(i)} = f^{(i)}\left(x_k\right) + \nu_k^{(i)}. \tag{18}$$

In matrix notation, the observation equation rewrites as

$$\mathbf{y}_k = \mathbf{f}\left(x_k\right) + \nu_k$$

where $\mathbf{f}\left(x_k\right) \equiv \left[f^{(1)}\left(x_k\right), \ldots, f^{(N)}\left(x_k\right)\right]^\top$.

For a detailed description of the filtering and estimation equations, the reader is referred to Hamilton (1994), Anderson & Moore (1979), Julier & Uhlmann (1997, 2002) and references therein.

[9] In Merton (1974) and Black & Cox (1976), the unobserved variable is the market value of assets.

6 Empirical study

6.1 Data and assumptions

The companies on which investigation is performed are the 225 firms of the CDX North American Investment Grade and High Yield indices (CDX.NA.IG.10 and CDX.NA.HY.10) provided by Markit as of June 18th, 2008. The selection of 225 firms spans multiple credit ratings and industrial sectors as well. The constituents of the portfolios are listed in Table 1.

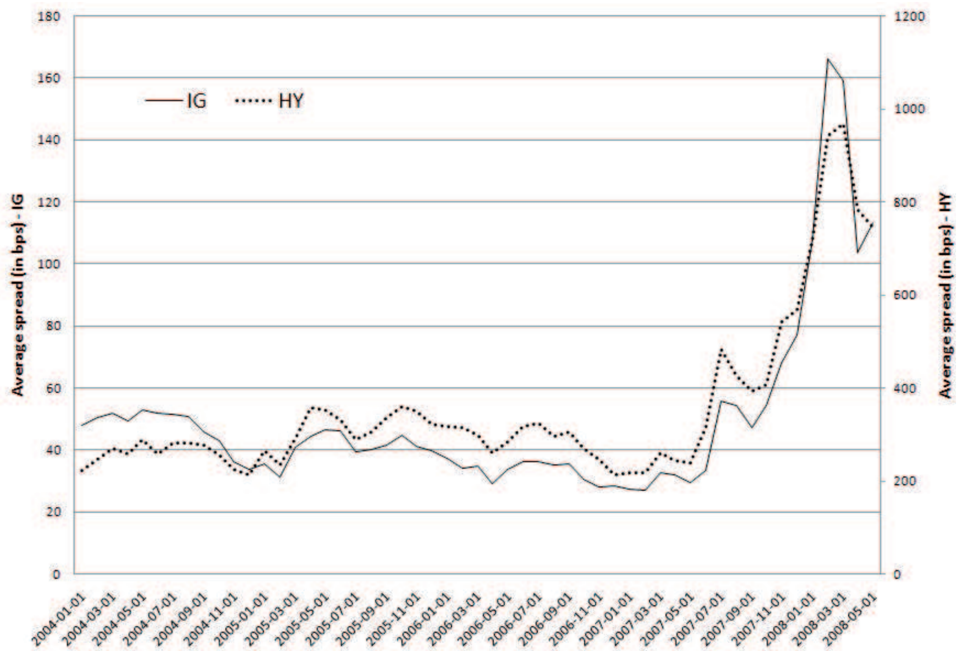
The monthly term structure of CDS prices from January 2004 to June 2008 is provided by DATASTREAM (95334 observations). Prices for maturities of one year through 10 years are available for most companies. However, only maturities of one to five years are used in the estimation process since CDS prices are computed using Schönbucher (2002) trees which considerably slows down estimation for large maturities. To illustrate how CDS premiums move over time, Figure 2 shows the evolution of the mean (taken across firms) CDS premium for maturities of five years, for both IG and HY portfolios.

In all experiments and unless stated otherwise, a constant risk-free rate of 4% has been assumed. This rate is consistent with the rounded average daily rates of one and three-months constant maturity Treasury

Table 1: List of the companies listed in the CDX NA IG (.10) and CDX NA HY (.10) portfolios as of June 13th, 2008 and obtained from Markit

CDX NA IG		CDX NA HY	
ACE Ltd	Intl Paper Co	Abitibi Consol Inc	Mirant North America LLC
Aetna Inc.	Istar Finl Inc	Advanced Micro Devices Inc	Mosaic Co
Alcoa Inc.	J C Penney Co Inc	The AES Corp	NALCO Co
Allstate Corp	Kohls Corp	AK Stl Corp	Neiman Marcus Gp Inc
Altria Gp Inc	Kraft Foods Inc	Allegheny Engy Supp Co LLC	Nortel Networks Corp
Amern Elec Pwr Co Inc	The Kroger Co.	Allied Waste North Amer Inc	NOVA Chems Corp
Amern Express Co	Ltd Brands Inc	ALLTEL Corp	NRG Energy Inc
Amern Intl Gp Inc	Liz Claiborne Inc	Amern Axle Mfg Inc	Owens IL Inc
Amgen Inc.	Lockheed Martin Corp	Amkor Tech Inc	Polyone Corp
Anadarko Pete Corp	Loews Corp	AMR Corp	Pride Intl Inc
Arrow Electrs Inc	M D C Hldgs Inc	ARAMARK Corp	Qwest Cap Fdg Inc
AT&T Inc	Macy's Inc	ArvinMeritor Inc	RH DONNELLEY Corp
AT&T Mobility LLC	Marriott Intl Inc	AVIS BUDGET CAR Rent LLC	RadioShack Corp
Autozone Inc	Marsh Mclennan Cos Inc	Beazer Homes USA Inc	Realogy Corp
Baxter Intl Inc	Masco Corp	Bombardier Inc	Reliant Energy Inc
Black Decker Corp	MBIA Ins Corp	Celestica Inc	Residential Cap LLC
Boeing Cap Corp	McDonalds Corp	Charter Comms Hldgs LLC	Rite Aid Corp
Bristol Myers Squibb Co	McKesson Corp	Chemtura Corp	Royal Caribbean Cruises Ltd
Brunswick Corp	MeadWestvaco Corp	Chesapeake Engy Corp	Sabre Hldgs Corp
Burlington Nthn Santa Fe Corp	MetLife Inc	Ctzns Comms Co	Saks Inc
Campbell Soup Co	Motorola Inc	Clear Channel Comms Inc	Sanmina SCI Corp
Cap One Bk	Natl Rural Utils Coop Fin Corp	CMS Engy Corp	Six Flags Inc
Cardinal Health Inc	NY Times Co	Cmnty Health Sys Inc	Smithfield Foods Inc
Carnival Corp	Newell Rubbermaid Inc	Constellation Brands Inc	Smurfit Stone Container Entp Inc
Caterpillar Inc	News America Inc	Cooper Tire Rubr Co	Std Pac Corp
CBS Corp	Nordstrom Inc	CSC Hldgs Inc	Sta Casinos Inc
CenturyTel Inc	Norfolk Sthn Corp	Dillards Inc	SUNGARD DATA Sys INC
Chubb Corp	Northrop Grumman Corp	DIRECTV Hldgs LLC	Tenet Healthcare Corp
Cigna Corp	Omnicom Gp Inc	Dole Food Co Inc	Tesoro Corp
CIT Gp Inc	Progress Engy Inc	Domtar Corp	Toys R Us Inc
Comcast Corp	Quest Diagnostics Inc	Dynegy Hldgs Inc	Tribune Co
Computer Sciences Corp	R R Donnelley Sons Co	Eastman Kodak Co	TRW Automotive Inc
ConAgra Foods Inc	Radian Gp Inc	EchoStar DBS Corp	Unisys Corp
ConocoPhillips	Raytheon Co	El Paso Corp	Utd Rents North Amer Inc
Constellation Engy Gp Inc	Rio Tinto Alcan Inc	Energy Future Hldgs Corp	Univision Comms Inc
Cox Comms Inc	Rohm Haas Co	Fairfax Finl Hldgs Ltd	Visteon Corp
CSX Corp	Safeway Inc	1st Data Corp	Windstream Corp
CVS Caremark Corp	Sara Lee Corp	Flextronics Intl Ltd	
Darden Restaurants Inc	Sempra Engy	Ford Mtr Co	
Deere Co	Sherwin Williams Co	Fst Oil Corp	
Devon Engy Corp	Simon Ppty Gp L P	Freeport McMoran Cop. Gold Inc	
Dominion Res Inc	Southwest Airls Co	Freescale Semiconductor Inc	
Dow Chem Co	Sprint Nextel Corp	Gen Mtrs Corp	
Duke Energy Carolinas LLC	Starwood Hotels Resorts Inc	GA PACIFIC LLC	
E I du Pont de Nemours Co	Target Corp	Goodyear Tire Rubr Co	
Eastman Chem Co	Textron Finl Corp	Harrahs Oper Co Inc	
Embarq Corp	Time Warner Inc	HCA Inc.	
Fed Home Ln Mtg Corp	Toll Bros Inc	Hertz Corp	
Fed Natl Mtg Assn	Transocean Inc	Host Hotels Resorts LP	
FirstEnergy Corp	Un Pac Corp	Idearc Inc	
Fortune Brands Inc	Unvl Health Svcs Inc	IKON Office Solutions Inc	
Gannett Co Inc DE	Valero Energy Corp	Intelsat Ltd	
Gen Elec Cap Corp	Verizon Comms Inc	Iron Mtn Inc	
Gen Mls Inc	Viacom	K Hovnanian Entpers Inc	
Goodrich Corp	Wal Mart Stores Inc	KB Home	
Halliburton Co	Walt Disney Co	L 3 Comms Corp	
Hartford Finl Svcs Gp Inc	WA Mut Inc	Lear Corp	
Hewlett Packard Co	Wells Fargo Co	Level 3 Comms Inc	
Home Depot Inc	Weyerhaeuser Co	Levi Strauss Co	
Honeywell Intl Inc	Whirlpool Corp	Liberty Media LLC	
Ingersoll Rand Co	Wyeth	Massey Engy Co	
Intl Business Machs Corp	XL Cap Ltd	Mediacom LLC	
Intl Lease Fin Corp		MGM MIRAGE	

Figure 2: Evolution of the mean five-year CDS premium (across firms) for both CDX.NA.IG (left axis) and CDX.NA.HY (right axis) portfolios, between January 2004 and May 2008



rates over the period of 2006–2008. The interest rates data are provided by the Federal Reserve of St. Louis' website via FRED (Federal Reserve Economic Data).

6.2 Estimation

This subsection discusses how non-linear filtering techniques have been applied in the specific context of the model. Descriptive statistics on the results of the latter estimation method are presented next.

6.2.1 Method

The UKF[10] filtering and estimation procedures have been applied on the whole sample of observations of CDS prices to the model. Following Section 4.1 and the constant interest rate assumption, the capital structure dynamics are such that the debt ratio follows

$$dX_t = \mu_X^{\mathbb{P}} X_t dt + \sigma_X X_t dW_t^{\mathbb{P}}. \tag{19}$$

The pricing of credit derivatives requires risk neutralization. Consistent with the capital structure models of Section 4.1, the risk premium on the $\{X_t, t \geq 0\}$ process is set as a constant, i.e.

$$\mu_X^{\mathbb{P}} - \xi_X = \mu_X^{\mathbb{Q}}$$

and ξ_X is the risk premium. Consequently, the state equation is

$$x_k = x_{k-1} + \left(\mu_X^{\mathbb{P}} - \frac{1}{2}\sigma_X^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_k \tag{20}$$

with $x_k \equiv \ln X_{t_k}$ and $\Delta t = 1/12$.

The theoretical price of each CDS as of a given date and debt ratio X_{t_k} is computed using the Schönbucher (2002) tree technique. With this approach, the default payoff is par minus recovery and the survival payoffs

[10] Similar computations have been performed using the EKF and results are very similar.

correspond to the quarterly payment of spreads. The recovery payment has been assumed to be a random fraction R_τ (see equation (4)) of a Treasury bond paid upon default.

Independence in the pricing error between each CDS price has also been assumed so that the matrix ρ is the identity matrix. This has been done to reduce the number of parameters to be estimated. In summary, the observation equation is

$$y_k^{(i)} = f^{(i)}(x_k) + \nu_k^{(i)}, i = 1, 2, \ldots, 5$$

with $\nu_k^{(i)}$ (having variance $\left(\delta^{(i)}\right)^2$) independent across maturities and $f^{(i)}(x_k) \equiv \ln\left(g^{(i)}(\exp(x_k))\right)$ where $g^{(i)}$ is the price of a i-year CDS computed with Schönbucher (2002). It is important to note that the dynamics of the debt ratio implied by the pricing function $g^{(i)}$ have to be under $\mathbb{Q}$ i.e. the drift of the debt ratio is $\mu_X^{\mathbb{Q}}$. Quasi-MLE has been performed on the available data on a firm-by-firm basis. The parameters of the UKF technique have been assumed to be $\kappa_{\text{UKF}} = 2, \alpha_{\text{UKF}} = 1$ and $\beta_{\text{UKF}} = 0$ as in Van der Merwe, Doucet, de Freitas & Wan (2000).

Finally, the initial state value ($\widehat{x}_{0|0}$) has been estimated and its variance ($P_{0|0}$) has been assumed very low (i.e. 0.001). Overall, the parameters to be estimated for each company are

$$\mu_X^{\mathbb{P}}, \xi_X, \sigma_X, \alpha, \theta, \kappa, \widehat{x}_{0|0}, \delta.$$

6.2.2 Results

This section shows statistics such as the mean, standard deviation and quantiles of the parameters obtained with the UKF estimation technique. This is meant to provide an overview of the range of parameters across companies. More thorough analyses will be conducted in the Analysis section.

Table 2 shows descriptive statistics on the estimated parameters using the UKF filtering technique. There are interesting differences between IG and HY firms. The drift of the debt ratio is higher for HY firms, meaning these companies have a tendency to contract more debt or their debt is more expensive (in terms of interest rate). It could also be that these companies have lower quality assets or simply a combination of the two. Their debt ratio is more volatile and their initial market debt ratio is greater. The latter results had to be expected given the nature and the separation of these companies in the two portfolios.

It can be seen in Table 2 that α is greater for HY companies, meaning that a slight change in the debt ratio will have more impact on the default probability, than for IG companies. Moreover, the critical debt ratio level (or θ) is lower for HY companies, meaning that investors have a lower tolerance toward the debt ratio for these companies. The effect is to further increase their default probability.

The standard error of the trading noise is the highest for the one-year CDS since it is more likely to contain elements not necessarily related to the true default risk of the company. This is by definition a

Table 2: Descriptive statistics on the distribution of parameters across the portfolio of firms of the CDX indices using the UKF filtering technique (A constant risk premium has been assumed)

	$\mu_X^{\mathbb{P}}$	ξ_X	σ_X	α	θ	κ	$\widehat{x}_{0\|0}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$
Mean	2,84%	2,71%	11,47%	14,0133	1,5139	49,23%	69,61%	24,89%	14,05%	6,61%	3,30%	8,94%
Stdev	8,24%	8,93%	9,14%	12,4408	0,4209	22,76%	32,01%	10,62%	7,47%	4,17%	6,32%	4,24%
10%	-1,98%	-2,17%	3,34%	4,0599	0,8559	14,23%	31,71%	12,23%	6,36%	0,10%	0,00%	4,79%
25%	-0,39%	-1,04%	5,29%	5,4141	1,2040	40,35%	45,33%	18,23%	9,49%	3,94%	0,00%	6,15%
50%	1,29%	0,66%	8,68%	9,7095	1,6087	49,97%	61,69%	23,83%	12,93%	6,37%	0,50%	7,86%
75%	3,59%	2,64%	14,39%	19,8965	1,9053	61,76%	87,99%	30,39%	18,13%	9,78%	5,27%	10,38%
90%	8,77%	8,85%	23,50%	29,7071	1,9913	75,34%	112,71%	37,58%	23,37%	11,84%	9,19%	15,25%
IG	2,02%	2,00%	10,46%	12,0145	1,5945	51,86%	64,55%	28,26%	16,10%	8,72%	3,11%	9,98%
HY	3,90%	3,61%	12,76%	16,5627	1,4112	45,87%	76,05%	20,62%	11,31%	3,95%	3,56%	7,62%

For each of the 225 firms, the parameters of the model have been estimated using monthly CDS prices of maturities of one to five years, using the UKF filtering technique. The data are provided by DATASTREAM and are available from January 2004 to May 2008. For risk neutralization, a constant risk premium has been assumed on the debt ratio process. This is consistent with the deterministic and random debts of Section 4.1. The mean, standard deviation and quantiles are computed across firms. The last two rows of each table computes the mean across firms of the CDX.NA.IG or CDX.NA.HY portfolios.

trading noise. When these standard deviations are compared with Duan & Fulop (2009), it can be seen that the ones presented in this paper are higher. This may be because CDS prices are used rather than equity prices but it is more conceivable that fitting five credit derivatives prices at a time has had an impact on the quality of the fit to each derivative.

Moreover, the parameters estimated show that the sensitivity function, which translates a debt ratio into a default intensity, should be increasing and convex, since the very large majority of α obtained in Table 2 are greater than 2.

Finally, Table 3 displays the absolute relative pricing error on the CDS premiums being used in the estimation procedure. For each firm, a single set of parameters has been estimated for the whole sampling period. As it contains some financial turmoil, the pricing error varies over time. One way to bypass this phenomenon is to estimate the model more frequently. The average (across firms and months) absolute relative pricing error is not a very good measure as it is driven by a single firm having extremely large pricing errors (around 3000%). The median absolute relative error gives a better picture of the overall behavior of the model. As noted by other authors, the pricing error is more important for the 1 year maturity as it may contain other effects than credit risk. For larger maturities, the model generally well performs for most of the studied firms.

Table 3: Absolute relative pricing errors across the portfolio of firms of the CDX indices

Maturity	1	2	3	4	5
All firms (2004–2008)					
Mean	37,7%	28,8%	23,1%	20,7%	23,9%
Median	16,2%	8,3%	4,3%	0,6%	5,3%
nb obs	9490	9192	9490	9192	9490
IG firms (2004–2008)					
Mean	23,1%	13,6%	7,9%	2,9%	7,7%
Median	18,2%	9,6%	5,9%	0,4%	5,9%
nb obs	5609	5570	5609	5570	5609
HY firms (2004–2008)					
Mean	58,7%	52,0%	45,2%	48,2%	47,3%
Median	13,8%	7,0%	2,5%	1,2%	4,5%
nb. obs	3881	3622	3881	3622	3881
2004 (all firms)					
Mean	25,6%	13,0%	7,5%	3,9%	9,6%
Median	21,9%	9,7%	4,3%	0,6%	5,8%
nb obs	1055	1008	1055	1008	1055
2005 (all firms)					
Mean	93,7%	84,5%	77,6%	77,9%	77,4%
Median	17,8%	8,7%	4,4%	0,5%	4,9%
nb obs	2158	2085	2158	2085	2158
2006 (all firms)					
Mean	19,0%	12,6%	6,7%	3,0%	6,6%
Median	13,9%	7,6%	4,2%	0,5%	4,9%
nb obs	2548	2464	2548	2464	2548
2007 (all firms)					
Mean	21,4%	11,8%	6,7%	4,0%	8,0%
Median	15,8%	8,1%	4,1%	0,7%	5,5%
nb obs	2634	2565	2634	2565	2634

For each of the 225 firms, the monthly time series of CDS prices were used to estimate the model parameters using UKF filtering technique. The data are provided by DATASTREAM and spans the period from January 2004 to May 2008. Using the parameters obtained with the UKF method, the filtered market debt ratio is computed and the CDS is priced for each available maturity and month. For each company, year and CDS maturity, the theoretical and observed CDS prices are compared. The absolute relative pricing error is computed as the absolute pricing error expressed as a percentage of the observed price.

6.3 Analysis

In this section, the relationship between the parameters of the model and the credit rating of the firm is analyzed. The stationarity of the parameters over two time intervals is also investigated, along with the expected recovery rate given.

6.3.1 Relationship with ratings

Table 4 shows the mean (and standard deviation) parameter value across firms having the same credit rating (S&P) as of May 31st, 2008. The rating of the firm (taken as the rating of the long-term debt) was taken from COMPUSTAT, as of the end of each year, from 2004 to 2006, and from Markit in 2008 (in both databases, the credit rating has been evaluated by S&P). Note that some companies were rated as BBB (investment-grade) but belonged to the HY portfolio which leads to slight differences with Table 2footnoteLiz Claiborne Inc. belongs to the IG portfolio but is rated BB (which is non investment-grade). Moreover, Allegheny Engy Supp Co LLC, Freeport McMoran Copper & Gold Inc and Mosaic Co were all rated BBB but belong to the HY portfolio.

Excluding the AAA class,[11] there are obvious trends that can be deduced from this table. Indeed, $\mu_X^{\mathbb{P}}$, σ_X, α and $\widehat{x}_{0|0}$ (θ) seem to be larger (smaller) with a decreasing credit quality but there is a lot of noise. Since there are only a few observations for AAA, AA and CCC ratings, groupings of companies into two categories (investment-grade (IG) and non-investment-grade (Non-IG)) are performed. For this section only, the designation "IG" means that the S&P rating is either AAA, AA, A or BBB while "Non-IG" represents a rating of BB, B or CCC. With more data in each category, statistical significances can be computed.

Linear regressions of the form

$$\widehat{p}^{(i)} = \beta_0 + \beta_1 \left(1 - \mathbb{I}_{\mathrm{IG}}^{(i)}\right) + \varepsilon^{(i)}, i = 1, 2, \ldots, 225 \tag{21}$$

[11] The three firms of the AAA class are GE Capital, FNMA (Fannie Mae) and FHLMC (Freddie Mac) and the last two practically defaulted in September 2008 despite their AAA rating. This may explain the discrepancy between this class and the others.

Table 4: Mean (and standard deviation) parameters across firms for various credit ratings

Averages across firms, over ratings

Rating	$\mu_X^{\mathbb{P}}$	σ_X	α	θ	κ	$\widehat{x}_{0\|0}$	Obs.
AAA	-3,99%	12,74%	16,4259	1,5853	60,56%	60,26%	3
AA	7,97%	12,83%	10,7647	1,3210	63,57%	42,68%	3
A	2,24%	10,68%	13,6111	1,6409	53,98%	67,95%	47
BBB	1,58%	9,87%	10,8530	1,5645	49,46%	63,67%	73
BB	3,65%	9,81%	16,6866	1,3976	42,97%	76,67%	45
B	5,76%	17,19%	17,6386	1,4504	46,17%	74,11%	36
CCC	0,84%	12,25%	14,6442	1,4101	52,72%	81,71%	16
IG	1,85%	10,31%	12,0124	1,5877	51,74%	64,69%	126
Non-IG	3,98%	13,02%	16,7034	1,4198	45,83%	76,55%	97

Standard deviations across firms, over ratings

Rating	$\mu_X^{\mathbb{P}}$	σ_X	α	θ	κ	$\widehat{x}_{0\|0}$	Obs.
AAA	10,16%	8,92%	14,6599	0,6972	20,60%	23,07%	3
AA	6,23%	6,25%	8,9698	0,2536	13,89%	18,88%	3
A	7,54%	8,61%	11,5302	0,3840	18,53%	30,69%	47
BBB	3,10%	4,86%	8,1726	0,3672	19,35%	27,68%	73
BB	11,60%	8,71%	13,4178	0,4481	24,56%	34,41%	45
B	10,98%	13,48%	18,6675	0,4686	27,58%	34,18%	36
CCC	6,16%	11,66%	9,5773	0,4259	30,96%	41,14%	16
IG	5,52%	6,58%	9,7145	0,3792	19,03%	28,61%	126
Non-IG	10,68%	11,61%	15,0296	0,4484	26,79%	35,23%	97

For each of the 225 firms, the monthly time series of CDS prices was used to estimate the model parameters using the UKF technique. The data are provided by DATASTREAM and spans the period from January 2004 to May 2008. Means and standard deviations are computed across firms for companies that had the same rating at the aforementioned date. The last column shows the number of companies in each rating class. The rating for two companies was not available (one in each portfolio).

are calculated where $\widehat{p}^{(i)}$ is the estimated value of some parameter p for company i and $\mathbb{I}_{\text{IG}}^{(i)}$ is an indicator that checks whether the firm qualifies as investment-grade (IG) (=1) or not (non-IG) (=0). Indicator values are taken from the S&P rating of the long-term debt in COMPUSTAT for 2004 to 2007 and the credit rating in 2008 is also according to S&P but comes from Markit. When we reject $\beta_1 = 0$ in favor of $\beta_1 \neq 0$, then there is a significant difference in the value of the parameter estimated for IG and non-IG firms. Finally, β_0 $(\beta_0 + \beta_1)$ is the mean value of $\{\widehat{p}^{(i)}, i = 1, 2, \ldots, 225\}$ for IG (non-IG) firms.

Table 5 shows the result of applying the regression of equation (21) on each of the six parameters $\mu_X^{\mathbb{P}}$, σ_X, α, θ, κ and $\widehat{x}_{0|0}$. It confirms what has been noticed in Section 6.2.2, that there are significant differences in the parameter values depending on the credit status (IG or non-IG) of the firm. The difference is statistically significant for σ_X, α, θ and $\widehat{x}_{0|0}$ in addition to being in the same direction than in Section 6.2.2. In other words, the debt ratio is more volatile for non-IG firms, the convexity of the transformation (α) is greater and the critical default threshold (θ) is lower.

The consequences of these findings are interesting. Suppose two firms have the same leverage dynamics and a common initial debt ratio, but for other reasons, one firm is rated as an IG and the other is not. Thus, for the poorly rated firm, two parameters will accelerate its default. First, the tolerance threshold is lower for the non-IG company, which means that leverage will attain θ faster. Moreover, once the threshold is attained, a higher value of α means that the default intensity will increase faster, further increasing future default likelihood.

6.3.2 Non-stationarity

Non-stationarity of the parameters is verified in this section to check how the onset of the credit crisis has influenced the evolution of CDS prices.

The sample of CDS prices from 2004 to 2008 is split in two, so that the first half contains the first 27 months (January 2004 to March 2006) while the second half contains the last 26 months (April 2006 to May 2008). One notes that the second half of the sample contains the onset of the 2007–2009 credit crisis so that a shift in the parameters should be observed.

Table 6 shows the distribution of the parameters in both halves of the sample along with the distribution of the difference (2006–2008 minus 2004–2006). The impact of the sudden rise in spreads in 2007–2008 is primarily seen in $\mu_X^{\mathbb{P}}$ (increase of 2.63%) and σ_X (increase of 4.60%) who considerably increased from one sample to the other. On the other hand, α decreased by 4 units, meaning that the debt ratio became less important in explaining default in this period. Another interpretation is that surprises became more relevant, so that the steep rise in spreads in 2007–2008 has been read by the model as both a deterioration of the solvency and an increase in surprises. The value of $\widehat{x}_{0|0}$, which represents the market debt ratio at the

Table 5: Results of the regression of each parameter on the investment grade status of the company

$\mu_X^{\mathbb{P}}$	Value	LB	UB
$\widehat{\beta}_0$	**1,85%**	0,42%	3,28%
$\widehat{\beta}_1$	2,13%	-0,05%	4,31%

θ	Value	LB	UB
$\widehat{\beta}_0$	**1,5877**	1,5157	1,6597
$\widehat{\beta}_1$	**-0,1679**	-0,2778	-0,0581

σ_X	Value	LB	UB
$\widehat{\beta}_0$	**10,31%**	8,72%	11,90%
$\widehat{\beta}_1$	**2,71%**	0,28%	5,14%

κ	Value	LB	UB
$\widehat{\beta}_0$	**51,74%**	47,76%	55,73%
$\widehat{\beta}_1$	-5,92%	-11,99%	0,16%

α	Value	LB	UB
$\widehat{\beta}_0$	**12,0124**	9,8561	14,1687
$\widehat{\beta}_1$	**4,6910**	1,4022	7,9798

$\widehat{x}_{0\|0}$	Value	LB	UB
$\widehat{\beta}_0$	**64,69%**	59,13%	70,24%
$\widehat{\beta}_1$	**11,86%**	3,39%	20,33%

For each of the 225 firms, the monthly time series of CDS prices was used to estimate the model parameters using the UKF technique. The data are provided by DATASTREAM and spans the period from January 2004 to May 2008. A regression of each parameter on the indicator of credit rating (IG or not) has been performed for each of the six parameters. The S&P rating is provided by Markit as of the date of the composition of the portfolio. The table shows the estimates of the slopes and intercept of this regression. LB (and UB) are the lower (upper) bounds of the 95% confidence interval for $\widehat{\beta}_0$ or $\widehat{\beta}_1$. When zero does not belong to the confidence interval, then there is a significant difference in the parameter value for IG and non-IG firms. Estimates that are significantly different than zero at a 5% level are denoted in **bold** face.

Table 6: Distribution of the parameters estimated with UKF in two samples: 2004–2006 and 2006–2008

2004–2006	$\mu_X^{\mathbb{P}}$	σ_X	α	θ	κ	$\widehat{x}_{0\|0}$
Mean	0,22%	8,10%	16,2585	1,3538	47,37%	67,46%
Stdev	6,30%	8,34%	15,7140	0,4404	19,05%	26,31%
10%	-4,33%	2,60%	4,0985	0,7771	20,30%	35,94%
25%	-1,96%	3,62%	6,2976	0,9856	37,58%	51,65%
50%	-0,45%	5,41%	12,1742	1,3438	49,82%	59,29%
75%	1,67%	8,67%	20,8080	1,7691	56,69%	84,30%
90%	4,61%	14,33%	30,0152	1,9727	71,40%	101,39%
IG	-0,24%	5,83%	15,0919	1,3604	48,11%	65,12%
HY	0,89%	11,48%	17,9939	1,3441	46,28%	70,94%

2006–2008	$\mu_X^{\mathbb{P}}$	σ_X	α	θ	κ	$\widehat{x}_{0\|0}$
Mean	2,84%	12,71%	12,2590	1,5890	47,33%	66,82%
Stdev	7,96%	9,74%	11,8369	0,3709	21,20%	32,42%
10%	-3,75%	3,35%	3,4912	1,0404	13,83%	30,83%
25%	-0,72%	6,42%	4,7222	1,3141	39,44%	41,25%
50%	1,53%	10,47%	7,0860	1,6732	49,96%	56,75%
75%	4,32%	15,69%	17,1842	1,9295	59,02%	87,03%
90%	9,64%	23,52%	27,5870	1,9991	71,93%	117,94%
IG	2,59%	11,71%	11,4623	1,6882	49,64%	65,43%
HY	3,22%	14,18%	13,4441	1,4415	43,90%	68,89%

Difference	$\mu_X^{\mathbb{P}}$	σ_X	α	θ	κ	$\widehat{x}_{0\|0}$
Mean	2,63%	4,60%	-3,9995	0,2351	-0,04%	-0,64%
Stdev	9,93%	9,59%	18,7842	0,5601	26,98%	36,71%
10%	-6,25%	-4,20%	-20,0337	-0,5346	-30,97%	-50,39%
25%	-1,67%	-0,29%	-11,2514	-0,1521	-15,65%	-23,93%
50%	2,05%	3,45%	-2,9527	0,2345	0,05%	-0,02%
75%	6,36%	9,30%	3,4440	0,6486	16,66%	21,95%
90%	11,42%	15,70%	13,5897	0,9642	30,56%	50,54%
IG	2,83%	5,88%	-3,6296	0,3278	1,53%	0,31%
HY	2,33%	2,70%	-4,5498	0,0974	-2,39%	-2,05%
Prop. > 0	65,33%	72,36%	39,20%	64,32%	50,75%	49,25%

For each of the 225 firms, the monthly time series of CDS prices were used to estimate the model parameters using the UKF technique. The data are provided by DATASTREAM and spans the period from January 2004 to May 2008. The sample is split approximately in two. The first half contains the first 27 months (January 2004 to March 2006) while the second half contains the last 26 months (April 2006 to May 2008). Estimation has been applied to both samples and the resulting parameters are compared. Means, standard deviations and quantiles are computed across firms. The top (middle) panel shows the parameters in the first (second) half. The bottom part of the table computes the variation in the parameter, i.e. the difference is taken as the parameter in the second sample (2006–2008) minus the parameter in the first (2004–2006). "Prop. > 0" is the proportion of companies for which the difference is positive.

beginning of the sample, did not change a lot since the conditions in April 2006 were similar to the ones observed in January 2004 (at the beginning of each sample). However, the filtered debt ratios increased during the time covered in the second sample so that θ increased accordingly. Consequently, this example shows that the parameters of the model may shift over time when important events occur on the markets.

It is also interesting to note that distinctions between the companies of the IG and HY portfolios still hold in both samples. Indeed, the drift, the volatility of the debt ratio and its initial value are higher for HY firms. Moreover, the sensitivity parameter α is also higher and the default threshold is lower for the riskiest companies. However, the differences look less important and regressions on the credit ratings (investment grade or not) are computed to check if those are statistically significant or not. The results are shown in Table 7. Note that because of transitions from IG to non-IG (and vice versa) between the two samples, there are slight differences between estimates shown in Table 7 and the ones shown in Table 6.

Table 7 shows that even though the direction of the relationships between the parameters and the credit status of the firm holds for the large majority of the cases, the differences (as noted in Section 6.3.2) are not large enough to be significant in many cases. The volatility of the debt ratio in both samples and the default threshold in the second sample are the exception, being significant at a 5% level.

Table 8 presents descriptive statistics about the absolute relative pricing errors for CDS prices for the two subsamples. For each firm and for each subperiod, the model's parameters have been estimated using UKF filtering technique. The theoretical CDS prices have been computed and compared to their market counterpart. A slight improvement is observed in terms of medians. However, the mean absolute relative

Table 7: Results of the regression of each parameter on the investment grade status of the company before and after the onset of the credit crisis

	2004–2006			2006–2008		
$\mu_X^{\mathbb{P}}$	Value	LB	UB	Value	LB	UB
$\widehat{\beta}_0$	-0,16%	-1,57%	1,25%	**2,59%**	1,14%	4,03%
$\widehat{\beta}_1$	0,52%	-1,36%	2,39%	0,72%	-1,56%	3,01%
σ_X	Value	LB	UB	Value	LB	UB
$\widehat{\beta}_0$	**5,97%**	4,17%	7,77%	**11,54%**	9,80%	13,29%
$\widehat{\beta}_1$	**3,52%**	1,12%	5,92%	**2,76%**	$>$ 0,00%	5,53%
α	Value	LB	UB	Value	LB	UB
$\widehat{\beta}_0$	**16,3951**	13,0946	19,6955	**11,4695**	9,3297	13,6093
$\widehat{\beta}_1$	-0,8723	-5,2623	3,5177	2,0895	-1,2981	5,4771
θ	Value	LB	UB	Value	LB	UB
$\widehat{\beta}_0$	**1,3862**	1,2891	1,4834	**1,6814**	1,6175	1,7453
$\widehat{\beta}_1$	-0,0622	-0,1915	0,0670	**-0,2353**	-0,3365	-0,1341
κ	Value	LB	UB	Value	LB	UB
$\widehat{\beta}_0$	**49,90%**	45,75%	54,04%	**49,62%**	45,81%	53,44%
$\widehat{\beta}_1$	-3,57%	-9,08%	1,94%	-5,86%	-11,90%	0,17%
$\widehat{x}_{0\|0}$	Value	LB	UB	Value	LB	UB
$\widehat{\beta}_0$	**64,92%**	59,08%	70,76%	**65,56%**	59,69%	71,42%
$\widehat{\beta}_1$	4,09%	-3,67%	11,86%	3,62%	-5,66%	12,91%

For each of the 225 firms, the monthly time series of CDS prices was used to estimate the model parameters using the UKF technique. The data are provided by DATASTREAM and spans the period from January 2004 to May 2008. The sample is split approximately in two. The first half contains the first 27 months (January 2004 to March 2006) while the second half contains the last 26 months (April 2006 to May 2008). A regression of each parameter on the indicator of credit rating (IG or not) has been performed for each of the six parameters for both halves of the sample. For the first sample, the S&P rating is the rating of the long-term debt provided by COMPUSTAT in 2006. For the second sample, the S&P rating is provided by Markit as of the date of the composition of the portfolio. The table shows the estimates of the slopes and intercept of this regression. LB (and UB) are the lower (upper) bounds of the 95% confidence interval for $\widehat{\beta}_0$ or $\widehat{\beta}_1$. When zero does not belong to the confidence interval, then there is a significant difference in the parameter value for IG and non-IG firms. Estimates that are significantly different than zero at a 5% level are denoted in **bold** face.

Table 8: Absolute relative pricing errors across the portfolio of firms of the CDX indices for 2004–2006 and 2006–2008 subsamples

	2004–2006					2006–2008				
Maturity	1	2	3	4	5	1	2	3	4	5
	All firms (2004–2006)					All firms (2006–2008)				
Mean	20,9%	12,3%	7,7%	3,9%	7,5%	15,6%	9,4%	5,6%	3,9%	8,0%
Median	15,4%	7,6%	4,2%	0,7%	4,1%	11,5%	5,8%	3,4%	1,4%	5,9%
nb obs	3981	3857	3981	3857	3981	2178	4769	4919	4769	4919
	IG firms (2004–2006)					IG firms (2006–2008)				
Mean	21,9%	13,1%	7,5%	2,7%	6,6%	16,7%	10,3%	6,8%	4,0%	9,2%
Median	17,5%	8,8%	5,0%	0,5%	4,5%	12,0%	6,1%	4,9%	1,2%	7,1%
	HY firms (2004–2006)					HY firms (2006–2008)				
Mean	19,3%	10,8%	8,0%	5,9%	9,0%	13,9%	7,9%	3,8%	3,9%	6,3%
Median	12,7%	6,1%	3,1%	1,2%	3,2%	10,7%	5,5%	1,8%	1,6%	4,6%

The CDS sample is divided in two subsamples, the first one, labeled 2004–2006, covers the period January 2004 to March 2006 (27 months) and the second one, labeled 2006–2008, goes from April 2006 to May 2008 (26 months). For each of the 225 firms and each subsample, the monthly time series of CDS prices were used to estimate the model parameters using UKF filtering technique. Using the parameters obtained with the UKF method, the filtered market debt ratio is computed and the CDS is priced for each available maturity and month. For each company, year and CDS maturity, the theoretical and observed CDS prices are compared. The absolute relative pricing error is computed as the absolute pricing error expressed as a percentage of the observed price.

error is much smaller, especially for HY firms. This is due to a few firms that have benefit of the sample split.

6.3.3 Recovery rate term structure

The value of the assets and liabilities at default can be used to provide a recovery rate that is consistent with the capital structure of the firm (see equation (4)). This approach is in line with Altman et al. (2004) and Altman (2006) who strongly suggest to represent recovery rates as being inversely proportional to default probabilities. In this section, the behavior of observed recovery rates at default are investigated with the model.

Given the parameters obtained in both samples (2004–2006 and 2006–2008), one million paths of the debt ratio have been generated under the real-world probability measure. When a default was generated, the result of equation (4) was computed. Defaults have then been sorted according to their moment of occurrence to measure the time-varying characteristics of the recovery rate. Thus, values of $R_\tau | \left(k \leq \tau < k+1\right), \, k = 0, 1, 2, \ldots, 9$ have been generated.

Figure 3 shows the term structure of expected recovery rates given default for companies of the CDX indices. The firms are grouped by their credit rating in the corresponding time period. It can be seen that recovery rates clearly vary over time and this is due to their time-varying solvency. The shape of the recovery rate term structure is explained by a reasoning similar to the one used for credit spread curves. Conditional upon survival, a highly-rated firm will only worsen so that its credit spread increases and its recovery rate should decrease over time. Similarly, given survival, a poorly-rated firm will improve its financial status, so that its credit spread will increase and its recovery rate will decrease.

As it is discussed in Altman (2006), recovery rates go down during economic crises. Figure 3 shows that in the second sample, comprising the recent credit crisis and current recession, recovery rates dropped down significantly. The drop has been more important for highly-rated firms as documented in Table 9. This is consistent with the important rise in CDS spreads of the IG portfolio compared with the spreads of the HY portfolio (see Figure 2).

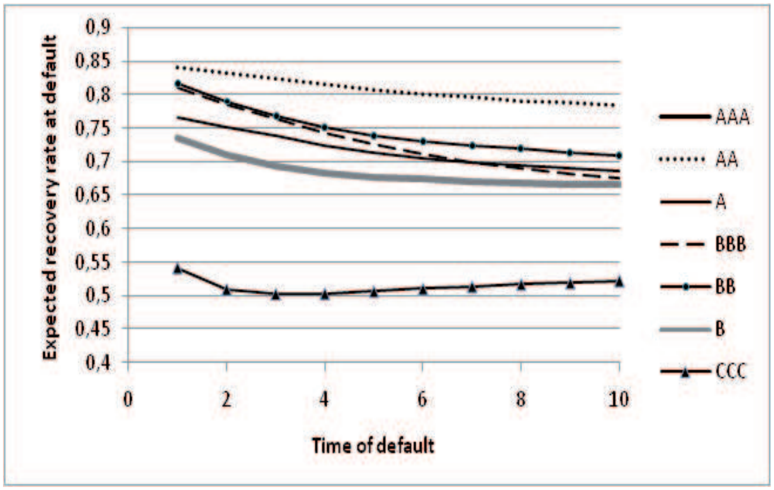
Figure 3 supports the fact that recovery rates are higher for highly-rated firms. This is especially obvious from the upper graph which depicts the term structure of recovery rates in the sample 2004–2006. Including the credit crisis in the second sample has the effect to reduce recovery rates, as discussed earlier. The AA class had a very important drop but there are few companies and among the three, two needed support from the U.S. government (TARP). In the four classes A, BBB, BB and B, the average drop is approximately

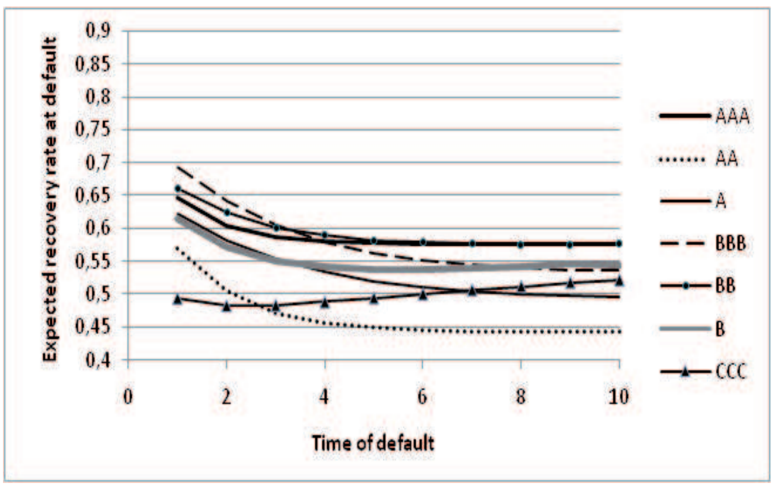
Table 9: Drop in the mean recovery rates from the first sample (2004–2006) to the second sample (2006–2008)

Time / Rating	AA	A	BBB	BB	B	CCC
1	27.11%	14.56%	11.68%	15.56%	12.05%	4.63%
2	32.52%	16.96%	14.39%	16.57%	13.82%	2.56%
3	35.05%	18.45%	15.85%	16.61%	14.30%	1.81%
4	35.74%	19.06%	16.36%	16.20%	14.21%	1.38%
5	35.76%	19.34%	16.30%	15.63%	13.89%	1.10%
6	35.35%	19.37%	15.94%	15.08%	13.52%	0.88%
7	35.12%	19.32%	15.46%	14.66%	12.97%	0.61%
8	34.47%	19.29%	14.94%	14.18%	12.57%	0.41%
9	34.38%	19.12%	14.50%	13.67%	12.03%	0.05%
10	33.89%	18.87%	13.86%	13.13%	12.11%	-0.01%
Nb (2004–2006)	2*	10	55	39	30	9
Nb (2006–2008)	3**	43	62	29	22	5

For each of the 225 firms, the monthly time series of CDS prices was used to estimate the model parameters using the UKF technique. The data are provided by DATASTREAM and spans the period from January 2004 to May 2008. The sample is split approximately in two. The first half contains the first 27 months (January 2004 to March 2006) while the second half contains the last 26 months (April 2006 to May 2008). For each of the companies, one million paths of the debt ratio over 30 years were simulated. The mean recovery rate given default is the average value of R_τ over the paths of default that occurred in a specific year. The results shown in this table computes the difference in the mean recovery rate from the sample 2004–2006 to 2006–2008, sorted by credit rating. The last two rows shows the number of companies considered in the analysis in each credit rating class and sample. There were no company rated AAA in the first sample. * The firms are GE Capital and FHLMC (Freddie Mac). ** The companies are AIG, Wal-Mart and Wells Fargo. Freddie Mac, AIG and Wells Fargo needed government support to get out of the financial crisis in September 2008.

Figure 3: Term structure of the mean recovery rate for all credit ratings in both samples (2004–2006 (top) and 2006–2008 (bottom))





The mean recovery rate given default has been computed for each company of the CDX indices. Recovery rates have been sorted by time of default to provide a term structure. Means are further computed over firms having a common credit rating. Parameters are estimated using maximum likelihood along with UKF to filter for trading noises. The rating is provided by S&P as of 2008.

15%. Moreover, the recovery rates in 2004–2006 are higher than the ones presented in Altman & Kishore (1998)[12] but the period 2004–2006 is one of exceptional growth. The level of the curves depicted in the sample 2006–2008 are closer to the ones presented in Altman & Kishore (1998) which is a blend of economic growth and severe turmoil.

In conclusion, recovery rates vary over time because the solvency of the firm changes and economic conditions as well. A CDS or a corporate bond should incorporate a different recovery rate for every period a default payment is due. Finally, as a consequence of the current recession, the mean recovery rates dropped significantly (by approximately 15%), especially for highly-rated firms.

[12] Their sample covered the time period 1971–1999.

7 Conclusion

The approach presented in this paper allows many structural models to become hybrid with a simple transformation of leverage to define the default intensity of the company. This transformation can be interpreted as the sensitivity of the credit risk of the company with respect to changes in leverage. As the default trigger is not purely specified by an exogenous barrier, the model produces endogenous random recovery rates that are negatively correlated to the default probabilities, which is consistent with the empirical findings. Thus the framework can be used to simultaneously model the two sources of credit risk i.e. the moment of default and the amount of loss at default, for risk management and pricing purposes.

The non-linearity between default intensity and leverage has been investigated empirically. To do so, the term structure of CDS premiums has been used to estimate the parameters of each company in the sample of 225 firms of the CDX NA IG and CDX NA HY indices. It was found that the credit risk of investment-grade firms is less sensitive to changes in their leverage than for non investment-grade firms. Moreover, the critical level of leverage is lower for the latter firms. These two effects work together to increase default probabilities of these companies. It was also found that the drift, the volatility and the initial level of the debt ratio are greater for riskier companies.

Estimating the model on two subsamples of CDS data, i.e. 2004–2006 and 2006–2008, it was discussed that parameters may be sensitive to major events such as the occurrence of the credit crisis in 2007. Many parameters shifted from one period to the other, leading to an increase in the credit risk of the companies. The credit crisis has been explained by the model as a general weakening of the solvency of companies (debt ratio grows faster and riskier) and an increase in the proportion of surprises in explaining defaults.

The expected recovery rates given default have been computed for each company in the sample over the two periods. Since recovery rates are inversely proportional to default probabilities, they show a decreasing (increasing) pattern for highly-(poorly-) rated firms. Due to the credit crisis, recovery rates dropped significantly and curves became steeper.

Based on the analysis conducted on the 225 firms of the CDX indices, it can be deduced there is a significant non-linear relationship between the default intensity and the leverage of the company. Moreover, this non-linearity may change over time and is more important for non investment-grade firms. Finally, a term structure of recovery rates can be deduced from the model and the curves have shifted down due to the credit crisis.

A Proofs

A.1 Survival probability

To compute the survival probability of the firm, one needs to evaluate the conditional expectation given by equation (5) where H_u is lognormal. It is possible to relate this expectation to a PDE, which in turn has been solved in a (quasi) closed-form solution in Dothan (1978). Since

$$Z_t \equiv S_t\left(T\right)\exp\left(-\int_0^t H_u du\right) = \mathrm{E}\left[\exp\left(-\int_0^T H_u du\right)\middle|\,\mathcal{G}_t\right]$$

is the conditional expectation of a $\mathcal{G}_T-$measurable bounded random variable with respect to an element of the filtration, it results that $\{Z_t : 0 \leq t \leq T\}$ is a $\{\mathcal{G}_t : 0 \leq t \leq T\}-$martingale. Now, assume that $S_t\left(T\right)$ is a doubly continuously differentiable function in H and t, that is, $S_t\left(T\right) = S\left(t, H_t; T\right)$. Applying Itô's lemma, one gets

$$dZ_t = \exp\left(-\int_0^t H_u du\right)\left[\begin{array}{c}\left(-H_t S + \frac{\partial S}{\partial t} + \mu_t^{(H)} H_t \frac{\partial S}{\partial H} + \frac{1}{2}\left(\sigma_t^{(H)}\right)^2 H_t^2 \frac{\partial^2 S}{\partial H^2}\right)dt \\ +\sigma_H\left(t\right) H_t \frac{\partial S}{\partial H} dB_t\end{array}\right].$$

The drift term of any martingale should be zero, meaning that the survival probability has to satisfy the partial differential equation

$$-H_t S + \frac{\partial S}{\partial t} + \mu_t^{(H)} H_t \frac{\partial S}{\partial H} + \frac{1}{2}\left(\sigma_t^{(H)}\right)^2 H_t^2 \frac{\partial^2 S}{\partial H^2} = 0$$

with the boundary conditions

$$\begin{aligned} S\left(T, H_T; T\right) &= S_T\left(T\right) = 1 \\ \lim_{H_t \to 0} S\left(t, H_t; T\right) &= 1 \\ \lim_{H_t \to \infty} S\left(t, H_t; T\right) &= 0. \end{aligned}$$

Applying the change of variable $t^* = T - t$ to the PDE, one obtains

$$\begin{aligned} H_{T-t^*} S\left(T - t^*, H_{T-t^*}; T\right) &= -\frac{\partial S}{\partial t^*}\left(T - t^*, H_{T-t^*}; T\right) + \mu_{T-t^*}^{(H)} H_{T-t^*} \frac{\partial S}{\partial H}\left(T - t^*, HT - t^*; T\right) \\ &\quad + \frac{1}{2}\left(\sigma_{T-t^*}^{(H)}\right)^2 H_{T-t^*}^2 \frac{\partial^2 S}{\partial H^2}\left(T - t^*, H_{T-t^*}; T\right). \end{aligned}$$

Letting $H_{t^*}^* = H_{T-t^*}$ and $S^*\left(t^*, H_{t^*}^*; T\right) = S\left(T - t^*, H_{t^*}^*; T\right)$, the PDE to solve becomes

$$-\frac{\partial S^*}{\partial t^*} + \mu_{T-t^*}^{(H)} H_{t^*}^* \frac{\partial S^*}{\partial H^*} + \frac{1}{2}\left(\sigma_{T-t^*}^{(H)}\right)^2 \left(H_{t^*}^*\right)^2 \frac{\partial^2 S^*}{\partial \left(H^*\right)^2} = H_{t^*}^* S^*$$

with the boundary conditions

$$\begin{aligned} S^*\left(0, H_{t^*}^*; T\right) &= 1 \\ \lim_{H_{t^*}^* \to \infty} S^*\left(t^*, H_{t^*}^*; T\right) &= 0 \\ \lim_{H_{t^*}^* \to 0} S^*\left(t^*, H_{t^*}^*; T\right) &= 1. \end{aligned}$$

The solution to this PDE is provided in Dothan (1978) when $\mu_t^{(H)}$ and $\sigma_t^{(H)}$ are constants.

A.2 Zero-coupon debt, random interest rates

A.2.1 Framework where the money market account is the numéraire

Recall the framework presented in Section 4.2. To price derivatives in such an environment, one must find an alternative probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, where the prices of all assets, discounted by the money market account, are martingales. Denote the time t value of the money market account by $D_t = \exp\left(\int_0^t r_u du\right)$ and define $\mathbf{B}_t^{\mathbb{Q}} \equiv \mathbf{B}_t^{\mathbb{P}} + \int_0^t \lambda_u du$, where $\{\lambda_t : t \geq 0\}$ is a $K+1$ dimensional predictable process. Using Itô's lemma, the dynamics of $A_t D_t^{-1}$ and $p_t\left(T\right) D_t^{-1}$ are

$$\begin{aligned} d\left(\frac{A_t}{D_t}\right) &= \left(\sigma_t^{(A)}\right)^\top A_t d\mathbf{B}_t^{\mathbb{Q}} \\ d\left(\frac{p_t\left(T\right)}{D_t}\right) &= p_t\left(T\right)\left(m_t\left(T\right) - r_t - \mathbf{\Sigma}_t^\top\left(T\right)\lambda_t\right) + p_t\left(T\right)\mathbf{\Sigma}_t\left(T\right) d\mathbf{B}_t^{\mathbb{Q}}. \end{aligned}$$

To obtain a probability measure $\mathbb{Q}$ such that $\left\{p_t\left(T\right) D_t^{-1} : t \geq 0\right\}$ and $\left\{A_t D_t^{-1} : t \geq 0\right\}$ are $\mathbb{Q}-$ martingales, the drifts of these processes should be 0 under $\mathbb{Q}$. Because of the choice of drift for the asset process (recall equation (16)), one immediately gets

$$dA_t = r_t A_t dt + \sigma_t^{\top(A)} A_t d\mathbf{B}_t^{\mathbb{Q}} \qquad (22a)$$

$$dp_t\left(T\right) = r_t p_t\left(T\right) + p_t\left(T\right)\mathbf{\Sigma}_t^\top\left(T\right) d\mathbf{B}_t^{\mathbb{Q}}. \qquad (22b)$$

A.2.2 Framework where the T-year zero-coupon bond is the numéraire

To compute the price of a defaultable zero-coupon bond, the dynamic of the default intensity under the $T-$forward measure $\mathbb{Q}_T$ is required.

Under the $T-$forward measure $\mathbb{Q}_T$, all assets, discounted by a zero-coupon bond with maturity T years, behave as martingales. Recall that under $\mathbb{Q}$, the dynamics of the risky asset and of the bond are given by the equations in (22a) and (22b). Applying Itô's lemma gives

$$d\left(\frac{A_t}{p_t\left(T\right)}\right) = \frac{A_t}{p_t\left(T\right)}\left(\mathbf{\Sigma}_t^{\top}\left(T\right)\mathbf{\Sigma}_t\left(T\right) - \sigma_t^{\top(A)}\mathbf{\Sigma}_t\left(T\right)\right)dt + \frac{A_t}{p_t\left(T\right)}\left(\sigma_t^{(A)} - \mathbf{\Sigma}_t\left(T\right)\right)^{\top} d\mathbf{B}_t^{\mathbb{Q}}.$$

Letting $\mathbf{B}_t^{\mathbb{Q}_T} \equiv \mathbf{B}_t^{\mathbb{Q}} + \int_0^t \gamma_u du$, one gets

$$\begin{aligned} d\left(\frac{A_t}{p_t\left(T\right)}\right) &= \frac{A_t}{p_t\left(T\right)}\left(\mathbf{\Sigma}_t^{\top}\left(T\right)\mathbf{\Sigma}_t\left(T\right) - \sigma_t^{\top(A)}\mathbf{\Sigma}_t\left(T\right) - \left(\sigma_t^{(A)} - \mathbf{\Sigma}_t\left(T\right)\right)^{\top}\gamma_t\right)dt \\ &\quad + \frac{A_t}{p_t\left(T\right)}\left(\sigma_t^{(A)} - \mathbf{\Sigma}_t\left(T\right)\right)d\mathbf{B}_t^{\mathbb{Q}_T}. \end{aligned}$$

Since $\left\{A_t p_t^{-1}\left(T\right) : 0 \leq t \leq T\right\}$ should be a $\mathbb{Q}_T-$martingale, its drift term is null:

$$\mathbf{\Sigma}_t^{\top}\left(T\right)\mathbf{\Sigma}_t\left(T\right) - \sigma_t^{\top(A)}\mathbf{\Sigma}_t\left(T\right) - \left(\sigma_t^{(A)} - \mathbf{\Sigma}_t\left(T\right)\right)^{\top}\gamma_t = 0,$$

implying that $\gamma_t = -\mathbf{\Sigma}_t\left(T\right)$. Since each element of γ_t is finite and deterministic, then the Novikov's condition

$$\mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T\left(\gamma_u^{(k)}\right)^2 du\right)\right] < \infty, k = 1, 2, \ldots, K+1$$

holds and, according to Cameron-Martin-Girsanov's theorem, $\mathbb{Q}$ and $\mathbb{Q}_T$ are equivalent and $\mathbf{B}_t^{\mathbb{Q}_T} \equiv \mathbf{B}_t^{\mathbb{Q}} + \int_0^t \gamma_u du$ is a $\mathbb{Q}_T-$Brownian motion. Consequently,

$$\begin{aligned} dp_t\left(\overline{T}\right) &= p_t\left(\overline{T}\right)\left(r_t + \mathbf{\Sigma}_t^{\top}\left(\overline{T}\right)\mathbf{\Sigma}_t\left(T\right)\right)dt + p_t\left(\overline{T}\right)\mathbf{\Sigma}_t^{\top}\left(\overline{T}\right)d\mathbf{B}_t^{\mathbb{Q}_T}, \\ dA_t &= \left(r_t + \mathbf{\Sigma}_t^{\top}\left(T\right)\sigma_t^{(A)}\right)A_t dt + \sigma_t^{\top(A)}A_t d\mathbf{B}_t^{\mathbb{Q}_T}, \end{aligned}$$

and, the dynamic of the debt ratio $X_t = p_t\left(\overline{T}\right)A_t^{-1}$ is

$$dX_t = X_t\mu_t^{(X)}\left(T,\overline{T}\right)dt + X_t\sigma_t^{(X)}\left(T,\overline{T}\right)^{\top} d\mathbf{B}_t^{\mathbb{Q}_T} \tag{23}$$

where

$$\begin{aligned} \mu_t^{(X)}\left(T,\overline{T}\right) &\equiv \mathbf{\Sigma}_t^{\top}\left(\overline{T}\right)\mathbf{\Sigma}_t\left(T\right) - \mathbf{\Sigma}_t^{\top}\left(T\right)\sigma_t^{(A)} + \sigma_t^{\top(A)}\sigma_t^{(A)} - \mathbf{\Sigma}_t^{\top}\left(\overline{T}\right)\sigma_t^{(A)} \\ \sigma_t^{(X)}\left(T,\overline{T}\right) &\equiv \mathbf{\Sigma}_t\left(\overline{T}\right) - \sigma_t^{(A)}. \end{aligned}$$

The final step is to find the behavior of the default intensity. Using the transformation given by (6) and Itô's lemma, one gets

$$\begin{aligned} dH_t &= H_t\left(\mu_t^{(X)}\left(T,\overline{T}\right)(\alpha-1) + \frac{1}{2}(\alpha-1)(\alpha-2)\sigma_t^{(X)}\left(T,\overline{T}\right)^{\top}\sigma_t^{(X)}\left(T,\overline{T}\right)\right)dt \\ &\quad + H_t(\alpha-1)\sigma_t^{(X)}\left(T,\overline{T}\right)d\mathbf{B}_t^{\mathbb{Q}_T}. \end{aligned}$$

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