Les Cahiers du GERAD ISSN: 0711–2440

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G–2007–22

March 2007

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Abstract

We consider the bandwidth coloring problem, a generalization of the well-known graph coloring problem. For the latter problem, a classical theorem, discovered independently by Gallai, Roy and Vitaver, states that the chromatic number of a graph is bounded from above by the number of vertices in the longest elementary path in any directed graph derived by orienting all edges in the graph. We generalize this result to the bandwidth coloring problem. Two proofs are given, a simple one and a more complex that is based on a series of equivalent mathematical programming models. These formulations can motivate the development of various solution algorithms for the bandwidth coloring problem.

Résumé

Nous considérons le problème de la coloration par bande, une généralisation de la coloration usuelle des sommets d'un graphe. Pour ce dernier, un théorème classique, énoncé indépendamment par Gallai, Roy et Vitaver, démontre que le nombre chromatique d'un graphe est borné supérieurement par le nombre de sommets sur le plus long chemin élémentaire dans un graphe orienté obtenu en choisissant une orientation pour chaque arête du graphe. Nous généralisons ce résultat au problème de la coloration par bande. Nous donnons deux preuves de ce résultat, une simple et une plus complexe qui est basée sur l'équivalence entre divers modèles de programmation mathématique pour la coloration par bande. Ces divers modèles peuvent motiver le développement de nouveaux algorithmes pour la résolution du problème de la coloration par bande.

1 Introduction

All graphs considered in this paper have no loops and no multiple edges. For a graph G, we denote V its vertex set and E its edge set. A strictly positive integer weight d_{ij} is associated to each edge $\{i,j\} \in E$. A k-coloring of G is a function $c : V \to \{1, 2, \ldots, k\}$, and it is said *d-legal* if $|c(i) - c(j)| \geq d_{ij}$ for all edges $\{i,j\} \in E$. The *d-chromatic number*, $\chi_d(G)$, is the smallest integer k such that a d-legal k-coloring exists for G. Finding the d-chromatic number of a graph is known as the *bandwidth coloring problem* [5, 6]. When $d_{ij} = 1$ for all $\{i,j\} \in E$, the problem reduces to the well-known graph coloring problem, which is NP-hard [4]. In this case, the d-chromatic number is simply the chromatic number, denoted $\chi(G)$.

An *orientation* of a graph G is a directed graph, denoted $\vec{G}$, obtained from G by orienting each edge $\{i,j\} \in E$ from i to j or from j to i. In other words, for each edge $\{i,j\} \in E$, there is one corresponding arc in $\vec{G}$, either (i,j) or (j,i). The weight of an arc (i,j) in an orientation $\vec{G}$ of G is the weight d_{ij} of the corresponding edge $\{i,j\} \in E$. An elementary *path* $\vec{P}$ in an orientation $\vec{G}$ of G is a sequence $(i_1, \ldots, i_p)$ of distinct vertices such that (i_l, i_{l+1}) $(l = 1, \cdots, p-1)$ is an arc in $\vec{G}$, and its *length* $L(\vec{P})$ is the total weight $\sum_{l=1}^{p-1} d_{i_l i_{l+1}}$. We denote $\Omega(G)$ the set of all orientations of G, and $\lambda(\vec{G})$ the length of a longest elementary path in $\vec{G}$.

In this paper, we give two proofs of the following theorem:

Theorem 1 $\chi_d(G) = 1 + \min_{\vec{G} \in \Omega(G)} \lambda(\vec{G})$.

As a direct corollary to this theorem, we obtain:

Corollary 2 *In any orientation $\vec{G}$ of G, $\chi_d(G) \leq \lambda(\vec{G}) + 1$.*

If $d_{ij} = 1$ for all $\{i,j\} \in E$, then the length $L(\vec{P})$ of an elementary path $\vec{P}$ in an orientation $\vec{G}$ of G is equal to its number of arcs, which means that $L(\vec{P}) + 1$ is the number of vertices on $\vec{P}$. Hence, by applying the above corollary to the special case of the graph coloring problem, we derive the following classical theorem, independently proved by Gallai [3], Roy [7] and Vitaver [8]:

Theorem 3 *The maximum number of vertices on an elementary path of an orientation $\vec{G}$ of G is at least equal to the chromatic number $\chi(G)$ of G.*

For variations on this theorem, the reader is referred to de Werra and Hansen (2005). The rest of this paper is dedicated to the proof of Theorem 1. The next section contains a simple proof, while Section 3 presents a more complex proof that uses a series of equivalent mathematical programming models (two models are equivalent if their optimal values are equal for all problem instances). Thus, we not only generalize the Gallai-Roy-Vitaver theorem to the bandwidth coloring problem, but we also suggest several equivalent mathematical programming formulations which can be used to develop various solution algorithms for the bandwidth coloring problem.

2 A Simple Proof of Theorem 1

A simple proof of Theorem 1 can be obtained with the help of the following lemma.

Lemma 4 *For every graph G there exists a circuit-free orientation $\vec{G}^*$ of G such that* $\lambda(\vec{G}^*) = \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$

Proof. Consider an orientation $\vec{G}'$ of G such that $\lambda(\vec{G}') = \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$. If $\vec{G}'$ contains circuits, then let C be one of them with a maximum number of arcs. Let (u, v) be an arc on C with minimum weight, $\vec{P}$ denote the path from v to u on C, and $\vec{G}''$ denote the orientation of G obtained from $\vec{G}'$ by changing the orientation of arc (u, v). The new arc (v, u) in $\vec{G}''$ does not belong to any circuit, else there would be a path with at least two arcs linking u to v in $\vec{G}'$, which, combined with $\vec{P}$, would constitute a circuit C' with strictly more arcs than C, a contradiction. Also, the longest elementary path $\vec{P}'$ in $\vec{G}''$ containing (v, u) has length $L(\vec{P}') \leq \lambda(\vec{G}')$ else, by replacing (v, u) with $\vec{P}$, one would get a path in $\vec{G}'$ of length strictly larger than $\lambda(\vec{G}')$, a contradiction. By optimality of $\vec{G}'$, we therefore have $\lambda(\vec{G}'') = \lambda(\vec{G}')$. By repeating this process a finite number of times, one obtains a circuit-free orientation $\vec{G}^*$ with $\lambda(\vec{G}^*) = \lambda(\vec{G}') = \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$. □

Proof of Theorem 1. Consider a circuit-free orientation $\vec{G}^*$ of G such that $\lambda(\vec{G}^*) = \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$. The existence of such an orientation follows from Lemma 4. For every $i \in V$, define $c(i)$ equal to 1 + the length of the longest path entering i in $\vec{G}^*$. Then $c(i) \in \{1, \ldots, 1 + \lambda(\vec{G}^*)\}$ for all $i \in V$, and $c(j) \geq c(i) + d_{ij}$ for all arcs (i, j) in $\vec{G}^*$, which means that c is a d-legal $(1+\lambda(\vec{G}^*))$-coloring of G. Hence, $\chi_d(G) \leq 1 + \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$.

Conversely, consider any d-legal $\chi_d(G)$-coloring c of G and define $\vec{G}^*$ as the orientation of G obtained by orienting every edge $\{i, j\} \in E$ from i to j if and only if $c(i) < c(j)$. Let $\vec{P} = (i_1, \ldots, i_p)$ be a longest elementary path in $\vec{G}^*$. We then have

$$L(\vec{P}) = \sum_{l=1}^{p-1} d_{i_l i_{l+1}} \leq \sum_{l=1}^{p-1} (c(i_{l+1}) - c(i_l)) = c(i_p) - c(i_1) \leq \chi_d(G) - 1.$$

Hence, $\min_{\vec{G}\in\Omega(G)} \lambda(\vec{G}) \leq \lambda(\vec{G}^*) = L(\vec{P}) \leq \chi_d(G) - 1$. □

3 Mathematical Programming Models for the Bandwidth Coloring Problem

We now give a more complex proof of Theorem 1 that is based on a series of equivalent mathematical programming models. For every model M, we denote $Z(M)$ its optimal value and $z_M(x)$ the value of a feasible solution x to M. The following nonlinear integer programming model, M_1, is based on the definition of the bandwidth coloring problem. It provides an optimal solution to the problem and an optimal value $Z(M_1)$ equal to $\chi_d(G)-1$.

$$M_1 \begin{cases} \text{minimize} & z_{M_1}(c,k) = k-1 & \\ \text{subject to} & |c_i - c_j| \geq d_{ij} & \forall \{i,j\} \in E \quad (1) \\ & 1 \leq c_i \leq k & \forall i \in V \quad (2) \\ & c_i \ \text{ integer} & \forall i \in V \quad (3) \end{cases}$$

By imposing constraints (1)-(3), it is clear that the variables c_i define a d-legal k-coloring, provided k is an integer, which is necessarily the case at optimality (otherwise, one could set $k = \max_{i\in V}\{c_i\}$ to obtain a feasible integer solution with a lower objective value). Since we are minimizing k, we have $k = \chi_d(G)$ in an optimal solution to this model.

Proposition 5 *Model M_1 is equivalent to its continuous relaxation M_2 obtained by dropping the integrality requirements (3):*

$$M_2 \begin{cases} \text{minimize} & z_{M_2}(c,k) = k-1 \\ \text{subject to} & \text{constraints (1) and (2)} \end{cases}$$

Proof. Since M_2 is a relaxation of M_1, we have $Z(M_1) \geq Z(M_2)$. Conversely, to show that $Z(M_2) \geq Z(M_1)$, it is sufficient to prove that from any optimal solution to M_2, we can construct a feasible solution to M_1 with the same objective value. Let (c,k) be an optimal solution to M_2 and define $(\overline{c},k)$ as follows: $\overline{c}_i = \lfloor c_i \rfloor$ for each $i \in V$. Clearly, this solution satisfies constraints (2) and (3). To show it also satisfies inequalities (1), let us assume the contrary: there exists an edge $\{i,j\} \in E$ such that $|\overline{c}_i - \overline{c}_j| < d_{ij}$. Without loss of generality, we can assume $c_i \geq c_j$, which implies $c_i - c_j \geq d_{ij}$ and $\overline{c}_i - \overline{c}_j < d_{ij}$. But then, we have: $c_i \geq c_j + d_{ij} \geq \overline{c}_j + d_{ij} > \overline{c}_i$, a contradiction, since $\overline{c}_j + d_{ij}$ is an integer that would be smaller than or equal to c_i but greater than the largest integer smaller than c_i. □

Proposition 6 *Model M_2 is equivalent to the following formulation M_3, where the notation a^+ stands for* $\max\{0,a\}$*:*

$$M_3 \begin{cases} \text{minimize} & z_{M_3}(c,k) = k - 1 + \sum\limits_{\{i,j\}\in E} (d_{ij} - |c_i - c_j|)^+ \\ \text{subject to} & \text{constraints (2)} \end{cases}$$

Proof. First note that if $|c_i - c_j| \geq d_{ij}$, then $(d_{ij} - |c_i - c_j|)^+ = 0$. Using this observation, we have $z_{M_2}(c,k) = z_{M_3}(c,k)$ for every feasible solution (c,k) to M_2. Hence, we can replace $z_{M_2}(c,k)$ by $z_{M_3}(c,k)$ in M_2 to obtain an equivalent model. Now, if we drop constraints (1), we obtain model M_3, which therefore provides a lower bound $Z(M_3)$ on $Z(M_2)$.

It remains to prove that $Z(M_3) \geq Z(M_2)$. Let (c,k) be an optimal solution to M_3. As observed above, if constraints (1) are satisfied, (c,k) is a feasible solution to M_2 with $z_{M_2}(c,k) = z_{M_3}(c,k)$. So, let us assume that at least one edge $\{u,v\} \in E$ violates constraints (1), i.e., $|c_u - c_v| < d_{uv}$, and, without loss of generality, that $c_u \geq c_v$. We then define $\delta_{uv} = d_{uv} - (c_u - c_v) > 0$ from which we derive the following new solution $(\overline{c}, \overline{k})$ to M_3:

$$\overline{c}_i = \begin{cases} c_i & \text{if } i = v \text{ or } c_i < c_u \\ c_i + \delta_{uv} & \text{otherwise,} \end{cases}$$

$$\overline{k} = k + \delta_{uv}.$$

We prove that $|\overline{c}_i - \overline{c}_j| \geq |c_i - c_j|$ for all edges $\{i,j\} \in E$. Consider any edge $\{i,j\} \in E$, and assume, without loss of generality, that $c_i \geq c_j$. If $\overline{c}_j = c_j$, then $\overline{c}_i \geq c_i \geq c_j = \overline{c}_j$, which implies $|\overline{c}_i - \overline{c}_j| = \overline{c}_i - \overline{c}_j \geq c_i - c_j = |c_i - c_j|$. Otherwise, $\overline{c}_j = c_j + \delta_{uv}$, which means that $c_i \geq c_j \geq c_u \geq c_v$. Then, there are two cases: 1) if $i = v$, then $\overline{c}_i = c_i = c_j$, which implies $|\overline{c}_i - \overline{c}_j| = \delta_{uv} > 0 = |c_i - c_j|$; 2) if $i \neq v$, then $\overline{c}_i = c_i + \delta_{uv}$, which implies $|\overline{c}_i - \overline{c}_j| = |c_i - c_j|$.

As a consequence, no constraint of type (1) satisfied by (c,k) is violated by $(\overline{c}, \overline{k})$, since $|\overline{c}_i - \overline{c}_j| \geq |c_i - c_j| \geq d_{ij}$, for all edges $\{i,j\} \in E$ satisfying (1). When $\{i,j\} = \{u,v\}$, we have $d_{uv} - |\overline{c}_u - \overline{c}_v| = d_{uv} - (c_u - c_v) - \delta_{uv} = 0$. This implies that constraint (1) for edge $\{u,v\}$ is no more violated in solution $(\overline{c}, \overline{k})$.

By optimality of (c,k), we have $z_{M_3}(c,k) - z_{M_3}(\overline{c}, \overline{k}) \leq 0$. But, we also have:

$$\begin{aligned} z_{M_3}(c,k) - z_{M_3}(\overline{c}, \overline{k}) &= (k-1) + \sum_{\{i,j\}\in E} (d_{ij} - |c_i - c_j|)^+ \\ &\quad -(\overline{k} - 1) - \sum_{\{i,j\}\in E} (d_{ij} - |\overline{c}_i - \overline{c}_j|)^+ \\ &= (k - \overline{k}) + \Big((d_{uv} - (c_u - c_v)) - (d_{uv} - (\overline{c}_u - \overline{c}_v)) \Big) \\ &\quad + \sum_{\{i,j\}\in E\setminus\{u,v\}} \Big(|\overline{c}_i - \overline{c}_j|^+ - |c_i - c_j|^+ \Big) \end{aligned}$$

$$\begin{aligned} &\geq \quad (k-(k+\delta_{uv})) + \Big((c_u-c_v)-(c_u-c_v)) - \delta_{uv}\Big) \\ &= \quad 0. \end{aligned}$$

Thus, the new solution $(\overline{c},\overline{k})$ is also optimal for M_3, but, compared to (c,k), it has at least one additional constraint of type (1) that is satisfied, and no further violated constraints of this type. Hence, by repeating the same argument a finite number of times, we would eventually derive a feasible solution to M_2 having the same objective value. □

For each edge $\{i,j\} \in E$ we now introduce two new variables a_{ij} and b_{ij} defined as follows:

$$a_{ij} = (d_{ij} - (c_{\min\{i,j\}} - c_{\max\{i,j\}}))^+ \tag{4}$$

$$b_{ij} = (d_{ij} - (c_{\max\{i,j\}} - c_{\min\{i,j\}}))^+. \tag{5}$$

Proposition 7 *Model M_3 is equivalent to the following formulation M_4:*

$$M_4 \left\{ \begin{array}{lll} \text{minimize} & z_{M_4}(c,k,a,b) = k - 1 + \sum\limits_{\{i,j\}\in E} \min\{a_{ij}, b_{ij}\} & \\ \text{subject to} & \text{constraints (2) and} & \\ & a_{ij} \geq (d_{ij} - (c_{\min\{i,j\}} - c_{\max\{i,j\}})) \quad \forall \{i,j\} \in E & (6) \\ & b_{ij} \geq (d_{ij} - (c_{\max\{i,j\}} - c_{\min\{i,j\}})) \quad \forall \{i,j\} \in E & (7) \\ & a_{ij}, b_{ij} \geq 0 \quad \forall \{i,j\} \in E & (8) \end{array} \right.$$

Proof. Consider any feasible solution (c,k,a,b) to M_4. Constraints (6)-(8) are equivalent to imposing $a_{ij} \geq (d_{ij} - (c_{\min\{i,j\}} - c_{\max\{i,j\}}))^+$ and $b_{ij} \geq (d_{ij} - (c_{\max\{i,j\}} - c_{\min\{i,j\}}))^+$. Hence, $Z(M_3) \leq Z(M_4)$ since (c,k) is a feasible solution to M_3, and the following inequality is valid for every edge $\{i,j\} \in E$:

$$\begin{aligned} (d_{ij} - |c_i - c_j|)^+ &= (\min\{d_{ij} - (c_j - c_i), d_{ij} - (c_i - c_j)\})^+ \\ &= \min\{(d_{ij} - (c_j - c_i))^+, (d_{ij} - (c_i - c_j))^+\} \\ &\leq \min\{a_{ij}, b_{ij}\}. \end{aligned}$$

The above inequality becomes an equality when a_{ij} and b_{ij} are defined according to (4) and (5). Hence, given any feasible solution (c,k) to M_3, the solution (c,k,a,b) obtained by using definitions (4) and (5) is feasible to M_4 and $z_{M_3}(c,k) = z_{M_4}(c,k,a,b)$, which means that $Z(M_4) \leq Z(M_3)$. □

Let A be the set of ordered pairs (i,j) with $\{i,j\} \in E$. Hence, for every edge $\{i,j\} \in E$, there are two elements (i,j) and (j,i) in A. Let $A^>$ be the subset of pairs $(i,j) \in A$ with

$i > j$, and let $A^<$ be the subset of pairs $(i,j) \in A$ with $i < j$. Definitions (4) and (5) are equivalent to

$$(d_{ij} - (c_i - c_j))^+ = \begin{cases} a_{ij} & \text{if } (i,j) \in A^< \\ b_{ij} & \text{if } (i,j) \in A^>. \end{cases}$$

Hence, by defining $t_{ij} = a_{ij}$ if $(i,j) \in A^<$ and $t_{ij} = b_{ij}$ if $(i,j) \in A^>$, definitions (4) and (5) are equivalent to

$$t_{ij} = (d_{ij} - (c_i - c_j))^+ \qquad \forall (i,j) \in A. \tag{9}$$

Proposition 8 *Model M_4 is equivalent to the following formulation M_5:*

$$M_5 \begin{cases} \text{minimize} & z_{M_5}(c,k,t,y) = k - 1 + \sum_{(i,j)\in A} y_{ij} t_{ij} & & \\ \text{subject to} & \text{constraints (2) and} & & \\ & t_{ij} \geq (d_{ij} - (c_i - c_j)) & \forall (i,j) \in A & (10) \\ & t_{ij} \geq 0 & \forall (i,j) \in A & (11) \\ & y_{ij} + y_{ji} = 1 & \forall \{i,j\} \in E & (12) \\ & y_{ij} \in \{0,1\} & \forall (i,j) \in A & (13) \end{cases}$$

Proof. Let (c,k,a,b) be a feasible solution to M_4, and let (c,k,t,y) be the feasible solution to M_5 obtained by defining variables t_{ij} according to (9), and by setting $y_{ij} = 1$ if $t_{ij} < t_{ji}$, or $t_{ij} = t_{ji}$ and $i < j$, and $y_{ji} = 0$ otherwise. We have $z_{M_4}(c,k,a,b) = z_{M_5}(c,k,t,y)$, since $\min\{a_{ij}, b_{ij}\} = \min\{t_{ij}, t_{ji}\} = t_{ij}y_{ij} + t_{ji}y_{ji}$ for every edge $\{i,j\} \in E$, which proves that $Z(M_5) \leq Z(M_4)$.

Consider now an optimal solution (c,k,t,y) to M_5. We necessarily have $t_{ij}y_{ij} + t_{ji}y_{ji} = \min\{t_{ij}, t_{ji}\}$, else a better solution could be obtained by permuting the values of y_{ij} and y_{ji}. Let (c,k,a,b) be the solution to M_4 obtained from (c,k,t,y) by setting $a_{ij} = t_{\min\{i,j\}\max\{i,j\}}$ and $b_{ij} = t_{\max\{i,j\}\min\{i,j\}}$ for every edge $\{i,j\} \in E$. The nonnegativity constraints (8) of M_4 are satisfied since $t_{ij} \geq 0$ for all $(i,j) \in A$. Constraints (6) and (7) of M_4 are also satisfied by (c,k,a,b) since

$$\begin{aligned} a_{ij} &= t_{\min\{i,j\}\max\{i,j\}} \geq d_{ij} - (c_{\min\{i,j\}} - c_{\max\{i,j\}}), \text{ and} \\ b_{ij} &= t_{\max\{i,j\}\min\{i,j\}} \geq d_{ij} - (c_{\max\{i,j\}} - c_{\min\{i,j\}}). \end{aligned}$$

Hence, (c,k,a,b) is a feasible solution to M_4, and $z_{M_4}(c,k,a,b) = z_{M_5}(c,k,t,y)$ since $t_{ij}y_{ij} + t_{ji}y_{ji} = \min\{t_{ij}, t_{ji}\} = \min\{a_{ij}, b_{ij}\}$ for every edge $\{i,j\} \in E$. This proves that $Z(M_5) \geq Z(M_4)$ □

Formulation M_5 can be viewed as a bilevel programming model. Indeed, let Y be the set of $|A|$-dimensional vectors satisfying constraints (12) and (13) of M_5. The problem of finding an optimal solution to M_5 for a fixed $y \in Y$ can be formulated using the following model $M_6(y)$:

$$M_6(y) \begin{cases} \text{minimize} & z_{M_6(y)}(c,k,t) = k - 1 + \sum_{(i,j)\in A} y_{ij} t_{ij} \\ \text{subject to} & \text{constraints (2), (10) and (11)} \end{cases}$$

Hence $Z(M_5) = \min_{y\in Y} Z(M_6(y))$, and $M_6(y)$ is equivalent to the following model, obtained by a simple change of variables, namely $\tilde{k} = k - 1$ and $\tilde{c}_i = c_i - 1$ for all $i \in V$:

$$M_6(y) \begin{cases} \text{minimize} & z_{M_6(y)}(\tilde{c},\tilde{k},t) = \tilde{k} + \sum_{(i,j)\in A} y_{ij} t_{ij} & & \\ \text{subject to} & \tilde{c}_i - \tilde{c}_j + t_{ij} \geq d_{ij} & \forall (i,j) \in A & (14) \\ & \tilde{k} - \tilde{c}_i \geq 0 & \forall i \in V & (15) \\ & t_{ij} \geq 0 & \forall (i,j) \in A & (16) \\ & \tilde{c}_i \geq 0 & \forall i \in V & (17) \end{cases}$$

This problem is feasible, since $\tilde{k} = 0$, $\tilde{c}_i = 0$ $(i \in V)$, and $t_{ij} = d_{ij}$ $((i,j) \in A)$ define a feasible solution to $M_6(y)$. As $Z(M_6(y)) \geq 0$, it also has a finite optimal value. Hence, it is equivalent to its dual, defined using the variables x_{ij} associated to constraints (14) and s_i corresponding to constraints (15):

$$M_7(y) \begin{cases} \text{maximize} & z_{M_7(y)}(x,s) = \sum_{(i,j)\in A} d_{ij} x_{ij} & & \\ \text{subject to} & \sum_{i\in V} s_i = 1 & & (18) \\ & \sum_{j|(i,j)\in A} x_{ij} - \sum_{j|(j,i)\in A} x_{ji} - s_i \leq 0 & \forall i \in V & (19) \\ & x_{ij} \leq y_{ij} & \forall (i,j) \in A & (20) \\ & s_i \geq 0 & \forall i \in V & (21) \\ & x_{ij} \geq 0 & \forall (i,j) \in A & (22) \end{cases}$$

Since $M_6(y)$ and $M_7(y)$ are dual problems, we have $Z(M_6(y)) = Z(M_7(y))$. Every $y \in Y$ corresponds to an orientation of G, denoted $\vec{G}_y$, obtained by choosing the orientation (i,j) for edge $\{i,j\} \in E$ if $y_{ij} = 1$, and (j,i) if $y_{ji} = 1$. By adding a nonnegative slack variable to each constraint (19), we obtain *flow conservation equations* having the following interpretation: each of these nonnegative slack variables correspond to the flow going from a *super-origin* q to each vertex $i \in V$. Hence, we denote x_{qi} these additional slack variables. Also, we can rewrite variables s_i as flow variables x_{ir} representing the flow coming into a *super-destination* r from each vertex $i \in V$. We denote by $\vec{G}_y^+$ the directed graph obtained from $\vec{G}_y$ by adding vertices q and r along with their incident arcs (i.e., there is an arc in

$\vec{G}_y^+$ from q to i and from i to r for every $i \in V$). With these transformations, we can reformulate $M_7(y)$ as follows:

$$M_8(y)\begin{cases} \text{maximize} & z_{M_8(y)}(x) = \sum_{(i,j)\in A} d_{ij}x_{ij} & \\ \text{subject to} & \sum_{i\in V} x_{qi} = 1 & (23)\\ & \sum_{i\in V} x_{ir} = 1 & (24)\\ & \sum_{j|(i,j)\in A} x_{ij} + x_{qi} - \sum_{j|(j,i)\in A} x_{ji} - x_{ir} = 0 \quad \forall i \in V & (25)\\ & x_{ij} \le y_{ij} \quad \forall (i,j)\in A & (26)\\ & x_{qi}, x_{ir} \ge 0 \quad \forall i \in V & (27)\\ & x_{ij} \ge 0 \quad \forall (i,j)\in A & (28)\end{cases}$$

Note that the redundant constraint (23) is derived by summing flow conservation equations (25) over $i \in V$. It is well-known that any feasible solution to this *network flow formulation* contains an elementary path from q to r in $\vec{G}_y^+$ (along with a finite number of elementary circuits) [1]. Since each such elementary path is formed of one arc going out of q, an elementary path in $\vec{G}_y$ and one arc going into r, the optimal value of this maximization problem is at least equal to the length $L(\vec{P})$ of the longest elementary path $\vec{P}$ in $\vec{G}_y$. If $\vec{G}_y$ is circuit-free, then $Z(M_8(y)) = L(\vec{P}) = \lambda(\vec{G}_y)$. Otherwise (i.e., if $\vec{G}_y$ contains a circuit), $Z(M_8(y))$ is possibly strictly larger than $\lambda(\vec{G}_y)$.

Proposition 9 $\min_{\vec{G}\in\Omega(G)} \lambda(\vec{G}) = \min_{y\in Y} Z(M_8(y))$

Proof. Consider a vector $y^* \in Y$ such that $Z(M_8(y^*)) = \min_{y\in Y} Z(M_8(y))$. Then $Z(M_8(y^*)) \ge \lambda(\vec{G}_{y^*}) \ge \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$.

Conversely, consider an orientation $\vec{G}^*$ such that $\lambda(\vec{G}^*) = \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G})$. According to Lemma 4, we may assume that $\vec{G}^*$ is circuit-free. Let y^* be the vector in Y such that $\vec{G}^* = \vec{G}_{y^*}$. We then have $\min_{\vec{G}\in\Omega(G)} \lambda(\vec{G}) = \lambda(\vec{G}_{y^*}) = Z(M_8(y^*)) \ge \min_{y\in Y} Z(M_8(y))$. □

It follows from all previous propositions that

$$\begin{aligned}\chi_d(G) - 1 &= Z(M_5)\\ &= \min_{y\in Y} Z(M_8(y))\\ &= \min_{\vec{G}\in\Omega(G)} \lambda(\vec{G}).\end{aligned}$$

Hence, Theorem 1 is proved.

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