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Abstract

In this paper, we study the *Lebesgue* property for convex risk measures for a class of càdlàg processes, extending previous work of Delbaen (2000) and Jouini et al. (2006). It is shown that *Lebesgue* property can be characterized in several equivalent ways. Application to allocation of risk capital is presented.

Key Words: Convex risk measure, Lebesgue property, càdlàg processes, capital allocation problem.

Résumé

Dans cet article, nous étudions la propriété de *Lebesgue* des mesures convexes de risque pour une classe de processus càdlàg, étendant les travaux antérieurs de Delbaen (2000) et Jouini et al. (2006). On montre que la propriété de *Lebesgue* peut être caractérisée de plusieurs manières équivalentes. Des applications liées à l'allocation de capital de risque sont données.

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1 Introduction

Coherent risk measures for finite probability spaces were introduced in Artzner et al. (1999) and were extended to general probability spaces in Delbaen (2002), where applications to risk measurement, premium calculation and capital allocation problems were discussed. Föllmer and Schied (2002) defined the more general notion of convex risk measures and they extended the representation results of Delbaen (2002). In Cheridito et al. (2004, 2005), the authors studied risk measures for stochastic processes, instead considering only random vectors.

As can be seen in Delbaen (2000), the key concept for obtaining representations of risk measures is the so-called *Fatou* property, which can be regarded as a form of *semi-continuity* in an appropriate space. The *Lebesgue* property is a stronger concept. In an appropriate space, it is related to a continuity property, allowing for approximations of risk measures. In the context of coherent risk measures for random variables, the *Lebesgue* property was studied in Delbaen (2002), while it was studied for convex risk measures of random vectors in Jouini et al. (2006).

In this paper we extend the definition of *Lebesgue* property to the space of bounded càdlàg processes. We characterize the risk measures with this property in several equivalent ways. For several examples we show when they have *Lebesgue* property, and finally the application in capital allocation will be discussed.

The paper is organized as follows. In Section 2, we recall basic definitions and results for convex risk measures of random vectors and for a class of càdlàg processes. In particular, we state two results, one related to the *Fatou* property for risk measures of bounded càdlàg processes, and another one related to the *Lebesgue* property for risk measures of random bounded vectors. The main results of the paper are presented in Section 3. In particular, we characterize relatively compact subsets of a given dual space and we characterize the *Lebesgue* property. Furthermore we present an extended version of James' Theorem. In Section 4, we give some examples of risk measures with *Lebesgue* property. In Section 5 applications in capital allocation problem will be discussed. The proof of the main results are given in the Appendix.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard and atom-less probability space and let $(\mathcal{F}_t)_{0\le t\le T}$ be a filtration with the usual conditions. Further assume that $L^1(\Omega, \mathcal{F})$ has a countable dense subset.

In Cheridito et al. (2004, 2005) the authors developed the theory of convex risk measures on the space of $\mathcal{R}^p$ consisting of stochastic processes on $[0,T]$ that are càdlàg, adapted and such that $X^* = \sup_{[0,T]} |X_t| \in L^p$, with $1 \le p \le \infty$.

Note that for any $1 \le p \le \infty$, $\mathcal{R}^p$, endowed with the norm $\|X\|_{\mathcal{R}^p} = \|X^*\|_{L^p}$, is a Banach space.

For $q \in [1,\infty]$, let $\mathcal{A}^q$ be the set of all $a = (a^{\mathrm{pr}}, a^{\mathrm{op}}) : [0,T]\times\Omega \to \mathbb{R}^2$ so that a^{pr} and a^{op} are right continuous, have finite variation in L^q, a^{pr} is predictable, $a_0^{\mathrm{pr}} = 0$, a^{op} is optional and purely discontinuous.

Denoting the variation of function f by $\mathrm{Var}(f)$, it follows that $\mathcal{A}^q$ is also a Banach space, when equipped with the norm $\|a\|_{\mathcal{A}^q} = \|\mathrm{Var(a)}\|_{L^q}$. Furthermore, if p and q satisfies $\frac{1}{p}+\frac{1}{q}=1$, there is a duality relation between $\mathcal{A}^q$ and $\mathcal{R}^p$, viz.

$$\langle X, a\rangle = \mathrm{E}\left[\int_{]0,T]} X_{t-}da_t^{\mathrm{pr}} + \int_{[0,T]} X_t da_t^{\mathrm{op}}\right], \quad (X,a) \in \mathcal{R}^p \times \mathcal{A}^q. \tag{2.1}$$

Note that

$$|\langle X, a\rangle| \le \|X\|_{\mathcal{R}^p}\|a\|_{\mathcal{A}^q}.$$

The subset $\mathcal{A}_+^q$ of $\mathcal{A}^q$ consisting of $a = (a^{pr}, a^{op})$ with both components non-negative and non-decreasing, will be important in the sequel.

Further let $\mathcal{D}_\sigma$ be the unit ball of $\mathcal{A}^1_+$, i.,e., the subset of $a \in \mathcal{A}^1_+$ so that

$$\|a\|_{\mathcal{A}^1} = E\,(a^{\text{pr}}_T + a^{\text{op}}_T - a^{\text{op}}_0) = 1.$$

We are now in a position to recall some important definitions. But before doing that, we interpret X as a loss, so our definition of a risk measure coincide with the definition of translation invariant submodular, see, e.g. Delbaen (2002).

Definition 1 *A convex risk measure ρ on $\mathcal{R}^p$ is a function from $\mathcal{R}^p \to \mathbb{R}$ such that for any $X, Y \in \mathcal{R}^p$:*

1. *$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$, for all $0 \le \lambda \le 1$.*
2. *$\rho(X + m) = \rho(X) + m$, for any $m \in \mathbb{R}$.*
3. *$\rho(X) \le \rho(Y)$, whenever $X \le Y$.*
   ρ is called a coherent risk measure if in addition
4. *$\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda > 0$.*

In Cheridito et al. (2004), the authors propose the following definition for the *Fatou* property.

Definition 2 *A convex risk measure ρ on $\mathcal{R}^\infty$ has Fatou property if for any bounded sequence $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}^\infty$, for which there exists $X \in \mathcal{R}^\infty$ so that $(X_n - X)^* \xrightarrow{\mathbb{P}} 0$, we have $\rho(X) \le \liminf \rho(X_n)$.*

The following characterization of the Fatou property for convex risk measures is taken from Cheridito et al. (2004). Recall that γ is a penalty function if $\gamma : \mathcal{D}_\sigma \to (-\infty, +\infty]$ is such that $-\infty < \inf_{a\in\mathcal{D}_\sigma} \gamma(a) < \infty$.

Theorem 2.1 *Let ρ be a mapping from $\mathcal{R}^\infty$ to $\mathbb{R}$. Then, following statements are equivalent.*

1-
$$\rho(X) = \sup_{a\in\mathcal{D}_\sigma} \left\{\langle X, a\rangle - \gamma(a)\right\}, \; X \in \mathcal{R}^\infty, \tag{2.2}$$
for some penalty function γ.

2- ρ is a convex risk measure on $\mathcal{R}^\infty$ such that $\left\{X \in \mathcal{R}^\infty | \rho(X) \le 0\right\}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed.

3- ρ is a convex risk measure on $\mathcal{R}^\infty$ with the Fatou property.

4- ρ is a convex risk measure on $\mathcal{R}^\infty$ which is continuous for bounded increasing sequences.

Moreover, in each case, the conjugate function ρ^*, restricted to $\mathcal{D}_\sigma$, and defined by

$$\rho^*(a) = \sup_{X\in\mathcal{R}^\infty} \{\langle X, a\rangle - \rho(X)\},$$

is a penalty function which is smaller than γ and γ can be replaced by ρ^* in (2.2).

The following corollary is also taken from Cheridito et al. (2004).

Corollary 2.2 *A coherent risk measure ρ on $\mathcal{R}^\infty$ has Fatou property if and only if there exists a subset $\mathcal{Q}$ of $\mathcal{D}_\sigma$ such that*

$$\rho(X) = \sup_{a\in\mathcal{Q}} \langle X, a\rangle. \tag{2.3}$$

In fact, due to positive homogeneity, one ends up with $\rho^*(a) = \lambda\rho^*(a)$ for any $\lambda > 0$, showing that $\rho^*(a) \in \{0, +\infty\}$.

Next, *Lebesgue* property for risk measures on L^∞ was studied in Jouini et al. (2006). Their definition can be extended to convex risk measures on $\mathcal{R}^\infty$ as follows.

Definition 3 *A convex risk measure ρ on $\mathcal{R}^\infty$ has Lebesgue property if for any bounded sequence $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}^\infty$, for which there exists $X \in \mathcal{R}^\infty$ so that $(X_n - X)^* \xrightarrow{\mathbb{P}} 0$, we have $\rho(X) = \lim \rho(X_n)$.*

The following result, proved in Jouini et al. (2006), is a characterization of convex risk measures on L^∞ with Lebesgue property.

Theorem 2.3 *Let ρ be a convex risk measure on L^∞ with Fatou property. The following conditions are equivalent.*

1- ρ has Lebesgue property.
2- $\{Y \in L^1_+ | \rho^*(Y) \leq c\}$ is a $\sigma(L^1, L^\infty)$-compact subset of L^1 for every $c \in \mathbb{R}$.
3- $\mathrm{dom}(\rho^*) = \{\rho^* < \infty\} \subseteq L^1$.

We are now justified to extend the definition of the *Lebesgue* property to $\mathcal{R}^p$ for $1 \leq p < \infty$ as follows.

Definition 4 *A convex risk measure ρ on $\mathcal{R}^p$, $1 \leq p < \infty$, has Lebesgue property if the set $\{a \in \mathcal{A}^q : \rho^*(a) \leq c\}$ is $\sigma(\mathcal{A}^q, \mathcal{R}^p)$-compact, where*

$$\rho^*(a) = \sup_{X \in \mathcal{R}^p} \{\langle X, a\rangle - \rho(X)\}, \quad a \in \mathcal{A}^q.$$

Remark 1 *We will see in the next section, proposition 3.4 that as long as ρ has a representation like 2.2 (with $\mathcal{A}^q \cap \mathcal{D}_\sigma$ instead of $\mathcal{D}_\sigma$) then for the case $1 \leq p < \infty$, ρ always has Lebesgue property. Theorem 3.5 shows that for the case $p = \infty$, ρ has Lebesgue property iff $\{a \in \mathcal{A}^1 : \rho^*(a) \leq c\}$ is $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-compact, which shows that definition 4 could also be extended for $p = \infty$.*

Before giving the main mathematical results of the paper we should give some explanations and remarks which will be used in next discussions.

Note that taking $T = 0$ and $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{R}^p$ can be identified with the space $L^p = L^p(\Omega, \mathcal{F})$, while $\mathcal{A}^q$ can be identified with L^q. However, for $T > 0$, the process $X_t \equiv Y \in L^p$ does not belong to $\mathcal{R}^p$, since X it is not adapted in general. Therefore, L^p is not a subset of $\mathcal{R}^p$. To include these trivial processes, one must enlarge $\mathcal{R}^p$ and $\mathcal{A}^q$ in the following way, as proposed in Cheridito et al. (2004).

Define

$$\hat{\mathcal{R}}^p = \left\{ X : [0,T] \times \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} X \text{ is càdlàg} \\ X^* \in L^p \end{array} \right\}, \tag{2.4}$$

and

$$\hat{\mathcal{A}}^q = \left\{ a : [0,T] \times \Omega \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} a = (a^{\mathrm{l}}, a^{\mathrm{r}}), a^l_0 = 0 \\ a^{\mathrm{l}}, a^{\mathrm{r}} \text{ measurable} \\ \text{ finite variation} \\ \text{and right continuous} \\ \mathrm{Var}(a^{\mathrm{l}}) + \mathrm{Var}(a^{\mathrm{r}}) \in L^q \end{array} \right\}. \tag{2.5}$$

Further extend the duality relation (2.1) by setting

$$\langle X, a\rangle = \mathrm{E}\left[\int_{]0,T]} X_{t-} da^{\mathrm{l}}_t + \int_{[0,T]} X_t da^{\mathrm{r}}_t \right], \quad (X,a) \in \hat{\mathcal{R}}^p \times \hat{\mathcal{A}}^q. \tag{2.6}$$

Let $\Pi^{\mathrm{op}}, \Pi^{\mathrm{pr}}$ be the optional and predictable projections. See, e.g., Dellacherie and Meyer (1980), Kannan and Lakshmikantham (2002) or Cheridito et al. (2004).

For $a = (a^l, a^r) \in \hat{\mathcal{A}}^q$, let $\tilde{a}^l = \Pi^{\mathrm{pr}} a^l$ and $\tilde{a}^r = \Pi^{\mathrm{op}} a^r$. One can split $\tilde{a}^r$ uniquely into a purely discontinuous finite variation part $\tilde{a}^r_d$ and a continuous finite variation part $\tilde{a}^r_c$ with $\tilde{a}^r_c(0) = 0$. Since $\tilde{a}^r_c$ is predictable, being continuous, one can define a map Π^* from $\hat{\mathcal{A}}^q$ to $\mathcal{A}^q$ by

$$\Pi^* a := (\tilde{a}^l + \tilde{a}^r_c, \tilde{a}^r_d).$$

Every predictable process is also optional, so $\tilde{a}^l, \tilde{a}^r_c, \tilde{a}^r_d$ are all optional. It follows from Cheridito et al. (2004) that

$$\langle X, a\rangle = \langle X, \Pi^* a\rangle, \qquad (X,a) \in \mathcal{R}^p \times \hat{A}^q. \tag{2.7}$$

Remark 2 (2.7) *implies that*

$$\Pi^* : (\hat{\mathcal{A}}^q, \sigma(\hat{\mathcal{A}}^q, \hat{\mathcal{R}}^p)) \to (\mathcal{A}^q, \sigma(\mathcal{A}^q, \mathcal{R}^p))$$

is continuous.

Next, since any predictable process is optional, it follows from Theorem 2.1.53 Kannan and Lakshmikantham (2002), that for any $a \in \mathcal{A}^q$, the measure $\mu_a(A) = \langle 1_A, a\rangle$ is optional and then we have $\langle X, a\rangle = \langle \Pi^{\mathrm{op}}(X), a\rangle$. That, together with (2.7), yield

$$\langle \Pi^{\mathrm{op}}(X), a\rangle = \langle \Pi^{\mathrm{op}}(X), \Pi^*(a)\rangle = \langle X, \Pi^*(a)\rangle, \qquad (X,a) \in \hat{\mathcal{R}}^p \times \hat{A}^q. \tag{2.8}$$

Let $X \in L^p(\Omega, \mathcal{F})$ be a random variable. By Doob's Stopping Theorem it is easy to see that the optional projection of a constant random process X is the martingale $M_t := \mathrm{E}[X|\mathcal{F}_t]$. So, using (2.8), it follows that for every $X \in L^p$ and every $a = (a^l, a^r) \in \hat{\mathcal{A}}^q$, one has

$$\mathrm{E}\big[(a^l_T + a^r_T - a^r_0)X\big] = \langle X, a\rangle = \langle M, a\rangle\,, \tag{2.9}$$

Definition 5 *To every convex risk measure ρ on $\mathcal{R}^p$, one can associate a convex risk measure on L^p, called the static risk, viz.*

$$\bar{\rho}(X) := \rho\Big(\mathrm{E}\big[X\big|\mathcal{F}_t\big]_{0\le t\le T}\Big), \quad X \in L^p.$$

Remark 3 *By Corollary 2.2, every coherent risk measure ρ on $\mathcal{R}^\infty$ and having Fatou property, can be identified with a subset $\mathcal{Q}$ of $\mathcal{D}_\sigma$. Let $\mathcal{P} = \mathrm{Var}(\mathcal{Q}) := \{\mathrm{Var}(a) : a \in \mathcal{Q}\}$. By relation (2.9) it is easy to see that for all $X \in L^\infty$,*

$$\bar{\rho}(X) = \sup_{f\in\mathcal{P}} \mathrm{E}\big[fX\big]. \tag{2.10}$$

3 Main results

We will now state our main results. Their proofs are given in a series of appendices.

Jouini et al. (2006) showed that having the *Lebesgue* property for a convex risk measure with the *Fatou* property is equivalent to the weak compactness of lower contour sets of conjugate function. In their proof they use the fact that for any uniformly integrable set $\mathcal{P} \subseteq L^1$ and uniformly bounded sequence X_n tending in probability to X we have:

$$\sup_{Y\in\mathcal{P}} \mathrm{E}\big[YX_n\big] \to \sup_{Y\in\mathcal{P}} \mathrm{E}\big[YX\big]. \tag{3.1}$$

See Jouini et al. (2006) for details.

In order to extend the *Lebesgue* property to bounded càdlàg process risk measures, we need to find an analog of (3.1) for the space of bounded càdlàg processes. Uniformly integrability is compactness in the weak topology for L^1, so we could use the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$ compact set of $\mathcal{A}^1$ instead. That would allow us to characterize the *Lebesgue* property for convex risk measures on $\mathcal{R}^\infty$. This characterization, when restricted to L^∞, yields the characterization in Jouini et al. (2006). On the other hand, we find that the compactness of a set $\mathcal{Q}$ in the topology $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$ is related to the compactness of $\mathrm{Var}(\mathcal{Q})$, variation of $\mathcal{Q}$.

In this section we start by characterizing compact subsets of $\mathcal{A}^q$ with respect to the compact subsets of L^q.

The first results is used to characterize compact sets of $\mathcal{A}^q$. That will be useful in applications.

Theorem 3.1 *Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $\mathcal{Q} \subset \mathcal{A}^q$. The following conditions are equivalent:*

(C1) *$\mathcal{Q}$ is relatively compact in $\sigma(\mathcal{A}^q, \mathcal{R}^p)$.*

(C2) $\mathrm{Var}(\mathcal{Q})$ *is relatively compact in $\sigma(L^q, L^p)$.*

Furthermore, when $p = \infty$, any of (C1) or (C2) is equivalent to

(C3) *$\mathcal{Q}$ is bounded and for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $X \in \mathcal{R}^\infty$ bounded by 1 and with* $\mathrm{E}[X^*] \leq \eta$, *we have*

$$\sup_{a \in \mathcal{Q}} \langle |X|, a \rangle < \varepsilon. \tag{3.2}$$

The following corollaries are immediate consequences of Theorem 3.1.

Corollary 3.2 *$\mathcal{Q} \subseteq \mathcal{A}^1$ is $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)-$ relatively compact if and only if* $\mathrm{Var}(\mathcal{Q})$ *is uniformly integrable.*

Corollary 3.3 *$\mathcal{Q} \subseteq \mathcal{A}^q$, with $p < \infty$, is $\sigma(\mathcal{A}^q, \mathcal{R}^p)-$ relatively compact if and only if it is sequentially relatively compact.*

In the following we consider always that the risk measures always have the robust representation such as (2.2) with $\mathcal{D}_\sigma \cap \mathcal{A}^q$ instead of $\mathcal{D}_\sigma$. By Theorem 2.1 for the case $p = \infty$, it is equivalent to assume that the convex risk measures has *Fatou* property.

Proposition 3.4 *For $1 \leq p < \infty$, every convex risk measure $\rho : \mathcal{R}^p \to \mathbb{R}$ having representation (2.2) has also the Lebesgue property.*

When $p = \infty$, we have the following result.

Theorem 3.5 *Let $\rho : \mathcal{R}^\infty \to \mathbb{R}$ be a convex risk measure. Then the following are equivalent:*

(L1) *ρ has Lebesgue property.*

(L2) *For all $c \in \mathbb{R}$, $\{a \in \mathcal{A}^1;\ \rho^*(a) \leq c\}$ is relatively compact in $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$.*

(L3) *$\bar{\rho}$ has Lebesgue property.*

(L4) *For all $c \in \mathbb{R}$, $\{f \in L^1;\ (\bar{\rho})^*(f) \leq c\}$ is relatively compact in $\sigma(L^1, L^\infty)$.*

We complete the section by stating a result that could be interpreted as James' Theorem for the duality $(\mathcal{A}^q, \mathcal{R}^p)$.

Theorem 3.6 (James' Theorem for $(\mathcal{A}^q, \mathcal{R}^p)$) *Let $\mathcal{Q}$ be a convex, $\sigma(\mathcal{A}^q, \mathcal{R}^p)$-closed subset of $\mathcal{A}^q$. The set $\mathcal{Q}$ is compact in $\sigma(\mathcal{A}^q, \mathcal{R}^p)$ iff for each member $X \in \mathcal{R}^p$ it takes its supremum on $\mathcal{Q}$.*

4 Examples of Risks with Lebesgue Property

In the following examples $\mathcal{P}_\sigma$ is a subset of $\tilde{\mathcal{D}}_\sigma \cap L^q$ for $1 \leq q \leq \infty$ where here $\tilde{\mathcal{D}}_\sigma$ is defined as follows

$$\tilde{\mathcal{D}}_\sigma = \left\{ f \in L^1 \middle| f \geq 0\,,\ \mathrm{E}[f] = 1 \right\}.$$

The coherent risk measure ρ_σ is defined on L^p viz.

$$\rho_\sigma(X) := \sup_{f \in \mathcal{P}_\sigma} \mathrm{E}\big[fX\big]. \tag{4.1}$$

For example, $\mathcal{P}_\sigma$ can be the set of all equivalent martingale measures for a Lévy process (Föllmer and Leukert, 2000). In the case of uncertainty, i.e., $\mathcal{P}_\sigma$ has more than one element, the risk measure ρ_σ can be interpreted as a measure of the uncertainty aversion associated with $\mathcal{P}_\sigma$. See, e.g., Cont (2006).

Example 1 *Let Θ be a set of stopping times and ρ be defined as follows*

$$\rho(Y) = \sup_{a \in \mathcal{Q}_\Theta} \langle Y, a \rangle, \tag{4.2}$$

where $\mathcal{Q}_\Theta = \left\{ \left(0, \mathrm{E}[f|\mathcal{F}_\theta] 1_{t \geq \theta}\right) \middle| f \in \mathcal{P}_\sigma, \theta \in \Theta \right\}$.

For example, Θ can be a ruin time or the time that insurance surplus hits a specific barrier, as in Asmussen (2000). Also, Θ can be the set of exercising times of an American option.

It is easy to see that

$$\rho(X) = \sup_{\theta \in \Theta} \rho_\sigma(X_\theta) \ , \ \forall X \in \mathcal{R}^p.$$

By (2.8) *and Remark 3, the static risk is calculated as*

$$\bar{\rho}(X) = \sup_{a \in \mathcal{Q}} \langle X, a \rangle = \sup_{f \in \mathcal{P}_\sigma, \theta \in \Theta} \mathrm{E}\Big[\mathrm{E}[X|\mathcal{F}_\theta] f\Big] = \sup_{f \in \mathcal{P}_\sigma, \theta \in \Theta} \mathrm{E}\Big[X \mathrm{E}[f|\mathcal{F}_\theta]\Big].$$

According to Theorem 2.3 when $p = \infty$, $\bar{\rho}$ has Lebesgue property iff

$$\{(\bar{\rho})^* < \infty\} = \left\{ \mathrm{E}[f|\mathcal{F}_\theta] \middle| \theta \in \Theta \, , \, f \in \mathcal{P}_\sigma \right\}$$

is uniformly integrable, so by Theorem 3.5 ρ has Lebesgue property iff the previous set is uniformly integrable.

In particular, it has Lebesgue property when $\mathcal{P}_\sigma$ is uniformly integrable. In other words, ρ has Lebesgue property if ρ_σ does.

Example 2 *First, for any random variable $f \in \mathcal{P}_\sigma \subseteq \hat{\mathcal{D}}_\sigma \cap L^q$ (for some $1 \leq q \leq \infty$) and stopping time $\theta \in \Theta$, define the random process f_θ as follows*

$$f_\theta(t) = \begin{cases} \frac{t}{\theta} \mathrm{E}[f|\mathcal{F}_t] & t \leq \theta, \\ \\ \mathrm{E}[f|\mathcal{F}_\theta] & \textit{otherwise.} \end{cases} \tag{4.3}$$

Then, on $\mathcal{R}^p$, set

$$\rho(X) = \sup_{a \in \mathcal{Q}} \langle X, a \rangle, \tag{4.4}$$

where $\mathcal{Q} = \left\{ (f_\theta, 0) \middle| f \in \mathcal{P}_\sigma \, , \, \theta \in \Theta \right\}$. *It is easy to see that*

$$\rho(X) = \sup_{\theta \in \Theta} \rho_\sigma \left(\frac{1}{\theta} \int_0^\theta X_t dt \right), \tag{4.5}$$

$$\mathrm{Var}(\mathcal{Q}) = \left\{ \mathrm{E}[f|\mathcal{F}_\theta] \ \middle| \ f \in \mathcal{P}_\sigma \, , \, \theta \in \Theta \right\}, \tag{4.6}$$

$$\bar{\rho}(X) = \sup_{f \in \mathcal{P}_\sigma \, , \, \theta \in \Theta} \mathrm{E}\Big[X \mathrm{E}[f|\mathcal{F}_\theta]\Big] \ , \ \textit{for } X \in L^p. \tag{4.7}$$

By part (C2) of Theorem 3.1 ρ has Lebesgue property iff

$$\mathrm{Var}(\mathcal{Q}) = \left\{ \mathrm{E}[f|\mathcal{F}_\theta] \ \middle| \ f \in \mathcal{P}_\sigma \, , \, \theta \in \Theta \right\}$$

is uniformly integrable when $p = \infty$. Also it has Lebesgue property if $\mathcal{P}_\sigma$ is uniformly integrable

Example 3 (Snell Envelope and American Option Price Stability) *Let $X \in \mathcal{R}^\infty$. For a stopping time S bounded by T and let*

$$\Theta_S = \big\{\theta \geq S \big| \theta \text{ is } [0,T]\text{-value stopping time}\big\}.$$

Set

$$\begin{aligned}\rho_S(X) &= \operatorname{ess}\sup_{a\in\mathcal{Q}_S} \langle X, a\rangle \\ &= \operatorname{ess\,sup}\left\{\mathrm{E}\big[X_\theta \big| \mathcal{F}_S\big]\middle|\theta \in \Theta_S\right\}.\end{aligned}$$

The process $\rho_t(X)$ is the smallest super-martingale larger than X which is called the Snell envelope of X, see e.g., Cheridito et al. (2005).

Now for any measurable set $A \in \mathcal{F}_S$ define

$$\rho_S^A(X) = \mathrm{E}\big[\rho_S(X)1_A\big]. \tag{4.8}$$

It is exactly equivalent to put $\mathcal{P}_\sigma = \big\{1_A\big\}$ and $\Theta = \Theta_S$ in Example 1. From Example 1 we know that ρ_S^A has the Lebesgue property. Since the choice of $A \in \mathcal{F}_S$ is arbitrary, then by (4.8), we have that for each stoping time S the Snell envelope $\rho_S(X)$ is continuous in the weak star topology. In particular, setting $\rho_t = \rho_t^\Omega$, then $\rho_t(X_n) \to \rho_t(X)$ when $(X_n - X)^* \to 0$. This shows how one can approximate the price of American option in continuous time by time discretization.

Example 4 (Cumulative-Stopping Risk) *Let ρ_σ be a risk measure on L^p. A natural way to assess the risk of a random process is the average of the risk over the time interval i.e. $\frac{1}{T}\int_0^T \rho_\sigma(X_s)ds$. On the other hand let us consider that there exists a stopping time (or a general random time) which shows the moments in which the financial position is at more risk. Then a way to measure the risk of a random process X in $\mathcal{R}^p$ is to calculate*

$$\rho(X) = \int_0^T \rho_\sigma(X_s) f_\theta(s)ds, \tag{4.9}$$

where f_θ is the density function of θ. This new convex risk measure is called the Cumulative-Stopping risk.

In fact, for any measure μ on $[0,T]$,

$$\int_0^T \rho_\sigma(X_s)\mu(ds)$$

will work and it is a mixture risk measure.

It is not very difficult to see that when the risk measure ρ_σ is $\sigma(L^p, L^q)$-lower semi-continuous then ρ is also lower semi-continuous. It means that when ρ_σ has a representation like (2.2) (with L^q instead of $\mathcal{A}^1$) then ρ has a representation like (2.2), (with $\mathcal{D}_\sigma \cap \mathcal{A}^q$ instead of $\mathcal{D}_\sigma$). On the other hand, when $p = \infty$, the convex risk ρ has the Lebesgue property iff ρ_σ does. Actually this follows from part (L3) of Theorem 3.5.

5 Applications in Capital Allocation Problem

In this section, we give an application of Theorem 3.5 to allocation of risk capital. This problem for one-period coherent risk measures was discussed in Delbaen (2002), where the weak star sub-gradient of a coherent risk measure was defined. It was shown that the existence of a solution for the capital allocation problem is equivalent to having a nonempty sub-gradient. James' Theorem played a key role in showing that the weak sub-gradient is not empty. In our setting, Theorem 3.6 plays about the same role. The same allocation problem for dynamic coherent risk measures for discrete times was studied in Cherny (2009). For coherent allocation of risk capital, see Denault (2001) and the references therein.

We begin by recalling the definition of capital allocation. For more details see Delbaen (2000), Aubin (1974) and Billera and Heath (1982).

Let $X_1, \dots, X_N$ be N random processes in $\mathcal{R}^p$ representing N financial positions, for example, the losses of N departments of a firm. The total capital required to face the risk of $X_1 + \dots + X_N$ is $\rho(\sum_{i=1}^{N} X_i) = k$. We want to find a "fair" allocation $(k_1, \dots, k_N)$ so that $k_1 + \dots + k_N = k$.

Definition 6 *An allocation $(k_1, \dots, k_N)$ with $k = k_1 + \dots + k_N$ is called fair in fuzzy game approach if for all α_j , $j = 1, \dots, N$, $0 \le \alpha_j \le 1$ we have*

$$\sum_j \alpha_j k_j \le \rho\left(\sum_j \alpha_j X_j\right).$$

Before moving on with our discussion we recall the definition of a sub-gradient.

Definition 7 *For a function $\rho : \mathcal{R}^p \to \mathbb{R}$, the weak sub-gradient of ρ at X is defined by*

$$\bigtriangledown \rho(X) := \{a \in \mathcal{A}^q | \rho(X+Y) \ge \rho(X) + \langle Y, a\rangle, \quad \forall Y \in \mathcal{R}^p\}. \tag{5.1}$$

When $p = \infty$ this set can be empty but for $p \neq \infty$ this set is always nonempty (Ruszczyński and Shapiro, 2006)[Proposition 3.1].

We have the following extension of Theorem 17, Section 8.2 Delbaen (2000) without proof. Actually if one looks at the proof of Theorem 17, Section 8.2 Delbaen (2000) every part of the proof can be stated with random process X instead of random variable X.

Theorem 5.1 *Let ρ be a coherent risk measure with representation (2.3) when $a \in \mathcal{A}^q \cap \mathcal{D}_\sigma$, given by a family $\mathcal{Q} \subseteq \mathcal{D}_\sigma \cap \mathcal{A}^q$. Then $a \in \bigtriangledown\rho(X)$ iff $a \in \mathcal{Q}$ and $\rho(X) = \langle X, a\rangle$.*

As a direct consequence of Theorems 3.5, 5.1 and 3.6, we have

Theorem 5.2 *Let $\rho : \mathcal{R}^p \to \mathbb{R}$ be a coherent risk measure with representation (2.3) when $a \in \mathcal{A}^q \cap \mathcal{D}_\sigma$ given by $\mathcal{Q} \subseteq \mathcal{D}_\sigma \cap \mathcal{A}^q$. The the following conditions are equivalent:*

- $\bigtriangledown\rho(X) \neq \emptyset$, $\forall X \in \mathcal{R}^p$;
- *$\mathcal{Q}$ is $\sigma(\mathcal{A}^q, \mathcal{R}^p)$-compact;*
- *$\mathrm{Var}(\mathcal{Q})$ is $\sigma(L^q, L^p)$-compact;*
- *ρ has the Lebesgue property;*
- *$\bar{\rho}$ has the Lebesgue property.*

Finally, we can state the solution of the optimal allocation problem, using Theorems 5.1, 3.6 and 5.2.

Theorem 5.3 *If $X = X_1 + \dots + X_N$ and if $a \in \bigtriangledown\rho$, then the allocation $k_i = \langle X_i, a\rangle$ is a fair allocation.*

5.1 Calculating the Sub-gradient

Before giving the examples we calculate the subgradient of the risk measure constructed in Example 1 by considering $\Theta = \{\theta\}$. Again we consider a subset $\mathcal{P}_\sigma \subseteq \tilde{\mathcal{D}}_\sigma \cap \mathcal{A}^q$ and we let $\mathcal{Q} = \left\{\left(0, \mathrm{E}[f|\mathcal{F}_\theta]\right)1_{\theta\ge t} \middle| f \in \mathcal{P}_\sigma\right\}$. It is easy to see that:

$$\rho(X) = \sup_{a\in\mathcal{Q}} \langle X, a\rangle = \sup_{f\in\mathcal{P}_\sigma} \mathrm{E}[X_\theta f] = \rho_\sigma(X_\theta). \tag{5.2}$$

Now consider that $f \in \bigtriangledown\rho_\sigma(X_\theta)$. Then we have

$$\rho(X) = \rho_\sigma(X_\theta) = \mathrm{E}\big[X_\theta f\big] = \big\langle X, \big(0, \mathrm{E}\big[f|\mathcal{F}_\theta\big]\big)1_{t\ge\theta}\big\rangle. \tag{5.3}$$

Since $\big(0, \mathrm{E}\big[f|\mathcal{F}_\theta\big]\big)1_{t\geq\theta} \in \mathcal{Q}$, then by Theorem 5.1, $\big(0, \mathrm{E}\big[f|\mathcal{F}_\theta\big]\big)1_{t\geq\theta} \in \triangledown\rho(X)$. Hence

$$\left\{\big(0, \mathrm{E}[f|\mathcal{F}_\theta]\big)1_{t\geq\theta}\middle| f \in \triangledown\rho_\sigma(X_\theta)\right\} \subseteq \triangledown\rho(X), \tag{5.4}$$

On the other hand, if $a \in \triangledown\rho(X)$, it follows from Theorem 5.1 that is must be of the form $a = \big(0, \mathrm{E}[f|\mathcal{F}_\theta]1_{t\geq\theta}\big)$. Therefore,

$$\rho_\sigma(X_\theta) = \rho(X) = \big\langle X, \big(0, \mathrm{E}[f|\mathcal{F}_\theta]1_{t\geq\theta}\big)\big\rangle = \mathrm{E}[X_\theta f]. \tag{5.5}$$

Since $f \in \mathcal{P}_\sigma$, it shows that $f \in \triangledown\rho_\sigma(X_\theta)$, which in turn yields

$$\triangledown\rho(X) \subseteq \big\{\big(0, \mathrm{E}[f|\mathcal{F}_\theta]\big)1_{t\geq\theta}\big| f \in \triangledown\rho_\sigma(X_\theta)\big\}.$$

Combining that with (5.4), we end up with

$$\triangledown\,\rho(X) = \left\{\big(0, \mathrm{E}[f|\mathcal{F}_\theta]\big)1_{t\geq\theta}\middle| f \in \triangledown\rho_\sigma(X_\theta)\right\}. \tag{5.6}$$

5.2 Examples of Capital Allocation

Example 5 (Quantile Base Allocation) *Let $X_1, \ldots, X_N$ be the random processes presenting the future evolution of the value of N financial positions. Let $X = X_1 + \cdots + X_N$, $\Theta = \{\theta\}$ and*

$$\mathcal{P}_\sigma = \left\{h \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})_+ \middle| \ \mathrm{E}[h] = 1, 0 \leq h \leq \frac{1}{1-\alpha}\right\},$$

for some confidence level $0 < \alpha < 1$. Here ρ_σ is $AVaR_\alpha^{\mathcal{F}_T}$. Since $\mathcal{P}_\sigma \subseteq L^\infty$ then ρ_σ is a risk measure on L^1 and the corresponding measure ρ is defined for $\mathcal{R}^1$. From Delbaen (2000)[Section 8], we know that if X_θ is continuous then

$$\triangledown\ AVaR_\alpha(X_\theta) = \left\{\frac{1}{1-\alpha}1_A\right\}, \tag{5.7}$$

where $A = \big\{X_\theta > q_\alpha(X_\theta)\big\}$.

From Theorem 5.3 and Example 4, the allocation $(k_1, \ldots, k_n)$ is given by:

$$k_i = \frac{1}{1-\alpha}\mathrm{E}\Big[X_{i,\theta}1_A\Big], \tag{5.8}$$

Now let $\mathbb{Q}$ be an equivalent measure to $\mathbb{P}$ under which X is a martingale. Then we have

$$A = \left\{\frac{\mathrm{E}[X_T\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_\theta]}{\mathrm{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_\theta]} > q_\alpha\left(\frac{\mathrm{E}[X_T\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_\theta]}{\mathrm{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_\theta]}\right)\right\}. \tag{5.9}$$

Example 6 *Let $W = (W^1, \ldots, W^M)$ be an M dimensional of independent Brownian motions. In the Example 5 let*

$$d\vec{X}_t = \mu_t dt + \sigma dW, \tag{5.10}$$

where each (possibly random) component μ_t^i of μ_t is a positive functions satisfying Novikov's conditions and σ is a deterministic $N \times M$ matrix. By applying Doob's inequality for martingales, one can see that $X_i \in \mathcal{R}^1$. Actually since the function $x \mapsto |x|$ is a convex function then $|W_t|$ is a sub-martingale. Then by Doob's martingale inequality we have

$$\mathbb{P}\left[W^* = \sup_{0\leq t\leq T}|W_t| \geq c\right] \leq \frac{\mathrm{E}\left[|W_T|\right]}{c^2}.$$

Now

$$\mathrm{E}\left[W^*\right] = \int_0^\infty \mathbb{P}\left[W^* \geq c\right]dc \leq 1 + \mathrm{E}\left[|W_T|\right]\int_1^\infty \frac{1}{c^2}dc < \infty.$$

Next, note that

$$dX = \left(\sum_{i=1}^{N} \mu_t^i\right)dt + \sum_{i=1}^{N}\sum_{j=1}^{M} \sigma_{ij} dW^j, \tag{5.11}$$

which can be rewritten as

$$dX = \tilde{\mu}_t dt + \tilde{\sigma} d\tilde{W}, \tag{5.12}$$

where $\tilde{\mu} = \sum_{i=1}^{n} \mu_t^i$, $\tilde{\sigma} = \left(\sum_{j=1}^{M}\left(\sum_{i=1}^{N}\sigma_j^i\right)^2\right)^{1/2}$ *and* $\tilde{W}$ *is a Brownian motion.*

So we have $\frac{X_t}{\tilde{\sigma}} = \frac{1}{\tilde{\sigma}}\int_0^t \tilde{\mu}_s ds + \tilde{W}_t$. *By Girsanov's Theorem* $\frac{X_t}{\tilde{\sigma}}$ *is a martingale under the measure* $\mathbb{Q}$ *defined as follows*

$$\mathrm{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \exp\left(-\int_0^t \frac{\tilde{\mu}_s}{\tilde{\sigma}} d\tilde{W} + \frac{1}{2}\int_0^t \left(\frac{\tilde{\mu}_s}{\tilde{\sigma}}\right)^2 ds\right).$$

Using (5.8) *we have*

$$k_i = \frac{1}{1-\alpha}\mathrm{E}\Big[1_A\ X_{i,\theta}\Big], \tag{5.13}$$

where

$$A = \left\{\mathrm{E}\left[X_T \exp\left(-\int_\theta^T \frac{\tilde{\mu}_s}{\tilde{\sigma}} d\tilde{W} + \frac{1}{2}\int_\theta^T \left(\frac{\tilde{\mu}_s}{\tilde{\sigma}}\right)^2 ds\right)\middle|\mathcal{F}_\theta\right] > q_\alpha\left(X_T \exp\left(-\int_\theta^T \frac{\tilde{\mu}_s}{\tilde{\sigma}} d\tilde{W} + \frac{1}{2}\int_\theta^T \left(\frac{\tilde{\mu}_s}{\tilde{\sigma}}\right)^2 ds\right)\right)\right\}.$$

Example 7 *Consider that we have an insurance company with* N *departments, the loss value of each is shown by* $X_{i,t}$. *Consider that* $\vec{X} = (X_{1,t}, \dots, X_{N,t})$ *is modeled by the following process:*

$$\vec{X}_t = \vec{c}(t) + \vec{L}_t, \tag{5.14}$$

where $\vec{c}(t)$ *is an decreasing process and* $\vec{L}_t$ *is an* N*-dimensional Lévy process and* $\vec{L}_t \in (\mathcal{R}^1)^N$. *For instance, by using Doob's martingale inequality (similar to what we have done in previous example) any Lévy process in which is in* $\mathcal{R}^1$. *The characteristic function* $\mathrm{E}\big[e^{\vec{\lambda}\cdot\vec{X}_t}\big]$ *can be expressed as follows*

$$\mathrm{E}\big[e^{i\vec{\lambda}\cdot\vec{X}_1}\big] = \exp\left\{i\vec{a}\cdot\vec{\lambda} + \frac{1}{2}\vec{\lambda}^T Q\vec{\lambda} + \int_{(-\infty,0)^N}\left(e^{i\vec{\lambda}\cdot\vec{x}} - 1 - i\vec{\lambda}\cdot\vec{x}1_{\{|\vec{x}|<1\}}\right)\Pi(d\vec{x})\right\},$$

where $i^2 = -1$, Q *is a positively definite* $N \times N$ *matrix,* $\vec{a}$ *is an* N*-dimensional drift vector and* Π *is a measure on* $(-\infty,\infty)^N$ *for which* $\int_{\mathbb{R}^N}(1 \wedge |x|^2)\Pi(d\vec{x}) < \infty$. *From this last relation we deduce* $X = X_1 + \cdots + X_N$ *where*

$$X_t = c(t) + L_t. \tag{5.15}$$

Here $c(t) = c_1(t) + \cdots + c_N(t)$ *is a decreasing function and* $L_t = \sum_j L_{j,t}$. *Let* $\mu_j = \mathrm{E}[L_{j,1}]$. *It is clear that* $L_{j,t} - \mu_j t$ *is a martingale. Let say* $d_j(t) = c_j(t) + \mu_j t$ *and* $d(t) = \sum d_j(t)$. *Then* $X_t - d(t)$ *and* $X_{j,t} - d_j(t)$ *for* $1 \le j \le N$ *are martingales.*

Now the quantile allocation is given by (5.8) *as follows*

$$k_j = \frac{1}{1-\alpha}\mathrm{E}\Big[1_A\big(X_{j,T} - d(T) + d(\theta)\big)\Big], \tag{5.16}$$

where

$$A = \Big\{\mathrm{E}[X_T|\mathcal{F}_\theta] + d_j(\theta) > q_\alpha\Big(\mathrm{E}[X_T|\mathcal{F}_\theta] + d_j(\theta)\Big)\Big\}. \tag{5.17}$$

Example 8 (Cumulative-Stopping Allocation) *In this example we consider an insurance company with N departments. To model the departments values we use an N-dimensional α-stable Lévy process. Actually let $(Z_{1,t},\dots,Z_{N,t})$ be a N-dimensional α-stable Lévy processes with $1<\alpha<2$. Again by Doob's martingale inequality we know that $Z_i \in \mathcal{R}^1$. Now for some negative numbers c^i and positive numbers ,a_1^i, a_2^i, $i=1,\dots,N$ let*

$$X_{i,t} = c^i t + a_1^i Z_{1,t} + \cdots + a_N^i Z_{N,t}\ ,\ i=1,\dots,N. \tag{5.18}$$

On the other hand the company is concerned with some financial position made in the market. There are some crucial moments at which this financial position is at risk. These moments are modeled with a random time θ. The company uses the risk measure AVaR_a to asses the risk at each single time $t\in[0,T]$ (to avoid any confusion between the α's in the definition of risk AVaR_α and the α in α-stable process we use the notation AVaR_a for some $0<a<1$ instead of AVaR_α). The question is what is the risk allocated to each department with respect to the Cumulative-Stopping risk $\mathrm{AVaR}_a^{\theta,CS}(X) = \int_0^T \mathrm{AVaR}_a(X_s) f_\theta(s)ds$.

Let $(k'_{1,t},\dots,k'_{N,t})$ be an allocation for the static problem of allocating the risk for $X_t = X_{1,t}+\dots,X_{N,t}$ using the risk measure AVaR_a. We define the random variables $K_{i,\theta}(\omega) = k'_{i,\theta(\omega)}$ and then we define $k_{i,\theta} = \mathrm{E}\left[K_{i,\theta}\right]$ for $i=1,\dots,N$. For $0\le\alpha_1,\dots,\alpha_N\le 1$ we have:

$$\begin{aligned}
\alpha_1 k_{1,\theta}+\dots,\alpha_N k_{N,\theta} &= \mathrm{E}\left[\alpha_1 K_{1,\theta}+\cdots+\alpha_N K_{N,\theta}\right]\\
&= \int_0^T (\alpha_1 k'_{1,s}+\cdots+\alpha_N k'_{N,s}) f_\theta(s)ds\\
&\le \int_0^T \mathrm{AVaR}_a(\alpha_1 X_{1,s}+\cdots+\alpha_N X_{N,s}) f_\theta(s)ds\\
&= \mathrm{AVaR}_a^{\theta,CS}(\alpha_1 X_1+\cdots+\alpha_N X_N),
\end{aligned}$$

and the inequality is equality when $\alpha_1=\cdots=\alpha_N=1$. It means that $(k_{1,\theta},\dots,k_{N,\theta})$ is an allocation for $(X_1,\dots,X_N)$.

Let $(l'_{1,t},\dots,l'_{N,t})$ be an allocation for $(Z_{1,t},\dots,Z_{N,t})$. It is clear that $(l'_{1,t},\dots,l'_{N,t}) = (c_1 t + k'_{1,t},\dots,c_N t + k'_{N,t})$. Since $Z_t = Z_{1,t}+\cdots+Z_{N,t}$ has the scaling property (i.e. $Z_t \overset{d}{=} t^{\frac{1}{\alpha}} Z_1$) and AVaR_a is positively homogeneous and law invariant we conclude that $l'_{i,t} = t^{\frac{1}{\alpha}} l'_{i,1}$. From this we have

$$k'_{i,t} = -c_i t + t^{\frac{1}{\alpha}}(k'_{i,1}+c_i)\ \ \textit{for}\ \ i=1,\dots,N.$$

Now

$$\begin{aligned}
k_{i,\theta} &= \mathrm{E}[K_{i,\theta}]\\
&= \int_0^T k'_{i,s} f_\theta(s)ds\\
&= \int_0^T \left(-c_i t + t^{\frac{1}{\alpha}}(k'_{i,1}+c_i)\right) f_\theta(s)ds\\
&= -c_i \mathrm{E}[\theta] + (k'_{i,1}+c_i)\mathrm{E}\left[\theta^{\frac{1}{\alpha}}\right].
\end{aligned}$$

But we know that $k'_{i,1} = \mathrm{E}\left[X_{i,1} | X_{1,1}+\cdots+X_{N,1} > q_a(X_{1,1}+\cdots+X_{N,1})\right]$ so:

$$k_{i,\theta} = -c_i\mathrm{E}[\theta] + \left(\mathrm{E}\left[X_{i,1}\middle| X_{1,1}+\cdots+X_{N,1} > q_a(X_{1,1}+\cdots+X_{N,1})\right] + c_i\right)\mathrm{E}\left[\theta^{\frac{1}{\alpha}}\right].$$

A Proofs of the main results

A.1 Proof of Theorem 3.1

$(C2) \Rightarrow (C1)$. We split this part into two cases.

Case 1: $p \neq \infty$. By Theorems 65,67 of Section VII Dellacherie and Meyer (1980) we know that when $p \neq \infty$, the set $\hat{\mathcal{A}}^q$ is the dual of $\hat{\mathcal{R}}^p$. Since $\hat{\mathcal{A}}^q$ is endowed with the weak* topology, then $\mathcal{Q}$ is relatively compact iff it is bounded and the latter is true iff $\mathrm{Var}(\mathcal{Q})$ is bounded. In other words, $\mathcal{Q}$ is relatively compact in $\sigma(\hat{\mathcal{A}}^q, \hat{\mathcal{R}}^p)$ iff $\mathrm{Var}(\mathcal{Q})$ is relatively compact in $\sigma(L^q, L^p)$. Now the assertion $(C2) \Rightarrow (C1)$ is true because of the continuity of $\Pi^* : \hat{\mathcal{A}}^q \to \mathcal{A}^q$ (Remark 2).

Case 2: $p = \infty$. We define a topology on $\mathcal{R}^\infty$, generated by semi-norms.

For any weakly relatively compact subset $\mathcal{P}$ in L^1 let

$$V(\mathcal{P}) := \{a \in \mathcal{A}^1 | \exists f \in \mathcal{P} \text{ s.t.} \mathrm{Var}(a) \leq |f|\}$$

and define the associated semi-norm for $\mathcal{P}$ on $\mathcal{R}^\infty$ by

$$P_{\mathcal{P}}(X) = \sup_{a \in V(\mathcal{P})} \langle X, a \rangle.$$

This topology is compatible with the vector structure because obviously the $V(\mathcal{P})$'s are bounded. We denote this topology by σ^1. Let $(\mathcal{R}^\infty)'$ be the dual of $\mathcal{R}^\infty$ with respect to the topology σ^1. It is clear that $\mathcal{A}^1 \subseteq (\mathcal{R}^\infty)'$. We want to show that $\mathcal{A}^1 = (\mathcal{R}^\infty)'$.

Let μ be an arbitrary element of $(\mathcal{R}^\infty)'$ and X_n be a non-negative sequence such that $(X_n)^* \xrightarrow{\mathbb{P}} 0$. By (3.1), we have

$$0 \leq P_{\mathcal{P}}(X_n) \leq \sup_{f \in H} \mathrm{E}[(X_n)^* |f|] \to 0. \tag{A.1}$$

That implies $X_n \xrightarrow{\sigma^1} 0$ and then $\mu(X_n) \to 0$. That fact and (5.1) of chapter VII Dellacherie and Meyer (1980) show that any μ can be decomposed into a difference of two positive functionals. Let μ^+ be the positive part. By definition of the positive part (relation (5.2) Section VII in Dellacherie and Meyer (1980)) for any $X \geq 0$, $\mu^+(X) = \sup_{0 \leq Y \leq X} \mu(Y)$. Let X_n be a positive and decreasing sequence for which $(X_n)^* \downarrow 0$ in probability. Let $0 \leq Y_n \leq X_n$ be such that $\mu^+(X_n) \leq \mu(Y_n) + \frac{1}{n}$. Then since $(Y_n)^* \xrightarrow{\mathbb{P}} 0$ by (A.1) we get:

$$0 \leq \mu^+(X_n) \leq \mu(Y_n) + \frac{1}{n} \to 0.$$

That fact and Theorem 2 of Chapter VII Dellacherie and Meyer (1980) give that $\mu^+ \in \mathcal{A}^1$. Similarly $\mu^- \in \mathcal{A}^1$, so then $\mu \in \mathcal{A}^1$. This shows that $\mathcal{A}^1 = (\mathcal{R}^\infty)'$.

The Corollary to Mackey's Theorem 9, Section 13, Chapter 2 Grothendieck (1973) leads us to $\sigma^1 \subseteq \tau(\mathcal{R}^\infty, \mathcal{A}^1)$, where $\tau(\mathcal{R}^\infty, \mathcal{A}^1)$ is the Mackey's topology. By this relation we get that for a relatively weakly compact subset $\mathcal{P}$ in L^1 there exists C, a compact disk in $(\mathcal{A}^1, \sigma(\mathcal{A}^1, \mathcal{R}^\infty))$, for which $\{X | \sup_{a \in C} \langle X, a \rangle < 1\} \subset \{X | P_{\mathcal{P}}(X) \leq 1\}$. By polarity $V(\mathcal{P}) \subseteq \{X | P_{\mathcal{P}}(X) \leq 1\}^\circ \subseteq \{X | \sup_{a \in C} \langle X, a \rangle < 1\}^\circ$. Using the generalized Bourbaki-Alaoglu Theorem we get that $\{X | \sup_{a \in C} \langle X, a \rangle < 1\}^\circ$ is compact in the topology $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$.

Let $\mathcal{P} = \mathrm{Var}(\mathcal{Q})$. By $\mathcal{Q} \subseteq V(\mathrm{Var}(\mathcal{Q}))$, the proof is complete.

$(C1) \Rightarrow (C2)$. Let $f_\alpha = \mathrm{Var}(a_\alpha) \in \mathrm{Var}(\mathcal{Q})$ be a net. Then by $(C1)$ there exists a subnet $a_\beta \in \mathcal{Q}$ and a such that $a_\beta \xrightarrow{\sigma(\mathcal{A}^q, \mathcal{R}^p)} a$. One can see $f_\beta = \mathrm{Var}(a_\beta) = \langle 1, a_\beta \rangle \to \langle 1, a \rangle = \mathrm{Var}(a) =: f$

$(C2) \Rightarrow (C3)$. We know every set is uniformly integrable iff its absolute value set is uniformly integrable so by assumption $(C2)$ there exist n such that $\mathrm{E}\left[1_{|\mathrm{Var}(a)|>n}|\mathrm{Var}(a)|\right] < \frac{\varepsilon}{2}$. By letting $\eta = \frac{\varepsilon}{2n}$ we have:

$$\begin{aligned}\langle |X|, a\rangle &\leq \langle \Pi^{\mathrm{op}}(X^*), a\rangle \\ &\leq \mathrm{E}\left[X^*|\mathrm{Var}(a)|\right] \\ &\leq n\mathrm{E}[X^*] + \mathrm{E}\left[1_{|\mathrm{Var}(a)|>n}|\mathrm{Var}(a)|\right] < \varepsilon.\end{aligned}$$

$(C3) \Rightarrow (C2)$. Let $X = \Pi^*(1_U)$ where U is a measurable set such that $\mathbb{P}(U) < \eta$. We have:

$$\mathrm{E}[1_U(a_T^{\pm} - a_0^{\pm})] = \langle |X|, a^{\pm}\rangle < \varepsilon,$$

which shows that $\mathrm{Var}(\mathcal{Q}_{\pm})$ and consequently $\mathrm{Var}(\mathcal{Q})$ is uniformly integrable. □

A.2 Proof of Proposition 3.4

We define the convex risk $\rho_1 : \hat{\mathcal{R}}^p \to \mathbb{R}$ for $p \neq \infty$ as follows

$$\rho_1(X) := \rho(\Pi^{\mathrm{op}}(X)).$$

It is not very difficult to see that every finite value and monotone convex function on a Banach lattice is continuous. For a proof see Proposition 3.1 Ruszczyński and Shapiro (2006).

So the convex risk ρ_1 is continuous. On the other hand by Theorems 65,67 of Section VII Dellacherie and Meyer (1980) we know that when $p \neq \infty$, the set $\hat{\mathcal{A}}^q$ is the dual of $\hat{\mathcal{R}}^p$. So by owing the Alaoglu Theorem we infer that the set $\{a \in \hat{\mathcal{A}}^q : \rho_1^*(a) \leq c\}$ is $\sigma(\hat{\mathcal{A}}^q, \hat{\mathcal{R}}^p)$-compact for every $c \in \mathbb{R}$. Let us assume that $a \in \mathcal{A}^q$. By relation (2.8) we have $\langle \Pi^{\mathrm{op}}(X), a\rangle - \rho(\Pi^{\mathrm{op}}(X)) = \langle X, a\rangle - \rho_1(X)$. This relation implies that $\rho_1^*(a) = \rho^*(a)$ for $a \in \mathcal{A}^q$ and then $\Pi^*\left(\{a \in \hat{\mathcal{A}}^q : \rho_1^*(a) \leq c\}\right) = \{a \in \mathcal{A}^q : \rho^*(a) \leq c\}$. Since $\Pi^* : \hat{\mathcal{A}}^q \to \mathcal{A}^q$ is continuous, the set $\{a \in \mathcal{A}^q : \rho^*(a) \leq c\}$ is $\sigma(\mathcal{A}^q, \mathcal{R}^p)$-compact. □

A.3 Proof of Theorem 3.5

$(L1) \Rightarrow (L3)$. Comes from the definition.

$(L3) \Rightarrow (L4)$. Is just Theorem 2.2.

$(L4) \Rightarrow (L2)$. Let $a \in \mathcal{A}^1_+$ be such that $\rho^*(a) \leq c$ for some real number c. Then by the definition of conjugate function, $\forall X \in \mathcal{R}^\infty$ we have $\langle X, a\rangle - \rho(X) \leq c$. In particular, this is true for every random process like $\Pi^{\mathrm{op}}(X)$ where $X \in L^\infty$. By (2.9) we infer $\mathrm{E}[\mathrm{Var}(a)X] - \bar{\rho}(X) \leq c$ for every $X \in L^\infty$. So we have $\mathrm{Var}(\{a \in \mathcal{A}^1_+ | \rho^*(a) \leq c\}) \subseteq \{\mu \in L^1_+ | \bar{\rho}^*(\mu) \leq c\}$. By the assumption $\mathrm{Var}(\{a \in \mathcal{A}^1_+ | \rho^*(a) \leq c\})$ is relatively compact in $\sigma(L^1, L^\infty)$ and by Theorem 3.1 $\{a \in \mathcal{A}^1_+ | \rho^*(a) \leq c\}$ is relatively compact in $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$.

$(L2) \Rightarrow (L1)$. First we assume that ρ is positively homogeneous. By this assumption, for every real number c, the set $\{a \in \mathcal{A}^1_+ | \rho^*(a) \leq c\}$ is equal to $\{a \in \mathcal{A}^1_+ | \rho^*(a) = 0\}$ which we denote with $\mathcal{Q}$.

Let X_n be a bounded sequence in $\mathcal{R}^\infty$ for which for some $X \in \mathcal{R}^\infty$, $(X_n - X)^* \xrightarrow{\mathbb{P}} 0$. Since ρ is positive homogeneous (therefore sub-additive) and decreasing we have :

$$|\rho(Z) - \rho(Y)| \leq \rho(-(Z-Y)^+) + \rho(-(Y-Z)^+)\ ,\ \forall Z, Y \in \mathbb{R}^\infty.$$

By the last relation we could consider $X_n \leq 0$, $X = 0$ and $(X_n)^* \xrightarrow{\mathbb{P}} 0$. Using hypothesis $(L2)$,$\mathcal{Q}$ is relatively compact in the topology $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$. So Theorem 3.1 gives that the close convex set $\mathrm{Var}(\mathcal{Q})$ is $\sigma(L^1, L^\infty)$-compact and as a consequence (by Theorem 2.3) the convex function $X \mapsto \sup_{f \in \mathrm{Var}(\mathcal{Q})} \mathrm{E}[fX]$ has the *Lebesgue* property. Now by relation (2.2) we have:

$$\rho(X_n) = \sup_{a \in \mathcal{Q}} \langle X_n, a\rangle \leq \sup_{f \in \mathrm{Var}(\mathcal{Q})} \mathrm{E}[(X_n)^* f] \xrightarrow{n} 0.$$

Let us consider that the convex function ρ is not necessarily positive homogeneous. Let X_n and X be bounded in $\mathcal{R}^\infty$ such that $(X_n - X)^* \xrightarrow{\mathbb{P}} 0$ (we adopt this part of the proof from the proof of Theorem 2.4 Jouini et al. (2006)). Since X_n is uniformly bounded then there is a bounded sequence $c_n \in \mathbb{R}^+$ and a positive number ε such that:

$$\rho(X_n) \leq \sup_{\rho^*(a)\leq c_n} \langle X_n, a\rangle - c_n + \varepsilon.$$

Let c be a cluster point of c_n and $I \subseteq \mathbb{N}$ such that $|c_n - c| < \varepsilon$ for all $n \in I$.

Let $\rho_1(X) := \sup\limits_{\{\rho^*(a)\leq c+\varepsilon\}} \langle X, a\rangle$. Since ρ_1 is positively homogeneous, it has the *Lebesgue* property. Now we have

$$\begin{aligned}
\rho(X) &\geq \sup_{\{\rho^*(\mu)\leq c+\varepsilon\}} \langle X, \mu\rangle - c - \varepsilon \\
&= \rho_1(X) - c - \varepsilon \\
&= \lim_{n\in I} \rho_1(X_n) - c - \varepsilon \\
&\geq \lim_{n\in I} \sup_{\rho^*(\mu)\leq c_n} \langle X_n, \mu\rangle - c - \varepsilon \\
&\geq \lim_{n\in I} \rho(X_n) - 3\varepsilon \\
&\geq \liminf \rho(X_n) - 3\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary the proof is complete. □

A.4 Proof of Theorem 3.6

(⇒) Is clear.

(⇐) Define the *convex function* ρ as follow:

$$\rho(X) := \sup_{a\in\mathcal{Q}} \langle X, a\rangle. \tag{A.2}$$

It is not difficult to see that $\mathrm{Var}(\mathcal{Q})$ is convex and weakly closed. Let $X \in L^\infty$. It is easy to see that $\bar{\rho}(X) = \sup\limits_{f\in \mathrm{Var}(\mathcal{Q})} \mathrm{E}[Xf]$. By hypothesis, for any $X \in L^\infty$ there exists an $a \in \mathcal{Q}$ such that

$$\rho((\mathrm{E}[X|\mathcal{F}_t])_{0\leq t\leq T}) = \langle (\mathrm{E}[X|\mathcal{F}_t])_{0\leq t\leq T}, a\rangle.$$

That gives $\bar{\rho}(X) = \mathrm{E}[\mathrm{Var}(a)X]$. This fact, with James' Theorem imply that $\mathrm{Var}(\mathcal{Q})$ is weakly compact. Now by Theorem 3.1 we deduce that $\mathcal{Q}$ is compact in $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$. □

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