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Abstract

In this paper the call admission control (CAC) and routing control (RC) problems for loss network systems are studied as optimal stochastic control (OSC) problems. The so-called pre-state process of the underlying system is a piecewise deterministic Markov process (PDMP) evolving deterministically between (random) event instants at which times the pre-state jumps to another value. The random events in the system correspond to the arrival of call requests or the departure of (active) connections. In the principal result the Hamilton-Jacobi-Bellman (HJB) equations are derived for the underlying stochastic optimal control problems; Unlike the usual single HJB scalar partial differential equation, we now have a finite collection of those, inducing coupling within a finite family of integer indexed value functions. The number of such coupled equations is equal to the number of admissible connection states within the network. Analytical expressions of optimal controls are derived for some simple loss network systems.

Key Words: Call admission control, Routing control, Loss network systems, HJB equations.

Résumé

Un formalisme de commande stochastique optimale est développé en vue de spécifier lorsqu'elles existent, des stratégies optimales d'admission et de routage dans un réseau de télécommunications sans possibilité de mise en attente. Pour ce faire, une représentation d'état du système, générale pour cette classe de réseaux, est d'abord développée. Elle comporte un pré-état appartenant à la classe des processus déterministes par morceaux et qui reste constant entre deux sauts aléatoires successifs. Les sauts aléatoires correspondent à l'arrivée de demandes d'appels ou la terminaison de connexions actives. Lorsque l'on adjoint au pré-état continu un processus ponctuel correspondant aux sauts aléatoires en question, on obtient le vecteur d'état. La représentation markovienne ci-dessus est utilisée pour obtenir les équations H-J-B (Hamilton-Jacobi-Bellman) associées. Contrairement au cas habituel pour lequel H-J-B est une équation aux dérivées partielles scalaire, nous obtenons ici une famille d'équations scalaires couplées. Le nombre d'équations ainsi couplées est égal au nombre différent d'états de connexion discrets admissibles qui peuvent tre observés dans le système. Des expressions analytiques de stratégies de commande optimale sont obtenues pour des cas particuliers de réseaux sans possibilité de mise en attente.

Mots clés : Admission d'appels, routage, réseaux sans possibilité de mise en attente, équations de Hamilton-Jacobi-Bellman.

1 Introduction

Call admission control (CAC) and routing control (RC) problems in telecommunication networks have been topics of active research for decades (see e.g. [1, 5, 12, 13]). In the 1960s, Benes [1] pioneered routing control in telephone networks, providing a general mathematical structure and deriving fundamental properties for telephone systems. The distinction between the work in this paper and that found in standard telecommunication texts and papers (see e.g. [1, 5, 12, 13]) is that here a network system is represented within a formal stochastic systems framework with a specified class of input stochastic processes and a stochastic hybrid state space process with a controlled evolution equation. This permits the formulation of an optimal stochastic control theory for loss network systems in [3, 7, 8, 9].

In particular, in this paper, CAC and RC problems for loss network systems are modeled as optimal stochastic control (OSC) problems.

Consider the loss network systems where the call request processes are general (not necessarily Poisson) renewal processes and the sojourn times of active connections are arbitrarily distributed, then the corresponding network state process has some particular characteristics: (1). the state processes have three parts: *the first part* n will be called the (active) connection state vector. It is non negative integer valued and can be expressed as the difference of two vector counting processes [2]; *the second part* ζ is a variable dimension real-valued piecewise deterministic process which will be called the age vector; while *the third part* e is an impulsive point process which will be called the event vector; (2). For any given such network system, the pre-state process, which is composed of n and ζ, is a piecewise deterministic Markov process (PDMP) [4] subject to an admissible state dependent control law, where the state value evolves deterministically between any two adjacent random event instants and jumps to some other state value at random event instants with some controlled state transition equation, i.e. the control law is measurable with respect to the state process.

The random events in the underlying system correspond to the arrival of call requests or the departure of active connections. However, there is a special aspect to this controlled model which can be stated as follows: if current time t happens to be a call request arrival instant, then the admission and routing controller has to instantaneously act on this information; thus at t, the new call is either accepted, or rejected, following which the state jumps to another deterministic phase ending at the next random event. As a result, the state will contain a mixture of integer and real valued components with right-continuous trajectories, as well as a vector point process component, where the only non-zero entries are in correspondence with the current instantaneous call request arrivals associated to origin destination pairs. Thus, what is usually considered driving exogenous noise in stochastic models becomes a part of the state process.

This methodology is different from [4] where the (random) events are considered as a pure disturbance process, the control law measurable with respect to the pre-state process is actually a policy and at any event instant the pre-state process is resampled by a past information (filtration) independent random variable with a controlled transition probability. When call request processes are Poisson and connection sojourn times are exponentially distributed, the latter framework is equivalent to that of Markov decision processes. In this paper we give the explicit relationship between our HJB equations and the Bellman equations for the Markov decision processes under a class of discounted infinite horizon cost functions.

Loss network systems may also be viewed as stochastic hybrid systems generalizing the class of deterministic hybrid systems as defined in [15] and the references therein; as indicated above, the state process is composed of three components: a discrete component, denoting the connections along the set of the routes in the loss network, and a continuous component constituting the vector of ages of the call requests and active connections in the loss network system; and an impulsive event process only in this case the state process includes, in addition to a discrete and a continuous part, an impulsive point process component.

Unlike infinite horizon discounted Markov decision processes where optimal controls are characterized by sets of coupled algebraic equations, the above classes of optimal control problems, the HJB equations for optimal CAC and RC control problems for loss network systems correspond in the most general case to a collection of coupled scalar first order partial differential equations relating a finite family of integer indexed value functions. The HJB equations in the stochastic point process case for stochastic manufacturing systems are studied in [14].

The paper is organized as follows. In Section 2, we formulate loss network systems and the Markov property of the state process is proved; in Section 3, we formulate the CAC and RC problems of loss network systems as optimal stochastic control (OSC) problems; the HJB equations for some particular simple loss network system controls are studied in Section 4; Section 5 contains the conclusions and outlines future work.

Symbols

- $\mathbb{Z}_1$ $= \{1, 2, 3, \cdots\}$;
- I_j j-dimension identity Matrix;
- $0_{h,j}$ $h \times j$-dimension zero Matrix;
- 1_A Indicator function;
- 1_i i-th unit vector in $\mathbb{R}^m$ with some proper value $m \in \mathbb{Z}_1$;
- 0_n a zero vector with the dimension same as the vector n;
- $\mathbf{1}_\zeta$ $\equiv (1, \cdots, 1)$ with the dimension same as the vector ζ.

2 State Space Structure Dynamics of Loss Network Systems

2.1 Networks of Loss Network Systems

The loss network of a loss network system is a capacitated network $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$ as formally defined below. Based upon this notion, a loss network system is defined in Definition 2.11.

Definition 2.1 A *network*, or *graph*, $Net(\mathbb{V}, \mathbb{L})$ consists of a set of vertices $\mathbb{V} = \{v_1, \cdots, v_V\}$, $V \in \mathbb{Z}_1$, with $\mathbb{Z}_1 \triangleq \{1, 2, \cdots\}$, and a set of links $\mathbb{L} = \{l_1, \cdots, l_L\}$, $L \in \mathbb{Z}_1$, where each link $l \in \mathbb{L}$ is an ordered pair $(v', v'') \in \mathbb{V} \times \mathbb{V}$ of distinct vertices.
A *network* $Net(\mathbb{V}, \mathbb{L})$ *with (link) capacities*

$$\mathbb{C} = \{c_s \equiv c(l_s) \,:\, 1 \le s \le L, c_s \in \mathbb{Z}_1\}$$

shall be denoted by $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$. □

Definition 2.2 A *route*, r in the network $Net(\mathbb{V}, \mathbb{L})$, connecting a vertex $o \in \mathbb{V}$ to a vertex $d \in \mathbb{V}$ is a finite sequence of vertices $r = (v'_1, \cdots, v'_k)$, such that

$$v'_1 = o,\ v'_k = d,$$

$$v'_i \ne v'_j,\ \text{for } i \ne j,$$

$$(v'_i, v'_{i+1}) \in \mathbb{L},\ \text{for } i = 1, \cdots, k-1.$$

The *set of routes* in the network $Net(\mathbb{V}, \mathbb{L})$ is denoted by $\mathcal{R}$, and we denote R as the cardinality of $\mathcal{R}$, i.e. $R = |\mathcal{R}|$. And the subset of routes with respect to a pair of vertices $\langle o, d\rangle$, denoted by $\mathcal{R}_{\langle o,d\rangle}$, is defined as

$$\mathcal{R}_{\langle o,d\rangle} \triangleq \Big\{ r = (v'_1, \cdots, v'_k);\ \text{such that } r \in \mathcal{R} \text{ and } v'_1 = o, v'_k = d \Big\} \tag{2.1}$$

□

Figure 2.1 is an illustration of three distinct routes between v_1 and v_8, which are $(v_1, v_2, v_5, v_4, v_8)$, (v_1, v_4, v_8) and (v_1, v_3, v_7, v_8) respectively in a network.

Definition 2.3 The *feasible set* of *origin-destination vertex pairs*, denoted by $\mathbb{V}^{\triangle}$, is defined as

$$\mathbb{V}^{\triangle} = \big\{ \langle o, d\rangle \in \mathbb{V} \times \mathbb{V};\ \exists\, r \in \mathcal{R},\quad \text{s.t. } r = (v'_1, \cdots, v'_j), v'_1 = o, v'_j = d,\ o \ne d \big\}.$$

□

We set, for each $\langle o, d\rangle \in \mathbb{V}^{\triangle}$, a unique index number i, $i \in \{1, 2, \cdots, |\mathbb{V}^{\triangle}|\}$, and may denote this $\langle o, d\rangle$ pair by $\langle o, d\rangle_i$, i.e. for each i, $i \in \{1, 2, \cdots, |\mathbb{V}^{\triangle}|\}$, there is a unique $\langle o, d\rangle \in \mathbb{V}^{\triangle}$ and $\langle o, d\rangle_i \equiv \langle o, d\rangle$.

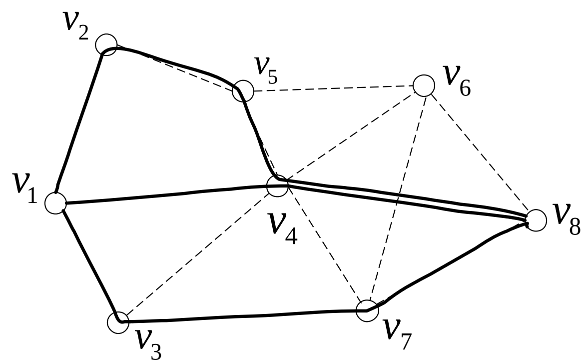

Figure 2.1: Distinct Routes in a Network

Definition 2.4 The *(admissible) set* of *connections*, denoted by $\mathcal{N}$, in $\mathcal{R}$ in the network with capacities $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$, is defined as

$$\mathcal{N} = \Big\{ n = (n_r) \in \mathbb{Z}_+^{SR} : \sum_{\substack{r \in \mathcal{R}; l_k \in r \\ s \in \{1 \cdots S\}}} b_s n_r^{(s)} \leq c_k, \ \forall \, k, 1 \leq k \leq L \Big\}, \tag{2.2}$$

where $n_r \equiv (n_r^{(1)}, \cdots, n_r^{(S)})$, $n_r^{(s)}$ denotes the number of s-class of connections at the route $r \in \mathcal{R}$ and $b_s \in \mathbb{Z}_+$, $s \in \{1 \cdots S\}$, denotes the number of units of link resource occupied by each s-class of connection. □

In the definition of $\mathcal{N}$, for each fixed l_k, the set of $r \in \mathcal{R}$ appearing in the sum is the set of routes each of which contains l_k as a link.
Since the routes in $\mathcal{R}$ are in one-to-one correspondence with the index of the components of a vector in $\mathbb{Z}_+^R \subset \mathbb{R}^R$, we shall by abuse of notation let $r \in \mathcal{R}$ also denote the integer indexing the corresponding vector component in $\mathbb{R}^R$, i.e.

$$n = (n_1, n_2, \cdots, n_R) \equiv (n_{r_1}, n_{r_2}, \cdots, n_{r_R})$$

Remark: For notational simplicity, in the following sections we only consider $S = 1$ and $b \equiv b_1 = 1$, i.e. we consider the loss network system where there only exists a single class of connections with each connection occupying one unit of resource during its sojourn in the system; while in [11], we study the sub-optimal CAC and RC control problems for large integrated loss network system where there exists multi-class of connections.

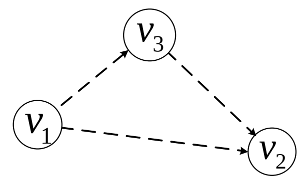

Figure 2.2: A Three-Vertex Capacitated Network

2.1.1 A Simple Example of a Capacitated Loss Network

Here we consider a simple network $Net(\mathbb{V},\mathbb{L},\mathbb{C})$, see Figure 2.2, where

$$\begin{aligned}
\mathbb{V} &= \{v_1, v_2, v_3\} \\
\mathbb{L} &= \{l_1 = (v_1, v_2), l_2 = (v_1, v_3), l_3 = (v_3, v_2)\} \\
\mathbb{C} &= \{c_l = 2;\ l\ \in \mathbb{L}\}.
\end{aligned}$$

Hence the set of routes, $\mathcal{R}$, is defined as

$$\mathcal{R} = \big\{r_1 = (v_1, v_3, v_2), r_2 = (v_1, v_2), r_3 = (v_1, v_3), r_4 = (v_3, v_2)\big\}$$

and the admissible connections set, $\mathcal{N}$, is defined as

$$\mathcal{N} = \Big\{n = (n_1, n_2, n_3, n_4) \in \mathbb{Z}_+^4; \quad \sum_{r_i \in \mathcal{R};\ l \in r_i} n_i \le 2,\ \forall l \in \mathbb{L}\Big\}.$$

2.2 Loss Network Systems

2.2.1 Loss Network System Framework

We consider a class of loss network systems w.r.t. the loss network $Net(\mathbb{V},\mathbb{L},C)$ specified in Definition 2.1, such that all call request processes are *renewal processes*, connection sojourn times are not necessarily exponentially distributed, and satisfies the following specifications:

(S1) The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carries the family of independent random processes and random variables

$$\{\mathrm{Rq}^+_{\langle o,d\rangle},\ \eta_m;\quad \text{such that } o, d \in \mathbb{V}, o \ne d,\ m \in \mathbb{Z}_1\} \tag{2.3}$$

(S2) The call request process *w.r.t.* $\langle o,d\rangle$, for all $o, d \in \mathbb{V}$, $o \ne d$, denoted by $\mathrm{Rq}^+_{\langle o,d\rangle}$, such that $\mathrm{Rq}^+_{\langle o,d\rangle} : \mathbb{R}_+ \times \Omega \to \mathbb{Z}_+$, is an autonomous *renewal process* with the arrival rate parameter of $\lambda^+_{\langle o,d\rangle}(\zeta^{\langle o,d\rangle})$, where $\zeta^{\langle o,d\rangle}$ denotes the elapsed time from the last call request event $e^+_{\langle o,d\rangle}$;

(S3) The random sojourn time of m-th established connection in the network system, denoted by η_m, $\eta_m : \Omega \to \mathbb{R}_+$, has a common arbitrary distribution $F(\zeta^c)$ with the density function as $f(\zeta^c)$ assumed to exist, where ζ^c denotes the age of the m-th connection.
Denote $\lambda^-(\zeta^c)$ as the departure rate of a connection w.r.t. the age of this connection of ζ^c, such that $\lambda^-(\zeta^c) = \frac{f(\zeta^c)}{1-F(\zeta^c)}$. Note that in case of exponential distribution of connection sojourn time, $\lambda^-(\zeta^c)$ is a constant. □

In Definition 2.5, the pre-state space is defined. Each pre-state value z, comprises 2 parts: the integer valued part n and the real valued part ζ with the variable dimension, where

(1) n specifies the number of active connections on each route $r \in \mathcal{R}$;
(2) ζ comprises the ages of all active connections and the elapsed time from the last call request arrival instant for each origin-destination pair $\langle o, d \rangle \in \mathbb{V}^{\triangle}$.

Definition 2.5 The *sub-state space* with respect to $n \in \mathcal{N}$, denoted by Z_n, is defined as the collection of index and age pairs:

$$Z_n = \{z \equiv (n,\, \zeta);\, \zeta \in \mathbb{R}_+^d\}, \quad \text{where } d \equiv d(n) \triangleq |\mathbb{V}^{\triangle}| + \sum_{r_i \in \mathcal{R}} n_i,$$

$$\zeta = \Big(\{\overbrace{\zeta^{\langle o,d\rangle_1}, \zeta^{\langle o,d\rangle_2}, \cdots, \zeta^{\langle o,d\rangle_{|\mathbb{V}^{\triangle}|}}}^{|\mathbb{V}^{\triangle}|}\},\ \{\overbrace{\zeta^{c_{1,1}}, \cdots, \zeta^{c_{1,n_1}}}^{n_1}\},$$
$$\cdots,\ \{\overbrace{\zeta^{c_{R,1}}, \cdots, \zeta^{c_{R,n_R}}}^{n_R}\}\Big), \qquad \text{if } n_i \geq 1,\ r_i \in \mathcal{R}, \tag{2.4}$$

with the constraints:

$$n \in \mathcal{N},\ \text{i.e. } n = (n_1, \cdots, n_R), \quad \text{s.t.} \sum_{r_i \in \mathcal{R};\, l_s \in r_i} n_i \leq c_s,\ \forall s, 1 \leq s \leq L;$$
$$\zeta^{c_{i,1}} > \zeta^{c_{i,2}} > \cdots > \zeta^{c_{i,n_i}} \geq 0, \quad \text{for any } i \in \{1, 2, \cdots, R\}, l$$

and define Γ_n as

$$\Gamma_n = \{\zeta \in \mathbb{R}_+^d;\ (n, \zeta) \in Z_n\} \tag{2.5}$$

The *pre-state space* Z is defined as: $Z \triangleq \dot{\bigcup}_{n \in \mathcal{N}} Z_n$, where $\dot{\bigcup}$ denotes the disjoint union of the indicated entities . □

Remarks: (1) Since there are n_i connections at the route r_i, then each of these connections can be uniquely denoted by c_{ij}, $j(i) \in \{1, \cdots, n_i\}$ and its age is denoted by $\zeta^{c_{ij}}$; (2) Specifically, the sequence of n_i connections at the route r_i is indexed by their age or birth

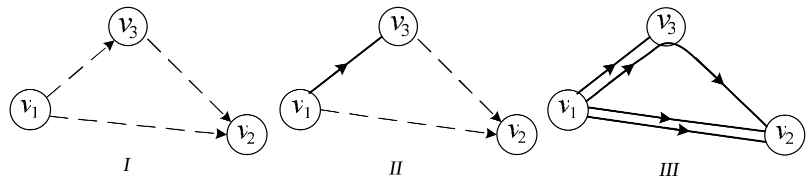

Figure 2.3: Pre-State Values with respect to Different Connection Allocations

time, such that $\zeta^{c_{i,1}} > \zeta^{c_{i,2}} > \cdots > \zeta^{c_{i,n_i}}$, *i.e. at the route* $r_i \in \mathcal{R}$ *the earlier a connection was established, the smaller is its index number.*

Here we display the form of the pre-state value $z \in Z$ with the network specified in Subsection (2.1.1).

In Case I in Figure 2.3 where there is no connection in the network, the pre-state value z takes the form:

$$z = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \zeta^{\langle v_1, v_2 \rangle} \\ \zeta^{\langle v_1, v_3 \rangle} \\ \zeta^{\langle v_2, v_3 \rangle} \end{bmatrix} \right)$$

In Case II in Figure 2.3 where there is only one active connection, which allocates on $r_3 \equiv (v_1, v_3)$, in the network, the pre-state value z takes the form:

$$z = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \zeta^{\langle v_1, v_2 \rangle} \\ \zeta^{\langle v_1, v_3 \rangle} \\ \zeta^{\langle v_2, v_3 \rangle} \end{bmatrix}, [\zeta^{c_{3,1}}] \right)$$

In Case III in Figure 2.3 where there are 1, 2, 1 and 0 connections on the route $r_1 \equiv (v_1, v_3, v_2)$, $r_2 \equiv (v_1, v_2)$, $r_3 \equiv (v_1, v_3)$ and $r_4 \equiv (v_3, v_2)$ respectively, the pre-state value z takes the form:

$$z = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \zeta^{\langle v_1, v_2 \rangle} \\ \zeta^{\langle v_1, v_3 \rangle} \\ \zeta^{\langle v_2, v_3 \rangle} \end{bmatrix}, \begin{bmatrix} \zeta^{c_{1,1}} \\ \zeta^{c_{2,1}} \\ \zeta^{c_{2,2}} \\ \zeta^{c_{3,1}} \end{bmatrix} \right)$$

Definition 2.6 The *(call request and connection departure) event set* induced by a given connection state vector $n \in \mathcal{N}$, denoted by E_n, is defined as:

$$E_n = e_n^0 \dot{\bigcup} E_n^+ \dot{\bigcup} E_n^-, \tag{2.6}$$

where $E_n^+ = \dot{\bigcup}_{\langle o,d \rangle_q \in \mathbb{V}^\triangle} e^+_{\langle o,d \rangle_q}$, $E_n^- = \dot{\bigcup}_{r_i \in \mathcal{R}} \left(\dot{\bigcup}_{j \in \{1, \cdots, n_i\}} e^-_{c_{ij}} \right)$ and

1) $e_n^0 \equiv 0$ denotes absence of a call request or a connection departure event, with 0 the zero vector in $\mathbb{R}^M$ and $M \equiv M(n) = |\mathbb{V}^\triangle| + \sum_{r_i \in R} n_i$

2) $e^{+}_{\langle o,d\rangle_q} \equiv 1_m \in \mathbb{R}^M$ denotes the call request (event) of $\langle o,d\rangle_q$ and 1_m is the m-th unit vector in $\mathbb{R}^M$, with $m = q$;
3) $e^{-}_{c_{ij}} \equiv 1_{m'} \in \mathbb{R}^M$ denotes the connection c_{ij} departure (event) and $1_{m'}$ is the m'-th unit vector in $\mathbb{R}^M$, with $m' = |\mathbb{V}^{\triangle}| + \sum_{k=1}^{i-1} n_k + j$.

The *(total) event set* E is defined as $E \triangleq \dot{\bigcup}_{n\in\mathcal{N}} E_n$. □

Definition 2.7 The *sub-state space* with respect to an index vector $n \in \mathcal{N}$, denoted by X_n, is: $X_n = Z_n \times E_n$.

The *state space* X is: $X = \dot{\bigcup}_{n\in\mathcal{N}} X_n$. □

Definition 2.8 The *(admissible) control (value) set*, with respect to a state value $x = (z,e) = (n,\zeta,e) \in X$, denoted by $U(x)$, is specified as:

$$U(x) = 0^{\langle o,d\rangle}(x) \dot{\bigcup} \mathbf{1}_r^{\langle o,d\rangle}(x) \dot{\bigcup} \mathbf{1}_c^{-}(x), \text{ where} \tag{2.7}$$
$$0^{\langle o,d\rangle}(x) = \dot{\bigcup}_{\langle o,d\rangle_q \in \mathbb{V}^{\triangle}} 0^{\langle o,d\rangle_q}(x),$$
$$\mathbf{1}_r^{\langle o,d\rangle}(x) = \dot{\bigcup}_{\langle o,d\rangle_q \in \mathbb{V}^{\triangle}} \Big(\dot{\bigcup}_{\substack{r\in\mathcal{R}_{\langle o,d\rangle_q} \\ n+1_r\in\mathcal{N}}} 1_r^{\langle o,d\rangle_q}(x) \Big),$$
$$\mathbf{1}_c^{-}(x) = \dot{\bigcup}_{r\in\mathcal{R}} \Big(\dot{\bigcup}_{j\in\{1,\cdots,n_r\}} 1^{-}_{c_{rj}}(x) \Big), \quad \text{with } 1^{-}_{c_{rj}}(x) \equiv 1^{-}_{c_{rj}(x)}(x),$$

where

1) $0^{\langle o,d\rangle_q}(x) \equiv 0 \in \mathbb{R}^R$ denotes that the call request $e^{+}_{\langle o,d\rangle_q}$ is rejected and 0 is the zero vector in $\mathbb{R}^R$;
2) $1_r^{\langle o,d\rangle_q}(x) \equiv 1_r \in \mathbb{R}^R$ denotes that the call request $e^{+}_{\langle o,d\rangle_q}$ is accepted and a connection is established on the route $r \in \mathcal{R}_{\langle o,d\rangle_q}$ under the link capacity constraints, i.e. $n + 1_r \in \mathcal{N}$ with 1_r is the r-th unit vector in $\mathbb{R}^R$;
3) $1^{-}_{c_{rj}}(x) \equiv -1_r \in \mathbb{R}^R$ denotes the departure of the j-th active connection on the route r.

The *control (value) set* U is defined as $U = \dot{\bigcup}_{x\in X} U(x)$. □

Here we give an example to display an admissible control with respect to a state value $x \in X$. See Figure 2.4, which is the loss network specified in Subsection (2.1.1).

Suppose that $z = \big(n,\zeta\big) \in Z$, with $n = (0,1,0,0)$, i.e. there is an active connection on the route $r_2 = (v_1,v_2)$, and $e = (1,0,0,0,0)$, i.e. a call request $\langle v_1,v_2\rangle$ occurs, see Definition (2.6), an admissible control, $u \in U(z,e)$, can be $1_{r_1}^{\langle v_1,v_2\rangle}(x) = (1,0,0,0)$, i.e. the call request $e^{+}_{\langle v_1,v_2\rangle}$ is accepted and a connection is established on the route $r_1 \equiv (v_1,v_3,v_2)$.

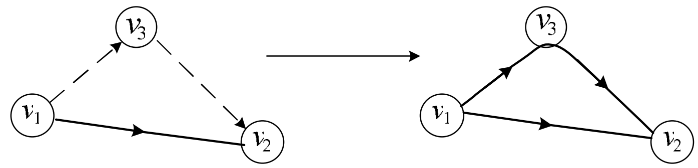

Figure 2.4: An Admissible Control with respect to A State Value

Definition 2.9 When z and e depend in a progressively measurable way on $(\Omega, \mathcal{F}, \mathbb{P})$, we refer to $z = \{z(t,\omega);\, t \in [0,T], \omega \in \Omega\}$, and $e = \{e(t,\omega);\, t \in [0,T], \omega \in \Omega\}$ as *pre-state* and *event processes*; and refer to x as the *state process*, such that

$$x = \Big\{x(t,\omega) \triangleq \big(z(t^-,\omega), e(t,\omega)\big),\ t \in [0,T], \omega \in \Omega\Big\}, \tag{2.8}$$

with $z(t^-,\omega) = \lim_{s\uparrow t} z(s,\omega)$. □

Define $\mathcal{F}_t \triangleq \bigvee_{s\in[0,t]} \sigma(x_s) \in \mathcal{F}$, i.e. $\mathcal{F}_.$ is the natural filtration extended by the process x. □

Definition 2.10 The set of admissible *state dependent*, or *Markov control laws* with respect to the interval $[0,T]$, $T \in \mathbb{R}_+$, is denoted by $\mathcal{U}[0,T]$, and is given by,

$$\mathcal{U}[0,T] = \big\{u : [0,T] \times X \to U; \quad \text{s.t. } u_t \text{ is } \sigma(x_t) \text{ measurable, } t \in [0,T]\big\} \tag{2.9}$$
$$\mathcal{U}[0,\infty) = \cup_{T\geq 0}\, \mathcal{U}[0,T] \tag{2.10}$$

□

Definition 2.11 A family of state processes $\{x_t \triangleq (z_{t^-}, e_t), t \in [0,T]\}$, taking values in X in a capacitated loss network $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$, subject to a family of admissible state dependent control laws $\mathcal{U}[0,T]$, is called a *loss network system.* □

Definition 2.12 For any instant $t \in \mathbb{R}_+$, we term a sequence of *event instants* $\{t_j(\omega)\}$ in $[t,\infty)$

$$t \leq t_1(\omega) < \cdots < t_j(\omega) < t_{j+1}(\omega) < \cdots, \quad (\Omega, \mathcal{F}, \mathbb{P}),\ \omega \in \Omega, \tag{2.11}$$

at which random call request or active connection departure event occurs as a sequence of *event instants* $t_. : \mathbb{Z}_+ \times \Omega \to \mathbb{R}_+$. The sequence $\tau_. : \mathbb{Z}_1 \times \Omega \to \mathbb{R}_+$, such that

$$\tau_{k+1}(\omega) \triangleq t_{k+1}(\omega) - t_k(\omega), \quad \text{with } t_0(\omega) \equiv t, \tag{2.12}$$

is defined as the sequence of *event intervals* (associated to $t_.(\omega)$). □

Definition 2.13 Consider a loss network system subject to an admissible state dependent control law $u \in \mathcal{U}[0,T]$, and the evolution of the state process as

$$x^u : [0,T] \times \Omega \to X, \tag{2.13}$$

with the pre-state value at 0 as $(n,\zeta) \in Z$. The *event* and *pre-state transition equations* are given by

$$e_t^u(\omega) = \begin{cases} 0 \in E_n, & \text{if } t_{i-1}(\omega) < t < t_i(\omega) \\ e \in E_n, & \text{if } t = t_i(\omega) \end{cases}, \qquad \text{where } n(\omega) \equiv n^u_{t_{i-1}}(\omega) \tag{2.14}$$

$$z_t^u(\omega) = \begin{cases} z^u_{t_{i-1}}(\omega) + \int_{t_{i-1}(\omega)}^t (0_n, \mathbf{1}_{\zeta_{t_{i-1}}})\, ds & \\ = \big(n_{t_{i-1}}(\omega), \zeta_{t_{i-1}}(\omega) + [t - t_{i-1}(\omega)]\mathbf{1}_{\zeta_{t_{i-1}}}\big), & \text{if } t \in (t_{i-1}(\omega), t_i(\omega)) \\ \Big(n^u_{t^-}(\omega) + u_t,\ A(\omega)[I_M(\omega) - e_t(\omega)e_t'(\omega)]\, \zeta_{t^-}(\omega)\Big) & \\ \equiv z^u_{t^-}(\omega) \circ u_t \equiv (n^u_{t^-}(\omega) + u_t, \zeta^u_{t^-}(\omega) \circ u_t), & \text{if } t = t_i(\omega) \end{cases}, \tag{2.15}$$

where the random matrix A will be defined in (2.17).

Hence by (2.14) and (2.15), the event process e^u is a *point process*; the pre-state process z^u is a *piecewise deterministic process* and the *state transition equation* is specified as

$$x_t^u(\omega) \triangleq (z^u_{t^-}(\omega), e^u_t(\omega)) = \begin{cases} (z^u_t(\omega), 0), & \text{if } t_{i-1}(\omega) < t < t_i(\omega), \\ (z^u_{t_i^-}(\omega), e^u_{t_i}(\omega)), & \text{if } t = t_i(\omega), \end{cases} \tag{2.16}$$

where e' denotes the transposition of vector e; $u \equiv u_t(x_t^u)$; $0_n \equiv (0, \cdots, 0)' \in \mathbb{Z}_+^R$ and $\mathbf{1}_\zeta \equiv (1, \cdots, 1)'$, where the dimension of ζ varies in accordance with the following specification of A.

$$A \equiv A(x_t^u, u_t(x_t^u)) = \begin{cases} A^+ \equiv A^+_{(M+1)\times M}, & \text{if } u_t(x_t^u) > 0 \\ A^- \equiv A^-_{(M-1)\times M}, & \text{if } u_t(x_t^u) < 0\ , \\ I_M, & \text{otherwise} \end{cases} \qquad \text{with } t = t_i, i = 1, 2, \cdots \tag{2.17}$$

$$A^+ = \begin{bmatrix} I_m & 0_{(m+1)\times(M-m)} \\ 0_{(M-m+1)\times m} & I_{(M-m)} \end{bmatrix},$$

$$\text{where } m = |\mathbb{V}^\Delta| + \sum_{j=1}^{l} n^{(j)}_{t_i^-} \text{ and } l \text{ is such that } u = 1^{r_l}_{\langle o,d\rangle_q}$$

$$A^- = \begin{bmatrix} I_m & 0_{m\times(M-m)} \\ 0_{(M-m-1)\times(m+1)} & I_{(M-m-1)} \end{bmatrix},$$

$$\text{where } m = |\mathbb{V}^\Delta| + \sum_{j=1}^{l-1} n^{(j)}_{t^-} + [k-1] \text{ and } l,\, k \text{ are such that } u = 1^-_{c_{lk}},$$

where M is the dimension of ζ_{t^-}; I_j and $0_{h\times j}$, $h, j \in \mathbb{Z}_1$, denote the j-dimensional identity matrix and $h \times j$-dimensional zero matrix respectively. □

In Definition 2.13, $u_t = 1^{r_l}_{\langle o,d\rangle_q}$ denotes that the call request $\langle o, d\rangle_q$ is accepted to the route r_l at t, while $u_t = 1^{-}_{c_{lk}}$ denotes that the k-th active connection on the route r_l terminates at t. And what the A^+ operator is doing, more specifically, is to increment the number of active connections on the chosen route, introduce a new age variable back of all others on the same route; the A^- operator is to decrease the number of active connections on the chosen route and delete the associated age value.

Remark: In the following sections, we may denote x^u and z^u by x and z respectively.

2.2.2 A State Process Realization of A Loss Network System

Consider a capacitated loss network specified in Section 2.1.1, we specify a realization of the controlled state process x during $[0, t_2)$.

Suppose that $z_0 = \left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}a\\b\\c\end{bmatrix}\right)$, $a, b, c \in \mathbb{R}_+$, then for $0 < t \le t_1$,

$$z_{t^-} = \left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}t+a\\t+b\\t+c\end{bmatrix}\right), \quad z_{t_1^-} = \left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}t_1+a\\t_1+b\\t_1+c\end{bmatrix}\right),$$

Remarks: during $[t_0, t_1)$, the dimension of the vector ζ_t is 3, since there is no active connection in the network during this interval.

Suppose at t_1, $e_{t_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $u_{t_1}(x_{t_1}) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, i.e. a call request $e^{+}_{\langle v_1,v_2\rangle}$ occurs at t_1 and this call request is accepted and a connection is established on the route r_2, then by the state transition equation specified in (2.15)

$$\begin{aligned} z_{t_1} &= \left(n_{t_1^-} + u_{t_1}(x_{t_1}),\ \mathbf{A}[I_3 - e_{t_1}e_{t_1}^T]\zeta_{t_1^-}\right) \\ &= \left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \mathbf{AB}\begin{bmatrix}t_1+a\\t_1+b\\t_1+c\end{bmatrix}\right), \quad \text{where} \quad \mathbf{A} = \mathbf{A}^{+}_{4\times 3} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}, \ \ \mathbf{B} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1&0&0\end{bmatrix} \\ &= \left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\t_1+b\\t_1+c\\---\\0\end{bmatrix}\right) \end{aligned}$$

Then we obtain that, for any $t_1 < t \le t_2$,

$$z_{t^-} = z_{t_1} + \int_0^{t-t_1} (0_n, \mathbf{1}_\zeta)dr = \left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}t-t_1\\t+b\\t+c\\---\\t-t_1\end{bmatrix}\right),$$

and

$$z_{t_2^-} = \left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}t_2-t_1\\t_2+b\\t_2+c\\---\\t_2-t_1\end{bmatrix}\right)$$

Remark: Here a connection is established in the network at t_1*, then* $Dim(\zeta_{t_1}) = Dim(\zeta_{t_1^-}) + 1 = 4$*. Hence during* $[t_1, t_2)$ *the dimension of the vector of system ages* ζ_t *is* 4. □

We consider the following notations,

- Denote $\tau_{m+1}(\omega)$ as the $(m+1)$-th event interval, for any $m \in \mathbb{Z}_+$, i.e.
$$\tau_{m+1}(\omega) = t_{m+1}(\omega) - t_m(\omega)$$
- $x_{t_m(\omega)}(\omega)$ and $\mathcal{F}_{t_m}$ are denoted by $x_m(\omega)$ and $\mathcal{F}_m$, respectively;
- Ω_m denotes a measurable set that exactly m events occur during $[t, t+s]$, i.e.
$$\Omega_m = \{\omega \in \Omega; t_m(\omega) \le t + s < t_{m+1}(\omega)\},$$
where t_m denotes the m-th event instant after the instant t. □

Lemmas 2.1 - 2.4, and Theorem 2.1 are dedicated to proving the Markov property of the state process x^u subject to any admissible state dependent control law $u \in \mathcal{U}[0,T]$, under Assumptions (S1) - (S3). More specifically,

- *Lemma 2.1.* Each event instant t_j, $j \in \mathbb{Z}_1$, is a stopping time of the filtration $\mathcal{F}_\cdot$;
- *Lemma 2.2.* Given the state value at the m-th event instant t_m, the probability distribution of the $(m+1)$-th event interval is independent of the information before t;
- *Lemma 2.3.* Given the state value at the m-th event instant t_m, the probability distribution of the state value at the $(m+1)$-th event instant is independent of the information before t;
- *Lemma 2.4.* Given the state value at the instant t, the expected value of $f(x_{t+s})1_{\Omega_m}$ is independent of the information before t, where $f : X \to \mathbb{R}$ is any bounded measurable function;
- *Theorem 2.1.* Subject to an admissible state dependent control law $u \in \mathcal{U}[0,T]$, the state process x^u is a Markov process. □

Lemma 2.1 For any $s \in \mathbb{R}_+$, denote t_j, $j \in \mathbb{Z}_1$, as the j-th event instant after s, then $\{\omega; t_j(\omega) \le t\} \in \mathcal{F}_t$, for any $t \in \mathbb{R}_+$, i.e. t_j is a stopping time of the filtration $\mathcal{F}_\cdot$

Proof. Denote $\{N_t; t \in [s,\infty)\}$ a point process counting the number of events occurring during $[s,t]$, then, by the state transition equation, we have

$$N_t = \sum_{\nu \in [s,t]} 1_{\{z_{\nu^-} \ne z_\nu\}}, \qquad \text{where } 1_A \text{ is an indicator function,}$$

i.e. N_t is a $\vee_{\nu\in[s,t]}\sigma(z_\nu)$ measurable. Hence N_t is $\mathcal{F}_t$ measurable and by the definition of N_t, we have

$$\{\omega; t_j(\omega) \le t\} = \{\omega; N_t(\omega) \ge j\} \in \mathcal{F}_t, \qquad \forall\, j \in \mathbb{Z}_+, \forall\, t \in \mathbb{R}_+,$$

which implies that each event instant t_j, $j \in \mathbb{Z}_1$, is a stopping time of the filtration $\mathcal{F}_\cdot$ □

Lemma 2.2 Denote $\tau_{m+1}(\omega)$ as the $(m+1)$-th event interval, i.e. $\tau_{m+1}(\omega) = t_{m+1}(\omega) - t_m(\omega)$, for any $m \in \mathbb{Z}_+$. We have

$$\mathbb{P}(\omega;\ \tau_{m+1}(\omega) \le t \,\big|\mathcal{F}_m) = \mathbb{P}(\omega;\ \tau_{m+1}(\omega) \le t \,\big|\sigma(x_m)), \qquad \forall\, t \in \mathbb{R}_+, \tag{2.18}$$

where, for notational simplicity, $x_{t_m(\omega)}(\omega)$ and $\mathcal{F}_{t_m}$ are denoted by $x_m(\omega)$ and $\mathcal{F}_m$ respectively.

Proof. By the state transition equation specified in Definition 2.13, we obtain that

$$x_{m^+}(\omega) = (n_m(\omega), \zeta_m(\omega), e_{m^+}(\omega)) = (n_m(\omega), \zeta_m(\omega), 0), \quad a.s., \tag{2.19}$$

where m denotes $t_m(\omega)$, the m-th event instant after s, and

$$n_m(\omega) = n_{m^-}(\omega) + u_m(n_{m^-}(\omega), \zeta_{m^-}(\omega), e_m(\omega)) \tag{2.20}$$
$$\zeta_m(\omega) = \zeta_{m^-}(\omega) \circ u_m(n_{m^-}(\omega), \zeta_{m^-}(\omega), e_m(\omega)) \tag{2.21}$$

By the definition of $\tau_{m+1}(\omega)$, we have

$$\tau_{m+1}(\omega) = \min_{e \in E_{n_m(\omega)}} \big\{\eta^e(\omega)\big\}, \tag{2.22}$$

where, in case of $e = e^+_{\langle o,d\rangle}$, for some $\langle o,d\rangle \in \mathbb{V}^\triangle$, $\eta^e(\omega)$ is the length of interval from $t_m(\omega)$ to the next $\langle o,d\rangle$ call request arrival instant $t_m(\omega) + \eta^e(\omega)$ after $t_m(\omega)$; in case of $e = e^-_c$, for some active connection c, $\eta^e(\omega)$ is the length of time from $t_m(\omega)$ to the connection c departure instant $t_m(\omega) + \eta^e(\omega)$ after $t_m(\omega)$.

Then by Assumptions (S1,S2), and (2.19) - (2.21), for any $t \in \mathbb{R}_+$, we obtain that

$$\mathbb{P}(\omega; \eta^c(\omega) \le t \mid \sigma(\zeta^c_m)) = \begin{cases} \mathbb{P}(\omega; \eta^c(\omega) \le t \mid \sigma(x_m)) \\ \mathbb{P}(\omega; \eta^c(\omega) \le t \mid \mathcal{F}_m) \end{cases}, \tag{2.23}$$

where $\zeta^c_m(\omega)$ is the age of connection c at the instant $t_m(\omega)$ and is a component of $x_{m^+}(\omega)$.

Thus we obtain that

$$\mathbb{P}(\omega; \eta^c(\omega) \le t \mid \mathcal{F}_m) = \mathbb{P}(\omega; \eta^c(\omega) \le t \mid \sigma(x_m)) \tag{2.24}$$

Similarly, by Assumptions (S1,S3), and (2.19) - (2.21), for any $\langle o,d\rangle \in \mathbb{V}^\triangle$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\omega; \eta^{\langle o,d\rangle}(\omega) \le t \mid \sigma(\zeta^{\langle o,d\rangle}_m)) = \begin{cases} \mathbb{P}(\omega; \eta^{\langle o,d\rangle}(\omega) \le t \mid \sigma(x_m)) \\ \mathbb{P}(\omega; \eta^{\langle o,d\rangle}(\omega) \le t' \mid \mathcal{F}_m) \end{cases} \tag{2.25}$$

where $\zeta^{\langle o,d\rangle}_m(\omega)$ is the length of interval $[t^{\langle o,d\rangle}(\omega), t_m(\omega)]$, where $t^{\langle o,d\rangle}(\omega)$ is the last $\langle o,d\rangle$ call request event instant before the instant $t_m(\omega)$, and $\zeta^{\langle o,d\rangle}_m(\omega)$ is a component of $x_{m^+}(\omega)$.

Thus we obtain that

$$\mathbb{P}(\omega; \eta^{\langle o,d\rangle}(\omega) \leq t \mid \mathcal{F}_m) = \mathbb{P}(\omega; \eta^{\langle o,d\rangle}(\omega) \leq t \mid \sigma(x_m)) \tag{2.26}$$

Then by (2.22), i.e. $\tau_{m+1}(\omega)$ is a measurable function of set of functions $\{\eta^e(\omega); e \in E_{n_m}\}$, and (2.24), (2.26), we have

$$\mathbb{P}(\omega; \tau_{m+1}(\omega) \leq t \mid \mathcal{F}_m) = \mathbb{P}(\omega; \tau_{m+1}(\omega) \leq t \mid \sigma(x_m)), \tag{2.27}$$

which is the conclusion. □

Lemma 2.3 For any bounded measurable function $f : X \to \mathbb{R}$, given the state value at $t_m(\omega)$, subject to an admissible state dependent control law $u \in \mathcal{U}$, the expected value of f at the $(m+1)$-th event instant t_{m+1}, is independent of the past information before t_m, $\mathcal{F}_m$, i.e.,

$$\mathbb{E}\{f(x_{m+1})\,|\mathcal{F}_m\} = \mathbb{E}\{f(x_{m+1})\,|\sigma(x_m)\}, \quad a.s. \tag{2.28}$$

Proof. For any $e' \in E_n$, with $n = n_{t_m}(\omega)$, we have

$$\begin{aligned} & \mathbb{P}\big(\omega;\ e_{m+1}(\omega) = e' \,\big|\, \mathcal{F}_m\big) \\ = \ & \mathbb{P}\big(\omega;\ \eta^{e'}(\omega) = \min_{e \in E_n}\{\eta^e(\omega)\} \,\big|\, \mathcal{F}_m\big) \\ = \ & \mathbb{P}\big(\omega;\ \eta^{e'}(\omega) = \min_{e \in E_n}\{\eta^e(\omega)\} \,\big|\, \sigma(x_m)\big), \quad \text{by (2.24), (2.26) in Lemma 2.2} \\ = \ & \mathbb{P}\big(\omega;\ e_{m+1}(\omega) = e' \,\big|\, \sigma(x_m)\big), \end{aligned} \tag{2.29}$$

where in case that e is a call request and connection departure event, $\eta^e(\omega)$ denotes the length of the interval between two adjacent call request and the sojourn time of an active connection, respectively. Also, by the state transition equation, we have

$$n_{(m+1)^-}(\omega) \equiv n_{t^-_{m+1}}(\omega) = n_m(\omega), \quad a.s. \tag{2.30}$$

$$\zeta_{(m+1)^-}(\omega) \equiv \zeta_{t^-_{m+1}}(\omega) = \zeta_m(\omega) + \tau_{m+1}(\omega)\mathbf{1}_{\zeta_m(\omega)}, \quad a.s. \tag{2.31}$$

Hence, by Lemma 2.2, (2.19)–(2.21), and (2.29)–(2.31), we obtain that

$$\mathbb{P}(\omega;\ x_{m+1}(\omega) \in C \,\big|\, \mathcal{F}_m) = \mathbb{P}(\omega;\ x_{m+1}(\omega) \in C \,\big|\, \sigma(x_m)), \quad \forall\, C \in \sigma(X), \tag{2.32}$$

where $x_{m+1}(\omega) \equiv (n_{m+1^-}(\omega), \zeta_{m+1^-}(\omega), e_{m+1}(\omega))$.

Then, for any bounded measurable function $f : X \to \mathbb{R}$, subject to an admissible state dependent control law $u \in \mathcal{U}$, we have

$$\mathbb{E}\{f(x_{m+1})\,|\mathcal{F}_m\} = \mathbb{E}\{f(x_{m+1})\,|\sigma(x_m)\}$$

□

Lemma 2.4 For all $t, s, t+s \in [0,T]$, and any bounded measurable function $f : X \to \mathbb{R}$,

$$\mathbb{E}\{f(x_{t+s})1_{\Omega_m} \,|\mathcal{F}_t\} = \mathbb{E}\{f(x_{t+s})1_{\Omega_m} \,|\sigma(x_t)\}, \tag{2.33}$$

where $\Omega_m = \{\omega \in \Omega; t_m(\omega) \le t+s < t_{m+1}(\omega)\}$, i.e. Ω_m denotes a measurable set that exactly $m \in \mathbb{Z}_+$ events occur during $[t, t+s]$.

Proof. By the definition of Ω_m, we have

$$\Omega_m = \{\omega; \sum_{i=1}^{m} \tau_i(\omega) \ \le s < \sum_{i=1}^{m+1} \tau_i(\omega)\}, \tag{2.34}$$

where τ_i denotes the i-th event interval. Then

$$f(x_{t+s}(\omega))1_{\Omega_m} = f(x_{t+s}(\omega))1_{\{\omega; \sum_{i=1}^{m} \tau_i(\omega) \le s < \sum_{i=1}^{m+1} \tau_i(\omega)\}}, \ \text{a.s.} \tag{2.35}$$

Also by the state transition equation and the definition of Ω_m, we have

$$z_{t+s}1_{\Omega_m} = \big[z_m + (0_n, [t+s-t_m]\mathbf{1}_{\zeta_m})\big]1_{\Omega_m} = \ \big[z_m + (0_n, [s - \sum_{i=1}^{m} \tau_i]\mathbf{1}_{\zeta_m})\big]1_{\Omega_m} \quad a.s. \tag{2.36}$$

$$e_{t+s}1_{\Omega_m} = e_m(\omega)1_{\{t_m(\omega)=t+s\}}1_{\Omega_m} = \ e_m(\omega)1_{\{\sum_{i=1}^{m} \tau_i(\omega)=s\}}1_{\Omega_m}, \quad a.s. \tag{2.37}$$

Hence, by (2.35), (2.36) and (2.37), we obtain that $f(x_{t+s}(\omega))I_{\Omega_m}$ is a measurable function of $x_m(\omega)$, $x_{m^+}(\omega)$, $\tau_1(\omega), \cdots, \tau_{m+1}(\omega)$, i.e. there exists a measurable function h, such that

$$f(x_{t+s}(\omega))1_{\Omega_m} = h(x_m(\omega), x_{m^+}(\omega), \tau_1(\omega), \cdots, \tau_{m+1}(\omega)), \quad a.s. \tag{2.38}$$

Then we have

$$\begin{aligned}
&\mathbb{E}\{f(x_{t+s})1_{\Omega_m} \,|\mathcal{F}_m\} \\
=\ &\mathbb{E}\{h(x_m, x_{m^+}, \tau_1, \cdots, \tau_{m+1}) \,|\mathcal{F}_m\}, && \text{by (2.38)} \\
=\ &\mathbb{E}\{h(x_m, x_{m^+}, \tau_1, \cdots, \tau_{m+1}) \,|\, \sigma(x_m, \tau_1, \cdots, \tau_m)\}, && \text{by Lemma 2.2} \\
\triangleq\ &h_m(x_m, \tau_1, \cdots, \tau_m)
\end{aligned} \tag{2.39}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E}\{h_m(x_m, \tau_1, \cdots, \tau_m) \,|\mathcal{F}_{m-1}\} \\
=\ &\mathbb{E}\{h_m(x_m, \tau_1, \cdots, \tau_m) \,|\sigma(x_{m-1}, \tau_1, \cdots, \tau_{m-1})\}, && \text{by Lemma 2.2 and 2.3} \\
\triangleq\ &h_{m-1}(x_{m-1}, \tau_1, \cdots, \tau_{m-1})
\end{aligned} \tag{2.40}$$

Hence for $k = 0, 1, \cdots, m-1$, we have

$$
\begin{aligned}
&\mathbb{E}\big\{h_{m-k}(x_{m-k}, \tau_1, \cdots, \tau_{m-k}) \,|\mathcal{F}_{m-k-1}\big\} \\
=\ &\mathbb{E}\big\{h_{m-k}(x_{m-k}, \tau_1, \cdots, \tau_{m-k}) \,|\, \sigma(x_{m-k-1}, \tau_1, \cdots, \tau_{m-k-1})\big\}, \quad \text{by Lemma 2.2 and 2.3} \\
\triangleq\ &h_{m-k-1}(x_{m-k-1}, \tau_1, \cdots, \tau_{m-k-1})
\end{aligned} \tag{2.41}
$$

Then we obtain that

$$
\begin{aligned}
&\mathbb{E}\big\{f(x_{t+s})1_{\Omega_m} \,|\mathcal{F}_t\big\} \\
&= \mathbb{E}\Big\{\mathbb{E}\Big\{\cdots \mathbb{E}\big\{f(x_{t+s})1_{\Omega_m} \,|\mathcal{F}_m\big\} \cdots |\mathcal{F}_1\Big\}|\mathcal{F}_t\Big\}, \qquad \text{by the smoothing property} \\
&= \mathbb{E}\Big\{\mathbb{E}\Big\{\cdots \mathbb{E}\big\{f(x_{t+s})1_{\Omega_m} \,|\sigma(x_m, \tau_1, \cdots, \tau_m)\big\}|\sigma(x_{m-1}, \tau_1, \cdots, \tau_{m-1}) \\
&\qquad\qquad \cdots |\sigma(x_1, \tau_1)\Big\}|\sigma(x_t)\Big\}, \qquad \text{by (2.39), (2.40) and (2.41)} \\
&= \mathbb{E}\Big\{\mathbb{E}\Big\{\cdots \mathbb{E}\big\{f(x_{t+s})1_{\Omega_m} \,|\mathcal{F}_m\big\} \cdots |\mathcal{F}_1\Big\}|\sigma(x_t)\Big\}, \quad \text{by (2.39), (2.40) and (2.41)} \\
&= \mathbb{E}\big\{f(x_{t+s})1_{\Omega_m} \,|\sigma(x_t)\big\}, \qquad \text{by the smoothing property,}
\end{aligned}
$$

which is the conclusion. □

Theorem 2.1 For all $t, s, t+s \in [0, T]$ and any bounded measurable function $f : X \to \mathbb{R}$, subject to any state dependent control law $u \in \mathcal{U}[0, T]$, we obtain that

$$
\mathbb{E}\big\{f(x_{t+s}) \,|\mathcal{F}_t\big\} = \mathbb{E}\big\{f(x_{t+s}) \,|\sigma(x_t)\big\}, \tag{2.42}
$$

i.e. state feedback causes the overall closed loop loss network system to generate a Markov state process.

Furthermore, in case that the state dependent control law u is time shift invariant,

$$
\mathbb{E}\big\{f(x_{t+s}) \,|\mathcal{F}_t\big\} = \mathbb{E}\big\{f(x'_s) \,|\sigma(x'_0)\big\}, \quad \text{with } x'_0(\omega) = x_t(\omega), \quad a.s., \tag{2.43}
$$

i.e. the state process x is a homogeneous Markov process when x is subject to a time shift invariant state dependent control law.

Proof. For any fixed instants $t, t+s \in [0, T]$, $s \geq 0$, we denote the measurable set that exactly m, $m \in \mathbb{Z}_+$ events occur during $[t, t+s]$ by Ω_m.

We observe that

$$
\Omega = \bigcup_{m \in \mathbb{Z}_+} \Omega_m, \quad \text{and} \quad \Omega_m \bigcap \Omega_n = \emptyset, \;\; \forall\, m \neq n,\; m, n \in \mathbb{Z}_+,
$$

i.e. Ω is the disjoint union of $\{\Omega_m; m \in \mathbb{Z}_+\}$.

Then we have

$$\mathbb{E}\big\{f(x_{t+s})\,|\mathcal{F}_t\big\} = \mathbb{E}\big\{f(x_{t+s}) \sum_{m\in\mathbb{Z}_+} 1_{\Omega_m}\,|\mathcal{F}_t\big\} = \mathbb{E}\big\{\sum_{m\in\mathbb{Z}_+} f(x_{t+s})1_{\Omega_m}\,|\mathcal{F}_t\big\}, \\ = \sum_{m\in\mathbb{Z}_+} \mathbb{E}\big\{f(x_{t+s})1_{\Omega_m}\,|\mathcal{F}_t\big\} \qquad (2.44)$$

Similarly

$$\mathbb{E}\big\{f(x_{t+s})\,|\sigma(x_t)\big\} = \sum_{m\in\mathbb{Z}_+} \mathbb{E}\big\{f(x_{t+s})1_{\Omega_m}\,|\sigma(x_t)\big\} \qquad (2.45)$$

Then by (2.44), (2.45) and Lemma 2.4, we get

$$\mathbb{E}\big\{f(x_{t+s})\,|\mathcal{F}_t\big\} = \mathbb{E}\big\{f(x_{t+s})\,|\sigma(x_t)\big\},$$

which is the conclusion.

In the case that the state dependent control law u is time shift invariant, the probability distribution of sequence of event intervals after t given the state value x_t, is independent of the information before t and the value of t, then parallel with the proof procedure of (2.47), one may prove that, subject to a time shift invariant state dependent control law u, the following holds

$$\mathbb{E}\big\{f(x_{t+s})\,|\mathcal{F}_t\big\} = \mathbb{E}\big\{f(x'_s)\,|\sigma(x'_0)\big\}, \quad \text{with } x'_0 = x_t, \quad a.s. \qquad (2.46)$$

i.e. the state process is a homogeneous Markov process subject to a time shift invariant state dependent control law. □

Lemma 2.5 For all $t, s, t+s \in [0,T]$ and any bounded measurable function $f : Z \to \mathbb{R}$, subject to any state dependent control law $u \in \mathcal{U}[0,T]$, we obtain that

$$\mathbb{E}\big\{f(z_{t+s})\,|\mathcal{F}_t\big\} = \mathbb{E}\big\{f(z_{t+s})\,|\sigma(z_t)\big\} \qquad (2.47)$$

Proof. The proof is parallel with that of Theorem 2.1. □

3 Optimal Control for Loss Network Systems

3.1 CAC and RC Problems for Loss Network Systems

The CAC and RC problems for the loss network systems can be formulated as optimal stochastic control problems, which require the specifications of i) the state dynamics and then ii) a (system) loss function covering a given interval.

Definition 3.1 The *(State Dependent) Optimal Stochastic Control (OSC) Problem* for finite horizon loss network systems is defined as follows:
For any instant $s \in [0,T)$ and the pre-state value at s as $z_s : \Omega \to Z$, consider a finite loss network system in the state transition equation specified in Definition (2.13), subject to an admissible state dependent control law $u \in \mathcal{U}[s,T]$:

$$z_t = z_t(z_s, u_s^t), \quad s \leq t \leq T, \tag{3.48}$$

with $z_t(z_s, u_s^t)$ denotes the pre-state value at t subject to the control law u with the pre-state value at s as z_s. And consider the cost function as

$$J(s, z_s; u) = \mathbb{E}\Big\{ \int_s^T g(t, z_t) dt \Big| \mathcal{F}_s \Big\}, \tag{3.49}$$

where the loss function $g : [s,T] \times Z \to \mathbb{R}$ is bounded and measurable with respect to (t, z).

Then the optimal stochastic control (OSC) problem (subject to admissible state dependent control laws) is given by the infimization:

$$V_{n_s}(s, \zeta_s) \equiv V(s, z_s) = \inf_{u \in \mathcal{U}[s,T]} J(s, \xi; u), \quad \text{with } z_s = (n_s, \zeta_s), \tag{3.50}$$

where the function $V \equiv \{V_n : [s,T] \times \Gamma_n \to \mathbb{R}; n \in \mathcal{N}\}$ is called the *value function of the OSC problems*. In case an infimizing function $u^0 \in \mathcal{U}[s,T]$ exists, u^0 shall be called an *optimal control law for OSC problems.*

The *Optimal Stochastic Control (OSC) Problem* for infinite loss network systems is defined same as the above with respect to the interval $[s, \infty)$. □

Theorem 3.1 [6] Consider a loss network system in the state transition equation subject to admissible state dependent control laws with the initial pre-state value at s as $z_s : \Omega \to Z$, $s \in [0,T)$; then the value function V satisfies

$$V_{n_s}(s, \zeta_s) = \inf_{u \in \mathcal{U}[s,t]} \mathbb{E}\Big\{ \int_s^t g(\nu, z_\nu) d\nu + V_{n_t}(t, \zeta_t) \,\Big|\mathcal{F}_s \Big\},$$

where $n_t \equiv n_t(z_s, u_s^t)$ and $\zeta_t \equiv \zeta_t(z_s, u_s^t)$. □

3.2 Dynamic Programming and the HJB Equation

3.2.1 The Strong Generator

Definition 3.2 For any measurable function $F : [0,T] \times Z \to \mathbb{R}$ and any $t \in [0,T]$, if the following exists

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}\big\{ F(t+\varepsilon, z_{t+\varepsilon}) - F(t, z_t) | z_t \big\}, \tag{3.51}$$

we call (3.51), denoted by $\mathcal{A}$, the *Strong Generator* of F at an instant t, i.e.

$$\mathcal{A}F(t,z_t) \triangleq \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) - F(t,z_t)|z_t\big\}. \tag{3.52}$$

And for any measurable function F, if (3.51) exists, it belongs to the *Domain of the Strong Generator*, denoted by $D(\mathcal{A})$. □

Remark: the measurable function $F : [0,T]\times Z \to \mathbb{R}$ may be considered as a group of measurable functions, such that $F = \{F_n : [0,T]\times\Gamma_n \to \mathbb{R};\ n\in\mathcal{N}\}$.

For any $n \in \mathcal{N}$, we consider the following assumptions:

(S4) For any event $e \in E_n$, denote $\lambda^e(\zeta)$ as the arrival rate of e w.r.t. an age vector value of ζ, such that $\lambda^e(\zeta) : \Gamma_n \to \mathbb{R}_+$, is bounded and continuous from right;

(S5) The loss function $g_n(t,\zeta) : [0,T]\times\Gamma_n \to \mathbb{R}$ is bounded and continuous from right with respect to (t,ζ). □

Remark: In Assumption (S4), by Assumptions (S2,S3), we have

$$\lambda^e(\zeta) = \lambda^e(\zeta^e), \qquad \textit{for all } \zeta \in \Gamma_n,$$

where ζ^e, the sojourn time of the event e, is a component of the age vector value ζ.

Recall: From Definition 2.13, we have, for any state value $x = (n,\zeta,e) \in X$ and an admissible state dependent control $u \in U(x)$, denote $\big(n+u, \mathbf{A}[I_M - ee']\zeta\big)$ by $z \circ u$.

Theorem 3.2 Suppose that a measurable function $F = \{F_n : [0,T]\times\Gamma_n \to \mathbb{R};\ n\in\mathcal{N}\}$ is such that $F_n \in C^1([0,T]\times\Gamma_n)$, for all $n \in \mathcal{N}$, then $F \in D(\mathcal{A})$, i.e. F belongs to the domain of the strong generator $\mathcal{A}$, and

$$\mathcal{A}F(t,z_t) = \Big[\frac{\partial}{\partial t} + \sum_{i=1}^{d(\zeta_t)} \frac{\partial}{\partial \zeta_t^i}\Big]\, F(t,z_t)\ + \sum_{e\in E_{n_t}} \lambda^e(\zeta_t)\big[F\big(t,\ z_t\circ u_t(z_t,e)\big) - F(t,z_t)\big], \tag{3.53}$$

where $t \in [0,T]$, $z_t \equiv (n_t,\zeta_t)$, $d(\zeta_t)$ and ζ_t^i denote the dimension and ith component of the vector ζ_t, respectively.

Proof. For all sufficient small positive value $\varepsilon > 0$, an event $e \in E_{n_t}$ occurs in $[t,t+\varepsilon]$ with probability $\lambda^e(\zeta_t)\varepsilon + o(\varepsilon)$; while by Assumptions (S1) - (S3), no event occurs during $[t,t+\varepsilon]$ with probability $(1 - \sum_{e\in E_{n_t}} \lambda^e(\zeta_t)\varepsilon + o(\varepsilon))$ and with probability $o(\varepsilon)$, more than 2 events occurs.

We denote Ω_e, Ω_0, Ω_2 as the measurable set that single event e, non event and more than two events occurs during $[t,t+\varepsilon]$ respectively, then by the fact that Ω is the disjoint union of Ω_0, Ω_2, and Ω_e, for all $e \in E_{n_t}$, we have

$$z_{t+\varepsilon} = z_{t+\varepsilon}1_{\Omega_0\cup_{e\in E_{n_t}}\Omega_e\cup\Omega_2} = z_{t+\varepsilon}1_{\Omega_0} + z_{t+\varepsilon}1_{\Omega_2} + \sum_{e\in E_{n_t}} z_{t+\varepsilon}1_{\Omega_e} \tag{3.54}$$

Suppose that single event e which occurs during $[t, t+\varepsilon]$ occurs at $t+\varepsilon' \in [t, t+\varepsilon]$, and an admissible state dependent control law $u \in \mathcal{U}[0,T]$ is implemented at the instant $t+\varepsilon'$. Then, by the state transition equation, we have

$$z_{t+\varepsilon} 1_{\Omega_e} = \big(z_{t+\varepsilon'} + (0_n, [\varepsilon - \varepsilon']\mathbf{1}_{\zeta_{t+\varepsilon'}})\big)\, 1_{\Omega_e} = \big(z_{(t+\varepsilon')^-} \circ u\big)\, 1_{\Omega_e} + (0_n, [\varepsilon - \varepsilon']\mathbf{1}_{\zeta_{t+\varepsilon'}})\, 1_{\Omega_e}, \tag{3.55}$$

where $z_{(t+\varepsilon')^-} = z_t + (0_n, \varepsilon' \mathbf{1}_\zeta)$, and $u \equiv u_{t+\varepsilon'}(z_{(t+\varepsilon')^-}, e)$.

In case that no event occurs during $[t, t+\varepsilon]$, by the state transition equation, we obtain that

$$z_{t+\varepsilon} 1_{\Omega_0} = \big(z_t + (0_n, \varepsilon \mathbf{1}_\zeta)\big) 1_{\Omega_0} \tag{3.56}$$

Hence, by (3.54), (3.55) and (3.56), we have

$$\begin{aligned}
&\mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) | z_t\big\} \\
&= \Big[1 - \sum_{e \in E_{n_t}} \lambda^e(\zeta_t)\varepsilon + o(\varepsilon)\Big]\, F\big(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)\big) \\
&\qquad + \sum_{e \in E_{n_t}} \Big[\lambda^e(\zeta_t)\,\varepsilon + o(\varepsilon)\Big]\, F\big(t+\varepsilon, z'\big) \\
&= \Big[1 - \sum_{e \in E_{n_t}} \lambda^e(\zeta_t)\varepsilon\Big]\, F\big(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)\big) \\
&\qquad + \sum_{e \in E_{n_t}} \lambda^e(\zeta_t)\,\varepsilon\, F\big(t+\varepsilon, z'\big) + o(\varepsilon), \qquad \text{since } F \text{ is bounded,}
\end{aligned}$$

where $z' = \big(z_{(t+\varepsilon')^-} \circ u\big)\, 1_{\Omega_e} + (0_n, [\varepsilon - \varepsilon']\mathbf{1}_{\zeta_{t+\varepsilon'}})$. Thus we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) - F(t, z_t) | z_t\big\} \\
&= \frac{1}{\varepsilon}\Big[F(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)) - F(t, z_t)\Big] \;\; - \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) F\big(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)\big) \\
&\quad + \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) F\big(t+\varepsilon, z'\big) \;\; + o(1) \\
&\equiv \alpha(F, t, z_t, \varepsilon) + \beta(F, t, z_t, \varepsilon) + \gamma(F, t, z_t, \varepsilon, e, u) + o(1)
\end{aligned} \tag{3.57}$$

Then in the case that ε converges to 0, we obtain that

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \alpha(F, t, z_t, \varepsilon) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big[F(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)) - F(t, z_t)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big[F(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)) - F(t, z_t + (0_n, \varepsilon \mathbf{1}_\zeta))\Big] \\
&\qquad\qquad + \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big[F(t, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)) - F(t, z_t)\Big] \\
&= \frac{\partial^+}{\partial t} F(t, z_{t^+}) + \sum_{i=1}^{d(\zeta_t)} \frac{\partial^+}{\partial \zeta_t^i} F(t, z_t) \\
&= \Big[\frac{\partial}{\partial t} + \sum_{i=1}^{d(\zeta_t)} \frac{\partial}{\partial \zeta_t^i}\Big] F(t, z_t), \quad \text{since } F \in C^1([0,T] \times Z),
\end{aligned}
\tag{3.58}
$$

where $z_t \equiv (n_t, \zeta_t)$ and $d(\zeta_t)$ denotes the dimension of ζ_t.

And, by Assumption (S4) and the continuity of the function F

$$
\lim_{\varepsilon \downarrow 0} \beta(F, t, z_t, \varepsilon) = -\lim_{\varepsilon \downarrow 0} \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) F\big(t+\varepsilon, z_t + (0_n, \varepsilon \mathbf{1}_\zeta)\big) = -\sum_{e \in E_{n_t}} \lambda^e(\zeta_t) F(t, z_t)
\tag{3.59}
$$

Also we have

$$
\begin{aligned}
&\lim_{\varepsilon \downarrow 0} z'(t, z_t, \varepsilon', e, \varepsilon) \\
&= \lim_{\varepsilon \downarrow 0} \big(z_{(t+\varepsilon')^-} \circ u_{t+\varepsilon'}(z_{(t+\varepsilon')^-}, e)\big) + (0_n, [\varepsilon - \varepsilon'] \mathbf{1}_{\zeta_{t+\varepsilon'}}), \quad \text{with } z_{(t+\varepsilon')^-} = z_t + (0_n, \varepsilon' \mathbf{1}_{\zeta_t}) \\
&= z_t \circ u_t(z_t, e), \quad \text{since } z_t + (0_n, \varepsilon' \mathbf{1}_{\zeta_t}) \xrightarrow{\varepsilon \downarrow 0} z_t \text{ and } [\varepsilon - \varepsilon'] \mathbf{1}_{\zeta_{t+\varepsilon'}} \xrightarrow{\varepsilon \downarrow 0} 0_{\zeta_t}
\end{aligned}
\tag{3.60}
$$

Hence, by (3.60), we obtain that

$$
\lim_{\varepsilon \downarrow 0} \gamma(F, t, z_t, \varepsilon, e, u) = \lim_{\varepsilon \downarrow 0} \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) F\big(t+\varepsilon,\ z'\big) = \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) F\big(t,\ z_t \circ u_t(z_t, e)\big)
\tag{3.61}
$$

Then, by (3.57) – (3.61), we obtain that

$$
\begin{aligned}
\mathcal{A}F(t, z_t) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) - F(t, z_t) | z_t\big\} \\
&= \Big[\frac{\partial}{\partial t} + \sum_{i=1}^{d(\zeta_t)} \frac{\partial}{\partial \zeta_t^i}\Big] F(t, z_t) \ + \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) \big[F(t,\ z_t \circ u_t(z_t, e)) - F(t, z_t)\big],
\end{aligned}
$$

which is the conclusion. □

Theorem 3.3 *(The Martingale Property)*

Consider a measurable function $F \in D(\mathcal{A})$ and C_t, for all $t \in [0,T]$, is specified as

$$C_t \triangleq F(t, z_t) - F(0, z) - \int_0^t \mathcal{A}F(\nu, z_\nu)d\nu \tag{3.62}$$

Then $\{C_t\}$ is a $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$-Martingale, i.e. $\mathbb{E}\{C_t|\mathcal{F}_s\} = C_s$, for any $s \in [0,t]$.

Proof. From (3.62) and Lemma 2.5,

$$\mathbb{E}\{(C_t - C_s)|\mathcal{F}_s\} = \mathbb{E}\{F(t, z_t)|z_s\} - F(s, z_s) - \mathbb{E}\{\int_s^t \mathcal{A}F(r, z_r)dr|z_s\} \tag{3.63}$$

Hence, to show $\{C_t\}$ is a $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$-Martingale, it is sufficient to prove that

$$\mathbb{E}\{F(t, z_t)|z_s\} - F(s, z_s) - \mathbb{E}\{\int_s^t \mathcal{A}F(r, z_r)dr|z_s\} = 0 \tag{3.64}$$

For any sequence of random variables, $\{X_\varepsilon(\omega)\}$ and $X(\omega)$ in $(\Omega, \mathcal{F}, P)$, such that $X_\varepsilon \xrightarrow{\varepsilon\downarrow 0} X$ a.s., $\mathbb{E}\{X_\varepsilon|\mathcal{G}\} \xrightarrow{\varepsilon\downarrow 0} \mathbb{E}\{X|\mathcal{G}\}$ a.s., where the σ-field $\mathcal{G} \subset \mathcal{F}$. Hence,

$$\begin{aligned}
&\mathbb{E}\big\{\mathcal{A}F(t, z_t)|z_s\big\} \\
&= \mathbb{E}\Big\{\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) - F(t, z_t)|z_t\big\}|z_s\Big\} \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\Big\{\mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) - F(t, z_t)|z_t\big\}|z_s\Big\} \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\big\{F(t+\varepsilon, z_{t+\varepsilon}) - F(t, z_t)|z_s\big\}, \qquad \text{by Lemma 2.5} \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\Big[\mathbb{E}\{F(t+\varepsilon, z_{t+\varepsilon})|z_s\} - \mathbb{E}\{F(t, z_t)|z_s\}\Big] \\
&= \frac{d^+}{dt}\mathbb{E}\{F(t, z_t)|z_s\}, \quad \text{where } \frac{d^+}{dt} \text{ denotes the right derivative.}
\end{aligned} \tag{3.65}$$

Similarly, $\mathbb{E}\big\{\mathcal{A}F(t, z_t)|z_s\big\} = \frac{d^-}{dt}\mathbb{E}\{F(t, z_t)|z_s\}$, where $\frac{d^-}{dt}$ denotes the left derivative.

So, for any $t \in [0,T]$,

$$\mathbb{E}\big\{\mathcal{A}F(t, z_t)|z_s\big\} = \frac{d^-}{dt}\mathbb{E}\{F(t, z_t)|z_s\} = \frac{d^+}{dt}\mathbb{E}\{F(t, z_t)|z_s\} = \frac{d}{dt}\mathbb{E}\{F(t, z_t)|z_s\}, \text{ a.s.}$$

Hence we obtain that

$$\begin{aligned} & \mathbb{E}\{F(t,z_t)|z_s\} - F(s,z_s) \\ = & \int_s^t \frac{d}{d\nu}\mathbb{E}\{F(\nu,z_\nu)|z_s\}d\nu = \int_s^t \mathbb{E}\big\{\mathcal{A}F(\nu,z_\nu)|z_s\big\}d\nu \end{aligned} \tag{3.66}$$

For an integrable random process $\{y_t, t \in [0,T]\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$ and any $A \in \mathcal{F}$, we have

$$1_A \int_s^t y_\nu d\nu = \int_s^t 1_A y_\nu d\nu, \text{ a.s.}, \tag{3.67}$$

where 1_A denotes an indicate function.

But, for any $A \in \sigma(z_s)$, and for any t, such that $0 \le s \le t \le T$,

$$\begin{aligned} \mathbb{E}\Big\{1_A \int_s^t \mathbb{E}\{y_\nu|z_s\}d\nu\Big\} &= \mathbb{E}\Big\{\int_s^t 1_A \mathbb{E}\{y_\nu|z_s\}d\nu\Big\} \\ &= \int_s^t \mathbb{E}\Big\{1_A \mathbb{E}\{y_\nu|z_s\}\Big\}d\nu \\ &= \int_s^t \mathbb{E}\big\{1_A y_\nu\big\}d\nu, \quad \text{since } 1_A \text{ is } \sigma(z_s) \text{ measurable} \\ &= \mathbb{E}\Big\{\int_s^t 1_A y_\nu d\nu\Big\} \\ &= \mathbb{E}\Big\{1_A \int_s^t y_\nu d\nu\Big\}, \text{ a.s.} \end{aligned} \tag{3.68}$$

By the definition of conditional expectation, (3.68) implies that

$$\int_s^t \mathbb{E}\{y_\nu|z_s\}d\nu = \mathbb{E}\big\{\int_s^t y_\nu d\nu|z_s\big\}, \ \ a.s. \tag{3.69}$$

Hence consider $y_\nu = \mathcal{A}F(\nu, z_\nu)$, by (3.69), we obtain that

$$\int_s^t \mathbb{E}\{\mathcal{A}F(\nu,z_\nu)|z_s\}d\nu = \mathbb{E}\big\{\int_s^t \mathcal{A}F(\nu,z_\nu)d\nu|z_s\big\}, \ \ a.s. \tag{3.70}$$

By (3.66) and (3.70), we obtain (3.64) which implies the conclusion. □

Remark: In case that the function F is time independent, the Martingale property shall be proved by applying the time-homogeneity property of the operator $\mathbb{E}_{|(0,z)}\{F(z_\cdot)\}$, see Proposition 14.13 *in [4].*

Corollary 3.1 *(The Dynkin Formula)*

For any measurable function $F \in D(\mathcal{A})$ and fixed initial pair (s,z), with $s \in [0,T]$, $z_s = z \in Z$,

$$\mathbb{E}_{|(s,z)}\{F(t,z_t)\} = F(s,z) + \mathbb{E}_{|(s,z)}\Big\{\int_s^t \mathcal{A}F(r,z_r)dr\Big\} \tag{3.71}$$

Proof. (3.71) is a special form of (3.64) with a deterministic pre-state value z at the initial instant s. □

3.2.2 HJB Equations for Loss Network System Control

Consider a class of OSC problems defined in Definition 3.1 with the hypothesis of the smoothness of the value function $V = \{V_n : [0,T] \times \Gamma_n; n \in \mathcal{N}\}$ such that $V_n \in C^1([0,T] \times \Gamma_n)$, for all $n \in \mathcal{N}$, then the value function is a classical solution of the first order PDEs which are indexed by the integer value $n \in \mathcal{N}$.

In Theorem 3.4, we consider the OSC problems for the loss network system such that the random call request processes are renewal processes which are not necessarily Poisson processes and the random sojourn times of the active connections are not necessarily exponentially distributed.

Theorem 3.4 *(The HJB Equation for Finite Horizon OSC Problems)*

Consider the OSC problems defined in Definition (3.1) with the finite horizon $[0,T]$. Consider that the value function $V = \{V_n \in C^1([0,T] \times \Gamma_n)$, for all $n \in \mathcal{N}\}$. Then V is a classical solution of the array of coupled first order PDEs with as many PDE's as the cardinality of the admissible connections space::

$$\begin{aligned} 0 = \Big[\frac{\partial}{\partial t} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big] V_n(t,\zeta) + g(t,z) \\ + \sum_{e \in E_n} \lambda^e(\zeta)\Big[\min_{u_t \in U(z,e)} V_{n+u_t(z,e)}(t,\, \zeta \circ u_t(z,e)) - V_n(t,\zeta)\Big], \end{aligned} \tag{3.72}$$

with the boundary condition $V_n(T,\zeta) = 0$, for any $z = (n,\zeta) \in Z$, and ζ^i denotes the i-th component of the vector ζ.

Proof. Consider the pre-state at instant s as $z_s = z \in Z$. Denote $\{z_t; t \in [s,T]\}$ as the controlled pre-state process subject to the control law $u \in \mathcal{U}[s,T]$ with the initial pre-state value at s as z.

(i) By Theorem 3.2, in case the value function V is such that $V_n \in C^1([0,T] \times \Gamma_n)$, for all $n \in \mathcal{N}$, then $V \in D(\mathcal{A})$. Also by the Dynkin formula, for any $t \in [s,T]$, we have

$$\mathbb{E}_{|(s,z)}\{V_{n_t}(t,\zeta_t)\} = V_n(s,\zeta) + \mathbb{E}_{|(s,z)}\Big\{\int_s^t \mathcal{A}V_{n_\nu}(\nu,\zeta_\nu)d\nu\Big\}, \tag{3.73}$$

where, for notational simplicity, we consider $\mathbb{E}\{.\} \equiv \mathbb{E}_{|(s,z)}\{.\}$.

By the Principle of Optimality, we obtain

$$V_n(s,\zeta) \leq \mathbb{E}_{|(s,z)}\Big\{ \int_s^t g\big(\nu, z_\nu\big)d\nu + V_{n_t}(t,\zeta_t)\Big\}, \ \ \forall\, t\ \in (s,T] \tag{3.74}$$

Thus, by (3.73) and (3.74), for any admissible state dependent control law $u \in \mathcal{U}[s,t]$,

$$\begin{aligned}
0\ &\leq \frac{1}{t-s}\ \mathbb{E}_{|(s,z)}\{V_{n_t}(t,\zeta_t) - V_n(s,\zeta)\} \ \ + \frac{1}{t-s}\mathbb{E}_{|(s,z)}\Big\{ \int_s^t g\big(\nu, z_\nu\big)d\nu\Big\} \\
&= \frac{1}{t-s}\mathbb{E}_{|(s,z)}\Big\{ \int_s^t \mathcal{A}V_{n_\nu}(\nu,\zeta_\nu) + g\big(\nu, z_\nu\big)d\nu\Big\} \\
&= \frac{1}{t-s}\mathbb{E}_{|(s,z)}\bigg\{ \int_s^t \Big[\Big[\frac{\partial}{\partial \nu} + \sum_{i=1}^{d(\zeta_\nu)} \frac{\partial}{\partial \zeta_\nu^i}\Big]\ V_{n_\nu}(\nu,\zeta_\nu) + g\big(\nu, z_\nu\big) \\
&\qquad\qquad + \sum_{e\in E_{n_\nu}} \lambda^e(\zeta_\nu)\big[V_{n_\nu + u_\nu(z_\nu,e)}(\nu,\zeta_\nu\circ u_\nu(z_\nu,e)) \ \ - V_{n_\nu}(\nu,z_\nu)\big]\Big] d\nu\bigg\} \\
&\xrightarrow{t\downarrow s} \Big[\frac{\partial}{\partial s} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big]\ V_n(s,\zeta) + g(s,z) \\
&\qquad\qquad + \sum_{e\in E_n} \lambda^e(\zeta)\big[\min_{u_s\in U(z,e)} V_{n+u_s(z,e)}(s,\zeta\circ u_s(z,e)) \ \ - V_n(s,\zeta)\big],
\end{aligned}$$

where the last convergence holds by the hypothesis of $V_n \in C^1([0,T]\times\Gamma_n)$, for all $n \in \mathcal{N}$, and Assumptions (S4) and (S5), i.e. $\lambda^e(.)$ and $g(.)$ is bounded and right continuous. Hence we obtain that

$$\begin{aligned}
0\ \leq &\Big[\frac{\partial}{\partial s} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big]\ V_n(s,\zeta) + g\big(s,z\big) \\
&+ \sum_{e\in E_n} \lambda^e(\zeta)\big[\min_{u_s\in U(z,e)} \big[V_{n+u_s(z,e)}(s,\zeta\circ u_s(z,e)) - V_n(s,\zeta)\big]
\end{aligned} \tag{3.75}$$

(ii) For any $\varepsilon > 0$, $s < t \leq T$, with $t-s$ small enough, there exists an admissible state dependent control law $u^{\varepsilon,t} \in \mathcal{U}([s,t])$, such that

$$V_n(s,\zeta) + \varepsilon[t-s] \geq \mathbb{E}_{|(s,z)}\Big\{ \int_s^t g\big(\nu, z_\nu\big)dt + V_{n_t}(t,\zeta_t)\Big\}$$

Thus by the Dynkin Formula, we obtain that

$$-\varepsilon \leq \ -\frac{1}{t-s}\mathbb{E}_{|(s,z)}\big\{V_{n_t}(t,\zeta_t) - V_n(s,\zeta)\big\} \ \ - \frac{1}{t-s}\mathbb{E}_{|(s,z)}\big\{ \int_s^t g\big(\nu, z_\nu\big)d\nu\big\}$$

$$
\begin{aligned}
&= -\frac{1}{t-s}\mathbb{E}_{|(s,z)}\Big\{\int_s^t \mathcal{A}V_{n_\nu}(\nu,\zeta_\nu) + g\big(\nu, z_\nu\big)d\nu\Big\} \\
&= \frac{1}{t-s}\int_s^t \Big[-\Big[\frac{\partial}{\partial \nu} + \sum_{i=1}^{d(\zeta_\nu)} \frac{\partial}{\partial \zeta_\nu^i}\Big] V_{n_\nu}(\nu,\zeta_\nu) - g\big(\nu, z_\nu\big) \\
&\qquad - \sum_{e\in E_{n_\nu}} \lambda^e(\zeta_\nu)\big[V_{n_\nu + u_\nu^{\varepsilon,t}(z_\nu,e)}\big(\nu,\ \zeta_\nu \circ u_\nu^{\varepsilon,t}(z_\nu,e)\big) \ - V_{n_\nu}(\nu,\zeta_\nu)\big]\Big] d\nu \\
&\overset{t\downarrow s}{\longrightarrow} \ -\Big[\frac{\partial}{\partial s} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big] V_n(s,\zeta) - g\big(s,z\big) \\
&\qquad - \sum_{e\in E_n} \lambda^e(\zeta)\big[V_{n+u_s^{\varepsilon,t}(z,e)}\big(s,\ \zeta\circ u_s^{\varepsilon,t}(z,e)\big) - V_n(s,\zeta)\big],
\end{aligned}
\tag{3.76}
$$

where the last convergence holds by the hypothesis of $V_n \in C^1([0,T]\times\Gamma_n)$, for all $n\in\mathcal{N}$, and Assumptions (S4) and (S5). Hence by (3.76), for any $\varepsilon > 0$, we have

$$
\begin{aligned}
-\varepsilon \leq \ &-\Big[\frac{\partial}{\partial s} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big] V_n(s,\zeta) - g\big(s,z\big) \\
&- \sum_{e\in E_n} \lambda^e(\zeta)\big[\min_{u_s\in U(z,e)} V_{n+u_s(z,e)}\big(s,\ \zeta\circ u_s(z,e)\big) - V_n(s,\zeta)\big],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
0 \geq \ &\Big[\frac{\partial}{\partial s} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big] V_n(s,\zeta) + g\big(s,z\big) \\
&+ \sum_{e\in E_n} \lambda^e(\zeta)\big[\min_{u_s\in U(z,e)} V_{n+u_s(z,e)}\big(s,\zeta\circ u_s(z,e)\big) - V_n(s,\zeta)\big]
\end{aligned}
\tag{3.77}
$$

Then by (3.75) and (3.77), for any pair of values $(s,z)\in[0,T]\times Z$, we obtain that

$$
\begin{aligned}
0 = &\Big[\frac{\partial}{\partial s} + \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i}\Big] V_n(s,\zeta) + g\big(s,z\big) \\
&+ \sum_{e\in E_n} \lambda^e(\zeta)\big[\min_{u_s\in U(z,e)} V_{n+u_s(z,e)}\big(s,\ \zeta\circ u_s(z,e)\big) \ - V_n(s,\zeta)\big],
\end{aligned}
$$

which is the conclusion. □

With the assumption of the uniqueness of the solution for the HJB equation developed in Theorem 3.4, if we obtain the value function by solving the HJB equation, then we could construct an optimal pair (z^0,u^0) for the underlying OSC problems.

But in general the smoothness of the value function V or uniqueness of the classical solutions to the HJB PDEs may not hold. Consequently, in [10] we develop the *viscosity solutions* to the HJB PDEs and show that, under some mild conditions, the continuous value function V is the unique viscosity solution to the collection of HJB PDEs.

In Corollary 3.2, we consider the OSC problems for the loss network system with Poisson call request processes and exponential distributed connection sojourn times. In this case, the pre-state value degenerates into $z = n \in \mathcal{N}$, *i.e.* the real valued part ζ disappears since the system is memoryless with Poisson call request processes and exponential distributed connection sojourn times; then the underlying HJB equation collapses into a group of coupled first order ODEs.

Corollary 3.2 If the value function $V = \{V_n : [0,T] \to \mathbb{R}\}$ is such that $V_n \in C^1([0,T])$, for all $n \in \mathcal{N}$, then V is a classical solution of the first order ODE system,

$$0 = \frac{d}{dt}V_n(t) + g(t,n) \; + \sum_{e\in E_n} \lambda^e(n)\big[\min_{u_t\in U(n,e)} V_{n+u_t(n,e)}(t) - V_n(t)\big], \tag{3.78}$$

with the boundary condition as $V_n(T) = 0$, for all $n \in \mathcal{N}$.

Proof. This is the special form of the HJB equation in Theorem 3.4 with $Z = \mathcal{N}$. □

Lemma 3.1 Consider the OSC problems with a class of infinite horizon integral cost functions with the loss function $g : [0,T] \times Z \to \mathbb{R}$, such that $g(t,z_t) = e^{-\beta t} g(z_t)$, hence, for any admissible state dependent control law $u \in \mathcal{U}[t,\infty)$, we have

$$J(t,z;u) = \mathbb{E}_{|(t,z)}\big\{\int_t^\infty g(\nu,z_\nu)d\nu\big\} = \mathbb{E}_{|(t,z)}\big\{\int_t^\infty e^{-\beta\nu} g(z_\nu)d\nu\big\},$$

with the pair of values $(t,z) \in [0,\infty) \times Z$. Then the value function V satisfies

$$V_n(t,\zeta) = \inf_{u\in\mathcal{U}[t,\infty)} J(t,z;u) = e^{-\beta t} V_n(0,\zeta)$$

Proof. Given any pre-state value $z \in Z$, denote $x^u_{(0)} : [0,\infty) \times \Omega \to X$ as the state process with the initial pre-state value at 0 as z subject to the admissible state dependent control law $u \in \mathcal{U}[0,\infty)$; and $x^{\widetilde{u}}_{(t)} : [t,\infty) \times \Omega \to X$ as the state process with the initial pre-state value at t as z subject to the admissible state dependent control law $\widetilde{u} \in \mathcal{U}[t,\infty)$.

Hence, by the state transition equations specified in (2.14, 2.15), together with Assumptions (S1)–(S3), i.e. the rates of the point processes are time shift invariant, subject to $\widetilde{u}(t+s) = u(s)$, for any $s \in \mathbb{R}_+$, the following holds

$$\text{Family of } \widetilde{u} \text{ controlled F.D.D.s of } x^{\widetilde{u}}_{(t)} \;=\; \text{Family of } u \text{ controlled F.D.D.s of } x^u_{(0)}, \tag{3.79}$$

where F.D.D.s of x denote the finite dimensional distributions of process x.

By (3.79) and the specification of the loss function $g(t,z) = e^{-\beta t} g(z)$, for any $t \in [0,\infty)$, $z \in Z$, we obtain that

$$J(t,z;\widetilde{u}) = e^{-\beta t} J(0,z;u) \tag{3.80}$$

Then by (3.80), we have $V_n(t,\zeta) = e^{-\beta t} V_n(0,\zeta) \equiv e^{-\beta t} V_n(\zeta)$, which is the conclusion. □

In the proof of the following theorem the proof strategy of establishing the two inequalities in parts (i) and (ii) follows that used in [16].

Theorem 3.5 *(The HJB Equation for Discounted Infinite Horizon OSC Problems)*

Suppose that the value function $V = \{V_n : \Gamma_n \to \mathbb{R}\}$ is such that $V_n \in C^1(\Gamma_n)$, for all $n \in \mathcal{N}$. Then V is a solution of the group of coupled first-order PDEs:

$$\begin{aligned} 0 = \beta V_n(\zeta) - \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i} V_n(\zeta) - g(z) \\ - \sum_{e \in E_n} \lambda^e(\zeta) \big[\min_{u \in U(z,e)} V_{n+u(z,e)}\big(\zeta \circ u(z,e)\big) - V_n(\zeta) \big], \end{aligned} \tag{3.81}$$

with the condition $|V_n(\zeta)| \le \int_0^\infty e^{-\beta s} L ds = \frac{L}{\beta}$, for any $z = (n,\zeta) \in Z$, and ζ^i denotes the i-th component of the vector ζ.

Proof. For any instant $t \in \mathbb{R}_+$, $z = (n,\zeta) \in Z$, let $\{z_t; t \in [0,\infty)\}$ be the pre-state trajectory subject to and a state dependent control law $u \in \mathcal{U}[t,\infty)$, with the initial state value at t as z. By Theorem 3.2 and Lemma 3.1, we have

$$\begin{aligned} &\mathcal{A} V_{n_t}(t,\zeta_t) \\ &= \Big[\frac{\partial}{\partial t} + \sum_{i=1}^{d(\zeta_t)} \frac{\partial}{\partial \zeta_t^i} \Big] V_{n_t}(t,\zeta_t) + \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) \big[V_{n_t + u_t(z_t,e)}\big(t,\ \zeta_t \circ u_t(z_t,e)\big) - V_{n_t}(t,\zeta_t) \big] \\ &= \Big[\frac{\partial}{\partial t} + \sum_{i=1}^{d(\zeta_t)} \frac{\partial}{\partial \zeta_t^i} \Big] e^{-\beta t} V_{n_t}(\zeta_t) + \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) \big[e^{-\beta t} V_{n_t + u_t(z_t,e)}(\zeta_t \circ u_t(z_t,e)\big) - e^{-\beta t} V_{n_t}(\zeta_t) \big] \\ &= e^{-\beta t} \Bigg[- \beta V_{n_t}(\zeta_t) + \sum_{i=1}^{d(\zeta_t)} \frac{\partial}{\partial \zeta_t^i} V_{n_t}(\zeta_t) \\ &\qquad + \sum_{e \in E_{n_t}} \lambda^e(\zeta_t) \big[V_{n_t + u_t(z_t,e)}\big(\zeta_t \circ u_t(z_t,e)\big) - V_{n_t}(\zeta_t) \big] \Bigg] \end{aligned} \tag{3.82}$$

(i). By the Principle of Optimality, for any $t, t \in (0,\infty)$, we obtain that

$$V_n(\zeta) \equiv V_n(0,\zeta) \le \mathbb{E}_{|(0,z)} \Big\{ \int_0^t e^{-\beta \nu} g(z_\nu) d\nu + V_{n_t}(t,\zeta_t) \Big\}, \tag{3.83}$$

Thus by (3.83), for any admissible state dependent control law $u \in \mathcal{U}[0,t]$, we obtain that

$$\begin{aligned}
0 \ &\geq -\frac{1}{t}\,\mathbb{E}_{|(0,z)}\{V_{n_t}(t,\zeta_t) - V_n(\zeta)\} - \frac{1}{t}\,\mathbb{E}_{|(0,z)}\Big\{\int_0^t e^{-\beta\nu} g(z_\nu)d\nu\Big\} \\
&= -\frac{1}{t}\,\mathbb{E}_{|(0,z)}\Big\{\int_0^t \Big[\mathcal{A}V_{n_\nu}(\nu,\zeta_\nu) + e^{-\beta\nu} g(z_\nu)\Big]d\nu\Big\} \\
&= \frac{1}{t}\,\mathbb{E}_{|(0,z)}\Bigg\{\int_0^t e^{-\beta\nu}\Bigg[\beta V_{n_\nu}(\zeta_\nu) - \sum_{i=1}^{d(\zeta_\nu)} \frac{\partial}{\partial \zeta_\nu^i} V_{n_\nu}(\zeta_\nu) - g(z_\nu) \\
&\qquad\qquad - \sum_{e\in E_{n_\nu}} \lambda^e(\zeta_\nu)\big[V_{n_\nu+u_\nu(z_\nu,e)}\big(\zeta_\nu \circ u_\nu(z_\nu,e)\big) - V_{n_\nu}(\zeta_\nu)\big]\Bigg]d\nu\Bigg\} \\
&\xrightarrow{t\downarrow 0} \beta\, V_n(\zeta) - \sum_{i=1}^{d(\zeta_\nu)} \frac{\partial}{\partial \zeta_\nu^i} V_{n_\nu}(\zeta_\nu) - g(z) - \sum_{e\in E_n} \lambda^e(\zeta)\big[V_{n+u(z,e)}\big(\zeta\circ u(z,e)\big) - V_n(\zeta)\big],
\end{aligned}$$

where the last convergence holds by the hypothesis of the value function $V_n \in C^1(\Gamma_n)$, for all $n \in \mathcal{N}$ and Assumptions (S4) and (S5), i.e. $\lambda^e(.)$ and $g(.)$ is bounded and right continuous. Hence for any $z \in Z$, we obtain that

$$0 \ \geq \beta\, V_n(\zeta) - \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i} V_n(\zeta) - g(z) - \sum_{e\in E_n} \lambda^e(\zeta)\big[\min_{u\in U(z,e)} V_{n+u(z,e)}\big(\zeta\circ u(z,e)\big) - V_n(\zeta)\big].$$

(ii). For any $\varepsilon > 0$, $0 < t < \infty$ with $t > 0$ small enough, there exists an admissible state dependent control $u^{\varepsilon,t} \in \mathcal{U}[0,t]$ such that,

$$V_n(0,\zeta) + \varepsilon(t) \geq \mathbb{E}_{|(0,z)}\Big\{\int_0^t e^{-\beta\nu} g(z_\nu)d\nu + V_{n_t}(t,\zeta_t)\Big\}$$

Thus from Theorem 3.2, we obtain that

$$\begin{aligned}
-\varepsilon \leq \ & -\frac{1}{t}\,\mathbb{E}_{|(0,z)}\big\{V_{n_t}(t,\zeta_t) - V_n(0,\zeta)\big\} - \frac{1}{t}\,\mathbb{E}_{|(0,z)}\{\int_s^t g\big(\nu, z_\nu\big)d\nu\} \\
= \ & -\frac{1}{t}\,\mathbb{E}_{|(0,z)}\Big\{\int_s^t \mathcal{A}V_{n_\nu}(\nu,\zeta_\nu) + g\big(\nu,z_\nu\big)d\nu\Big\} \\
= \ & \frac{1}{t}\,\mathbb{E}_{|(0,z)}\Bigg\{\int_s^r te^{-\beta\nu}\Bigg[\beta V_{n_\nu}(\zeta_\nu) - \sum_{i=1}^{d(\zeta_\nu)} \frac{\partial}{\partial \zeta_\nu^i} V_{n_\nu}(\zeta_\nu) - g(z_\nu) \\
&\qquad\qquad - \sum_{e\in E_{n_\nu}} \lambda^e(\zeta_\nu)\big[V_{n_\nu+u_\nu(z_\nu,e)}\big(\zeta_\nu \circ u_\nu(z_\nu,e)\big) - V_{n_\nu}(\zeta_\nu)\big]\Bigg]d\nu\Bigg\}
\end{aligned}$$

$$\xrightarrow{t\downarrow 0} \beta V(z) - \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i} V_n(\zeta) - g(z) - \sum_{e\in E_n} \lambda^e(\zeta) \big[V_{n+u(z,e)}\big(\zeta \circ u(z,e)\big) - V_n(\zeta)\big],$$

where the last convergence holds by the hypothesis of the value function $V_n \in C^1(\Gamma_n)$, for all $n \in \mathcal{N}$ and Assumptions (S4) and (S5). Hence for any $z \in Z$, we have

$$-\varepsilon \leq \beta V_n(\zeta) - \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i} V_n(\zeta) - g(z) - \sum_{e\in E_n} \lambda^e(\zeta) \big[\min_{u\in U(z,e)} V_{n+u(z,e)}\big(\zeta \circ u(z,e)\big) - V_n(\zeta)\big],$$

which holds for any $\varepsilon > 0$, so

$$0 \leq \beta V_n(\zeta) - \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i} V_n(\zeta) - g(z) - \sum_{e\in E_n} \lambda^e(\zeta) \big[\min_{u\in U(z,e)} V_{n+u(z,e)}\big(\zeta \circ u(z,e)\big) - V_n(\zeta)\big],$$

Since the conclusions from **(i)** and **(ii)** hold for any fixed $z = (n, \zeta) \in Z$, hence,

$$\beta V_n(\zeta) = \sum_{i=1}^{d(\zeta)} \frac{\partial}{\partial \zeta^i} V_n(\zeta) + g(z) + \sum_{e\in E_n} \lambda^e(\zeta) \big[\min_{u\in U(z,e)} V_{n+u(z,e)}\big(\zeta \circ u(z,e)\big) - V_n(\zeta)\big],$$

which is the conclusion. □

Consider the OSC problems in Theorem 3.5, with the Poisson call request processes and exponentially distributed connection sojourn times, the first order PDE HJB equation degenerates into the group of integer indexed piecewise linear equations.

Corollary 3.3 The value function $V : \mathcal{N} \to \mathbb{R}$ is a solution of the following equations:

$$\beta V_n = g_n + \sum_{e\in E_n} \lambda^e(n) \Big[\min_{u\in U(n,e)} \big\{V_{n+u(n,e)}\big\} - V_n\Big], \quad \text{for all } n \in \mathcal{N} \tag{3.84}$$

Proof. It is a special form of the HJB equation of (3.81) without the partial derivative parts. □

Definition 3.3 *(Policy Value Set)*

The *policy (value)* $\mathbf{u}(z)$ with respect to the pre-state value $z \equiv (n, \zeta) \in Z$ is defined as

$$\mathbf{u}(z) = \big\{u(z,e) \in U(z,e); \quad e \in E_n\big\} \tag{3.85}$$

We denote the *policy set* $\{u(z)\}$, with respect to $z \in Z$, by $\mathbf{U}(z)$.

$$\mathbf{U}(z) = \big\{\mathbf{u}(z); \quad e \in E_n,\ u(z,e) \in U(z,e)\big\} \tag{3.86}$$

□

Corollary 3.4 The HJB equation, in Corollary 3.3, for discounted infinite horizon OSC Problems with Poisson call request processes and exponentially distributed connection sojourn times, is equivalent to the *Bellman Equation* for discrete-time discounted infinite horizon OSC Problems with the discounted factor α, controlled transition probability $\widetilde{\mathbb{P}}$ and loss function $\widetilde{g}$

$$V_n = \widetilde{g}(n) + \alpha \min_{\mathbf{u}\in\mathbf{U}(n)} \{ \sum_{n'\in\mathcal{N}} \widetilde{\mathbb{P}}(n,n';\mathbf{u})V_{n'} \}, \ \ n\in\mathcal{N},$$

where

$$\begin{aligned}
&\alpha = \frac{\lambda}{\lambda+\beta}, \ \ \lambda\in\mathbb{R}_+, \ \text{such that } \lambda > \lambda(n) \equiv \sum_{e\in E_n} \lambda^e(n), \ \text{for any } \ n \\
&\widetilde{g}(n) = \frac{g(n)}{\lambda+\beta} \\
&\widetilde{\mathbb{P}}(n,n';\mathbf{u}) = \begin{cases} \frac{\lambda(n)}{\lambda}\,\mathbb{P}(n,n';\mathbf{u}), & \text{if } n'\neq n \\ \frac{\lambda(n)}{\lambda}\,\mathbb{P}(n,n;\mathbf{u}) + 1 - \frac{\lambda(n)}{\lambda}, & \text{if } n' = n \end{cases}
\end{aligned}$$

where $\mathbb{P}(n,n';\mathbf{u})$ denotes the controlled transition probability.

Proof.

$$\begin{aligned}
0 &= \beta\, V_n - g(n) + \sum_{e\in E_n} \lambda^e(n)V_n \ \ - \sum_{e\in E_n} \lambda^e(n) \min_{u\in U(n,e)} \ \ V_{n+u(n,e)} \\
&= g(n) + \Big[\sum_{e\in E_n} \lambda^e(n) \min_{u\in U(n,e)} V_{n+u(n,e)} \Big] \ - \big[\beta + \sum_{e\in E_n} \lambda^e(n)\big] V_n
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= \frac{g(n)}{\lambda+\beta} \ + \left[\sum_{e\in E_n} \frac{\lambda^e(n)}{\lambda(n)} \frac{\lambda(n)}{\lambda} \frac{\lambda}{\lambda+\beta} \min_{u\in U(n,e)} V_{n+u(n,e)} \right] \ - V_n + \frac{\lambda-\lambda(n)}{\lambda+\beta} V_n, \\
&= \widetilde{g}(n) + \alpha \left[\frac{\lambda(n)}{\lambda} \Big[\sum_{e\in E_n} \frac{\lambda^e(n)}{\lambda(n)} \min_{u\in U} V_{n+u(n,e)} \Big] \ + [1 - \frac{\lambda(n)}{\lambda}] V_n \right] - V_n,
\end{aligned}$$

Then, for any $n\in\mathcal{N}$, with the controlled transition probability $\mathbb{P}(n,n';\mathbf{u})$ satisfying

$$\sum_{e\in E_n} \frac{\lambda^e(n)}{\lambda(n)} V_{n+u(n,e)} = \sum_{n'\in\mathcal{N}} \mathbb{P}(n,n';\mathbf{u})\ V_{n'},$$

we obtain that

$$\begin{aligned}
V_n &= \widetilde{g}(n) + \alpha \left[\frac{\lambda(n)}{\lambda} \Big[\min_{\mathbf{u}\in\mathbf{U}(n)} \sum_{n'\in\mathcal{N}} \mathbb{P}(n,n';\mathbf{u})V_{n'} \Big] \ + [1 - \frac{\lambda(n)}{\lambda}] V_n \right] \\
&= \widetilde{g}(n) + \alpha \ \ \min_{\mathbf{u}\in\mathbf{U}(n)} \sum_{n'\in\mathcal{N}} \widetilde{\mathbb{P}}(n,n';\mathbf{u})V_{n'},
\end{aligned}$$

which is the conclusion. □

4 Examples of Loss Network System Control

4.1 A Single-Link Network with One Unit Capacity

In this section, we study a class of OSC problems for a loss network system with a single link network with one unit capacity.

(i) Consider a class of OSC problems where the call request processes are renewal processes which are not necessarily Poisson processes, the sojourn times of active connections are not necessarily exponentially distributed, and a discounted infinite horizon integral cost function as $\mathbb{E}\{\int_0^\infty e^{-\beta t} g(n_t, \zeta_t) dt\}$, with $\mathcal{N} = \{0, 1\}$. Then the corresponding HJB equation is specified as:
In case $n = 0$: Suppose that the age of system is $\zeta \equiv \zeta^{\langle o,d \rangle}$, and we know that the event set with respect to $n = 0$ is $E_0 = \{e^+ \equiv e^+_{\langle o,d \rangle}\}$, then the HJB equation attached to $n = 0$ is:

$$\beta V_0(\zeta) = \frac{\partial}{\partial \zeta} V_0(\zeta) + g(0, \zeta) + \lambda^{e^+} \Big[\min_{u \in U(z, e^+)} V_{n+u(z,e^+)}\big(\zeta \circ u(z, e^+)\big) - V_0(\zeta) \Big],$$

where

$$\min_{u \in U(z, e^+)} V_{n+u(z,e^+)}\big(\zeta \circ u(z, e^+)\big) = \min\big\{V_0(0), V_1((0,0))\big\}.$$

In case $n = 1$: Suppose that the age of system is $\zeta = (\zeta^+, \zeta^-) \equiv (\zeta^{\langle o,d \rangle}, \zeta^{c_1})$, and we have $E_1 = \{e^+, e^- \equiv e^-_{c_1}\}$, then the HJB equation attached to $n = 1$ is:

$$\begin{aligned} \beta V_1(\zeta) = & \Big(\frac{\partial}{\partial \zeta^+} + \frac{\partial}{\partial \zeta^-}\Big) V_1(\zeta) + g(1, \zeta) \\ & + \lambda^{e^+} \Big[\min_{u \in U(z, e^+)} V_{n+u(z,e^+)}\big(\zeta \circ u(z, e^+)\big) - V_1(\zeta) \Big] + \lambda^{e^-} \Big[V_0(\zeta^+) - V_1(\zeta) \Big], \end{aligned}$$

where

$$\min_{u \in U(z, e^+)} V_{n+u(z,e^+)}\big(\zeta \circ u(z, e^+)\big) = \min\big\{V_1(0, \zeta^{c_1})\big\} = V_1(0, \zeta^-),$$

since, under the link capacity constraint, the call request $e^+ \equiv e^+_{\langle o,d \rangle}$ has to be rejected at the state value $n = 1$, i.e. there already exists an active connection in the link.

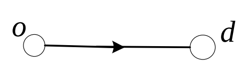

Figure 4.5: A Single-Link Network with One Unit Capacity

In summary, the HJB equation is:

$$\begin{cases} [\beta + \lambda^+(\zeta^+)]V_0(\zeta^+) &= \frac{\partial}{\partial \zeta^+} V_0(\zeta^+) + \lambda^+(\zeta^+) \min\{V_0(0), V_1(0,0)\}, \\ [\beta + \lambda^+(\zeta^+) + \lambda^-(\zeta^-)]V_1(\zeta) &= \frac{\partial}{\partial \zeta^+} V_1(\zeta) + \frac{\partial}{\partial \zeta^-} V_1(\zeta) + g(1,\zeta) \\ & \quad +\lambda^+(\zeta^+)V_1(0,\zeta^-) + \lambda^-(\zeta^-)V_0(\zeta^+), \end{cases}$$

with the condition $|V_0(\zeta^+)|$, $|V_1(\zeta^+,\zeta^-)| \le \frac{L}{\beta}$, for all ζ^+, $\zeta^- \in \mathbb{R}_+$.

(ii) Consider a class of OSC problems where the call request process is a Poisson process with the rate parameter as $\lambda^+ \equiv \lambda^{e^+_{\langle o,d \rangle}}$ and the active connection sojourn times have the departure rate as $\lambda^-(\zeta) \equiv \lambda^{e^-_{c_1}}(\zeta)$, $\forall \zeta \in \mathbb{R}_+$, and the discounted infinite horizon cost function is specified as

$$\mathbb{E}\{\int_0^\infty e^{-\beta t} g(n_t,\zeta_t)dt\}, \quad \text{with } g(n,\zeta) = \begin{cases} 0, & n=0 \\ -1, & n=1 \end{cases}, \quad \text{for all } \zeta \in \mathbb{R}_+$$

Then the HJB equation is:

$$\begin{cases} \beta V_0 &= \lambda^+ \big[\min\{V_0, V_1(0)\} - V_0 \big], \\ \beta V_1(\zeta) &= \frac{\partial}{\partial \zeta} V_1(\zeta) - 1 + \lambda^+[V_1(\zeta) - V_1(\zeta)] + \lambda^-(\zeta)[V_0 - V_1(\zeta)], \end{cases} \tag{4.87}$$

with the condition $|V_0|$, $|V_1(\zeta^-)| \le \frac{L}{\beta}$, for all $\zeta^- \in \mathbb{R}_+$.

Lemma 4.1 $V_1(0) < V_0$, i.e. the optimal control is to accept a coming call request whenever the network system is empty.

Proof.

1) Suppose that $V_1(0) > V_0$ then one obtains that $[\beta + \lambda^+]V_0 = \lambda^+ V_0$, hence $V_0 = 0$ and $V_1(0) > 0$, but we observe that $V_1(0) \le 0$;

2) Suppose that $V_1(0) = V_0$ then one obtains that $V_0 = V_1(0) = 0$, hence

$$\frac{d}{d\zeta} V_1(\zeta) = [\beta + \lambda^-(\zeta)]V_1(\zeta) + 1,$$

with the initial value as $V_1(0) = 0$. Then we obtain that $V_1(\zeta) \ge 0, \forall \zeta \in \mathbb{R}_+$, but from the definition of cost function, we observe that $V_1(\zeta) \le 0$, which is the contradiction. Then from 1) and 2) we obtain the conclusion, $V_1(0) < V_0$. □

Then by (4.87) and Lemma 4.1, the HJB equation is specified as

$$\begin{cases} [\beta + \lambda^+]V_0 &= \lambda^+ V_1(0), \\ \frac{d}{d\zeta} V_1(\zeta) &= [\beta + \lambda^-(\zeta)]V_1(\zeta) + [1 - \lambda^-(\zeta)V_0], \end{cases}$$

with the boundary condition $\lim_{\zeta \to \infty} V_1(\zeta) < \infty$.

The solution is

$$V_1(0) = [1 + \frac{\beta}{\lambda^+}]V_0 \tag{4.88}$$

$$V_1(\zeta) = \exp(\int_0^\zeta [\beta + \lambda^-(r)]dr)V_1(0) + \int_0^\zeta \exp(\int_s^\zeta [\beta + \lambda^-(r)]dr)[1 - \lambda^-(s)V_0]ds \tag{4.89}$$

Hence by (4.88) and (4.89), we obtain that

$$V_0 = \frac{V_1(\zeta) - e^{\beta\zeta}e^{\int_0^\zeta \lambda^-(s)ds}\int_0^\zeta e^{-\beta s - \int_0^s \lambda^-(r)dr}ds}{e^{\beta\zeta}e^{\int_0^\zeta \lambda^-(s)ds}[1 + \frac{\beta}{\lambda^+}] - e^{\beta t}e^{\int_0^\zeta \lambda^-(s)ds}\int_0^\zeta e^{-\beta s - \int_0^s \lambda^-(r)dr}\lambda^-(s)ds}$$

$$= \frac{e^{-\beta\zeta}e^{-\int_0^\zeta \lambda^-(s)ds}V_1(\zeta) - \int_0^\zeta e^{-\beta s - \int_0^s \lambda^-(r)dr}ds}{1 + \frac{\beta}{\lambda^+} - \int_0^\zeta e^{-\beta s}f(s)ds}, \qquad \text{for all } \zeta \in \mathbb{R}_+, \tag{4.90}$$

with f as the density function of connection sojourn times

Hence by (4.90), the boundary condition of $\lim_{\zeta\to\infty}|V_1(\zeta)| < \infty$, and the fact of $\lim_{\zeta\to\infty}\int_0^\zeta e^{-\beta s}f(s)ds = \mathbb{E}\{e^{-\beta\tau}\}$, we obtain that

$$V_0 = -\frac{\lim_{\zeta\to\infty}\int_0^\zeta e^{-\beta s - \int_0^s \lambda^-(r)dr}ds}{1 + \frac{\beta}{\lambda^+} - \lim_{\zeta\to\infty}\int_0^\zeta e^{-\beta s}f(s)ds} = -\frac{\int_0^\infty e^{-\beta s - \int_0^s \lambda^-(r)dr}ds}{1 + \frac{\beta}{\lambda^+} - \mathbb{E}\{e^{-\beta\tau}\}}, \tag{4.91}$$

where τ denotes the sojourn time of connections.

Then (4.91) together with (4.88) and (4.89) is a solution of the associated HJB equation. □

(iii) Consider a class of OSC problems with a Poisson call request process and exponentially distributed connection sojourn time, and a discounted infinite horizon integral cost function as $\mathbb{E}\{\int_0^\infty e^{-\beta t}g(n_t)dt\}$, with the loss function same as that in case **ii)**. Then the HJB equation is:

$$\begin{cases} \beta V_0 & = \lambda^+ \min\{V_0, V_1\} - \lambda^+ V_0 \\ \beta V_1 & = -1 + \lambda^+[V_1 - V_1] + \lambda^-[V_0 - V_1] \end{cases},$$

where $\min\{V_0, V_1\} = V_1$, then the solution is

$$\begin{cases} V_0 & = -\frac{\lambda^+}{\beta[\lambda^+ + \lambda^- + \beta]} \\ V_1 & = -\frac{\lambda^+ + \lambda^- + \beta[\lambda^+ + \lambda^- + \beta]}{\beta[\lambda^- + \beta][\lambda^+ + \lambda^- + \beta]} \end{cases}$$

(iv) Consider a class of OSC problems with Poisson call request process and exponentially distributed connection sojourn time, and a finite horizon integral cost function as $\mathbb{E}\{\int_0^T g(n_t)dt\}$, with the loss function same as that in **ii)**. Then the HJB equation is:

$$\begin{cases} \frac{d}{dt}V_0(t) &= \lambda^+ V_0(t) - \lambda^+ V_1(t) \\ \frac{d}{dt}V_1(t) &= \lambda^- V_1(t) - \lambda^- V_0(t) + 1 \end{cases}, \tag{4.92}$$

with the boundary condition $V_0(T) = V_1(T) = 0$.
Then the solution to the HJB equation (4.92) is

$$\begin{cases} V_0(t) &= \frac{\lambda^+}{\lambda}[t-T] - \frac{\lambda^+}{\lambda^2}[e^{\lambda[t-T]} - 1] \\ V_1(t) &= \frac{\lambda^+}{\lambda}[t-T] + \frac{\lambda^-}{\lambda^2}[e^{\lambda[t-T]} - 1] \end{cases}, \quad \text{for any } t \in [0,T],$$

where $\lambda = \lambda^+ + \lambda^-$.

4.2 A Single-Link Network with Finite Unit Capacity

Consider a class of OSC problems, with respect to a single link network with M-unit capacity, which possesses Poisson call request process and exponentially distributed connection sojourn times with the rates as λ^+ and λ^- respectively, and a discounted infinite horizon cost function as $\int_0^\infty e^{-\beta t} g(n_t)dt$, such that $g(n) = -n$, for all $n \in \mathcal{N} = \{0, 1, \cdots, M\}$. Then the HJB equation is a group of integer indexed piecewise linear equations,

$$\begin{cases} \beta V_0 &= \lambda^+ \min\{V_0, V_1\} - \lambda^+ V_0 \\ \beta V_1 &= -1 + \lambda^+ \min\{V_1, V_2\} + \lambda^- V_0 - [\lambda^+ + \lambda^-]V_1 \\ & \cdots \quad \cdots \\ \beta V_{M-1} &= -[M-1] + \lambda^+ \min\{V_{M-1}, V_M\} \\ & \qquad +[M-1]\lambda^- V_{M-2} - [\lambda^+ + [M-1]\lambda^-]V_{M-1} \\ \beta V_M &= -M + \lambda^+ V_M + M\lambda^- V_{M-1} - [\lambda^+ + M\lambda^-]V_M \end{cases} \tag{4.93}$$

Corollary 4.1 The optimal control is to accept the call request to the network whenever the system is not fully occupied, *i.e.*

$$V_{i+1} < V_i, \quad \text{for all} \quad i \in \{0, \cdots, M-1\} \tag{4.94}$$

Proof.

1) $V_1 < V_0$
Suppose that $V_1 \geq V_0$, then by (4.93), we have $\beta V_0 = \lambda^+ V_0 - \lambda^+ V_0$, hence $V_0 = 0$. Also

$$\beta V_1 = -1 + \lambda^+ \min\{V_1, V_2\} + \lambda^- V_0 - [\lambda^+ + \lambda^-]V_1 \leq -1 + \lambda^+ V_1 - [\lambda^+ + \lambda^-]V_1,$$

which implies $V_1 < 0$, which is a contradiction. So we get the conclusion that $V_1 < V_0$ and also $V_0 = \frac{\lambda^+}{\beta + \lambda^+} V_1$.

2) $V_2 < V_1$
Suppose that $V_2 \geq V_1$, then by (4.93), we have

$$\beta V_1 = -1 + \lambda^+ V_1 + \lambda^- V_0 - [\lambda^+ + \lambda^-]V_1 = -1 + \lambda^-[V_0 - V_1],$$

which implies that $V_1 \geq -\frac{1}{\beta}$. Also we have

$$\begin{aligned}\beta V_2 &= -2 + \lambda^+ \min\{V_2, V_3\} + 2\lambda^- V_1 - [\lambda^+ + 2\lambda^-]V_2 \\ &\leq -2 + \lambda^+ V_2 + 2\lambda^- V_1 - [\lambda^+ + 2\lambda^-]V_2 = -2 + 2\lambda^-[V_1 - V_2] \leq -2\end{aligned}$$

which implies $V_2 \leq -\frac{2}{\beta}$. So we obtain $-\frac{1}{\beta} \leq V_1 \leq V_2 \leq -\frac{2}{\beta}$ which is a contradiction. Hence $V_2 < V_1$.
Suppose that $V_0 > V_1 > V_2 > \cdots > V_{k-1}$, for any $k \in \{1, \cdots, M-1\}$.

3) $V_k < V_{k-1}$
Suppose that $V_k \geq V_{k-1}$, then we have

$$\begin{aligned}\beta V_k &= -k + \lambda^+ \min\{V_{k+1}, V_k\} + k\lambda^- V_{k-1} - [\lambda^+ + k\lambda^-]V_k \\ &\leq -k + \lambda^+ V_k + k\lambda^- V_{k-1} - [\lambda^+ + k\lambda^-]V_k \\ &= -k + k\lambda^-[V_{k-1} - V_k] \leq -k\end{aligned}$$

which implies that $V_k \leq -\frac{k}{\beta}$. But, by $V_k \geq V_{k-1}$, we obtain that

$$\begin{aligned}\beta V_{k-1} &= -[k-1] + \lambda^+ \min\{V_k, V_{k-1}\} + [k-1]\lambda^- V_{k-2} - [\lambda^+ + [k-1]\lambda^-]V_{k-1} \\ &= -[k-1] + \lambda^+ V_{k-1} + [k-1]\lambda^- V_{k-2} - [\lambda^+ + [k-1]\lambda^-]V_{k-1} \\ &= -[k-1] + [k-1]\lambda^-[V_{k-2} - V_{k-1}] \\ &\geq -[k-1], \qquad \text{since } V_{k-2} \geq V_{k-1},\end{aligned}$$

which implies that $V_{k-1} \geq -\frac{k}{\beta} + \frac{1}{\beta}$. Hence we have

$$-\frac{k}{\beta} + \frac{1}{\beta} \leq V_{k-1} \leq V_k \leq -\frac{k}{\beta},$$

which is a contradiction. Hence $V_k < V_{k-1}$. □

Hence, by Corollary 4.1, The HJB equation (4.93) is equivalent to the group of linear equations:

$$\begin{cases} \beta V_0 &= \lambda^+ V_1 - \lambda^+ V_0 \\ \beta V_1 &= -1 + \lambda^+ V_2 + \lambda^- V_0 - [\lambda^+ + \lambda^-]V_1 \\ & \cdots \quad \cdots \\ \beta V_{M-1} &= -[M-1] + \lambda^+ V_M + [M-1]\lambda^- V_{M-2} - [\lambda^+ + [M-1]\lambda^-]V_{M-1} \\ \beta V_M &= -M + \lambda^+ V_M + M\lambda^- V_{M-1} - [\lambda^+ + M\lambda^-]V_M \end{cases} \tag{4.95}$$

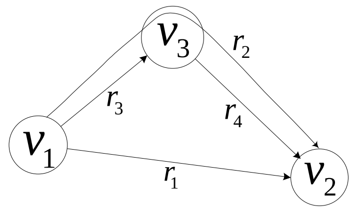

Figure 4.6: A Three-Vertex Network with One-Unit Link Capacity

4.3 A Triangle Network with One Unit Capacity

Consider a network $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$, see Figure (4.6), with

$$\begin{aligned} \mathbb{V} &= \{v_1, v_2, v_3\} \\ \mathbb{L} &= \{l_1 = (v_1, v_2), l_2 = (v_1, v_3), l_3 = (v_3, v_2)\} \\ \mathbb{C} &= \{c_l = 1;\ l \in \mathbb{L}\} \end{aligned} \tag{4.96}$$

Then the route set $\mathcal{R}$ and admissible connection set $\mathcal{N}$ are

$$\begin{aligned} \mathcal{R} &= \{r_1 = (v_1, v_2), r_2 = (v_1, v_3, v_2), r_3 = (v_1, v_3), r_4 = (v_3, v_2)\} \\ \mathcal{N} &= \{n = ((n_1, n_2, n_3, n_4) \in \mathbb{Z}_+^4; \sum_{r_i \in \mathcal{R};\, l \in r_i} n_i \le 1,\ \forall l \in \mathbb{L}\} \\ &= \left\{ \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\} \end{aligned}$$

For notational simplicity, the value n is considered as a 4-bit binary number, i.e.

$$\mathcal{N} = \{0, 1, 2, 3, 4, 8, 9, 10, 11, 12\}.$$

Consider a loss network system, with the loss network $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$ as (4.96), which possesses Poisson call request processes and exponential distributed connection sojourn times, and a discounted infinite horizon cost function as $\mathbb{E}\{\int_0^\infty e^{-\beta t} g(n_t) dt\}$ with the loss function specified as

$$g(n) = -\sum_{i=1}^{4} n_i, \qquad \text{for all } n = (n_1, n_2, n_3, n_4) \in \mathcal{N},$$

i.e.

$$g(0) = 0; g(1) = g(2) = g(4) = g(8) = -1; g(3) = g(9) = g(10) = g(12) = -2; g(11) = -3,$$

The corresponding HJB equation is specified as:

Firstly we develop the HJB equation attached to $n = 0$.

The event set with $n = 0$ is $E_0 = \{e^+_{\langle 1,2\rangle}, e^+_{\langle 1,3\rangle}, e^+_{\langle 3,2\rangle}\}$, then, by Corollary (3.3), the HJB equation with respect to $n = 0$ is specified as:

$$\beta V_0 = g(0) + \sum_{e\in E_0} \lambda^e \min_{u\in U(0,e)} \{V_{0+u(0,e)}\} - \sum_{e\in E_0} \lambda^e V_0, \quad \text{where}$$
$$\min_{u\in U(0,e^+_{\langle 1,2\rangle})} \{V_{0+u(0,e^+_{\langle 1,2\rangle})}\} = \min\{V_0, V_4, V_8\}, \quad \text{with } U(0, e^+_{\langle 1,2\rangle}) = \{0, 4, 8\};$$
$$\min_{u\in U(0,e^+_{\langle 1,3\rangle})} \{V_{0+u(0,e^+_{\langle 1,3\rangle})}\} = \min\{V_0, V_2\}, \quad \text{with } U(0, e^+_{\langle 1,3\rangle}) = \{0, 2\};$$
$$\min_{u\in U(0,e^+_{\langle 3,2\rangle})} \{V_{0+u(0,e^+_{\langle 3,2\rangle})}\} = \min\{V_0, V_1\}, \quad \text{with } U(0, e^+_{\langle 3,2\rangle}) = \{0, 1\},$$

which is equivalent to the following

$$[\beta + \sum_{e\in E_0} \lambda^e]V_0 = \lambda^{e^+_{\langle 1,2\rangle}} \min\{V_0, V_4, V_8\} + \lambda^{e^+_{\langle 1,3\rangle}} \min\{V_0, V_2\} + \lambda^{e^+_{\langle 3,2\rangle}} \min\{V_0, V_1\}$$

The HJB equation attached to other index numbers can be developed parallel with the HJB equation attached to $n = 0$. Hence the HJB equation for the underlying OSC problem is the group of indexed coupled piecewise linear equations,

$$\Big[\beta + \sum_{e\in E_0} \lambda^e\Big]V_0 = \lambda^{e^+_{\langle 1,2\rangle}} \min\{V_0, V_4, V_8\} + \lambda^{e^+_{\langle 1,3\rangle}} \min\{V_0, V_2\} + \lambda^{e^+_{\langle 3,2\rangle}} \min\{V_0, V_1\} \tag{4.97a}$$

$$\Big[\beta + \sum_{e\in E_1} \lambda^e\Big]V_1 = -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\{V_1, V_9\} + \lambda^{e^+_{\langle 1,3\rangle}} \min\{V_1, V_3\} + \lambda^{e^+_{\langle 3,2\rangle}} V_1 + \lambda^- V_0 \tag{4.97b}$$

$$\Big[\beta + \sum_{e\in E_2} \lambda^e\Big]V_2 = -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\{V_2, V_{10}\} + \lambda^{e^+_{\langle 1,3\rangle}} V_2 + \lambda^{e^+_{\langle 3,2\rangle}} \min\{V_2, V_3\} + \lambda^- V_0 \tag{4.97c}$$

$$\Big[\beta + \sum_{e\in E_3} \lambda^e\Big]V_3 = -2 + \lambda^{e^+_{\langle 1,2\rangle}} \min\{V_3, V_{11}\} + \lambda^{e^+_{\langle 1,3\rangle}} V_3 + \lambda^{e^+_{\langle 3,2\rangle}} V_3 + \lambda^- V_1 + \lambda^- V_2 \tag{4.97d}$$

$$\Big[\beta + \sum_{e\in E_4} \lambda^e\Big]V_4 = -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\{V_4, V_{12}\} + \lambda^{e^+_{\langle 1,3\rangle}} V_4 + \lambda^{e^+_{\langle 3,2\rangle}} V_4 + \lambda^- V_0 \tag{4.97e}$$

$$\Big[\beta + \sum_{e\in E_8} \lambda^e\Big] V_8 = -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\big\{V_8, V_{12}\big\} + \lambda^{e^+_{\langle 1,3\rangle}} \min\big\{V_8, V_{10}\big\} + \lambda^{e^+_{\langle 3,2\rangle}} \min\big\{V_8, V_9\big\} + \lambda^- V_0 \tag{4.97f}$$

$$\Big[\beta + \sum_{e\in E_9} \lambda^e\Big] V_9 = -2 + \lambda^{e^+_{\langle 1,2\rangle}} V_9 + \lambda^{e^+_{\langle 1,3\rangle}} \min\big\{V_9, V_{11}\big\} + \lambda^{e^+_{\langle 3,2\rangle}} V_9 + \lambda^- V_1 + \lambda^- V_8 \tag{4.97g}$$

$$\Big[\beta + \sum_{e\in E_{10}} \lambda^e\Big] V_{10} = -2 + \lambda^{e^+_{\langle 1,2\rangle}} V_{10} + \lambda^{e^+_{\langle 1,3\rangle}} V_{10} + \lambda^{e^+_{\langle 3,2\rangle}} \min\big\{V_{10}, V_{11}\big\} + \lambda^- V_2 + \lambda^- V_8 \tag{4.97h}$$

$$\Big[\beta + \sum_{e\in E_{11}} \lambda^e\Big] V_{11} = -3 + \lambda^{e^+_{\langle 1,2\rangle}} V_{11} + \lambda^{e^+_{\langle 1,3\rangle}} V_{11} + \lambda^{e^+_{\langle 3,2\rangle}} V_{11} + \lambda^- V_3 + \lambda^- V_9 + \lambda^- V_{10} \tag{4.97i}$$

$$\Big[\beta + \sum_{e\in E_{12}} \lambda^e\Big] V_{12} = -2 + \lambda^{e^+_{\langle 1,2\rangle}} V_{12} + \lambda^{e^+_{\langle 1,3\rangle}} V_{12} + \lambda^{e^+_{\langle 3,2\rangle}} V_{12} + \lambda^- V_4 + \lambda^- V_8 \tag{4.97j}$$

Lemma 4.2 For the control problem for the loss network specified in (4.96), $V_8 < V_4$, i.e. with respect to the state value $(n, e^+_{1,2})$, it is better to accept the call request $e^+_{1,2}$ to the direct route $r_1 = (v_1, v_2)$ than to the two-hop route $r_2 = (v_1, v_3, v_2)$.

Proof. By the HJB equation, we obtain that

$$[\beta + \sum_{e\in E_4} \lambda^e] V_4 = -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\big\{V_4, V_{12}\big\} + \lambda^{e^+_{\langle 1,3\rangle}} V_4 + \lambda^{e^+_{\langle 3,2\rangle}} V_4 + \lambda^- V_0$$

$$\begin{aligned}[\beta + \sum_{e\in E_8} \lambda^e] V_8 &= -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\big\{V_8, V_{12}\big\} \\ &\quad + \lambda^{e^+_{\langle 1,3\rangle}} \min\big\{V_8, V_{10}\big\} + \lambda^{e^+_{\langle 3,2\rangle}} \min\big\{V_8, V_9\big\} + \lambda^- V_0 \\ &\le -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\big\{V_8, V_{12}\big\} + \lambda^{e^+_{\langle 1,3\rangle}} V_8 + \lambda^{e^+_{\langle 3,2\rangle}} V_8 + \lambda^- V_0\end{aligned}$$

$$\Longrightarrow \begin{cases} [\beta + \lambda^{e^+_{\langle 1,2\rangle}} + \lambda^-] V_4 = -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\big\{V_4, V_{12}\big\} + \lambda^- V_0 \\ [\beta + \lambda^{e^+_{\langle 1,2\rangle}} + \lambda^-] V_8 \le -1 + \lambda^{e^+_{\langle 1,2\rangle}} \min\big\{V_8, V_{12}\big\} + \lambda^- V_0 \end{cases},$$

which implies $V_8 < V_4$. □

In Lemma 4.3, we display that, in case that the direct route r_1 is occupied, the acceptance, of the coming call request $e^+_{\langle 1,2\rangle}$ to the two-hop route (v_1, v_3, v_2), depends upon the values of $\lambda^{e^+_{\langle 1,3\rangle}}, \lambda^{e^+_{\langle 3,2\rangle}}$ and λ^-.

Lemma 4.3 When the direct route r_1 is occupied, the optimal control value with respect to the call request $e^+_{\langle 1,2\rangle}$ depends upon the values of $\lambda^{e^+_{\langle 1,3\rangle}}, \lambda^{e^+_{\langle 3,2\rangle}}$ and λ^-.

Proof.

1) In case of $\lambda^{e^+_{\langle 1,3\rangle}} = \lambda^{e^+_{\langle 3,2\rangle}} = 0$, i.e. there is no $e^+_{\langle 1,3\rangle}$, $e^+_{\langle 3,2\rangle}$ call requests, then we have

$$[\beta + \lambda^+ + \lambda^-]V_8 = -1 + \lambda^+ \min\{V_8, V_{12}\} + \lambda^- V_0, \qquad \text{with } \lambda^+ \equiv \lambda^+_{\langle 1,2\rangle} \quad (4.98)$$

$$[\beta + \lambda^-]V_{12} = -1 + \Big[\lambda^-[V_4 + V_8] - 1\Big] \quad (4.99)$$

Suppose that $V_8 \leq V_{12}$, then $[\beta + \lambda^-]V_8 = -1 + \lambda^- V_0$. Hence we have

$$\lambda^- V_0 \leq \lambda^-[V_4 + V_8] - 1 \qquad or \qquad V_0 \leq -\frac{1}{\lambda^-} + V_4 + V_8.$$

Also we observe that $V_0 > V_8$, since

$$[\beta + \lambda^+]V_0 = \lambda^+ \min\{V_0, V_4, V_8\} = \lambda^+ \min\{V_0, V_8\}$$

So $V_8 < V_0 \leq V_8 - \frac{1}{\lambda^-} + V_4$, which is a contradiction.

Hence $V_{12} < V_8$ in the case of $\lambda^{e^+_{\langle 1,3\rangle}} = \lambda^{e^+_{\langle 3,2\rangle}} = 0$. i.e. when the direct route r_1 is occupied, it is better to accept the call request $e^+_{1,2}$ to the multi-link route r_2 than to reject it in the case that there is no $e^+_{\langle 1,3\rangle}$, $e^+_{\langle 3,2\rangle}$ call requests arrival in the future.

2) In case of $\lambda^+ \equiv \lambda^{e^+_{\langle 1,2\rangle}} = \lambda^{e^+_{\langle 1,3\rangle}} = \lambda^{e^+_{\langle 3,2\rangle}} \to \infty$

$$[\beta + 3\lambda^+ + \lambda^-]V_8 = -1 + \lambda^+ \min\{V_8, V_{12}\} + \lambda^+ \min\{V_8, V_{10}\} + \lambda^+ \min\{V_8, V_9\} + \lambda^- V_0,$$

$$[\beta + \lambda^-]V_{12} = -2 + \lambda^-[V_4 + V_8].$$

Hence $[\beta + \lambda^+ + \lambda^-]V_8 \leq -1 + \lambda^+ V_{12} + \lambda^- V_0$, then we obtain that

$$V_8 \leq \frac{\lambda^+}{\beta + \lambda^+ + \lambda^-} V_{12} + \frac{-1 + \lambda^- V_0}{\beta + \lambda^+ + \lambda^-} \quad \to \quad V_{12}, \qquad \text{with } \lambda^+ \to \infty,$$

which implies $V_8 \leq V_{12}$, when $\lambda^+ \to \infty$. □

5 Conclusion and future work

In this paper the CAC and RC problems for loss network systems have been formulated as OSC problems. A feature of the resulting controlled loss network systems is that they are hybrid stochastic systems with variable dimension state processes; for these processes certain properties, such as the Markov property under state feedback control laws and piecewise continuity, have been established.

The HJB equation for the underlying OSC problems is derived. In the case that the value function $V = \{V_{\mathbf{n}} : [0,T] \times \Gamma_{\mathbf{n}}); \mathbf{n} \in \mathcal{N}\}$ is continuously differentiable, i.e. $V_{\mathbf{n}} \in C^1([0,T] \times \Gamma_{\mathbf{n}})$, for all $\mathbf{n} \in \mathcal{N}$, V is a classical solution of the HJB equation; however such smoothness requirements may be too restrictive and the uniqueness of the classical solution of the HJB equation may also not be satisfied. So viscosity solutions, [10], of the HJB equation need to be considered. For Poisson call request processes, exponentially distributed sojourn times, and discounted infinite horizon cost functions, the network HJB equations have been shown to be equivalent to those associated with Markov decision processes, and the numerical techniques developed for the latter can be employed for obtaining numerical solutions.

Furthermore, the optimal control framework in this paper, provides the foundation for currently ongoing work [11], on computational more viable decentralized sub-optimal control of the large loss network systems based upon on a combination of game theory with the ideas of statistical mechanics.

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