Les Cahiers du GERAD ISSN: 0711–2440

The Steiner Equivalent Subgraph Polyhedra for Series-Parallel Digraphs

A. Hadjar

G–2005–72

September 2005

Les textes publiés dans la série des rapports de recherche HEC n'engagent que la responsabilité de leurs auteurs. La publication de ces rapports de recherche bénéficie d'une subvention du Fonds québécois de la recherche sur la nature et les technologies.

The Steiner Equivalent Subgraph Polyhedra for Series-Parallel Digraphs

Ahmed Hadjar

GERAD and Département de mathématiques et de génie industriel
École Polytechnique de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec) Canada H3C 3A7
ahmed.hadjar@gerad.ca

September 2005

Les Cahiers du GERAD

G–2005–72

Copyright © 2005 GERAD

Abstract

We give complete descriptions of the Steiner equivalent subgraph polytope and its dominant when the underlying digraph is strongly connected and series-parallel. These descriptions show that the Steiner equivalent subgraph problem is polynomially solvable on such digraphs and lead as well to the description of the dominant of the Steiner dicut poytope for this class of digraphs.

Key Words: Steiner equivalent subgraphs, Steiner dicuts, Steiner strongly connected subgraphs, series-parallel graphs, polyhedra.

Résumé

Nous considérons, dans ce papier, le problème du sous-graphe équivalent Steiner de coût minimum. Nous montrons que, dans le cas des graphes séries-parallèles fortement connexes, les inégalités triviales et celles associées aux coupes Steiner minimales sont suffisantes pour décrire complètement l'enveloppe convexe des solutions réalisables ainsi que son dominant. Cela nous permet de déduire que le problème se résout en un temps polynomial, sur cette classe de graphes, et d'obtenir également une description complète du dominant du polytope des coupes Steiner pour cette même classe.

1 Introduction

Let $G = (V, A)$ be a connected loopless digraph and $S \subseteq V$ a subset of distinguished nodes of G, called *terminals*. A *Steiner equivalent subgraph* $T = (V, A_T)$ of G is a subgraph such that, for any pair of distinct terminals s_1 and s_2, there exists a dipath from s_1 to s_2 in T if and only if there exists one in G. When $S = V$, a Steiner equivalent subgraph is nothing less than a classical equivalent subgraph. If G is strongly connected then a minimal Steiner equivalent subgraph is a strongly connected subgraph that spans S. Given a cost vector c associated with the arcs of G, the Steiner equivalent subgraph problem consists of finding the minimum cost Steiner equivalent subgraph of G. This problem is NP-hard since, when G is strongly connected and $S = V$, an hamiltonian circuit of G is a Steiner equivalent subgraph with $|V|$ arcs.

The incidence vector x of a subgraph (V, A') of G is a $\{0,1\}$-vector of R^A such that $x(e) = 1$ if and only if $e \in A'$. The Steiner equivalent subgraph polytope associated with G relatively to S, and denoted by $SESP(G, S)$, is the convex hull of incidence vectors of all Steiner equivalent subgraphs of G. The dominant of the Steiner equivalent subgraph polytope is $DSESP(G, S) = SESP(G, S) + R^A_+$. In the case where $S = V$, these polyhedra are simply denoted by $ESP(G)$ and $DESP(G)$ respectively. The Steiner equivalent subgraph problem can then be formulated as $min\{cx : x \in SESP(G, S)\}$.

Let $W \subset V$ such that $\emptyset \neq W \neq V$. With $\overline{W}$ we mean the node set $V \setminus W$. Let $\delta^+(W)$ be the set of arcs having their tails in W and their heads in $\overline{W}$. $\delta^+(W)$ is called *dicut* (note that $\delta^+(\overline{W})$ is not necessarily empty). We shall say that dicut $\delta^+(W)$ is induced by W. When $|W|$ is equal to 1 or $|V| - 1$, the dicut is called *trivial*. A *Steiner dicut* is a dicut $\delta^+(W)$ such that both W and $\overline{W}$ have nonempty intersections with the terminal set S. A Steiner dicut is *minimal* if it does not include another Steiner dicut strictly. Let us denote by $\mathcal{W}(S)$ the family of all subsets of V that induce nonempty minimal Steiner dicuts of G. If x is a vector of R^A and A' is a subset of edges of G then $x(A') = \sum_{e \in A'} x(e)$. Clearly, an integer vector $x \in R^A$ is the incidence vector of a Steiner equivalent subgraph of G if and only if x satisfies the following inequalities :

$$x(\delta^+(W)) \geq 1, \quad \forall\, W \in \mathcal{W}(S). \tag{1}$$

$$x(e) \geq 0, \quad \forall\, e \in A. \tag{2}$$

$$x(e) \leq 1, \quad \forall\, e \in A. \tag{3}$$

Inequalities (1) are the minimal *Steiner dicut inequalities* while inequalities (2) and (3) are called *trivial inequalities*. Observe that if $|S| < 2$ then $SESP(G, S)$ is described by the trivial inequalities and $DSESP(G, S) = R^A_+$. Let us denote by $P(G, S)$ and $DP(G, S)$ the polyhedra given respectively by (1)-(3) and (1)-(2). Similarly we will simply denote them by $P(G)$ and $DP(G)$ when $S = V$.

To the best of our knowledge the Steiner equivalent subgraph problem has been studied only for $S = V$ (i.e., the equivalent subgraph problem). Moyles and Tompson [13], Hsu [9], Martello [11] and Martello and Toth [12] developed algorithms for the minimum cardinality equivalent subgraph problem. Parker and Rardin [15] proposed a linear time algorithm to solve the equivalent subgraph problem on series-parallel digraphs and Hadjar [8] showed that this problem is also polynomially solvable on a class of digraphs that contains directed Halin graphs. Polyhedral investigations were as well reported in the literature for this case. Chopra [3] described families of facet defining inequalities for $DESP(G)$. For strongly connected series-parallel digraphs, Chopra [4] and Margot and Schaffers [10] proved respectively that $DESP(G) = DP(G)$ and $ESP(G) = P(G)$. In [8] Hadjar gave some properties of $ESP(G)$ and its dominant and characterized digraphs for which these polyhedra are given by the trivial dicut inequalities and the trivial inequalities. He also showed that $DESP(G) = DP(G)$ and $ESP(G) = P(G)$ for a class of digraphs that includes the directed Halin graph one.

Our main purpose, in this paper, is to generalize the results of Chopra [4] for the Steiner case; i.e., for any $S \subseteq V$. We show in Section 2 that, for strongly connected series-parallel digraphs and for any set of terminals, $SESP(G, S) = P(G, S)$ and $DSESP(G, S) = DP(G, S)$. A direct consequence is that the Steiner equivalent subgraph problem is polynomially solvable on such digraphs. We derive as well, in Section 3, a complete description of the dominant of the convex hull of the incidence vectors of the Steiner dicuts for this class of digraphs. Section 4 draws some concluding remarks.

To end this section, we list some other definitions and notations. The contraction of a given edge (or arc) consists of identifying its endnodes into a single node and of removing the resulting loops. An undirected graph is said to be *series-parallel* [5] if it can not be reduced, by successive applications of edge removal and edge contraction operations, to a complete graph on four nodes (K_4). A series-parallel digraph is obtained by orienting the edges of a series-parallel undirected graph. An arc $e \in A$ with tail u and head v will be denoted by $e = (u, v)$. We say that tow arcs $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are *parallel* if $u_1 = u_2$ and $v_1 = v_2$ and *anti-parallel* if $u_1 = v_2$ and $v_1 = u_2$. By uv-dipath we mean a dipath with tail u and head v. We shall write $\delta^+(v)$ instead of $\delta^+(\{v\})$, where $v \in V$. A_{uv} will denote the set of all arcs of G with endnodes u and v. Let $x \in R^A$ and $A' \subset A$, by $x_{A'}$ we mean the restriction of x to A'. If $x \in DP(G, S)$ and $\delta^+(W)$ is a Steiner dicut of G, such that its corresponding inequality $x(\delta^+(W)) \geq 1$ is tight for x, then we shall say that $\delta^+(W)$ is tight for x. Finally, our graphs will be connected and loopless.

2 $SESP(G, S)$ and $DSESP(G, S)$ for series-parallel digraphs

Before stating the main result of this paper, let us mention some simple and useful remarks.

As we are considering strongly connected series-parallel digraphs, we assume for the sake of simplicity, that our series-parallel digraphs are obtained from undirected series-parallel graphs by replacing each edge by two anti-parallel arcs. This assumption makes no further restrictions given that any series-parallel digraph $G = (V, A)$ is a subgraph of a series-parallel digraph $\widehat{G} = (V, \widehat{A})$ obtained in that way and clearly $SESP(G, S)$ and $DSESP(G, S)$ are faces of respectively $SESP(\widehat{G}, S)$ and $DSESP(\widehat{G}, S)$ that can be obtained by making tight inequalities $x(g) \geq 0$ for $g \in \widehat{A} \setminus A$.

We shall then use the following definition of minors : a digraph G' is said to be a *minor* of a given digraph G if it can be obtained from G by repeatedly removing a pair of anti-parallel arcs and/or contracting an arc.

Remark 1 *By the previous assumption, any connected minor of a strongly connected series-parallel digraph is a strongly connected series-parallel digraph.*

It is well known that an arbitrary undirected series-parallel graph with neither parallel edges nor a node of degree one contains a node of degree two. This property can then be extended to series-parallel digraphs as follows.

Lemma 1 *A series-parallel digraph with neither parallel arcs nor a node having exactly one neighbor contains a node with exactly two neighbors.*

We will also use the following easy lemma.

Lemma 2 *Let G be a digraph and S the set of its terminals. Let $x \in DP(G, S)$ and let $\delta^+(W)$ and $\delta^+(W')$ be two Steiner dicuts of G tight for x. If $\delta^+(W \cap W')$ and $\delta^+(W \cup W')$ are Steiner dicuts then they are tight for x as well.*

We state now our main result.

Theorem 1 *Let G be a strongly connected series-parallel digraph and S the set of its terminals. $DSESP(G, S) = DP(G, S)$.*

Proof. Clearly the result is true for digraphs with two nodes. Suppose that there exists a series-parallel digraph $G = (V, A)$, with a set of terminals S, for witch the theorem does not hold. Assume that G and S are chosen so that :

(i) G is minimal for this property; i.e., the theorem holds for any minor of G with any set of terminals;

(ii) S is maximal; i.e., the result is true for G with any terminal set strictly includes S.

Trivially, $|S| \geq 2$, $DESP(G, S) \subseteq DP(G, S)$ and any integer point of $DP(G, S)$ is in $DSESP(G, S)$. Suppose that $DP(G, S)$ has a fractional extreme point $\overline{x}$. So there exists a family of $|A \setminus A_0|$ node subsets $\widetilde{\mathcal{W}}(S) \subseteq \mathcal{W}(S)$ such that $\overline{x}$ is the unique solution of the following equation system

$$\begin{cases} x(\delta^+(W)) = 1 & \forall\ W \in \widetilde{\mathcal{W}}(S) \\ x(e) = 0 & \forall\ e \in A_0. \end{cases} \tag{4}$$

where $A_0 = \{e \in A : \overline{x}(e) = 0\}$. We will say that system (4) defines $\overline{x}$. Clearly, $0 \leq \overline{x}(e) \leq 1$, $\forall e \in A$.

Claim 1 *Let $u, v \in V$. $\overline{x}(A_{uv}) \geq 1$.*

Proof. Suppose the contrary and consider the digraph $G' = (V, A')$, where $A' = A \setminus A_{uv}$. If G' is not connected then all terminals of G are in a same connected component, say $H = (V_H, A_H)$, since otherwise A_{uv} would be a Steiner dicut whose corresponding inequality is not satisfied by $\overline{x}$. In this case, any minimal Steiner dicut of G is contained in A_H and then $(A \setminus A_H) \subseteq A_0$. So $\overline{x}_{A_H}$ is the unique solution of the system obtained from system (4) by removing equations $x(g) = 0$ for $g \in A \setminus A_H$. Thus $\overline{x}_{A_H}$ is a fractional extreme point of $DP(H, S)$; but this contradicts the minimality of G since H is a minor of G that can be obtained by contraction of all arcs in $A \setminus A_H$. Suppose now that G' is connected. By Remark 1, G' is a strongly connected minor of G. Again if we remove equations $x(g) = 0$, $\forall g \in A_{uv}$, from system (4) then we obtain a system having $\overline{x}_{A'}$ as the unique solution; hence $\overline{x}_{A'}$ is a fractional extreme point of $DP(G', S)$, a contradiction. □

Claim 2 *G does not have parallel arcs.*

Proof. Assume that $e_1 = (u, v)$, $e_2 = (u, v)$, $f_1 = (v, u)$, $f_2 = (v, u)$ are arcs of G. Suppose, by Claim 1, that $\overline{x}(e_1) > 0$. Considerer the point $\overline{x}'$ obtained from $\overline{x}$ by substituting $\overline{x}(e_1) - \epsilon$ and $\overline{x}(e_2) + \epsilon$ for $\overline{x}(e_1)$ and $\overline{x}(e_2)$ respectively, where ϵ is a sufficiently small positive scalar. Since e_1 and e_2 belong to the same Steiner dicuts of G, the point $\overline{x}'$ belongs to $DP(G, S)$ and satisfies all equations of system (4); therefore $\overline{x}$ is not an extreme point, a contradiction. □

Claim 3 *G does not contain a node having exactly one neighbor*

Proof. Suppose that G has a node v with only one neighbor u. As, by Claim 2, G is parallel arcs free, v is incident with exactly two arcs $e = (u, v)$ and $f = (v, u)$. Given that $\overline{x}(e) + \overline{x}(f) > 0$ (by Claim 1), at least one of these arcs must belong to a minimal Steiner dicut of G, in which case, v must be in S. Actually, the only minimal Steiner dicuts intersecting $\{e, f\}$ are $\{e\}$ and $\{f\}$; so $\overline{x}(e) = \overline{x}(f) = 1$. Therefore, if we denote by $G'' = (V'', A'')$ the minor of G obtained by contracting e and by w the node arising from this contraction then $\overline{x}_{A''}$ is a fractional extreme point of $DP(G'', S'')$, where $S'' = S \cup \{w\}$, a contradiction. □

The digraph G has neither parallel arcs nor a node with only one neighbor, so by Lemma 1, G contains a node v_2 adjacent to exactly two nodes v_1 and v_3. That is v_2 is incident with exactly 4 arcs $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $f_1 = (v_3, v_2)$ and $f_2 = (v_2, v_1)$ (see Figure 1).

In the sequel we will denote by $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ the minors of G obtained by contracting respectively arcs e_1 and e_2 and by w_1 and w_2 the nodes arising from these contractions. The terminal sets of G_1 and G_2 will be denoted respectively by

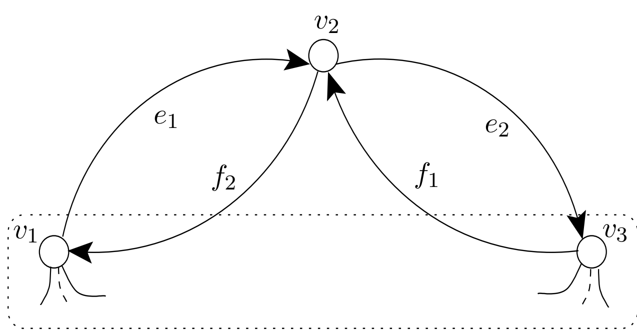

Figure 1: Node v_2 adjacent to exactly two nodes v_1 and v_3.

S_1 and S_2 with $S_1 = (S \setminus \{v_1, v_2\}) \cup \{w_1\}$ if $S \cap \{v_1, v_2\} \neq \emptyset$ or $S_1 = S$ otherwise and $S_2 = (S \setminus \{v_2, v_3\}) \cup \{w_3\}$ if $S \cap \{v_2, v_3\} \neq \emptyset$ or $S_2 = S$ otherwise. Note that, for the sake of simplicity, if g denotes an arc (u, v) (resp. (v, u)) of G where v is an endnode of e_j, with $j \in \{1, 2\}$, then g will also denote the arc (u, w_j) (resp. (w_j, u)) of G_j; for instance, e_2 will denote as well the arc (w_1, v_3) of G_1.

For $j = 1, 2$, observe that, by Remark 1, G_j is a strongly connected series-parallel digraph; thus, by the minimality assumption, $DSESP(G_j, S_j) = DP(G_j, S_j)$. As well, since any Steiner dicut of G_j with respect to S_j is a Steiner dicut of G with respect to S, $\overline{x}_{A_j} \in DP(G_j, S_j)$.

Claim 4 *At least one of the arcs e_1 and f_2 (resp. e_2 and f_1) appears in at least two equations of any system defining $\overline{x}$.*

Proof. Assume that each of e_1 and f_2 appears in exactly one equation of system (4). We can also assume that $S \setminus \{v_1, v_2\} \neq \emptyset$, since otherwise the Claim would not be true for e_2 and f_1 too, in which case, we consider these two arcs instead of e_1 and f_2. Consider the minor $G_1 = (V_1, A_1)$ of G whose terminal set S_1 is of cardinality at least two. The point $\overline{x}_{A_1}$ is fractional because if $\overline{x}(e_1)$ or $\overline{x}(f_2)$ is fractional then e_1 or f_2 belongs to a Steiner dicut tight for $\overline{x}$ that contains at least one arc $g \in A_1$ such that $\overline{x}(g)$ is fractional. Let Γ^1 denote the system obtained from system (4) by removing the equation containing $x(e_1)$ and that containing $x(f_2)$. Then $\overline{x}_{A_1}$ is the unique solution of system Γ^1 any equation of which is either of the form $x(g) = 0$, with $g \in A_1$, or corresponds to a Steiner dicut of G_1 tight for $\overline{x}_{A_1}$. Hence $\overline{x}_{A_1}$ is a fractional extreme point of $DP(G_1, S_1)$, a contradiction. $\square$

Claim 5 *$v_2 \in S$.*

Proof. Assume that $v_2 \notin S$. Note that, in this case, no minimal Steiner dicut of G contains both arcs e_1 and e_2. On the one hand, if $\overline{x}(e_1) > 0$ then e_1 belongs to a Steiner dicut $\delta^+(W_{e_1})$ of G tight for $\overline{x}$ and, as $\delta^+(W_{e_1} \cup \{v_2\})$ is a Steiner dicut, $\overline{x}(e_1) \leq \overline{x}(e_2)$. On the other hand, when $\overline{x}(e_2) > 0$, e_2 is an element of a Steiner dicut $\delta^+(W_{e_2})$ of G tight for $\overline{x}$ and, since $\delta^+(W_{e_2} \setminus \{v_2\})$ is also a Steiner dicut, $\overline{x}(e_1) \geq \overline{x}(e_2)$. Thus $\overline{x}(e_1) = \overline{x}(e_2)$. Similarly, $\overline{x}(f_1) = \overline{x}(f_2)$. It follows that one can substitute $x(e_2)$ (resp. $x(f_1)$) for $x(e_1)$ (resp. $x(f_2)$) in all equations of system (4), except in one, and get a system defining $\overline{x}$ that contradicts Claim 4. $\square$

Claim 6 *If* $\overline{x}(e_1) \neq \overline{x}(e_2)$ *or* $\overline{x}(f_1) \neq \overline{x}(f_2)$ *then* $\overline{x}(\delta^+(v_2)) = \overline{x}(\delta^+(V \setminus \{v_2\})) = 1$

Proof. As $v_2 \in S$ (by Claim 5), $\delta^+(v_2)$ and $\delta^+(V \setminus \{v_2\})$ are Steiner dicuts of G; so $\overline{x}(e_1) + \overline{x}(f_1) \geq 1$ and $\overline{x}(e_2) + \overline{x}(f_2) \geq 1$. Let us suppose that $\overline{x}(e_1) + \overline{x}(f_1) > 1$. In such a case, $\overline{x}(e_1) > 0$, $\overline{x}(f_1) > 0$ and then there exist two Steiner dicuts $\delta^+(W_{e_1})$ and $\delta^+(W_{f_1})$, with $W_{e_1} \neq V \setminus \{v_2\} \neq W_{f_1}$, that are tight for $\overline{x}$ and contain respectively e_1 and f_1. Note that v_2 is neither in W_{e_1} nor in W_{f_1}. Observe also that $W_{e_1} \cap W_{f_1} \cap S = \emptyset$ since otherwise both $\delta^+(W_{e_1} \cap W_{f_1})$ and $\delta^+(W_{e_1} \cup W_{f_1})$ would be Steiner dicuts of G and, by Lemma 2, $\delta^+(W_{e_1} \cup W_{f_1})$ would be a tight dicut that includes $\{e_1, f_1\}$ which means $\overline{x}(e_1) + \overline{x}(f_1) \leq 1$. So $(W_{e_1} \setminus W_{f_1}) \cap (S \setminus \{v_2\}) \neq \emptyset$ and accordingly $\delta^+(W_{e_1} \cup \{v_2\})$ is a Steiner dicut of G that contains e_2 instead of e_1 which in turn implies $\overline{x}(e_2) \geq \overline{x}(e_1)$. As well $(W_{f_1} \setminus W_{e_1}) \cap (S \setminus \{v_2\}) \neq \emptyset$, $\delta^+(W_{f_1} \cup \{v_2\})$ is a Steiner dicut of G that contains f_2 instead of f_1 and $\overline{x}(f_2) \geq \overline{x}(f_1)$. We deduce that if $\overline{x}(e_1) + \overline{x}(f_1) > 1$ then $\overline{x}(e_2) \geq \overline{x}(e_1)$, $\overline{x}(f_2) \geq \overline{x}(f_1)$ and $\overline{x}(e_2) + \overline{x}(f_2) \geq \overline{x}(e_1) + \overline{x}(f_1) > 1$.

Assume now that $\overline{x}(e_2) + \overline{x}(f_2) > 1$. By using analogous arguments as for the previous case, one can show that $\overline{x}(e_1) \geq \overline{x}(e_2)$, $\overline{x}(f_1) \geq \overline{x}(f_2)$ and $\overline{x}(e_1) + \overline{x}(f_1) \geq \overline{x}(e_2) + \overline{x}(f_2) > 1$.

We conclude that if $\overline{x}(\delta^+(v_2)) > 1$ or $\overline{x}(\delta^+(V \setminus \{v_2\})) > 1$ then $\overline{x}(e_1) = \overline{x}(e_2)$ and $\overline{x}(f_1) = \overline{x}(f_2)$. $\square$

Claim 7 *If* $\overline{x}(e_1) \neq \overline{x}(e_2)$ *or* $\overline{x}(f_1) \neq \overline{x}(f_2)$ *then* $0 < \overline{x}(g) < 1$, $\forall g \in \{e_1, e_2, f_1, f_2\}$.

Proof. Suppose (w.l.o.g.) that $\overline{x}(e_1) > \overline{x}(e_2)$. From Claim 6, $\overline{x}(e_1) + \overline{x}(f_1) = \overline{x}(e_2) + \overline{x}(f_2) = 1$ and hence $\overline{x}(f_2) > \overline{x}(f_1)$. Notice that at most one arc among e_1, e_2, f_1 and f_2 is in A_0; in fact, if $\overline{x}(e_2) = 0$ then, by Claim 1, $\overline{x}(f_1) > 0$ and if $0 < \overline{x}(e_2) < \overline{x}(e_1) \leq 1$ then $\overline{x}(f_2) = 1 - \overline{x}(e_2)$ is positive.

Assume (w.l.o.g.) that $\overline{x}(e_2) = 0$. We have then $0 < \overline{x}(e_1) < 1$, $0 < \overline{x}(f_1) < 1$ and $\overline{x}(f_2) = 1$. We distinguish two cases.

Case 1 : $S \setminus \{v_1, v_2\} \neq \emptyset$.

Consider the minor $G_1 = (V_1, A_1)$ whose terminal set S_1 has at least two elements one of which is w_1. So $\overline{x}_{A_1}$ is a fractional point of $DSESP(G_1, S_1)$ that can be decomposed as follows :

$$\overline{x}_{A_1} = \sum_{i=1}^{t_1} \alpha_i y^i + \gamma^1$$

where $\alpha_i \geq 0$ for $i = 1, ..., t_1$, $\sum_{i=1}^{t_1} \alpha_i = 1$, $\gamma^1 \in R_+^{A_1}$ and each y^i is the incidence vector of a minimal Steiner equivalent subgraph of G_1 that does not contain arc e_2.

For, $i = 1, ..., t_1$, define $x^i \in R^A$ as follows

$$x^i(g) = \begin{cases} y^i(g) & if\ g \in A \setminus \{e_1, f_2\} \\ 1 & if\ g = f_2 \\ 1 - y^i(f_1) & if\ g = e_1 \end{cases}$$

(this corresponds to the extensions shown in Figure 2) and let $\gamma \in R^A$ such that

$$\gamma(g) = \begin{cases} \gamma^1(g) & if\ g \in A \setminus \{e_1, f_2\} \\ \overline{x}(g) - \sum_{i=1}^{t_1} \alpha_i & if\ g \in \{e_1, f_2\} \end{cases}$$

One can check easily that each point x^i is the incidence vector of a Steiner equivalent subgraph of G and

$$\overline{x} = \sum_{i=1}^{t_1} \alpha_i x^i + \gamma.$$

But this is impossible given that $\overline{x}$ is an extreme point of $DP(G, S)$.

Case 2 : $S = \{v_1, v_2\}$.

First, observe that $(\delta^+(v_3) \backslash \{f_1\}) \subseteq A_0$ since $\delta^+(\{v_2, v_3\})$ is the unique Steiner dicut of G that intersects $\delta^+(v_3) \backslash \{f_1\}$ and it contains f_2 with $\overline{x}(f_2) = 1$. Therefore $\overline{x}(\delta^+(\{v_2, v_3\})) = 1$.

Now look at the minor $G_2 = (V_2, A_2)$ and its terminal set S_2, where $w_2 \in S_2$ and $|S_1| = 2$. Thus $\overline{x}_{A_2}$ is a fractional point of $DSESP(G_2, S_2)$ and can be decomposed as follows :

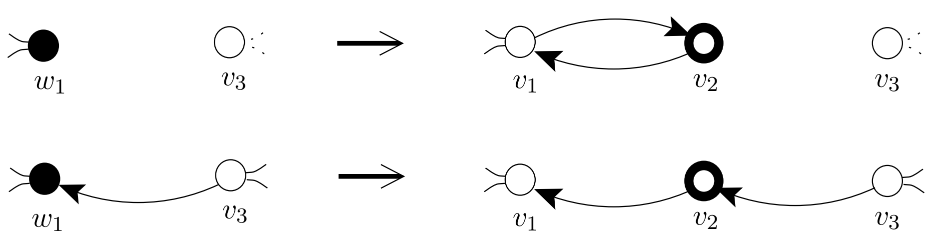

Figure 2: Extensions of the y^i vectors.

$$\overline{x}_{A_2} = \sum_{i=1}^{t_2} \alpha_i z^i + \gamma^2$$

where $\alpha_i \geq 0$ for $i = 1, ..., t_2$, $\sum_{i=1}^{t_2} \alpha_i = 1$, $\gamma^2 \in R_+^{A_2}$ and each z^i is the incidence vector of a minimal Steiner equivalent subgraph of G_2.

Note that since $\overline{x}_{A_2}(\delta^+(w_2)) = 1$, at most one of the points z^i, $i = 1, ..., t_2$, corresponds to a Steiner equivalent subgraph that has exactly two arcs, e_1 and f_2, while the other points correspond to Steiner equivalent subgraphs containing f_2 but not e_1.

For, $i = 1, ..., t_2$, define $x^i \in R^A$ as follows

$$x^i(g) = \begin{cases} z^i(g) & if\ g \in A \setminus \{e_2, f_1\} \\ 0 & if\ g = e_2 \\ 1 - z^i(e_1) & if\ g = f_1 \end{cases}$$

(see Figure 3) and let $\gamma \in R^A$ such that

$$\gamma(g) = \begin{cases} \gamma^2(g) & if\ g \in A \setminus \{e_2, f_1\} \\ \overline{x}(g) - \sum_{i=1}^{t_2} \alpha_i & if\ g \in \{e_2, f_1\} \end{cases}$$

Thus each point x^i is the incidence vector of a Steiner equivalent subgraph of G and $\overline{x}$ can be expressed as

$$\overline{x} = \sum_{i=1}^{t_2} \alpha_i x^i + \gamma.$$

But this contradicts the fact that $\overline{x}$ is an extreme point of $DP(G, S)$.

This shows that $\overline{x}(e_1)$, $\overline{x}(e_2)$, $\overline{x}(f_1)$ and $\overline{x}(f_2)$ are positive. Furthermore, as $\overline{x}(e_1) + \overline{x}(f_1) = \overline{x}(e_2) + \overline{x}(f_2) = 1$, these components are less than 1. □

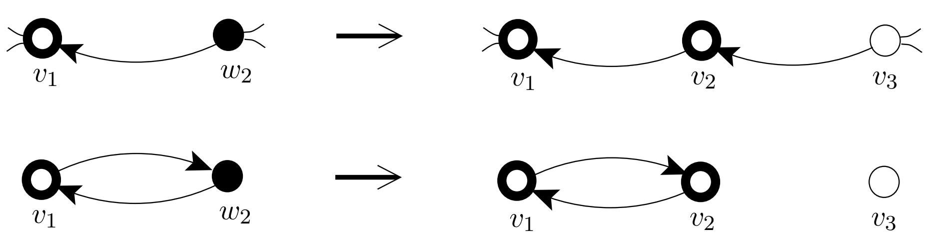

Figure 3: Extensions of the z^i vectors.

Claim 8 *$\overline{x}(e_1) = \overline{x}(e_2)$ and $\overline{x}(f_1) = \overline{x}(f_2)$.*

Proof. Assume (w.l.o.g.) that $\overline{x}(e_1) > \overline{x}(e_2)$. So, by Claims 6 and 7, $0 < \overline{x}(e_2) < \overline{x}(e_1) < 1$ and $0 < \overline{x}(f_1) < \overline{x}(f_2) < 1$. As well, from Claim 4, it follows that at least one arc of e_1 and f_2 belongs to a Steiner dicut that is tight for $\overline{x}$ and different from $\{e_1, f_1\}$ and $\{e_2, f_2\}$. Assume (w.l.o.g.) that e_1 belongs to such a Steiner dicut denoted by $\delta^+(W_{e_1})$. Then all terminals of G except v_2 are in W_{e_1}, since otherwise $\delta^+(W_{e_1} \cup \{v_2\})$ would be also a Steiner dicut (obtained from $\delta^+(W_{e_1})$ by replacing e_1 by e_2) which implies $\overline{x}(e_1) \le \overline{x}(e_2)$. Observe that if, instead of e_1, f_2 belongs to a Steiner dicut $\delta^+(W_{f_2})$ tight for $\overline{x}$ and different from $\delta^+(v_2)$ then analogously, as $\overline{x}(f_1) < \overline{x}(f_2)$, $S \cap W_{f_2} = \{v_2\}$.

The set S_2 of terminals of G_2 is of cardinality at least two and contains w_2. The fractional point $\overline{x}_{A_2}$ belongs to $DSESP(G_2, S_2)$, so it can be decomposed as follows :

$$\overline{x}_{A_2} = \sum_{i=1}^{\widehat{t}} \alpha_i \widehat{y}^i + \widehat{\gamma}$$

where $\alpha_i \ge 0$ for $i = 1, ..., \widehat{t}$, $\displaystyle\sum_{i=1}^{\widehat{t}} \alpha_i = 1$, $\widehat{\gamma} \in R_+^{A_2}$ and each $\widehat{y}^i$ is the incidence vector of a minimal Steiner equivalent subgraph of G_2. Evidently, as $\overline{x}_{A_2}(\delta^+(W_{e_1})) = 1$, we have $\widehat{y}^i(\delta^+(W_{e_1})) = 1$ for $i = 1, ..., \widehat{t}$.

Let us extend each vector $\widehat{y}^i$, $i = 1, ..., \widehat{t}$, to a vector $x^i \in R^A$ as follows :

$$\begin{array}{ll}
x^i(g) = \widehat{y}^i(g) & \text{for } \; g \in A \setminus \{e_2, f_1\} \\
x^i(e_2) = 1 \;\text{ and }\; x^i(f_1) = 1 & \text{if } \; \widehat{y}^i(e_1) = 0 \;\text{ and }\; \widehat{y}^i(f_2) = 0 \\
x^i(e_2) = 1 \;\text{ and }\; x^i(f_1) = 0 & \text{if } \; \widehat{y}^i(e_1) = 1 \;\text{ and }\; \widehat{y}^i(f_2) = 0 \\
x^i(e_2) = 0 \;\text{ and }\; x^i(f_1) = 1 & \text{if } \; \widehat{y}^i(e_1) = 0 \;\text{ and }\; \widehat{y}^i(f_2) = 1 \\
x^i(e_2) = 0 \;\text{ and }\; x^i(f_1) = 0 & \text{if } \; \widehat{y}^i(e_1) = 1 \;\text{ and }\; \widehat{y}^i(f_2) = 1
\end{array}$$

These extensions are illustrated in Figure 4.

Let T_i and $\widehat{T}_i$ be respectively the subgraphs of G and G_2 corresponding to x^i and $\widehat{y}^i$ for some $i \in \{1, ..., t\}$. Actually $\widehat{T}_i$ is a Steiner equivalent subgraph of G_2. If $\widehat{T}_i$ does not contain both e_1 and f_2 then plainly T_i is a Steiner equivalent subgraph of G. Assume that e_1 and f_2 are in $\widehat{T}_i$. In this case, we claim that w_2 is incident with exactly two arcs e_1 and f_2 in $\widehat{T}_i$. Indeed, since $\widehat{y}^i(\delta^+(W_{e_1})) = 1$, e_1 is the unique arc leaving W_{e_1}. Suppose that there exists a w_2s_1-dipath of $\widehat{T}_i$, with $s_1 \in S_2 \setminus \{w_2\}$, that does not include e_1 and f_2. Because $\widehat{T}_i$ is minimal, arc f_2 belongs to a w_2s_2-dipath in $\widehat{T}_i$ with $s_2 \in S_2 \setminus \{w_2, s_1\}$. Consequently, as $S_2 \setminus \{w_2\} \subseteq W_{e_1}$, $\widehat{T}_i$ does not contain dipaths linking s_1 and s_2; but this is impossible seeing that $\widehat{T}_i$ is a Steiner equivalent subgraph. Hence, since $v_3 \notin S_2$, T_i is a Steiner equivalent subgraph of G.

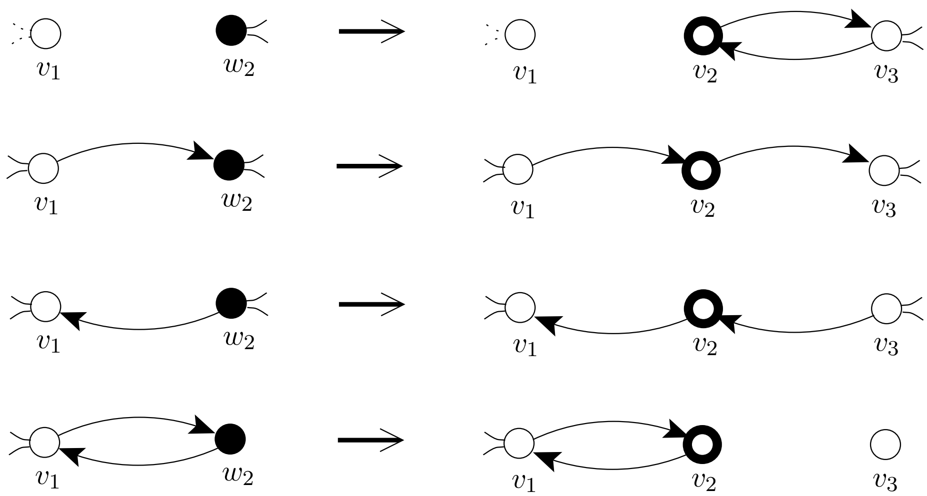

Figure 4: Extensions of the $\widehat{y}^i$ vectors.

Let $\gamma \in R^A$ such that

$$\gamma(g) = \begin{cases} \widehat{\gamma}(g) & if \;\; g \in A \setminus \{e_2, f_1\} \\ \overline{x}(g) - \sum_{i=1}^{\widehat{t}} \alpha_i & if \;\; g \in \{e_2, f_1\} \end{cases}$$

We can then write

$$\overline{x} = \sum_{i=1}^{\widehat{t}} \alpha_i x^i + \gamma$$

contradicting the fact that $\overline{x}$ is an extreme point of $DP(G, S)$. □

Claim 9 v_1, $v_3 \in S$.

Proof. Assume that $v_1 \notin S$ and let $W \subset V$ such that $v_1 \in W$ and $W \cap S = \emptyset$. We will show that $\overline{x}$ satisfies the inequalities $x(\delta^+(W)) \geq 1$ and $x(\delta^+(\overline{W})) \geq 1$. If $v_3 \in W$ then $\delta^+(v_2) \subseteq \delta^+(\overline{W})$ and $\delta^+(V \setminus \{v_2\}) \subseteq \delta^+(W)$ and consequently $\overline{x}(\delta^+(W)) \geq 1$ and $\overline{x}(\delta^+(\overline{W})) \geq 1$. Suppose now that $v_3 \notin W$ and let $W' = W \cup \{v_2\}$. Trivially, as $|S| \geq 2$, $W' \cap S \neq \emptyset$ and $(\overline{W'}) \cap S \neq \emptyset$. Since, by Claim 8, $\overline{x}(e_1) = \overline{x}(e_2)$ and $\overline{x}(f_1) = \overline{x}(f_2)$, we have

$$\begin{aligned} \overline{x}(\delta^+(W)) = \overline{x}(\delta^+(W')) \geq 1 \\ \overline{x}(\delta^+(\overline{W})) = \overline{x}(\delta^+(\overline{W'})) \geq 1. \end{aligned}$$

Hence it is clear that $\overline{x}$ is a fractional extreme point of $DP(G, S')$, with $S' = S \cup \{v_1\}$. But this contradicts the fact that S is maximal. Similarly, one can prove that $v_3 \in S$. □

Let $\delta^+(W)$ be a dicut of G such that $\{v_2\} \neq W \neq V \setminus \{v_2\}$ and $\delta^+(W) \cap \{e_1, e_2\} \neq \emptyset$. Denote by $\delta^+(W')$ the dicut obtained from $\delta^+(W)$ by substituting e_2 for e_1 if $e_1 \in \delta^+(W)$ or e_1 for e_2 otherwise. Since $\{v_1, v_2, v_3\} \subseteq S$ (by Claims 5 and 9), $\delta^+(W)$ is a Steiner dicut if and only if $\delta^+(W')$ does so. Furthermore, as $\overline{x}(e_1) = \overline{x}(e_2)$ (by Claim 8), $\delta^+(W)$ is tight for $\overline{x}$ if and only if $\delta^+(W')$ does so. This also holds for f_1 and f_2.

Let E_{e_1} (resp. E_{f_2}) denote the equation $x(e_1) + x(e_2) = 1$ (resp. $x(f_1) + x(f_2) = 1$) if $\{v_2\}$ (resp. $V \setminus \{v_2\}$) is an element of $\widetilde{\mathcal{W}}(S)$ or any equation of system (4) where $x(e_1)$ (resp. $x(f_2)$) appears.

Thus one can substitute $x(e_2)$ (resp. $x(f_1)$) for $x(e_1)$ (resp. $x(f_2)$) in all equations of system (4), except in E_{e_1} (resp. E_{f_2}), and get a system defining $\overline{x}$ that contradicts Claim 4. This completes the proof of our theorem. □

Let us turn to $SESP(G)$. In [14], Rais showed that if the feasible solutions of a binary polytope are closed under supersets, then the polytope is the intersection of its dominant with the unit cube. Actually, this property holds for the Steiner equivalent subgraphs; thus the following corollary is a straightforward consequence of Theorem 1 and Rais' result.

Corollary 1 *Let G be a strongly connected series-parallel digraph and S the set of its terminals. $SESP(G,S) = P(G,S)$.*

Theorem 1 and Corollary 1 show that inequalities (1), (2) and (3) are sufficient to describe $SESP(G,S)$ and $DSESP(G,S)$ when G is a strongly connected series-parallel digraph. Of course, this does not hold for general digraphs; when $S = V$, Chopra [4] showed that $DESP(K_4^d) \neq DP(K_4^d)$, where K_4^d is the non-series-parallel digraph obtained from K_4 (the complete graph on four nodes) by replacing each edge by two anti-parallel arcs. However, Hadjar [8] proved that $DESP(G) = DP(G)$ for a class of non-series-parallel digraphs such as directed Halin graphs. He also showed that one can orient the edges of any 2-connected graph to obtain a strongly connected digraph whose associated equivalent subgraph polyhedra are completely described by the trivial dicut inequalities and the trivial inequalities. Note that the digraphs considered in [8] are anti-parallel arcs free.

We conclude this section by pointing out that the separation problem associated with Steiner dicut inequalities can easily be reduced to the maximum flow problem and hence it can be solved in polynomial time. Thus, from Corollary 1, it follows (see Grötschel, Lovász and Schrijver [7])

Corollary 2 *The Steiner equivalent subgraph problem is polynomially solvable on strongly connected series-parallel digraphs.*

3 The Steiner dicuts in series-parallel digraphs

Let $G = (V, A)$ be a digraph and S the set of its terminals. Let us denote by $SCP(G, S)$ the Steiner dicut polytope of G with respect to S (i.e. the convex hull of the incidence vectors of all Steiner dicuts of G) and by $DSCP(G, S)$ its dominant. The extreme points of $DSCP(G, S)$ correspond to minimal Steiner dicuts of G. Let $\mathcal{T}(S)$ be the set of the minimal Steiner equivalent subgraphs of G. Consider the polyhedron $DP(G, S)$ defined in Section 1 and let $DQ(G, S)$ denote the polyhedron ($\subset R^A$) given by

$$\begin{cases} y(\delta^+(T)) \geq 1 & \forall\ T \in \mathcal{T}(S) \\ y(e) \geq 0 & \forall\ e \in A. \end{cases}$$

The extreme points of $DSCP(G, S)$ are in bijection with the non trivial facets of $DP(G, S)$ while those of $DSESP(G, S)$ are in bijection with the non trivial facets of $DQ(G, S)$. Hence, whenever all extreme points of $DP(G, S)$ (resp. $DQ(G, S)$) are integer, $DP(G, S)$ and $DQ(G, S)$ form what Fulkerson [6] called *a blocking pair of polyhedra*. The following corollary is a direct consequence of Fulkerson's results [6].

Corollary 3 *$DSESP(G, S) = DP(G, S)$ if and only if $DSCP(G, S) = DQ(G, S)$.*

Thus, Corollary 4 below is the Steiner version of Chopra's result [4] and follows from Theorem 1 and Corollary 3.

Corollary 4 *Let G be a strongly connected series-parallel digraph and S the set of its terminals. $DSCP(G, S) = DQ(G, S)$.*

4 Concluding remarks

When the underlying digraph is strongly connected and series-parallel, the minimal Steiner dicut inequalities and the trivial inequalities are sufficient to describe the Steiner equivalent subgraph polyhedra. By this characterization, the equivalent subgraph problem is polynomially solvable on strongly connected series-parallel digraphs. It also led to a description of the dominant of the Steiner dicut polytope. Actually these results generalize those of Chopra [4] for the Steiner case.

A closely related polytope is that of the Steiner strongly connected subgraphs (i.e., the convex hull of the incidence vectors of all strongly connected subgraphs that span S), denoted by $SSCSP(G, S)$. Trivially, when $G = (V, A)$ is strongly connected and $S = V$, $SSCSP(G, S) = SESP(G, S)$. However, if $S \subset V$ with $|S| \geq 2$ then the two polytopes can be completely different. This is the case for the example illustrated in Figure 5. A strongly connected series-parallel digraph G, with two terminals s_1 and s_2, and its arc costs are given in Figure 5(a); Figure 5(b) and Figure 5(c) show respectively the minimum cost Steiner equivalent subgraph and the minimum cost Steiner strongly connected subgraph of G.

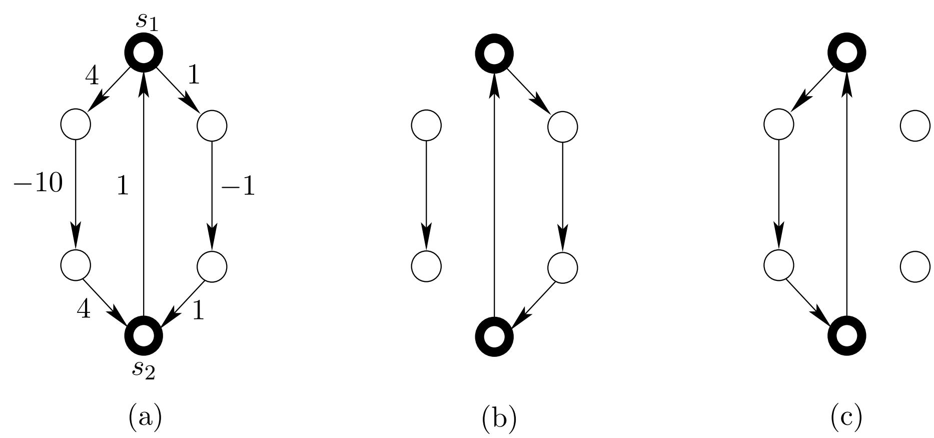

Figure 5: Digraph G, with $S = \{s_1, s_2\}$, and the optimal extreme points of $SESP(G, S)$ and $SSCSP(G, S)$.

To ensure the strong connectivity of the Steiner equivalent subgraphs, one has to include the following inequalities

$$\begin{array}{l} x(\delta^+(W)) - x(e) \geq 0 \\ x(\delta^+(\overline{W})) - x(e) \geq 0 \end{array} \quad \forall W \subset V \;\; s.t. \;\; S \subseteq W, \;\; \forall e \notin A(W). \tag{5}$$

where $A(W)$ is the set of all arcs having both endnodes in W. Hence an integer vector $x \in R^A$ is the incidence vector of a Steiner strongly connected subgraph of G if and only if x satisfies inequalities (1), (2), (3) and (5). Note that inequalities (5) are similar to those used by Baïou and Mahjoub [1] for the Steiner 2-edge connected subgraph polytope and by Coullard, Rais, Rardin and Wagner [2] for the Steiner 2-connected subgraph polytope on series-parallel graphs.

Although, since the minimal Steiner equivalent subgraphs (of a strongly connected digraph) are minimal Steiner strongly connected subgraphs, the dominant of $SSCSP(G, S)$ and that of $SESP(G, S)$ coincide. Thus Theorem 1 gives as well a description of the dominant of $SSCSP(G, S)$ for strongly connected series-parallel digraphs.

References

[1] M. Baïou and A. R. Mahjoub, Steiner 2-edge connected subgraph polytopes on series-parallel graphs, *SIAM J. Disc. Math.*, 10 (1997), pp. 505–514.

[2] C. R. Coullard, A. Rais, R. L. Rardin and D. K. Wagner, The 2-connected-Steiner-subgraph polytope for series-parallel graphs, Report CC–91–23, School of Industrial Engineering, Purdue University, West Lafayette, IN, 1991.

[3] S. Chopra, Polyhedra of the equivalent subgraph problem and some edge connectivity problems, *SIAM J. Disc. Math.*, 5 (1992), pp. 321–337.

[4] S. Chopra, The equivalent subgraph and directed cut polyhedra on series-parallel graphs, *SIAM J. Disc. Math.*, 5 (1992), pp. 475–490.

[5] R. J. Duffin, Topology of series-parallel networks, *J. Math. Anal. Appl.*, 10 (1965), pp. 303–318.

[6] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Math. Programming*, 1 (1971), pp. 168–194.

[7] M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica*, 1 (1981), pp. 70–89.

[8] A. Hadjar, Polyhedra for the equivalent subgraph problem, *Les Cahiers du GERAD*, G–2005–57, HEC Montréal (2005).

[9] H. T. Hsu, An algorithm for finding a minimum equivalent graph of a digraph, *J. Assoc. Comput. Mach.*, 22 (1975), pp. 11–16.

[10] F. Margot and M. Schaffers, Integrality proof with a silicon flavor for polytopes on graphs definable by composition, Report RO910524, E.P.F. Lausanne (1993).

[11] S. Martello, An algorithm for finding a minimum equivalent graph of a strongly connected digraph, *Computing*, 15 (1979), pp. 183–194.

[12] S. Martello and P. Toth, Finding a minimum equivalent graph of a digraph, *Networks*, 12 (1982), pp. 89–100.

[13] D. M. Moyles and G. L. Thompson, An algorithm for finding a minimum equivalent graph of a digraph, *J. Assoc. Comput. Mach.*, 16 (1969), pp. 455–460.

[14] A. Rais, *The 2-connected Steiner subgraph problem*, Ph.D. dissertation, Purdue University (1992).

[15] M. B. Richey, R. C. Parker and R. L. Rardin, An efficiently solvable case of the minimum weight equivalent subgraph problem, *Networks*, 15 (1985), pp. 217–228.