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H. Assa, A. Balbás,
R. Balbás

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Hirbod Assa

GERAD & Département de mathématiques et de statistique
Université de Montréal
Montréal (Québec) Canada, H3C 3J7
assa@dms.umontreal.ca

Alejandro Balbás

Department of Business Administration
Universidad Carlos III de Madrid
Madrid, Spain

Raquel Balbás

Department of Actuarial and Financial Economics
Complutense University of Madrid
Madrid, Spain

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Abstract

In this paper we study the situation when a market might be destabilized in the presence of Good Deals. A Good Deal is in general a financial position while making no cost, does not produce any risk. We study Good Deals while a firm deals with a coherent risk measure and the market prices are ruled with a sub-linear pricing rule. The most important observation of this work is that the existence of a Good Deal is equivalent to the incompatibility between the pricing rule and the risk measure. Incompatibility has been introduced and studied in Balbás and Balbás (2009). We look into this situation from regulatory point of view in order to rule out Good Deals, purposing to stabilize financial markets. We propose some practical ways of modifying a risk measure in a minimal way, for regulating financial institutions to reserve more capital, in order to place financial institutions in a safer position.

Key Words: Compatibility, Compatible Extension, No Good Deal, CVaR, CCVaR, CAPM, Stochastic Discount Factor, External Risk, No Better Choice, Global Risk, Global/Local ratio.

Résumé

Dans cet article nous étudions la situation dans laquelle un marché peut être déstabilisé en présence de bonnes affaires. Une bonne affaire est en général une situation financière qui, tout en ne faisant pas de frais, ne produit pas de risque. Nous étudions de bonnes affaires lorsqu'une firme traite avec une mesure de risque cohérente et que les prix du marché sont régis par une règle de tarification sous-linéaire. L'observation la plus importante de ce travail est que l'existence d'une bonne affaire est équivalente à l'incompatibilité entre la règle de tarification et la mesure du risque. L'incompatibilité a été introduite et étudiée dans Balbás et Balbás (2009). Nous nous penchons sur cette situation du point de vue réglementaire afin d'exclure les bonnes affaires de l'intention de stabiliser les marchés financiers. Nous proposons quelques moyens pratiques de modifier une mesure du risque de façon minimale et de réglementer les institutions financières pour qu'elles mettent en réserve plus de capital afin se placer dans une position plus sûre.

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1 Introduction

Stability of financial markets is one of the biggest concern of regulators in particular, central banks. In the last century, world has witnessed many financial crises which provoked regulators to establish some rules in order to make safer and more stable markets. For example, in European Union, Basle II (finance) and Solvency II (insurance) contains a set of rules which the industry section should respect in order to place corporations in a safer position. Following these rules, any corporation computes its "capital reserves", i.e., additional capital that will be devoted to overcome mainly their loss periods in their economical activities. The appropriate size of reserve could be considered as the risk level associated to the firm activities. The importance of these rules and accordingly the "capital reserves" is to keep the markets in a safer and more stable state. It is generally accepted that stability of a market is mainly reached while the market is in equilibrium. The general theory of market equilibrium has been developed during the last century (see Debreu (1959)). It is also known that equilibrium is a balance between market participant needs and their preferences. In general, the state of stability is an outcome of a fair allocation of available resources, among market participants. But one cannot always rely on the existence of equilibrium while there are financial opportunities which destabilize a market. Most of the time, market destabilizers are financial positions deemed to be simultaneously safe and profitable. The most known example of such positions are an Arbitrages. An Arbitrage is easily detectable and cannot survive for a long time in a market. But Arbitrages are not the only positions which destabilize a market. In recent years, different risk measures have been used in financial institutions and regulatory sectors in order to assess the risk of financial positions and in order to calculate the capital requirement in reserve. Sometimes, these risk measures provoke a new generation of market destabilizers. These financial positions are the major objectives we will study in this paper. In this paper we study a kind of pathological positions called Good Deals. These positions are introduced and studied in Cochrane and Saa-Requejo (2000) and Cerný and Hodges (2002). Here we have our definition of a Good Deal which in general means a financial position without making any cost, never produces risk. Obviously, these kind of positions are in a high demand. We study these positions when we deal with a coherent risk measure in the presence of a sub-linear pricing rule (in the sense of Jouini and Kallal (1995b) and Jouini and Kallal (1995a)). We discover that the existence of a Good Deal is an outcome of incompatibility between the pricing rule and the risk measure. Incompatibility between the pricing rule and the risk measure is introduced and studied in Balbás and Balbás (2009).

A Good deal is not a rare phenomena. We show that dealing with the predominate class of law invariant risk measures, in very known models such as Black-Scholes model, always Good Deals exist. Therefore, our main goal in this paper is to find a recovery of the risk measure when a Good Deal does exist. This is mainly done for regulatory purposes while this can be also used for hedging and pricing. Our observation is that an underestimation in assessing the capital requirement for an under-questioned financial position, produces Good Deals. This provokes the regulator to ask financial institutions who hold that position, for more capital in reserve. Recovering this situation is carried out with modifying the risk measure to one which always dominates the first risk measure, taking into account the price of short sales. This is an important observation that the price of short sales should be seen in the assessment of capital requirement in reserve (see Corollary 5.1, ($E1$)).

For the reader convenience we enumerate the article's achievement as follows:

1 Concept of Good Deal and incompatibility are two folds of a single fact.

2 Good Deals do exist in very known models.

3 Two major direction of risk recovery is introduced.

4 In particular, a recovery for CVaR is presented.

In Section 2 we will present the notations and the general framework we are going to deal with. The concept of compatibility will be introduced in Section 3. We will consider a (maybe incomplete and/or imperfect) Arbitrage-free market with pricing rule π and a coherent risk measure ρ. In this section we also define the concept of a Good Deal, inspired by definitions in Cerný and Hodges (2002) and Cherny (2006). We will show that the lack of compatibility is equivalent to the existence of Good Deals. The most important result of this section is Theorem 3.2, which establishes that the necessary and sufficient condition to ensure the compatibility between the pricing rule and the risk measure is to rule out the Good Deals. Also, it is equivalent to belonging at least one Stochastic Discount Factor (hence SDF) to the sub-gradient of ρ.

In Section 4 we show that Good Deals almost always exsit when we deal with law invariant coherent risk measures.

In Section 5 we show that given an incompatible couple (π, ρ), how one can construct a minimal coherent risk measure ρ_m^{min} (or just ρ_m), compatible with π while also $\rho \leq \rho_m$. We will see that the existence of the minimal modification is tied to the existence of a minimal point of a partial order on SDF. The most important contribution of this section is Corollary 5.1, where ρ_m is constructed.

Section 6 discusses some ways to modify the minimal modification, among many possible ways. In particular, we discuss two ways, the first way is regarding some external criteria, and the second is regarding with the No Better Choice pricing technique. We focus on concrete risk functions and pricing models in Section 6. Special attention is devoted to the CVaR because this coherent risk measure is becoming very popular among researchers, managers and practitioners, due to its favorable properties. We apply the findings of Section 5 to CVaR so as to build the Compatible Conditional Value at Risk (CCVaR) in a general incomplete markets, where SDF is the set of all Equivalent Martingale Measures (hence EMM). This modifies the discussion in Balbás and Balbás (2009) where compatible CVaR; CCVaR, has been introduced. Hence, it seems our treatment overcomes the shortcoming of CVaR in producing Good Deals with preserving the good properties of the CVaR.

Finally at the end, Appendix contains some lemmas and the proof of the theorems, which need more explanation and might interrupt the flow of discussions.

2 Preliminaries and notations

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$ algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. Consider also a couple of conjugate numbers $p \in [1, \infty]$ and $q \in [1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual L^p (L^q) denotes the space of $\mathbb{R}-$valued random variables y on Ω such that $\mathbb{E}(|Y|^p) < \infty$, $\mathbb{E}()$ representing the mathematical expectation ($\mathbb{E}(|Y|^q) < \infty$, or y essentially bounded if $q = \infty$). According to the Riesz Representation Theorem, we have that L^q is the dual space of L^p when $p \neq \infty$.

In this paper we consider only two period of time, today and tomorrow, represented with 0 and T, respectively. Every random variable presents the pay-off of a financial position at time T. Whenever we talk about risk or price of a financial position we mean the present value of the price and the present risk associated to the financial position.

Let us assume that $Y \subset L^p$ is a closed convex cone containing $\mathbb{R}$ and is composed of all viable pay-offs, *i.e.*, for every $Y \in \mathcal{Y}$ there is a price associated to y. Consider a sub-linear arbitrage free pricing rule $\pi : Y \rightarrow \mathbb{R}$ in the sense of Jouini and Kallal (1995b) and Jouini and Kallal (1995a) i.e. a sub-additive, positive homogeneous function while also $\pi(k) = k$ for every $k \in \mathbb{R}$.

Remark 2.1 *The pricing rule π for example can be consider the super-replication price while also Y consists of all random variables like y such that there exists a viable self financing process which can super hedge y.*

Let

$$\rho : L^p \longrightarrow \mathbb{R},$$

be a general risk function that a trader uses in order to control the risk level of his final wealth at T. Assume that ρ is continuous and satisfies:

1. $\rho(Y + k) = \rho(Y) - k$,for every $Y \in L^p$ and $k \in \mathbb{R}$.
2. $\rho(\alpha Y) = \alpha\rho(Y)$, for every $Y \in L^p$ and $\alpha > 0$.
3. $\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2)$, for every $Y_1, Y_2 \in L^p$.

4. $\rho(X) \leq \rho(Y)$ for every $X, Y \in L^p$ and $X \geq Y$.

Particular interesting examples are the Conditional Value at Risk (CVaR) of Rockafellar et al. (2006).

Consider a continuous ρ satisfying 1), 2), 3) and 4). Denote by

$$\Delta_\rho = \{Z \in L^q | -\mathbb{E}(YZ) \leq \rho(Y), \forall Y \in L^p\}. \tag{2.1}$$

The set Δ_ρ is obviously convex. Bearing in mind the Representation Theorem 2.4.9 in Zalinescu (2002) for $p \neq \infty$, and using a proof similar to that of the Representation Theorem of a risk measure in Rockafellar et al. (2006), it may be stated that Δ_ρ is also $\sigma(L^q, L^p)$ −compact

$$\rho(Y) = \max\{-\mathbb{E}(YZ) : Z \in \Delta_\rho\}, \tag{2.2}$$

holds for every $Y \in L^p$. Furthermore by 1 (cash-invariance) and 4 (monotonicity) one can see that,

$$\Delta_\rho \subset \left\{Z \in L^q_+ | \mathbb{E}(Z) = 1\right\}. \tag{2.3}$$

Finally, by means of the Hahn Banach Separation Theorem, one may easily prove that if $\Delta_\rho \subset L^q$ is a convex and $\sigma(L^q, L^p)$ −compact while also Δ_ρ satisfies (2.3), then there exists a unique continuous ρ satisfying 1), 2), 3) and 4) such that (2.2) holds.

As for $p = \infty$, in order to have the same representation we need that ρ fulfills the Fatou property defined in Delbaen (2002). We say that ρ has Fatou property if for any bounded sequence $\{X_n\}_n \subseteq L^\infty$ converging in probability to X we have that $\rho(X) \leq \liminf_n \rho(X_n)$. With this assumption Δ_ρ defined as above is a subset of L^1. In general, Δ_ρ is not $\sigma(L^1, L^\infty)$-compact, hence in the seminal for $p = \infty$ we also add the assumption that Δ_ρ is $\sigma(L^1, L^\infty)$-compact, which with the aid of Dunford-Pettis Theorem means that Δ_ρ is Uniformly Integrable. This assumption is verified while dealing with many known risk measures, for instance CVaR_α, where we have $\Delta_{\text{CVaR}_\alpha} = \{f : \Omega \to \mathbb{R} |\ 0 \leq f \leq \frac{1}{\alpha}, \mathbb{E}[f] = 1\}$. We add the compactness assumption helping us to better explore the idea of this paper avoiding very elaborated functional analysis discussions while covering the most important examples.

3 Compatibility and Good Deals

This section will be devoted to introduce and characterize the notion of compatibility between risk measures and pricing rules and its relation with Good Deals.

3.1 Compatibility

Definition 3.1 *The pricing rule π and the risk measure ρ are said to be compatible if there is no sequence $(Y_n)_{n=1}^\infty \subset \mathcal{Y}$ such that $\pi(Y_n) \leq 0$ for every $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \rho(Y_n) = -\infty, \tag{3.1}$$

simultaneously hold. □

Although the economical interpretation of the above definition is quite clear we bring the following discussion made in Balbás and Balbás (2009)

If π and ρ are not compatible, then every manager who uses ρ to assess the risk (capital requirement) can make the capital requirements as negative as he/she wihes, which does not make any economical sense. Indeed, suppose a random variable $Y_0 \in \mathcal{Y}$ represents the T-value of a financial position. The Y_0's final risk is given by $\rho(Y_0)$, justifying that this quantity is an adequate final value (at T) of the capital requirement. Indeed, by translation- invariance of ρ we have

$$\rho(Y_0 + \rho(Y_0)) = 0,$$

and the risk vanishes if the amount $\rho(Y_0)$ is invested in the risk-less security. But subadditivity of ρ and the existence of the sequence $(Y_n)_{n=1}^{\infty} \subset \mathcal{Y}$ above imply that

$$\rho(Y_0 + Y_n) \longrightarrow -\infty,$$

while

$$\pi(Y_0 + Y_n) \leq \pi(Y_0),$$

which means that no capital has to be added and the risk level may be reduced as desired if the manager buys Y_n. Thus, the capital requirement $\rho(Y_0)$ does not have to be added. On the contrary, by adding Y_n the trader may even borrow an arbitrary amount of money $-\rho(Y_0 + Y_n) \longrightarrow \infty$, since, according to translation-invariance of ρ,

$$\rho(Y_0 + Y_n + \rho(Y_0 + Y_n)) = 0.$$

The following propositions helps to understand better the state of incompatibility between ρ and π which are taken from Balbás and Balbás (2009)

Proposition 3.1 *The pricing rule π and the risk measure ρ are incompatible if and only if for every $a \in \mathbb{R}$ there exists a sequence $(Y_n)_{n=1}^{\infty} \subset \mathcal{Y}$ such that $\pi(Y_n) \leq a$ for every $n \in \mathbb{N}$ and (3.1) simultaneously hold.*

Proposition 3.2 *The pricing rule π and the risk measure ρ are not compatible if and only if for every $a \in \mathbb{R}$ there exists a sequence $(Y_n)_{n=1}^{\infty} \subset \mathcal{Y}$ such that $\rho(Y_n) \leq a$ for every $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \pi(Y_n) = -\infty,$$

simultaneously hold.

The interpretation of Propositions 3.1 and 3.2 seems to be clear. If π and ρ are incompatible then there is a significant lack of balance between prices and risks. This lack may provoke pathological situations, as described above, that cannot be accepted economically .

Now we consider a more practical discussion when we want to hedge a financial position x with all possible choices we can make subject to a given budget constrain on the set Y. Therefore, consider the following problem

$$\begin{cases} \min \rho(Y - X) \\ \pi(Y) \leq c \\ Y \in \mathcal{Y} \end{cases} . \tag{3.2}$$

where $c \in \mathbb{R}$ is a budget constrain on the price. This problem has been studied in Balbás et al. (2010), Balbás et al. (2009a) and Balbás et al. (2009b), hence we refer the reader to read this reference for more details.

Following the methods in Balbás et al. (2009b), bearing in mind (2.2), (3.2) is equivalent to the following infinite-dimensional linear optimization problem

$$\begin{cases} \min \theta \\ \theta + \mathbb{E}(YZ) \geq 0, \quad \forall Z \in \Delta_\rho \\ \pi(Y) \leq c \\ \theta \in \mathbb{R},\, Y \in \mathcal{Y} \end{cases} \tag{3.3}$$

It is shown in Balbás et al. (2009a) that the dual of (3.3) is given as

$$\begin{cases} \max\ \mathbb{E}[XZ] \\ \lambda\pi(Y) - \mathbb{E}(YZ) \geq 0, \quad \forall Y \in \mathcal{Y} \\ \lambda \in \mathbb{R}, \lambda \geq 0, Z \in \Delta_\rho \end{cases} \tag{3.4}$$

is the dual of (3.3), $\lambda \in \mathbb{R}$ and $Z \in \Delta_\rho$ being the decision variables. Again following discussions in the same reference we have that (3.4) is equivalent to the following

$$\begin{cases} \max\ \mathbb{E}[XZ], \\ Z \in \Delta_\rho \cap \mathcal{R}, \end{cases} \tag{3.5}$$

where

$$\mathcal{R} := \left\{ Z \in L^q \,\middle|\, \pi(Y) - \mathbb{E}(YZ) \geq 0, \forall Y \in \mathcal{Y} \right\}. \tag{3.6}$$

3.2 Good Deals

A Good Deal is a financial position which in general makes no cost while the associated risk is also negative. Such a position is in high demand and may destabilize a market in its presence. The term "Good Deal" for the first time has been used in Cochrane and Saa-Requejo (2000) when the author defined a Good Deal as a financial position with a reasonably high Sharpe ration. Then the term "Good Deal" appeared in Cerný and Hodges (2002) where the author defined the concept of "desirable" positions, and introduced a Good Deal as a desirable position with non positive price. In Cherny (2006) the author showed how the two previous concepts can rejoin when we use a generalized Sharp ratio, using a coherent risk measure instead of standard deviation. Here we adapt their definition to our setting as follows

Definition 3.2 *A Good Deal is a random variable y such that $\pi(Y) \leq 0$ and $\rho(Y) < 0$. No Good Deal is an assumption where there is not any Good Deal in the market.*

We also define the set $\mathcal{R}(A)$ for any subset $A \subseteq L^p$ as follows:

$$\mathcal{R}\big(A\big) := \left\{ Z \in L^q_+ \middle| \mathbb{E}(Z) = 1 \ , \ \mathbb{E}(ZY) \leq 0 \ \forall Y \in A \right\}. \tag{3.7}$$

Similar to Cherny (2006) Theorem 3.4 we have the following theorem:

Theorem 3.1 *There exists No Good Deal if and only if*

$$\Delta_\rho \cap \mathcal{R} \neq \emptyset.$$

Proof. The result is easily concluded following the proof of Theorem 3.4 in Cherny (2006) and observing that

$$\Delta_\rho \cap \mathcal{R}\big(\{X|\pi(X) \leq 0\}\big) = \Delta_\rho \cap \mathcal{R}.$$

□

Now we have the following theorem.

Theorem 3.2 *The following conditions are equivalent:*

1. *π and ρ are compatible.*
2. *There is No Good Deal.*
3. *$\mathcal{R} \cap \Delta_\rho \neq \emptyset$.*
4. *$\rho \geq -\pi$.*
5. *Problem (3.2) is bounded.*
6. *Problem (3.5) has a feasible solution.*
7. *There is no duality gap between (3.2) and (3.5).*

Proof. With the aid of Theorems 3.1 and following Balbás et al. (2009a) and Balbás et al. (2009b), we have $7 \Leftrightarrow 6 \Leftrightarrow 5 \Leftrightarrow 1 \Leftrightarrow 3 \Leftrightarrow 2$.
$(2 \Rightarrow 4)$. From the definition it is obvious that there is No Good Deal iff $\forall Y \in \mathcal{Y}$, $\pi(Y) \leq 0$ implies $\rho(Y) \geq 0$. Since $\pi(Y - \pi(Y)) = 0$ then $\rho(Y - \pi(Y)) \geq 0$ and because ρ is translation invariant then $\rho(Y) \geq -\pi(Y)$.
$(4 \Rightarrow 2)$. Let us consider that there exists a Good Deal $Y \in \mathcal{Y}$. So by definition $\rho(Y) < 0$ and $\pi(Y) \leq 0$ which obviously implies $\rho(Y) < -\pi(Y)$. □

What we deduce from the emptiness of the intersection $\mathcal{R} \cap \Delta_\rho$ is either the risk or the price is underestimated. In the both case, we are led to enlarging either Δ_ρ or $\mathcal{R}$. It is always possible to find an modification Δ_ρ (or $\mathcal{R}$) in a minimal way, which is the subject is studied in the next sections. But before that in the upcoming short section we want to show that Good Deals are not rare positions.

4 Law Invariant Coherent Risk Measures and Good Deals

In this section we show that in many known models such as Black- Scholes model, using a law invariant risk measure produces Good Deals. A law invariant risk measure ρ is a risk measure while $\rho(X) = \rho(Y)$ for any two random variable x, y with identical distributions.

It has been recently proven in Filipovic and Svindland (2008) the effective domain of every law invariant coherent risk measure ρ is L^1. This shows that one can consider that Δ_ρ, which is a subset of dual space, is in L^∞. For instance we know from Delbaen (2002) that:

$$\Delta_{\mathrm{CVaR}_\alpha} = \left\{ Z \in L^0 \,\middle|\, 0 \le z \le \frac{1}{\alpha}\,,\, \mathbb{E}[z] = 1 \right\}, \tag{4.1}$$

which is obviously a subset of L^∞. This implies that every pricing rule like one is given by Black-Scholes model, with unbounded stochastic discount factors cannot meet the statement 3 in Theorem 3.2.

5 Recovering Incompatibility

Discussions in last section show that compatibility may fail in very important cases. Now it is natural to analyze whether modifications of a risk measure allows us to recover this situation. The existence of a minimal recovery is tied to some mathematical concepts which appears in the following.

In this section we propose the recovery of risk measure to a *minimal compatible* one. This modification is always possible, which we are going to discuss in this section.

Definition 5.1 *Let π be a pricing rule and ρ be a coherent risk measure. Suppose that there exists a continuous $\tilde{\pi} : L^p \longrightarrow \mathbb{R}$ modifying π. Then the minimal compatible modification, ρ_m^{min} (or in brief ρ_m) of ρ is a measure such that:*

a) π and ρ_m^{min} are compatible, and $\rho \le \rho_m^{min}$

b) ρ_m^{min} is minimal, i.e., if π and $\tilde{\rho}$ are compatible, $\rho \le \tilde{\rho}$ and $\tilde{\rho} \le \rho_m^{min}$ then $\tilde{\rho} = \rho_m^{min}$.

From now on unless mentioning, otherwise ρ_m means ρ_m^{min}

Now we discuss the existence of ρ_m. We begin with the definition of $\preceq$ for two members $Z_1, Z_2 \notin \Delta_\rho$ as follows:

$$Z_1 \preceq Z_2 \;\Leftrightarrow\; C(Z_1) \subseteq C(Z_2), \tag{5.1}$$

where $C(Z_i)$ is the closed convex hall of $\Delta_\rho \cup \{Z_i\}$ for $i = 1, 2$. It is clear that since Δ_ρ is $\sigma(L^q, L^p)$-compact, then $C(Z_i) = \big\{(1-\lambda)Z_i + \lambda Z | \, \lambda \in [0,1]\;,\; Z \in \Delta_\rho\big\}$ for $i = 1, 2$. So we can give the following equivalent definition for $\preceq$:

$$Z_1 \preceq Z_2 \;\Leftrightarrow\; Z_1 \in C(Z_2).$$

In the following theorem, which is proved in Appendix, we see that the minimal member of this partial ordering always exists.

Theorem 5.1 *There exists $Z \in \mathcal{R}$ such that if for some $\tilde{Z} \in \mathcal{R}$, $\tilde{Z} \preceq Z$ then $\tilde{Z} = Z$.*

The proof of the following theorem is straightforward

Theorem 5.2 *The modified risk measure ρ_m is minimal if and only if*

$$\Delta_{\rho_m} = C(Z),$$

for some Z minimal in $\big(\mathcal{R}, \preceq\big)$.

By Theorems 5.1 and 5.2 we have the following corollary:

Corollary 5.1 (Minimal Modification of Risk Measure) *Let π be a pricing rule and ρ be a coherent risk measure. Suppose that there exists a continuous $\tilde{\pi} : L^p \longrightarrow \mathbb{R}$ modifying π. Then the minimal modification ρ_m always exists and we have the following statements:*

(E1) $\rho_m(Y) = \max\{\rho(Y), -\mathbb{E}(ZY)\}$ *for some Z minimal in $(\mathcal{R}, \preceq)$.*

(E2) ρ_m *is coherent if and only if ρ is coherent and Z is nonnegative.*

(E3) $\rho_m(Y) \geq \max\{\rho(Y), -\pi(Y)\}$.

(E4) If the market is perfect (i.e., if π is linear and continuous) then

$$\rho_m(Y) = \max\{-\pi(Y), \rho(Y)\}, \tag{5.2}$$

holds for every $Y \in \mathcal{Y}$.

Remark 5.1 *Notice that the existence of the modification $\tilde{\pi}$ above frequently holds. For instance, if the market is perfect, i.e. if Y is a subspace and π is linear and continuous, then the existence of $\tilde{\pi}$ follows from the Hahn Banach Theorem. On the other hand, if the market is complete and perfect then π will be increasing so as to prevent the existence of arbitrage (Duffie (1988)).*

6 Modification Rules

In the following we propose some methods of finding a minimal modification ρ_m of ρ. Modifying a risk measure is a way to better assess the risk in order to rule out Good Deals and mainly is carried out for regulatory purposes. Since a regulator should be concerned with all possibilities, it might consider that every position could be priced. This is why in the sequel we consider that $\mathcal{Y} = L^p$.

Here we propose two main method to modify the risk measure. The first method relies on minimizing a third function ϕ, which is interpreted as an external criteria (or maximizing an external utility). This new measure ϕ regards the fundamentals of ρ-user, for example we will see by considering $\phi(.) = \|.\|_{L^1}$ it does not allow that the set Δ_ρ spread out far much.

As for the second method of modifying the risk measure, our method is an outcome of finding the No Better Choice price of the *Global/Local Efficiency Ratio* (see Cherny (2006)).

6.1 External Criteria

In this section we let only Δ_{ρ_m} spreads out upon touching $\mathcal{R}$ in a minimum of an external criteria function ϕ. More precisely, let Z_{min} be a point in $\mathcal{R}$ such that for some $Z^* \in \Delta_\rho$ the following inequality holds

$$\phi(Z_{min} - Z^*) \leq \phi(Z - Z_1)\ ,\ \forall (Z, Z_1) \in \mathcal{R} \times \Delta_\rho.$$

Before moving on with our discussion, we should give the exact definition of ϕ.

Definition 6.1 *A function $\phi : (\mathcal{R} - \Delta_\rho) \subseteq L^q \to \mathbb{R}$ is an external criteria if*

(ϕ1) ϕ is positive and convex.

(ϕ2) $(Z, Z_1) \mapsto \phi(Z - Z_1)$ attains its minimum at $(Z_{min}, Z^*) \in \mathcal{R} \times \Delta_\rho$.

(ϕ3) $\phi(Z) = 0$ if and only if $Z = 0$.

Now we have the following theorem

Theorem 6.1 *Consider there is a Good Deal. With the notation above Z_{min} is a minimal for $(\mathcal{R}, \preceq)$.*

Proof. Since there is at least a Good Deal then by Theorem 3.2 we know that $\mathcal{R} \cap \Delta_\rho = \emptyset$. To prove the theorem's statement we consider the contrary, considering that there exists $\tilde{Z} \in C(Z_{min}) \cap \mathcal{R}$ such that $\tilde{Z} \neq Z_{min}$. Since $Z_{min} \neq \tilde{Z} \in C(Z_{min})$ there exists $\lambda \in (0, 1]$ and $Z_1 \in \Delta_\rho$ such that

$$\tilde{Z} = (1 - \lambda) Z_{min} + \lambda Z_1.$$

By convexity of Δ_ρ we deduce that $Z_2 = (1-\lambda)Z^* + \lambda Z_1 \in \Delta_\rho$. By conditions $(\phi 1), (\phi 3)$ we have that

$$\begin{aligned}\phi(\tilde{Z} - Z_2) &= \phi\Big((1-\lambda)Z_{min} + \lambda Z_1 - ((1-\lambda)Z^* + \lambda Z_1)\Big) \\ &= \phi\Big((1-\lambda)(Z_{min} - Z^*)\Big) \\ &\leq (1-\lambda)\phi(Z_{min} - Z^*).\end{aligned}$$

Since $0 \leq 1-\lambda < 1$, by definition of Z_{min} we should have $\phi(Z_{min} - Z^*) = 0$. By condition $(\phi 3)$ we get that $Z_{min} = Z^*$ which contradicts our Good Deal assumption. □

Remark 6.1 *In an incomplete market, there are more than one equivalent martingale measure (EMM) for the price of an underlined stock. Among many choices, the right pick is always an important question. For example, the minimal martingale measure provided by Föllmer Schweizer decomposition, the one which is the nearest in L^q-norm to the historical measure $\mathbb{P}$ or the one which has the least entropy. For a stock price modeled with a geometric Lévy process, the family of all martingale measures and different methods to pick an appropriate EMM is discussed in* Chan (1999)*. We can add another to this list, which concerns the existence of Good Deals. In the next section we will see how with an appropriate choice for ϕ, one can find a new criteria in choosing an EMM.*

6.1.1 Compatible Conditional Value at Risk; CCVaR

In this section we extend discussions in Balbás and Balbás (2009) to find a modified CVaR_α which we will call Compatible CVaR.

For that, in the following we are going to use the theory we have developed in the last section by implementing the external criteria $\phi(X) = \int_\Omega |X|$ and $\rho = \text{CVaR}_\alpha$ in Theorem 6.1.

Lemma 6.1 *For a given $g \in L^1_+$ with $\mathbb{E}[g] = 1$, the L^1-distance between g and $\Delta_{\text{CVaR}_\alpha}$ equals $2\int_\Omega \left(g - \frac{1}{\alpha}\right)^+$, i.e.,*

$$\min_{z_0 \in \Delta_{\text{CVaR}_\alpha}} \int_\Omega |g - z_0| = 2\int_\Omega \left(g - \frac{1}{\alpha}\right)^+.$$

Furthermore, the minimum is attained only in points Z^* defined as follows

$$Z^* = \frac{1}{\alpha} 1_{\{g \geq \frac{1}{\alpha}\}} + (g+h) 1_{\{g < \frac{1}{\alpha}\}}, \tag{6.1}$$

where h is a non-negative function in which $(g+h)1_{\{g<\frac{1}{\alpha}\}} \leq \frac{1}{\alpha}$ and

$$\int_{\{g<\frac{1}{\alpha}\}} h = \int_\Omega \left(g - \frac{1}{\alpha}\right)^+.$$

Proof. See the Appendix. □

Now from Theorem 6.1 and Lemma 6.1 the following theorem turns out,

Theorem 6.2 *Let SDF be the set of all Stochastic Discount Factors (e.g., EMM in an incomplete market). Consider that the minimum of $2\mathbb{E}[(\cdot - \frac{1}{\alpha})^+]$ over SDF, is attained at $g^* \in$SDF. Then g^* is a minimal point of* $(\text{SDF}, \preceq)$.

6.2 Global Risk and Performance Maximization

In this section we study partially the following coherent risk measure

$$y \mapsto \max\{\rho(Y), \pi(-Y)\}.$$

Following the two upcoming discussions we think that studying this risk measure is of a great importance. Studying this risk measure leads us in the sequel to a minimal modification of ρ, in the presence of Good Deals.

Discussion 1. For a moment let us revisit the hedging problem 3.3 for a general couple (ρ, π). For a general couple (ρ, π) the pricing and hedging of type 3.3, for any position y, is possible if the intersection $\mathcal{R} \cap \Delta_\rho$ is not empty (Theorem 3.2). Hence, the price is maximum of $\mathbb{E}[YZ]$ over all $Y \in \mathcal{R} \cap \Delta_\rho$. Therefore, one can associate a pricing rule π_ρ to the couple (ρ, π) as follows

$$\pi_\rho(Y) = \max\left\{\mathbb{E}[YZ] \,\middle|\, Z \in \Delta_\rho \cap \mathcal{R}\right\}. \tag{6.2}$$

According to Theorem 3.2 the associated pricing rule exists if there is no duality gap between 3.5 and 3.3, hence pricing and hedging is possible. With the same theorem again the associated pricing rule exists if there is No Good Deal.

We know that regardless the emptiness of intersection $\mathcal{R} \cap \Delta_\rho$, the price π always exists. A natural question is what is the risk measure which its associated pricing rule is π?

We give the following definition then

Definition 6.2 *The Global Risk $GR : L^p \to \mathbb{R}$ is the smallest risk measure dominating ρ such that the pricing rule π_{GR} associated with the couple (GR, π) is equal to π i.e. $\pi_{GR} = \pi$.*

Discussion 2. We have seen in $(E1)$, Corollary 5.1, that a minimal modification of a risk measure is a minimal way of assessing risk, taking into account a short sell price. We say *a short sell price* because there is not always one way of finding a minimal modification (i.e. choosing Z in $(E1)$). This can be interpreted as *the least* conservative modification of risk measure in order to rule out Good Deals.

On the contrary, one can ask what is *the most* conservative modification of risk in order to avoid generating Good Deals. The answer is of course the following modification

$$y \mapsto \max\left\{\rho(Y), \pi(-Y)\right\}. \tag{6.3}$$

This risk measure is counting the risk of the ρ-user, taking into account *the short price* i.e. $\pi(-Y)$. *The short price* can be interpreted as the risk which market associates to y. Obviously, it is an upper bound for the capital reserve when Good Deals are ruled out.

Here we have the following proposition relating Discussions 1 and 2, the proof is quite obvious so we leave it to the reader

Proposition 6.1 *For the couple (ρ, π) we have*

$$GR(Y) = \max\left\{\rho(Y), \pi(-Y)\right\}. \tag{6.4}$$

This proposition shows that the Global Risk does not only assess the trader's risk but also it assesses the market response to the shortening y which could be interpreted as the market risk.

As it is usual in the literature of coherent risk theory, in the sequel, we will denote the function $-\rho$ by u, and we will call it *monetary utility* associated to ρ.

What may seem very valuable to assess is the utility against the Global Risk. That means the ratio $\frac{u(Y)}{GR(Y)}$, which in some sense shows how worthwhile is keeping the financial position y. It is quite obvious that when the utility is positive, $u(Y) > 0$, and the Global Risk in non positive, $GR(Y) \leq 0$, the financial position y is very well performed, consequently we attribute $+\infty$ to it. On the contrary, the ratio should be 0 if $GR(Y) > 0$ and $u(Y) \leq 0$. So we have the following definition

Definition 6.3 *For a couple (π, ρ) the Global/Local performance ratio GL is defined as follows*

$$GL(Y) = \begin{cases} +\infty & \text{if } GR(Y) < 0, \\ \frac{u(Y)}{GR(Y)} & \text{if } GR(Y) \geq 0 \text{ and } u(Y) > 0, \\ 0 & \text{if } GR(Y) \geq 0 \text{ and } u(Y) \leq 0, \end{cases} \tag{6.5}$$

when $\frac{\text{positive}}{0} = +\infty$.

It is very easy to show that

$$GL(Y) = \begin{cases} +\infty & \text{if } u(Y) > 0 \text{ and } \pi(-Y) \leq 0, \\ \frac{u(Y)}{\pi(-Y)} & \text{if } u(Y) > 0 \text{ and } \pi(-Y) > 0, \\ 0 & \text{if } u(Y) \leq 0. \end{cases} \tag{6.6}$$

Here we can see that GL is a performance ration of keeping y. Let us look at the first line. If the utility of y is positive while the cost of shortening it is non-positive, obviously we should keep y. The second line indicates if the utility of keeping y is positive while the price of shortening is also positive, the higher the utility the more worthwhile keeping y, and the higher the shortening price the less worthwhile keeping y. Finally if the utility is non-positive it is not worth to keep y.

Now let us consider we are in a market without any Good Deal. Let y be a financial position y such that $\pi(Y) \leq 0$. It is clear since $\mathcal{R} \cap \Delta_\rho \neq \emptyset$ then $u(Y) \leq 0$ and by (6.6) we have $GL(Y) = 0$. This is interpreted as GL does not rank the market better than π does. But in the case that $\mathcal{R} \cap \Delta_\rho = \emptyset$ we always have $\sup_{\pi(Y) \leq 0} GL(Y) > 0$. This number shows the distance of the market with a market in the absence of Good Deals. We summarize previous discussions in the following proposition which the proof is now clear

Proposition 6.2 *There is No Good Deal if and only if $GL(Y) = 0$ for all y such that $\pi(Y) \leq 0$.*

Here we lead the discussion to the No Better Choice pricing associated with the performance ration GL defined by Cherny (2006). The definition of NBC is adapted for our setting from Cherny (2006).

Definition 6.4 *For any financial position g the NBC pricing is a real number x such that*

$$\sup_{\{Y + h(g-x) \,|\, \pi(Y) \leq 0\,,\, h \in \mathbb{R}\}} GL\Big(Y + h(g - x)\Big) = \sup_{\{y \,|\, \pi(Y) \leq 0\}} GL(Y). \tag{6.7}$$

Actually it is the cost for g in which the maximum efficiency ratio does not increase by adding the new product g. The set of all NBC prices are denoted by I_{NBC}.

We denote the supremum in (6.7) with R^* i.e.

$$R^* = \sup_{\{y \,|\, \pi(Y) \leq 0\}} GL(Y).$$

In Cherny (2006) it is shown that

$$R^* = \inf \left\{ R \geq 0 \,\middle|\, \left(\frac{1}{1+R} \Delta_\rho + \frac{R}{1+R} \bar{co}\big(\Delta_\rho \cup \mathcal{R}\big) \right) \cap \mathcal{R} \neq \emptyset \right\}.$$

Before moving on further to our discussion we impose the following assumption on $\mathcal{R}$

$$\mathcal{R} \text{ is } \sigma(L^q, L^p) - \text{compact}. \tag{6.8}$$

Since Δ_ρ and $\mathcal{R}$ are $\sigma(L^q, L^p)$-compact and both Δ_ρ and $\mathcal{R}$ are convex we have

$$\bar{co}\big(\Delta_\rho \cup \mathcal{R}\big) = co\big(\Delta_\rho \cup \mathcal{R}\big). \tag{6.9}$$

To show (6.9) let us consider $X \in co\big(\Delta_\rho \cup \mathcal{R}\big)$. Therefore, $X = \sum_{i=1}^{k} \mu_i Y_i + \sum_{j=1}^{l} \lambda_j Z_j$, where μ_i, λ_js are positive such that $\sum \mu_i + \sum \lambda_j = 1$ and also $(Y_i, Z_j) \in \Delta_\rho \times \mathcal{R}$ for $1 \le i \le k$, $1 \le j \le l$. Letting $\mu = \sum \mu_i$ and $\lambda = \sum \lambda_j$ we have that

$$X = \mu\Big(\sum \frac{\mu_i}{\mu} Y_i\Big) + \lambda\Big(\sum \frac{\lambda_j}{\lambda} Z_j\Big).$$

By convexity of Δ_ρ and $\mathcal{R}$ it yields that every member of $co\big(\Delta_\rho \cup \mathcal{R}\big)$ can be written as $X = \mu Y + \lambda Z$ for $(Y, Z) \in \Delta_\rho \times \mathcal{R}$ where $\lambda + \mu = 1$ for nonnegative λ and μ. Now let us consider that $X_n \in co\big(\Delta_\rho \cup \mathcal{R}\big)$ converges in $\sigma(L^q, L^p)$ to X. Therefore, there exist $0 \le \lambda_n \le 1$, $Y_n \in \Delta_\rho$ and $Z_n \in \mathcal{R}$ such that $X_n = (1-\lambda_n) Y_n + \lambda_n Z_n$. Since Δ_ρ and $\mathcal{R}$ are $\sigma(L^q, L^p)$-compact, upon a subsequence one can consider that Y_n, Z_n and λ_n converge to Y, Z and λ respectively in $\Delta_\rho, \mathcal{R}$ and $[0,1]$. This implies that $X = (1-\lambda)Y + \lambda Z \in co\big(\Delta_\rho \cup \mathcal{R}\big)$.

Now by (6.9) and $\Delta_\rho \cap \mathcal{R} = \emptyset$ it is clear that if for some $R > 0$, $Z_1 \in \Delta_\rho$ and $Z \in co(\Delta_\rho \cup \mathcal{R})$ we have $\frac{1}{1+R} Z_1 + \frac{R}{1+R} Z \in \mathcal{R}$ then $Z \in \mathcal{R}$. Hence we can rewrite it as

$$R^* = \inf\left\{ R \ge 0 \,\middle|\, \left(\frac{1}{1+R}\Delta_\rho + \frac{R}{1+R}\mathcal{R}\right) \cap \mathcal{R} \ne \emptyset \right\}.$$

Let

$$\mathcal{D}^* = \frac{1}{1+R^*}\Delta_\rho + \frac{R^*}{1+R^*}\bar{co}\big(\Delta_\rho \cup \mathcal{R}\big).$$

In Cherny (2006) it is shown that

$$I_{NBC}(g) = \big\{\mathbb{E}(Zg) \big|\, Z \in \mathcal{D}^*\big\}.$$

On the other hand in the same reference it is discussed that $\mathcal{D}^* \cap \mathcal{R}$ contains of the closest points of $\mathcal{R}$ to the set Δ_ρ with respect to the following distance:

$$\begin{aligned} d(\Delta_\rho, Z) &= \inf\left\{ R \ge 0 \,\middle|\, \exists (Z_1, \tilde{Z}) \in \Delta_\rho \times co(\Delta_\rho \cup \mathcal{R}),\, \frac{1}{1+R} Z_1 + \frac{R}{1+R}\tilde{Z} = Z \right\} \\ &= \inf\left\{ R \ge 0 \,\middle|\, \exists (Z_1, \tilde{Z}) \in \Delta_\rho \times \mathcal{R},\, \frac{1}{1+R} Z_1 + \frac{R}{1+R}\tilde{Z} = Z \right\}. \end{aligned}$$

This equality follows from the previous arguments. By Corollary 3.10 [Cherny (2006)] one can deduce that

$$\mathcal{D}^* \cap \mathcal{R} = \big\{ Z \in \mathcal{R} \,|\, d(\Delta_\rho, Z) \text{ is minimum } \big\}. \tag{6.10}$$

To justify this distance we first define the following distance on $(\Delta_\rho, \mathcal{R})$

$$\begin{cases} d : \Delta_\rho \times \mathcal{R} \to [0, +\infty], \\ d(Z_1, Z) = \inf\big\{ R \ge 0 \,|\, \exists \tilde{Z} \in \mathcal{R},\, \frac{1}{1+R} Z_1 + \frac{R}{1+R}\tilde{Z} = Z \big\}. \end{cases} \tag{6.11}$$

Geometrically, we connect Z_1 to Z and continue until hitting the last point in $\mathcal{R}$, named $\tilde{Z}$ (since $\mathcal{R}$ is $\sigma(L^q, L^p)$-compact the last point exists). So there is $R \ge 0$ such that $Z = \frac{1}{1+R} Z_1 + \frac{R}{1+R}\tilde{Z}$. Then $d(Z_1, Z) = R$. In the case that the continuation of the semi line $\overrightarrow{Z_1 z}$ hits $\mathcal{R}$ only in Z (i.e. $Z = \tilde{Z}$) we put $d(Z_1, Z) = +\infty$.

The function d is lower semi-continuous in $\sigma(L^q, L^p)$ topology; see Lemma 7.2 in the Appendix.

So we can find $(Z_1^{min}, Z^{min}) \in \Delta_\rho \times \mathcal{R}$ such that

$$d(Z_1^{min}, Z^{min}) = \inf \{d(Z_1, Z) \,|\, (Z_1, Z) \in \Delta_\rho \times \mathcal{R}\}.$$

And again by Lemma 7.2 and this last relation

$$\begin{aligned}\mathcal{D}^* \cap \mathcal{R} &= \{Z \in \mathcal{R} \,|\, d(\Delta_\rho, Z) \text{ is minimum } \} \\ &= \{Z \in \mathcal{R} \,|\, \exists Z_1 \in \Delta_\rho \,,\, d(Z_1, Z) \text{ is minimum } \}.\end{aligned}$$

The members of the set $\mathcal{D}^* \cap \mathcal{R}$ are the discount factors for the No Better Choice pricing technique. But interestingly the members of this set are also minimal for $(\mathcal{R}, \preceq)$ (following theorem) which by Theorem 5.2 leads us to a good choice of the risk recovery. The proof of the following theorem is in Appendix.

Theorem 6.3 *All members of $\mathcal{D}^* \cap \mathcal{R}$ are minimal for $(\mathcal{R}, \preceq)$.*

7 Appendix

Proof of Theorem 5.2. Before proving the theorem we need to prove the following lemma

Lemma 7.1 *Let $\{Z_n\}$ be a sequence in $\mathcal{R}$ such that $Z_1 \succeq Z_2 \succeq Z_3 \succeq \dots$ and $Z_n \to Z$ in $\sigma(L^q, L^p)$. Then*

$$\cap_{i\in\mathbb{N}} C(Z_i) = C(Z).$$

Proof. For every arbitrary $N \in \mathbb{N}$, we have $Z_n \preceq Z_N$, $\forall n \geq N$. Hence, $Z_n \in C(Z_N)$, $\forall n \geq N$. By closeness of $C(Z_N)$'s we deduce $Z \in C(Z_N)$. That gives $C(Z) \subseteq C(Z_N)$, $\forall N \geq 1$ and therefore, $C(Z) \subseteq \cap_{i\in\mathbb{N}} C(Z_i)$.

For the other implication let us consider $\tilde{Z} \in \cap_{i\in\mathbb{N}} C(Z_i)$. By definition for every N there exists $\lambda_N \in [0, 1]$ and $Z_N^* \in \Delta_\rho$ such that

$$\tilde{Z} = (1 - \lambda_N) Z_n + \lambda_N Z_N^*.$$

Since Δ_ρ is $\sigma(L^q, L^p)$-compact and $[0, 1]$ is bounded, one can extract a convergent sub-sequence from Z_N^* and λ_N converging to Z^* and λ, respectively. In limit we have

$$\tilde{Z} = (1 - \lambda) Z + \lambda Z^*.$$

By definition this gives that $\tilde{Z} \in C(Z)$ and therefore the proof is complete. □

Proof of the theorem. Let $\bar{Z} \in \mathcal{R}$ be fixed. Let

$$\mathcal{A} = \left\{ Z \in C(\bar{Z}) \cap \mathcal{R} \,\middle|\, Z \preceq \bar{Z} \right\}. \tag{7.1}$$

We show that $(\mathcal{A}, \succeq)$ satisfies the conditions of Zorn's lemma. Since $\bar{Z} \in C(\bar{Z})$, the set $\mathcal{A}$ is obviously nonempty. On the other hand let $\{Z_n\}_n$ be a chain in $\mathcal{A}$, that means $Z_1 \succeq Z_2 \succeq \dots$. Since $\mathcal{A}$ is $\sigma(L^q, L^p)$-compact, there exists a subsequence $\{z_{n_k}\}_k$ such that $z_{n_k} \to z$ in $\sigma(L^q, L^p)$, for some $Z \in \mathcal{A}$. By applying Lemma 7.1 and knowing that $\dots \supseteq C(Z_i) \supseteq C(Z_{i+1}) \supseteq \dots$ we have that $\cap_{i\in\mathbb{N}} C(Z_i) = C(Z)$. This means that Z is a supremal point of the chain. Now by applying Zorn's lemma there exists a maximal (minimal for $\preceq$) point $Z \in \mathcal{A}$.

Now we claim that Z is a minimal point for $\mathcal{R}$. Let us consider $\tilde{Z} \preceq Z$ and $\tilde{Z} \in \mathcal{R}$. Since $\bar{Z} \succeq Z \succeq \tilde{Z}$, by definition $\tilde{Z}$ is in $\mathcal{A}$ and consequently $Z = \tilde{Z}$. This means Z is minimal for $\mathcal{R}$. □

Proof of Lemma 6.1. Let $Z \in \Delta_{\mathrm{CVaR}_\alpha} = \{f | 0 \leq f \leq \frac{1}{\alpha}, \mathbb{E}[f] = 1\}$ and define

$$\begin{aligned}Z_1 &:= (Z - g) 1_{z \geq g}, \\ Z_2 &:= (g - Z) 1_{\{g \geq z, g < \frac{1}{\alpha}\}},\end{aligned}$$

$$Z_3 := \min(Z, g),$$
$$Z_4 := (\frac{1}{\alpha} - Z)1_{g \geq \frac{1}{\alpha}},$$
$$Z_5 := (g - \frac{1}{\alpha})1_{g \geq \frac{1}{\alpha}}.$$

It is clear that

$$Z_1 + Z_3 = Z,$$
$$g = Z_2 + Z_3 + Z_4 + Z_5.$$

Therefore,

$$\int Z_1 + \int Z_3 = 1, \tag{7.2}$$
$$1 = \int Z_2 + \int Z_3 + \int Z_4 + \int Z_5. \tag{7.3}$$

On the other hand since z_2 and Z_4 are nonnegative we have

$$2Z_2 + Z_4 \geq 0. \tag{7.4}$$

Taking integration from (7.4) and adding it up to the summation of (7.2) and (7.3), it turns out

$$\int Z_1 + \int Z_2 \geq \int Z_5,$$

which by adding one more $\int Z_5$ the last relation becomes

$$\int Z_1 + \int Z_2 + \int Z_5 \geq 2\int Z_5.$$

With this, it is now easy to see that

$$\int |Z - g| = \int Z_1 + \int Z_2 + \int_{g \geq \frac{1}{\alpha}} (g - Z) \tag{7.5}$$
$$\geq \int Z_1 + \int Z_2 + \int Z_5 \tag{7.6}$$
$$\geq 2\int Z_5 = 2\int \left(g - \frac{1}{\alpha}\right)^+. \tag{7.7}$$

Therefore, $2\int \left(g - \frac{1}{\alpha}\right)^+$ is smaller than $\int |Z - g|$ for all Z.

Now we take three steps to conclude the proof: 1-First, we show every Z^* introduced in Lemma 6.1 is a minimum. 2- Second, we show at least one Z^* exists. 3- Third, we prove every minimum has the same structure as in (6.1).

Step 1. Let Z^* be defined as

$$Z^* = \frac{1}{\alpha}1_{g \geq \frac{1}{\alpha}} + (g + h)1_{g < \frac{1}{\alpha}}. \tag{7.8}$$

We show that $Z_2^* = Z_4^* = 0$. It is very easy to see that $Z_4^* = 0$. As for $Z_2^* = 0$ just observe that by definition of Z^*, $\{g < \frac{1}{\alpha}, g \geq Z^*\} = \{h = 0\}$ and therefore

$$Z_2^* = (g - Z^*)1_{\{g \geq Z^*, g < \frac{1}{\alpha}\}} = -h1_{\{g \geq Z^*, g < \frac{1}{\alpha}\}} = -h_{\{h=0\}} = 0.$$

Now since $Z_2^* = Z_4^* = 0$, we have equality in (7.6) and (7.7), which implies that Z^* is a minimum.

Step 2. We show that there exist a function h which satisfies the conditions in Lemma 6.1 and can be plugged into (6.1). By step 1. this should be a minimum.

First observe that since $0 > 1-\frac{1}{\alpha} = \int(g-\frac{1}{\alpha}) = \int_{g\geq\frac{1}{\alpha}}(g-\frac{1}{\alpha})+\int_{g<\frac{1}{\alpha}}(g-\frac{1}{\alpha})$ it turns out that $\int_{g\geq\frac{1}{\alpha}}(g-\frac{1}{\alpha}) < \int_{g<\frac{1}{\alpha}}(\frac{1}{\alpha}-g)$. Let $\lambda = \frac{\int_{g\geq\frac{1}{\alpha}}(g-\frac{1}{\alpha})}{\int_{g<\frac{1}{\alpha}}(\frac{1}{\alpha}-g)}$. Therefore, $\lambda < 1$. Defining $h := \lambda(\frac{1}{\alpha}-g)1_{g<\frac{1}{\alpha}}$, it is clear that h fulfills condition of Lemma 6.1.

Step 3. From step 1. it is clear that the amount of the minimum is $2\int\left(g-\frac{1}{\alpha}\right)^+$. This along with (7.6) and (7.7) show that for any minimal point $Z^*\in\Delta_\rho$ we must have $Z_2^* = Z_4^* = 0$.
Let us denote a minimum with Z^*. Since Z^* is a minimum, in (7.6) and (7.7) the inequalities must become equality. From (7.4), this implies that $Z_2^* = Z_4^* = 0$. $Z_4^* = 0$ implies that $Z^*1_{g\geq\frac{1}{\alpha}} = \frac{1}{\alpha}$. This is the first part of (6.1). On the other hand from $0 = Z_4^* = (g-Z^*)1_{\{g\geq Z^*, g<\frac{1}{\alpha}\}}$ it turns out that g cannot be larger than Z^* on $\{g<\frac{1}{\alpha}\}$. This gives that the function $h := (Z^*-g)1_{\{g<\frac{1}{\alpha}\}}$ is non-negative. Since $Z^*\leq\frac{1}{\alpha}$, it is obvious that $(g+h)1_{\{g<\frac{1}{\alpha}\}} \leq \frac{1}{\alpha}$.
Now

$$\begin{aligned}2\int\left(g-\frac{1}{\alpha}\right)^+ &= \int|g-Z^*|\\ &= \int_{g\geq\frac{1}{\alpha}}\left(g-\frac{1}{\alpha}\right)+\int_{g<\frac{1}{\alpha}}h\\ &= \int\left(g-\frac{1}{\alpha}\right)^+ + \int_{g<\frac{1}{\alpha}}h,\end{aligned}$$

which shows Z^* fulfills the conditions of Lemma 6.1 . □

Proof of Theorem 6.3. To prove the theorem first we need to prove the following lemma

Lemma 7.2 *d is $\sigma(L^q, L^p)$-lower semi-continuous.*

Proof. Let $a\in[0,+\infty]$. Then we must prove that

$$\left\{(Z_1, Z)\in\Delta_\rho\times\mathcal{R}\,\middle|\, d(Z_1,Z)\leq a\right\},$$

is $\sigma(L^q, L^p)$-closed. The case $a = +\infty$ is trivial. The case $a = 0$ is never applied , since we considered that $\Delta_\rho\cap\mathcal{R} = \emptyset$. So let $a\in(0,+\infty)$. Let $\{(Z_1^n, Z^n)\}_n$ be a sequence which fulfills $d(Z_1^n, Z^n)\leq a$ and converges to (Z_1, Z) in $\sigma(L^q, L^p)$. For each n there exists $\tilde{Z}^n$ such that $z^n = \frac{1}{1+d(Z_1^n,Z^n)}z_1^n + \frac{d(Z_1^n,Z^n)}{1+d(Z_1^n,Z^n)}\tilde{Z}^n$. Since $d(Z_1^n, Z^n)$ is bounded, by $\sigma(L^q, L^p)$-compactness of $\mathcal{R}$ one can find subsequence n_k such that $d(Z_1^{n_k}, Z^{n_k})$ and $\tilde{Z}^{n_k}$ converge respectively to d $(0\leq d\leq a)$ and $\tilde{Z}\in\mathcal{R}$. In limit we have that $\frac{1}{1+d}Z_1 + \frac{d}{1+d}\tilde{Z} = Z$, which by definition in turn yields $d(Z_1, Z)\leq d\leq a$. □

Proof of the theorem. Let $Z\in\mathcal{D}^*\cap\mathcal{R}$. By Lemma 7.2 we can consider there exist $Z_1^{min}\in\Delta_\rho$ such that $d(Z_1^{min}, Z)$ is minimum over $\Delta_\rho\times\mathcal{R}$. Let us show Z with Z^{min}. We know that there exists $\tilde{\tilde{Z}}\in\mathcal{R}$ such that

$$Z^{min} = \frac{1}{1+d(Z_1^{min}, Z^{min})}Z_1^{min} + \frac{d(Z_1^{min}, Z^{min})}{1+d(Z_1^{min}, Z^{min})}\tilde{\tilde{Z}}.$$

Now let us consider the contrary, that means there exists $\tilde{Z}\in C(Z^{min})\cap\mathcal{R}$ and $\tilde{Z}\neq Z^{min}$. By definition there exists $Z_2\in\Delta_\rho$ and $R\in[0,+\infty)$ such that $\frac{1}{1+R}Z_2 + \frac{R}{1+R}Z^{min} = \tilde{Z}$. From this relation it turns out that $d(Z_2,\tilde{Z})\leq R$ which yields $d(Z_1^{min}, Z^{min})\leq R<+\infty$. It assures us that $Z^{min}\neq\tilde{\tilde{Z}}$.

The point $\tilde{Z}$ cannot be on the line passes through $Z_1^{min}, Z^{min}, \tilde{\tilde{Z}}$. Actually since $\tilde{Z}\prec Z^{min}\prec\tilde{\tilde{Z}}$ then $\tilde{Z}\notin\overrightarrow{Z^{min}\tilde{\tilde{Z}}}$. It remains two possibilities: either $\tilde{Z}\in[Z_1^{min}, Z^{min})$ or $Z_1^{min}\in[\tilde{Z}, Z^{min})$. The first is ruled out since obviously then $d(Z_1^{min}, \tilde{Z}) < d(Z_1^{min}, Z^{min})$. The second possibility is also ruled out since then by convexity of $\mathcal{R}$, we get $Z_1^{min} \in \mathcal{R}$. Now we have four different points $Z_1^{min}, Z^{min}, \tilde{\tilde{Z}}, \tilde{z}$ which are not in the same direction and three of them $Z_1^{min}, Z^{min}, \tilde{\tilde{Z}}$ are in the same direction. So the convex combination of this four points is involved in a two dimensional affine space P. It is clear that $Z_2 \in P$. $Z_2 \neq Z_1^{min}$ since otherwise $\tilde{Z}$ is on the the line passing through $Z_1^{min}, Z^{min}, \tilde{\tilde{Z}}$. In the affine space P, the side $Z^{min}Z_2$ of the triangular $\triangle Z_1^{min}Z^{min}Z_2$ is hit by the semi-line $\overrightarrow{\tilde{\tilde{Z}}\tilde{Z}}$ in point $\tilde{Z}$. So the continuation should hit the other side $Z_1^{min}Z_2$ (the other side is impossible since again it puts $\tilde{Z}$ on the line passing through $Z_1^{min}, Z^{min}, \tilde{\tilde{Z}}$). Denote the hit point with Z_3 which by convexity belongs to Δ_ρ. Now on the side $Z_1^{min}Z^{min}$ of triangular $\triangle Z_1^{min}Z^{min}Z_2$ we find a point Z_4 in which Z_3Z_4 is parallel to Z_2Z^{min}. Obviously $Z_4 \in (Z_1^{min}, Z^{min})$. Since Z_3Z_4 and Z_2Z^{min} are parallel we have:

$$\frac{|Z_3\tilde{Z}|}{|\tilde{Z}\tilde{\tilde{Z}}|} = \frac{|Z_4Z^{min}|}{|Z^{min}\tilde{\tilde{Z}}|} < \frac{|Z_1^{min}Z^{min}|}{|Z^{min}\tilde{\tilde{Z}}|} = d(Z_1^{min}, Z^{min}). \quad (7.9)$$

But from definition $d(Z_3, \tilde{Z}) \leq \frac{|Z_3\tilde{Z}|}{|\tilde{Z}\tilde{\tilde{Z}}|}$. Therefore, $d(Z_3, \tilde{Z}) < d(Z_1^{min}, Z^{min})$ which is a contradiction. □

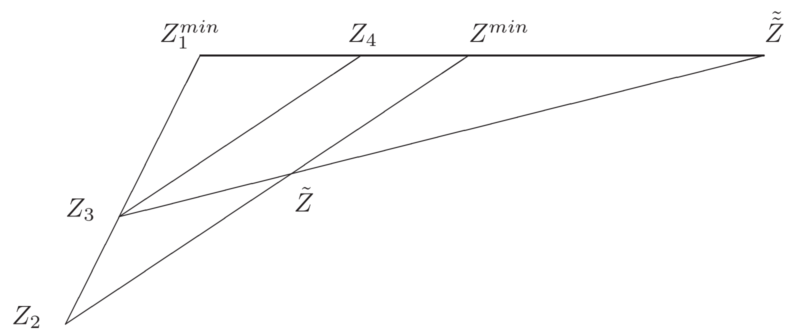

Figure 1: The proof illustration of Theorem 6.3

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