**Filtering for Detecting Multiple
Targets Trajectories Using
Noisy Images**

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Abstract

The aim of this paper is to present efficient algorithms for the detection of multiple targets in noisy images of a finite region. The algorithms are based on the optimal filter of a multidimensional Markov chain signal. Simulations are used to illustrate the efficiency of the method for detecting the positions of the many targets moving on a torus.

Key Words: Filtering, Markov chains, multiple targets.

Résumé

Le but de cet article est de présenter des algorithmes efficaces pour la recherche de cibles dans des images bruitées. Les algorithmes reposent sur le filtre optimal pour une chaîne de Markov multidimentionnelle représentant le signal. Nous présentons aussi des simulations dans le cas d'une, deux, trois et quatre cibles, démontrant l'efficacité de la méthode proposée pour la détection de positions de cibles se déplaçant sur un tore.

1   Introduction

The problem of trying to find targets moving in some area using noisy images is a fundamental problem of filtering. Think for example of trying to locate a person lost at sea on a dingy when there are bad weather conditions. See for example Ballantine et al. (2005) and Gentil et al. (2005).

In this paper, one is interested in locating m targets moving according to independent regime switching Markov chains on a finite set $\mathcal{T}$, where their exact positions $z^1, \ldots, z^m$ are hidden in noisy black-and-white images Y of $\mathcal{T}$. Note that the proposed model is a particular case of Hidden markov models, e.g. Elliott et al. (1995).

To describe the dynamics involved, set $X_k = (Z_k^1, \ldots, Z_k^m, R_k^1, \ldots, R_k^m)$, with Z_k^j and R_k^j standing respectively for the position and the regime of the j-th particle at time k. Suppose that $(X_k)_{k \geq 0}$ is a homogeneous Markov chain on $\mathcal{T}^m \times \mathcal{R}^m$, $\mathcal{R} = \{1, \ldots, d\}$, and setting $x = (z^1, \ldots, z^m, r^1, \ldots, r^m)$, $x' = (w^1, \ldots, w^m, s^1, \ldots, s^m)$, its transition matrix $\mathcal{P}$ is given by

$$
\begin{aligned}
\mathcal{P}(x, x') &= P\left\{ X_k = (z^1, \ldots, z^m, r^1, \ldots, r^m) \, | X_{k-1} = (w^1, \ldots, w^m, s^1, \ldots, s^m) \right\} \\
&= \prod_{j=1}^m P\left(Z_k^j = z^j | Z_{k-1}^j = w^j, R_k^j = r^j \right) P\left(R_k^j = r^j | R_{k-1}^j = s^j \right).
\end{aligned}
$$

Note that $\left(Z_k^j \right)_{k \geq 0}$ is not a Markov chain in general, but $\left(R_k^j \right)_{k \geq 0}$ is a Markov chain and so is $\left(Z_k^j, R_k^j \right)_{k \geq 0}$.

Because X_k is not observed, the best tool to use is thus filtering theory. Note that, in addition to the unobserved regime, the fact that more than one target may be at the same location complicates the tracking.

Since it is assumed that the targets are not identifiable, the only information available at time k about the location and regime is supposed to be given by the pair (S_k, ξ_k), where $\xi_k(x) = 1$ or 0 according to if a target is at site x or not, and $S_k \in \mathcal{A}$, where $\mathcal{A}$ is the set of all sets A of the form $\bigcup_{j=1}^l \{(x_j, r_j, n_j)\}$, where n_j is the number of targets at site x_j with regime r_j and $n_1 + \ldots + n_l = m$. Obviously, ξ_k is a function of S_k, so one can write $\xi_k = \mathcal{L}(S_k)$.

Since $(X_k)_{k \geq 0}$ is a homogeneous Markov chain, $(S_k)_{k \geq 0}$ is also a homogeneous Markov chain, with state space $\mathcal{A}$ and its transition matrix M on $\mathcal{A} \times \mathcal{A}$ is given by

$$
M(A, B) = P(S_k = A | S_{k-1} = B), \qquad k \geq 1,
$$

which can be easily calculated from the transition matrix $\mathcal{P}$ of the Markov chain $(X_k)_{k \geq 0}$.

Next, assume that the observations $Y_k \in \{0,1\}^{\mathcal{T}}$ at time k consists of random perturbed images according to the following scheme: Given $S_0, \ldots, S_k$, $(Y_k(x))_{x \in \mathcal{T}}$ are independent and

$$P(Y_k(x) = 0 | \xi_k(x) = 0) = p_0, \tag{1}$$
$$P(Y_k(x) = 1 | \xi_k(x) = 1) = p_1, \tag{2}$$

where $0 < p_0, p_1 < 1$.

The filtering problem consists in computing

$$P(S_k = A | Y_1 = y_1, \ldots, Y_k = y_k)$$

for any $A \in \mathcal{A}$ and any $k \geq 1$.

In the next section, one finds the optimal filter, while in section 3, illustrations and simulations are used to assess the performance of the algorithm for one, two, three and four targets.

2   Description of the optimal filter

Throughout this article, one assumes that the transition matrix M is known. Otherwise one could adapt the methodology developed Gentil et al. (2005). In what follows, one wants to find an easy algorithm for computing

$$P(S_k = A | \mathcal{Y}_k),$$

where $A \in \mathcal{A}$ and $\mathcal{Y}_k$ is the sigma-algebra generated by observations $Y_1, \ldots, Y_k$. This probability describes the estimation of the position of the m targets in $\mathcal{T}$ together with the associated regimes.

For any $A = \{(z^1, r^1, n^1), \ldots, (z^l, r^l, n^l)\} \in \mathcal{A}$, define its projection onto the "position space" $\mathcal{A}_P$ by

$$\mathrm{Proj}(A) = \left\{ (z^1, n^1), \ldots, (z^l, n^l) \right\},$$

If one is interested in the positions only, i.e. finding $P(\mathrm{Proj}(S_k) = B | \mathcal{Y}_k)$, then it is given by

$$P(\mathrm{Proj}(S_k) = B | \mathcal{Y}_k) = \sum_{A \in \mathcal{A};\ \mathrm{Proj}(A) = B} P(S_k = A | \mathcal{Y}_k).$$

The first step in finding a recursive formula for $P(S_k = A | \mathcal{Y}_k)$, is to compute, for any $y, \xi \in \{0,1\}^{\mathcal{T}}$, the conditional probability

$$P(Y_k = y | \xi_k = \xi) = P\left(\cap_{x \in \mathcal{T}} \{Y_k(x) = y(x)\} | \xi_k = \xi \right).$$

To this end, for any $y, \xi \in \{0, 1\}^{\mathcal{T}}$, set

$$\Lambda(y, \xi) = (1 - p_1)^{|\mathcal{T}|} \left(\frac{(1 - p_0)(1 - p_1)}{p_0 p_1} \right)^{\langle y, \xi \rangle} \left(\frac{p_1}{1 - p_1} \right)^{\langle y \rangle} \left(\frac{p_0}{1 - p_1} \right)^{\langle \xi \rangle},$$

where $|\mathcal{T}| = \operatorname{card}(\mathcal{T})$, $\langle y \rangle = \sum_{x \in \mathcal{T}} y(x)$ and $\langle y, \xi \rangle = \sum_{x \in \mathcal{T}} y(x) \xi(x)$.

Using the independence assumption, together with (1)–(2), one can check that

$$P(Y_k = y | \xi_k = \xi) = \Lambda(y, \xi).$$

Let P be the joint law of $(X_k)_{k \geq 0}$ with initial distribution ν and the observations (Y_k) with law given by (1)–(2), and let Q be the joint law of $(X_k)_{k \geq 0}$ with initial distribution ν, and independent Bernoulli observations with mean $1/2$, i.e corresponding with $p_0 = p_1 = 1/2$.

Further let $\mathcal{G}_k$ be the sigma-algebra generated by $Y_1, \ldots, Y_k, X_0, \ldots, X_k$. Then it is easy to check that with respect to $\mathcal{G}_k$, P is equivalent to Q and

$$\left. \frac{dP}{dQ} \right|_{\mathcal{G}_k} = \prod_{j=1}^{k} 2^{|\mathcal{T}|} \Lambda(Y_j, \xi_j). \tag{3}$$

Further define

$$L_k = \prod_{j=1}^{k} \Lambda(Y_j, \xi_j). \tag{4}$$

It follows that for any $\mathcal{G}_k$-measurable random variable Z and for any sigma-algebra $\mathcal{F} \subset \mathcal{G}_k$,

$$E_P(Z | \mathcal{F}) = \frac{E_Q(ZL_k | \mathcal{F})}{E_Q(L_k | \mathcal{F})}. \tag{5}$$

While this formula is an easy consequence of the properties of conditional expectations, in the context of filtering, (5) is known as the Kallianpur-Striebel formula. The key observation here is to note that expectations relative to Q are much easier to evaluate since the signal and the observations are independent. Moreover all variables $\{Y_i(x)\}_{1 \leq i \leq k, x \in I}$ are independent and identically distributed Bernoulli with mean $1/2$.

For any $A \in \mathcal{A}$, define

$$q_k(A) = E_Q(\mathbb{I}(S_k = A) L_k | \mathcal{Y}_k).$$

Note that according to (5), one has, for any $A \in \mathcal{A}$,

$$P(S_k = A | \mathcal{Y}_k) = \frac{q_k(A)}{\sum_{B \in \mathcal{A}} q_k(B)}. \tag{6}$$

Therefore the conditional law of S_k given $\mathcal{Y}_k$ is completely determined by the $\{q_k(A); \ A \in \mathcal{A}\}$.

It only remains to find a recursive formula for the unnormalized measures q_k, $k \geq 1$. To this end, for any $y \in \{0,1\}^{\mathcal{T}}$ and any $A \in \mathcal{A}$, set

$$D_A(y) = \Lambda(y, \mathcal{L}(A)).$$

Using independence under Q together with (3), one obtains

$$
\begin{aligned}
q_{k+1}(A) &= E_Q(\mathbb{I}(S_{k+1} = A)L_{k+1}|\ \mathcal{Y}_{k+1}) \\
&= E_Q[E_Q(\mathbb{I}(S_{k+1} = A)\Lambda(Y_{k+1}, \xi_{k+1})|\ Y_{k+1}, S_k)L_k|\ \mathcal{Y}_k] \\
&= D_A(Y_{k+1}) \sum_{B \in \mathcal{A}} M(A, B)E_Q\left\{\mathbb{I}(S_k = B)L_k|\ \mathcal{Y}_k\right\} \\
&= D_A(Y_{k+1}) \sum_{B \in \mathcal{A}} M(A, B)q_k(B).
\end{aligned}
$$

Therefore we obtain the so-called "Zakai" equation, namely

$$q_{k+1}(A) = D_A(Y_{k+1}) \sum_{B \in \mathcal{A}} M(A, B)q_k(B). \tag{7}$$

Since $\mathcal{A}$ is finite, Zakai equation (7) can be evaluated, at least theoretically. Note that by definition, q_0 is determined by the initial law of the targets. Having observed Y_1, one can calculate the measure q_1, and so on.

According to (6), the most probable value of S_k given the first k observations can be estimated by choosing $A \in \mathcal{A}$ such that $q_k(A) = \max_{B \in \mathcal{A}} q_k(B)$. Moreover, the most probable "position" $B \in \mathcal{A}_P$ can be estimated by choosing B maximizing

$$\sum_{A \in \mathcal{A};\ \mathrm{Proj}(A)=B} q_k(B).$$

REMARK. In view of applications, one can either consider that the parameters p_0 and p_1 have been estimated or one can use the maximum likelihood method to estimate them from the observation of images.

3   Illustrations and simulations

In what follows, it is assumed that $\mathcal{T} = \{1, \ldots, N\}$ is a torus, that is $N+1$ is identified with 1 and so on. Further assume that the there are only two regimes, and that the targets are

independent and move to their nearest neighbors (right or left), with probabilities α_1 and $\alpha_2 = 1 - \alpha_1$ for the first regime, and with probabilities α_3 and $\alpha_4 = 1 - \alpha_3$ for the second regime. The dynamic of the (independent) regimes is given by the following Markov chain with values in $\{1, 2\}$:

$$\begin{cases} P(R_{k+1} = 2 | \ R_k = 1) = a \\ P(R_{k+1} = 1 | \ R_k = 2) = b, \end{cases}$$

$0 \leq a, b \leq 1$.

The simulations were restricted to the cases of one, two, three and four targets, and torus with length $N = 500$ in the case of one or two targets, length $N = 150$ for three targets and length $N = 25$ for four targets.

In order to estimate the efficiency of the algorithm, the mean error over several time intervals was computed. After only 5 to 10 steps, the positions predictions were already quite good.

The error made at each iteration was calculated in the following way: in case of just one estimate A, the L^1-distance between the targets and the estimate was calculated; in case of several estimates, the largest L^1-distance was kept. The first iteration was never considered.

Various values of parameters p_0 and p_1 were taken into account, while the other parameters were given the values: $\alpha_1 = \alpha_3 = 0.2$, $\alpha_2 = \alpha_4 = 0.8$, $a = b = 0.9$. Finally, the initial distribution q_0 was chosen to be the uniform law on all possible configurations. The results are reported in Tables 1–4.

Note that, in each case, the results are quite satisfactory. From the 5th or 10th iteration, depending on the choice of parameters p_0 and p_1, the distance between the estimation and the targets is about one or three pixels. This is due to the fact that the algorithm provides an exact solution to the resolution of the optimal filter. The fact that one seems to lose precision when there are many targets, is mainly due to the fact that one used the L^1 distance to calculate the error.

Finally, animations representing the results of the simulations described below, as in Figure 1 for two targets, can be obtained at the web site

http://www.ceremade.dauphine.fr/~gentil/ensimulations2.html.

Calculations were done using C and MATLAB.

Table 1: Mean absolute error for one target on a torus of length 500

Parameters	Time intervals		
	$[2, 100]$	$[10, 100]$	$[30, 100]$
$p_0 = p_1 = 0.9$	3.9	0.2	0.1
$p_0 = p_1 = 0.95$	2.8	0.4	0.3

Table 2: Mean absolute error for one target on a torus of length 500

Parameters	Time intervals		
	$[2, 100]$	$[10, 100]$	$[30, 100]$
$p_0 = p_1 = 0.9$	6.5	0.8	0.8
$p_0 = p_1 = 0.95$	5.5	0.2	0.3

Table 3: Mean absolute error for one target on a torus of length 150

Parameters	Time intervals		
	$[2, 100]$	$[10, 100]$	$[30, 100]$
$p_0 = p_1 = 0.9$	2.2	1.8	1.3
$p_0 = p_1 = 0.95$	1.0	0.4	0.4

Table 4: Mean absolute error for one target on a torus of length 25

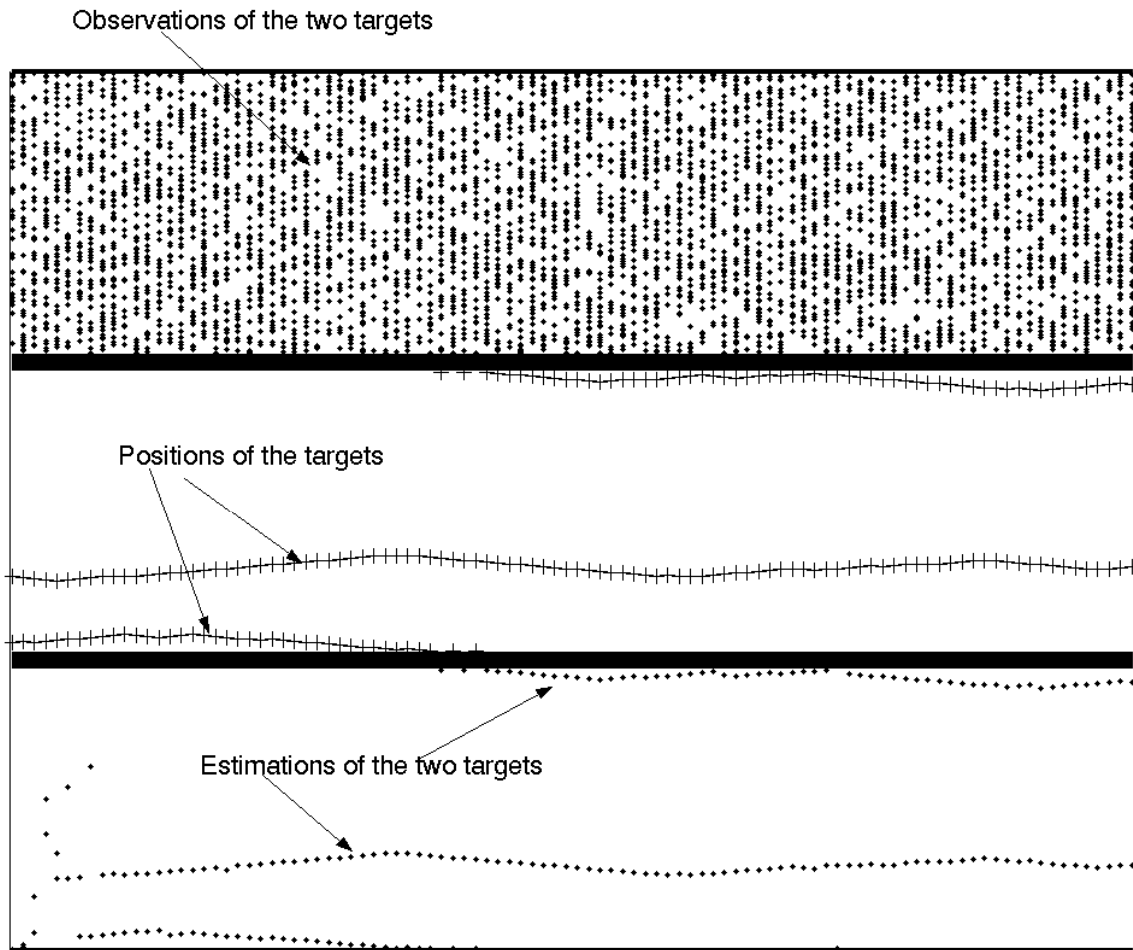| Parameters	Time intervals		
	$[2, 100]$	$[10, 100]$	$[30, 100]$
$p_0 = p_1 = 0.9$	5.1	3.0	3.2
$p_0 = p_1 = 0.95$	0.6	0.5	0.6

Figure 1: MATLAB film for two targets on a torus on length 300

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