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A. Hadjar

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Ahmed Hadjar

GERAD and
Département de mathématiques et de génie industriel
École Polytechnique de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec) Canada H3C 3A7
ahmed.hadjar@gerad.ca

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Abstract

We give some properties of the equivalent subgraph polytope and its dominant. We characterize those digraphs whose corresponding polyhedra are completely described by bound and trivial dicut inequalities. We give complete descriptions of the equivalent subgraph polyhedra for a class of digraphs which includes directed Halin graphs. As well we derive the description of the dominant of the dicut polytope and we show that the equivalent subgraph problem is polynomially solvable for this class.

Key Words: Equivalent subgraphs, dicuts, polyhedra, facets.

Résumé

Dans cet article, nous présentons quelques propriétés du polytope des sous-graphes équivalents et de son dominant. Nous donnons une caractérisation des graphes orientés dont les polyèdres des sous-graphes équivalents associés sont entièrement décrits par les inégalités de bornes et les inégalités de coupes triviales. Nous décrivons également ces polyèdres pour une large classe de graphes qui contient celle des graphes de Halin orientés. Nous déduisons ensuite la description du dominant du polytope des coupes et nous montrons que le problème du sous-graphe équivalent de coût minimum se résout en temps polynomial, pour cette classe de graphes.

1 Introduction

Let $G = (V, E)$ be a digraph with node set V and arc set E. A subgraph $T = (V, E_T)$ of G is called *equivalent subgraph* if, for any pair of distinct nodes $u, v \in V$, there exists a dipath from u to v in T if and only if there exists one in G. If G is strongly connected then an equivalent subgraph is nothing less than a strongly connected subgraph of G.

Given an arc cost vector c for G, the equivalent subgraph problem consists of finding the minimum cost equivalent subgraph of G. This problem is NP-hard, since G has an hamiltonian circuit if and only if it is strongly connected and has an equivalent subgraph with $|V|$ arcs.

Relatively, there is little known about the equivalent subgraph problem. Moyles and Tompson [13], Hsu [9], Martello [11] and Martello and Toth [12] studied the minimum cardinality equivalent subgraph problem. In [12] the authors give a branch and bound algorithm. Richey, Parker and Rardin [15] show that the equivalent subgraph problem can be solved in linear time when G is a directed series-parallel graph.

When G is strongly connected, the equivalent subgraph problem is equivalent to the minimum cost strong connectivity augmentation problem : given a subgraph $H = (V, E_H)$ of G, the latter problem consists of finding a minimum cost subset E' of $E \setminus E_H$ such that $(V, E_H \cup E')$ is a strongly connected subgraph of G; the equivalent subgraph problem is then the particular case where $H = (V, \emptyset)$ and, conversely, one can modify the cost vector c by substituting a sufficiently small value for the costs of the arcs of E_H and solve the equivalent subgraph problem in G. Eswaran and Tarjan [6] give a linear time algorithm for the minimum cardinality strong connectivity augmentation for complete digraphs.

The incidence vector of an equivalent subgraph T of G is a vector $x \in R^E$ such that $x(e) = 1$ if $e \in T$ and $x(e) = 0$ if $e \notin T$. The equivalent subgraph polytope $ESP(G)$ is the convex hull of incidence vectors of all equivalent subgraphs of G. Thus the equivalent subgraph problem can be formulated as $min\{cx : x \in ESP(G)\}$. Let $DESP(G)$ denotes the dominant of $ESP(G)$, i.e., $DESP(G) = ESP(G) + R^E_+$.

Of course, in view of the NP-hardness characteristic of the equivalent subgraph problem, one can not expect to describe completely (by linear inequality systems) $ESP(G)$ for any given digraph G. To the best of our knowledge, relatively few polyhedral investigations were reported in the literature. Chopra [2] gives some families of facet defining inequalities for $DESP(G)$. Chopra [3] and Margot and Schaffers [10] prove respectively that if G is a strongly connected directed series-parallel graph then $DESP(G)$ and $ESP(G)$ are completely described by dicut inequalities and bound inequalities.

It is interesting to mention the close relationship between $ESP(G)$ and the polytope associated with the asymmetric traveling salesman problem (a well known NP-hard problem that consists to determine a minimum cost hamiltonian circuit of a given strongly connected digraph with edge costs). Actually, since an hamiltonian circuit is an equivalent subgraph of G in which every node has exactly one successor and one predecessor, the asymmetric traveling salesman polytope $ATSP(G)$ is a face of $ESP(G)$, i.e., $ATSP(G) = ESP(G) \cap \{x \in R^E : x(\delta^-(v)) = x(\delta^+(v)) = 1,\ \forall v \in V\}$.

In this paper we study the equivalent subgraph polytope and its dominant and we describe them for some classes of digraphs. In Section 2, we give some properties of $ESP(G)$ and $DESP(G)$ and the relationship between them. We show that one can focus on 2-connected strongly connected digraphs. Section 3 gives a characterization of ESP-trivial digraphs which are digraphs whose corresponding polyhedra are completely described by bound and trivial dicut inequalities. In the last section, we consider a graph decomposition scheme introduced by Cornuéjols, Naddef and Pulleyblank [4] for the symmetric traveling salesman problem and, by following a parallel approach, we derive complete descriptions of the equivalent subgraph polyhedra for a large class of digraphs that includes directed Halin graphs. This leads, as a direct consequence, to the description of the dominant of the dicut polytope for this class. We show as well that the equivalent subgraph problem can be solved in polynomial time on such digraphs. The rest of this section is devoted to more definitions and notations.

Let $G = (V, E)$ be a digraph. n and m will denote respectively the number of nodes and the number of arcs of G. An arc $e \in E$ with tail u and head v will be denoted by $e = (u, v)$. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be arcs of G, we will say that e_1 and e_2 are *parallel* (resp. *anti-parallel*) if $u_1 = u_2$ and $v_1 = v_2$ (resp. $u_1 = v_2$ and $v_1 = u_2$), as well, when $u_1 \neq v_2$ and $v_1 = u_2$ we will say that e_1 and e_2 are *in series*. A uv-dipath is a dipath with tail u and head v. If $A \subset V$ then $E(A)$ will be the set of the arcs of G with both endnodes in A and $G[A]$ will be the subgraph of G induced by the nodes which belong to A. We will denote by $G[B]$, where $B \subset E$, the subgraph of G induced by the endnodes of the arcs of B. Let $S \subset V$ such that $\emptyset \neq S \neq V$. We denote $\overline{S} = V \setminus S$. Let $\delta_G(S)$ be the set of arcs having exactly one end-node in S, $\delta^+_G(S)$ the set of arcs leaving S and $\delta^-_G(S)$ the set of arcs entering S; only when these sets are indexed by letter G, we simply denote them by $\delta(S)$, $\delta^+(S)$ and $\delta^-(S)$. Thus $\delta(S) = \delta^+(S) \cup \delta^-(S)$, $\delta(S) = \delta(\overline{S})$ and $\delta^+(S) = \delta^-(\overline{S})$. $\delta(S)$ is called a *cut* while $\delta^+(S)$ and $\delta^-(S)$ are called *dicuts*. A k-cut (resp. k-dicut) is a cut (resp. a dicut) with k arcs. A dicut $\delta^+(S)$ is *minimal* if there does not exist $S' \subset V$ such that $\delta^+(S') \subset \delta^+(S)$. When $S = \{v\}$, $v \in V$, we shall write $\delta(v)$, $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta(\{v\})$, $\delta^+(\{v\})$ and $\delta^-(\{v\})$. Furthermore, When $|S|$ is equal to 1 or $n-1$, the cut and dicuts induced by S will be called *trivial*. The *condensed digraph* of G, denoted by $G^\star = (V^\star, E^\star)$, is the digraph obtained from G by shrinking each strongly connected component of G to a single node (i.e., contracting all the arcs of that component) and removing the resulting loops. The *simplified digraph* of G, denoted by $G^s = (V^s, E^s)$, is the digraph obtained from G by removing all arcs but one from each set of parallel arcs. $G^{\star^s} = (V^{\star^s}, E^{\star^s})$ will be called the *simplified condensed digraph* of G. If x is a vector of R^E and B is a subset of edges of G then $x(B) = \sum_{e \in B} x(e)$. Finally, our digraphs will be connected and loopless and we will not distinguish sometimes an equivalent subgraph of G from neither its incidence vector nor its edge set.

2 Basic properties of the equivalent subgraph polyhedra

Let $G = (V, E)$ be a digraph. An integer vector $x \in R^E$ is the incidence vector of an equivalent subgraph of G if and only if x satisfies the following inequalities:

$$x(\delta^+(S)) \geq 1, \quad \forall\, S \subset V \ s.t.\ \delta^+(S) \neq \emptyset \tag{1}$$

$$0 \leq x(e) \leq 1, \quad \forall\, e \in E. \tag{2}$$

Inequalities (1) are called *dicut inequalities* while inequalities (2) are called *trivial inequalities* or *bound inequalities*. *Trivial facets* will be facets induced by trivial inequalities.

An arc $e \in E$ will be called *essential* if e belongs to any equivalent subgraph of G, i.e., $G \setminus \{e\}$ is not an equivalent subgraph of G. Trivially, an arc $e = (u, v) \in E$ is essential if and only if there does not exist a uv-dipath in $G \setminus \{e\}$. Equivalently, if e is essential then there exists $S \subset V$ such that $\delta^+(S) = \{e\}$. Hence one can determine all essential arcs of G in polynomial time. We denote by $\Psi(G)$ the set of essential arcs of G and by ψ its cardinality.

Theorem 1 *Let $G = (V, E)$ be a digraph.*

(i) The dimension of $ESP(G)$ is $m - \psi$.

(ii) Inequality $x(e) \geq 0$, $e \in E$, induces a facet of $DESP(G)$ if and only if $e \notin \Psi(G)$.

(iii) Inequality $x(e) \geq 0$, $e \in E$, induces a facet of $ESP(G)$ if and only if e belongs to neither $\Psi(G)$ nor a 2-dicut that does not intersect $\Psi(G)$.

(iv) Inequality $x(e) \leq 1$, $e \in E$, induces a facet of $ESP(G)$ if and only if $e \notin \Psi(G)$.

Proof.

(i) $ESP(G) \subset \{x \in R^E : x(e) = 1,\ e \in \Psi(G)\}$. Since the equations $x(e) = 1$, $e \in \Psi(G)$, are linearly independent, $dim(ESP(G)) \leq m - \psi$. Let $\mathcal{T} = \{G\} \cup \{T_e = G \setminus \{e\} : e \in (E \setminus \Psi(G))\}$ be a set of equivalent subgraphs of G. As, for every $e \in (E \setminus \Psi(G))$, T_e is the only equivalent subgraph of $\mathcal{T}$ that does not contain e, $\mathcal{T}$ is a subset of $m - \psi + 1$ affinely independent equivalent subgraphs of $ESP(G)$. Thus $dim(ESP(G)) = m - \psi$.

(ii) Let $\mathcal{F}_0$ be the face of $DESP(G)$ induced by the inequality $x(e) \geq 0$. We have $\mathcal{F}_0 = \emptyset$ if and only if $e \in \Psi(G)$. Assume that $e \notin \Psi(G)$. Let x^e be the incidence vector of the equivalent subgraph $G \setminus \{e\}$. For $f \in E \setminus \{e\}$, let x^f be the vector obtained from x^e by adding 1 to $x^e(f)$. The m vectors $\{x^g : g \in E\}$ of $DESP(G)$ are affinely independent and belong to $\mathcal{F}_0$, so $\mathcal{F}_0$ is a facet of $DESP(G)$.

(iii) The face $\mathcal{F}'_0$ of $ESP(G)$ induced by inequality $x(e) \geq 0$ is empty if and only if $e \in \Psi(G)$. Assume that $e \notin \Psi(G)$. At first, note that $dim(\mathcal{F}'_0) = dim(ESP(G \setminus \{e\})) = m - 1 - |\Psi(G \setminus \{e\})|$, so $\mathcal{F}'_0$ induces a facet if and only if $\Psi(G) = \Psi(G \setminus \{e\})$. One can show easily that e does not belong to a 2-dicut that does not intersect $\Psi(G)$ if and only if $\Psi(G) = \Psi(G \setminus \{e\})$.

(*iv*) Let $\mathcal{F}_1$ be the face of $ESP(G)$ defined by $x(e) \leq 1$. If $e \in \Psi(G)$ then $ESP(G) \subset \{x \in R^E : x(e) = 1\}$ and $\mathcal{F}_1 = ESP(G)$. Assume that $e \notin \Psi(G)$. As $G \setminus \{e\}$ is an equivalent subgraph of G that does not belong to $\mathcal{F}_1$, $\mathcal{F}_1$ is a proper face of $ESP(G)$. Let $\mathcal{T}$ be the set of equivalent subgraphs of G defined in the proof of assertion (i). We have then $\mathcal{T} \setminus \{T_e\} \subset \mathcal{F}_1$ and $dim(\mathcal{T} \setminus \{T_e\}) = m - \psi - 1$. Thus the inequality $x(e) \leq 1$ defines a facet of $ESP(G)$. □

Using blocking polyhedra theory, Chopra [3] proved the follow.

Theorem 2 [3] *Let $G = (V, E)$ be a digraph and let $S \subset V$ such that $1 \leq |S| \leq n-1$. The inequality $x(\delta^+(S)) \geq 1$ induces a facet of $DESP(G)$ if and only if $\delta^+(S)$ is minimal.*

In Theorem 3 we characterize facet inducing dicuts for $ESP(G)$. For that, we need the two following lemmas.

Lemma 1 *Let $G = (V, E)$ be a digraph and let $S \subset V$ such that $1 \leq |S| \leq n - 1$. The dicut $\delta^+(S)$ is minimal if and only if all arcs of $\delta^+(S)$ have their tails in a same strongly connected component of $G[S]$ and their heads in a same strongly connected component of $G[\overline{S}]$.*

Proof. Trivial. □

Lemma 2 *Let $G = (V, E)$ be a digraph and let $\delta^+(S)$ be a minimal dicut of G with $S \subset V$ and $1 \leq |S| \leq n-1$. For every arc $f = (u, v)$ of $E \setminus (\Psi(G) \cup \delta^+(S))$ there exists a uv-dipath of G that contains at most one arc of $\delta^+(S)$.*

Proof. Let $f = (u, v) \in E \setminus (\Psi(G) \cup \delta^+(S))$. Let $P_f = (u = w_1, w_2, ..., w_r = v)$ be a uv-dipath of $G \setminus \{f\}$ (such a dipath exists since $f \notin \Psi(G)$). Suppose that $|P_f \cap \delta^+(S)| \geq 2$. Let (w_i, w_{i+1}) and (w_j, w_{j+1}) be respectively the first arc and the last arc of $\delta^+(S)$ on P_f (when one goes from u to v along P_f). Denote by $P_{w_{i+1}w_{j+1}}$ the subpath of P_f linking w_{i+1} and w_{j+1}. As $\delta^+(S)$ is minimal, it follows by Lemma 1 that w_{i+1} and w_{j+1} belong to a strongly connected component of $G[\overline{S}]$. Let L be a $w_{i+1}w_{j+1}$-dipath of $G[\overline{S}]$. We have $L \cap \delta^+(S) = \emptyset$. Thus $(P_f \setminus P_{w_{i+1}w_{j+1}}) \cup L$ is a uv-dipath of G that intersects $\delta^+(S)$ at exactly one arc which is (w_i, w_{i+1}). □

Theorem 3 *Let $G = (V, E)$ be a digraph and let $S \subset V$ such that $1 \leq |S| \leq n - 1$. The inequality $x(\delta^+(S)) \geq 1$ defines a facet of $ESP(G)$ if and only if the dicut $\delta^+(S)$ is minimal and contains at least two arcs.*

Proof. If $\delta^+(S) = \{e\}$, with $e \in E$, then $e \in \Psi(G)$ and $ESP(G) \subset \{x \in R^E : x(e) = 1\}$.

Suppose that $\delta^+(S)$ is not minimal; i.e, there exists $S' \subset V$ such that $\delta^+(S') \subset \delta^+(S)$. Thus $x(\delta^+(S)) \geq x(\delta^+(S')) \geq 1$ and the inequality $x(\delta^+(S)) \geq 1$ does not induce a facet of $ESP(G)$.

Assume now that $\delta^+(S)$ is minimal and contains at least two arcs. Let $\mathcal{F}$ be the face of $ESP(G)$ defined by $x(\delta^+(S)) \geq 1$. Let $T_0 = G$. Since $\delta^+(S) \subset T_0$, T_0 is an equivalent subgraph of G that does not belong to $\mathcal{F}$. So $\mathcal{F}$ is a proper face of $ESP(G)$. We shall show that $dim(\mathcal{F}) = m - \psi - 1$.

Note that, by Lemma 1, $\forall\, g \in \delta^+(S)$, $(T_0 \setminus \delta^+(S)) \cup \{g\}$ is an equivalent subgraph of G. Note also that if $g = (u, v)$ is an arc of an equivalent subgraph T of G and P is a uv-dipath of G then $(T \setminus \{g\}) \cup P$ is an equivalent subgraph of G.

For every arc $f \in \delta^+(S)$, consider the equivalent subgraph $T_f = (T_0 \setminus \delta^+(S)) \cup \{f\}$.

Let e be an arc of $\delta^+(S)$. For every arc $f' = (u', v') \in E \setminus [\delta^+(S) \cup \Psi(G)]$, let $P_{f'}$ be a $u'v'$-dipath of $G \setminus \{f'\}$ such that $|P_{f'} \cap \delta^+(S)| \leq 1$ (such a dipath exists by Lemma 2) and consider the following equivalent subgraph

$$T_{f'} = \begin{cases} [T_0 \setminus (\delta^+(S) \cup \{f'\})] \cup [P_{f'} \cap \delta^+(S)] & , \ if\ |P_{f'} \cap \delta^+(S)| = 1 \\ [T_0 \setminus (\delta^+(S) \cup \{f'\})] \cup \{e\} & , \ if\ |P_{f'} \cap \delta^+(S)| = 0. \end{cases}$$

Let $\mathcal{T} = \{T_g : g \in (E \setminus \Psi(G))\}$. One can check easily that $\mathcal{T} \subset \mathcal{F}$ and that the $m - \psi$ elements of $\mathcal{T}$ are affinely independent. Hence $\mathcal{F}$ is a facet of $ESP(G)$. □

As for other combinatorial optimization problems, the equivalent subgraph problem has the property that a complete description of $ESP(G)$ can be easily derived from that of its dominant. Rais [14] showed that if feasible solutions for a binary polytope are closed under supersets, then the polytope is the intersection of its dominant with the unit cube. Clearly this property holds for equivalent subgraphs and Rais' result implies the following one

Theorem 4 *Let $G = (V, E)$ be a digraph. $ESP(G)$ is defined by the system obtained from that defining $DESP(G)$ by adding the inequalities $x(e) \leq 1$, $\forall e \in E$.*

By Theorem 4, we will restrict our attention, in the rest of the paper, to the characterization of $DESP(G)$. Furthermore, the following propositions show that the description of that polyhedron, for a given digraph, reduces to its description for strongly connected digraphs.

Proposition 1 *Let G_1, ..., G_l be the strongly connected component of a digraph G and let $G^\star = (V^\star, E^\star)$ be the condensed digraph of G. Then $DESP(G)$ is (minimally) defined by the union of the (minimal) inequality systems defining $DESP(G_1)$, ..., $DESP(G_l)$ and $DESP(G^\star)$.*

Proof. The arc sets E_1,..., E_l and $E^\star$ are disjoint. Thus, the polyhedron defined by the union of the systems has all its extreme points integer if and only each system defines an integer polyhedron.

Let $T \subseteq E$. Consider $T_1 = T \cap E_1$,..., $T_l = T \cap E_l$ and $T^\star = T \cap E^\star$. One can see easily that T induces an equivalent subgraph of G if and only if T_1,..., T_l and $T^\star$ induce equivalent subgraphs of respectively G_1,..., G_l and $G^\star$. □

Actually, the condensed digraph $G^\star$ of Proposition 1 is acyclic and the description of its corresponding polyhedron is given in Proposition 2 bellow.

Lemma 3 *Let $G = (V, E)$ be a digraph and let e_1 and e_2 be two parallel arcs of G. Let G' be the digraph obtained from G by removing arc e_2. Then $DESP(G)$ is described by the system obtained from that one defining $DESP(G')$ by substituting $x(e_1) + x(e_2)$ for $x(e_1)$ and adding the nonnegativity inequalities $x(e_1) \geq 0$ and $x(e_2) \geq 0$.*

Proof. Obvious. □

Proposition 2 *Let $G = (V, E)$ be an acyclic digraph and let*

$$\mathcal{S} = \{S \subset V : \delta^+_{G^s}(S) = \{e\},\ e \in \Psi(G^s)\}.$$

The system below is minimal and defines $DESP(G)$

$$\begin{cases} x(\delta^+(S)) \geq 1\ , & \forall\ S \in \mathcal{S} \\ \quad x(e) \geq 0\ , & \forall\ e \in E \setminus \Psi(G) \end{cases} \tag{3}$$

Proof. At first let us show that $DESP(G^s)$ is given by

$$\begin{cases} x(e) \geq 1\ , & \forall\ e \in \Psi(G^s) \\ x(e) \geq 0\ , & \forall\ e \in E \setminus \Psi(G^s) \end{cases} \tag{4}$$

Let $P(G^s)$ be the polyhedron defined by system (4). Trivially, $DESP(G^s) \subseteq P(G^s)$ and all extreme points of $P(G^s)$ are $\{0, 1\}$-vectors.

As G^s is acyclic and do not have parallel arcs, by Lemma 1, any minimal dicut of G^s consists of an essential arc of G^s. Consequently, by Theorem 2, every extreme point of $P(G^s)$ satisfies inequalities (1) and (2) and corresponds then to an equivalent subgraph of G^s. Thus, $P(G^s) = DESP(G^s)$.

Since, by Lemma 1, any minimal dicut of G consists of a set of parallel arcs one of which is essential in G^s, system (3) can be derived from system (4) by repeated applications of Lemma 3.

The minimality of system (3) is trivially implied by Theorem 1 and Theorem 2. □

Obviously, from an algorithmic point of view, the solution of the equivalent subgraph problem on G reduces to the computation of the minimum cost equivalent subgraphs of the strongly connected components G_1,..., G_l and the condensed digraph $G^\star$. For the condensed digraph $G^\star = (V^\star, E^\star)$, Moyles and Thompson [13] proved that $\Psi(G^{\star^s})$ is the minimum cardinality equivalent subgraph of $G^\star$. If we denote by $\widetilde{G^\star}$ the digraph obtained from $G^\star$ by removing all arcs but the minimum cost one from each set of parallel arcs then the minimum cost equivalent subgraph of $G^\star$ is induced by $\Psi(\widetilde{G^\star}) \cup \{e \in E^\star \setminus \Psi(\widetilde{G^\star}) : c(e) < 0\}$.

Observe finally that one can as well focus on 2-connected digraphs since it is easy to show that if G'_1,..., $G'_{l'}$ are the 2-connected components of a given digraph G then $DESP(G)$ is (minimally) defined by the union of the (minimal) inequality systems defining $DESP(G'_1)$,..., $DESP(G'_{l'})$.

3 ESP-trivial digraphs

For strongly connected directed series-parallel graphs, Chopra [3] proved that $DESP(G)$ is completely described by dicut and nonnegativity inequalities. He showed also that for the directed non-series-parallel graph, obtained from K_4 by substituting two anti-parallel arcs for each edge, the result does not hold. In this section and the next one, we give directed non-series-parallel graphs whose equivalent subgraph polyhedra are described by a polynomial number of dicut and nonnegativity inequalities.

Consider the following polyhedron associated with a (not necessarily strongly connected) digraph $G = (V, E)$

$$P(G) = \begin{cases} x(\delta^+(v)) \geq 1 \ , & \forall \ v \in V \ s.t. \ \delta^+(v) \neq \emptyset \\ x(\delta^-(v)) \geq 1 \ , & \forall \ v \in V \ s.t. \ \delta^-(v) \neq \emptyset \\ x(e) \geq 0 \ , & \forall \ e \in E \setminus \Psi(G) \end{cases} \tag{5}$$

We will say that G is *ESP-trivial* if $DESP(G) = P(G)$; i.e, if the equivalent subgraph polytope and its dominant are completely described by trivial inequalities and trivial dicut inequalities. Let us mention that ESP-trivial digraphs are analogous to elementary graphs introduced by Cornuéjols, Naddef and Pulleyblank [4] for the symmetric traveling salesman problem. In Theorem 5 below we characterize ESP-trivial digraphs, so we need the following lemma.

Lemma 4 *All extreme points of $P(G)$ are $\{0, 1\}$-vectors.*

Proof. Let $\widehat{G} = (\widehat{V}, \widehat{E})$ be the undirected bipartite graph, with $2|V|$ nodes and $|E|$ edges, obtained from G such that

$$v^+, \ v^- \in \widehat{V} \Longleftrightarrow v \in V$$

$$u^+ \text{ and } v^- \text{ are adjacent in } \widehat{G} \Longleftrightarrow (u, v) \in E.$$

$\{v^+ : v \in V\}$ and $\{v^- : v \in V\}$ are stable sets of $\widehat{G}$.

Since there is a bijection between E and $\widehat{E}$, we have $\forall \ v \in V$, the dicuts $\delta^+(v)$ and $\delta^-(v)$ of G correspond respectively to the cuts $\delta_{\widehat{G}}(v^+)$ and $\delta_{\widehat{G}}(v^-)$ of $\widehat{G}$.

Denote by M the node-edge incidence matrix associated with the graph $\widehat{G}$. For any $w \in \widehat{V}$, the corresponding row of M is the incidence vector of $\delta_{\widehat{G}}(w)$. Hence the polyhedron

$P(G)$ can be expressed as follows

$$P(G) = \begin{cases} Mx \geq 1 \\ x \geq 0 \end{cases} \tag{6}$$

As $\widehat{G}$ is bipartite, the matrix M is totally unimodular (see for instance [7]). Therefore $P(G)$ is a $\{0,1\}$-polyhedron. □

Theorem 5 *A digraph $G = (V, E)$ is ESP-trivial if and only if the three following properties are satisfied :*

(i) For any strongly connected component $G[W]$, $W \subseteq V$, and for any $S \subset W$ such that $2 \leq |S| \leq |W| - 2$ and $G[S]$ is strongly connected, $G[W \setminus S]$ has a node with no predecessor;

(ii) Any node w of a strongly connected component $G[W]$, with $W \subset V$ and $|W| \geq 2$, which is a successor (resp. predecessor) of a node of $V \setminus W$ is the unique successor (resp. predecessor) of a node of W in G.

(iii) For each essential arc e of the simplified condensed digraph $G^{\star^s}$ of G there exists a node v of $G^{\star^s}$, corresponding to a 1-node connected component of G, such that $\delta^+_{G^{\star s}}(v) = \{e\}$ or $\delta^-_{G^{\star s}}(v) = \{e\}$.

Proof. Assume that G is ESP-trivial. Suppose that condition (i) does not hold for a strongly connected component $G[W]$, with $W \subseteq V$, and a node subset $S \subset W$ such that $2 \leq |S| \leq |W| - 2$ and $G[S]$ is strongly connected. Let $T = E \setminus [\delta^+(S) \cap E(W)]$. The fact that the only arcs of G which are not in T are those going from S to $W \setminus S$, that $T[W]$ (=$G[W]$) is strongly connected and that each node of $W \setminus S$ has a predecessor in $T[W \setminus S]$ (=$G[W \setminus S]$) guarantees that any node of V has a predecessor and a successor in T. The incidence vector of T belongs to $P(G)$. However T is not an equivalent subgraph of G (since $T \cap G[W]$ is not strongly connected) and hence $DESP(G) \neq P(G)$, a contradiction.

Consider a strongly connected component $G[W]$ of G, with $W \subset V$ and $|W| \geq 2$, and a node $w \in W$ which is a successor of a node of $V \setminus W$ in G. By Proposition 1, all facet inducing inequalities for $DESP(G[W])$ define facets of $DESP(G)$. It follows that $\delta^-_{G[W]}(w) = \delta^-(w) \cap E(W)$ does not induce a facet of $DESP(G[W])$. Hence, by Theorem 2, $\delta^-_{G[W]}(w)$ is not minimal and contains a minimal dicuts of $G[W]$ which must be trivial since G is ESP-trivial. So there exists a node v of $W \setminus \{w\}$ such that $\delta^+_{G[W]}(v) \subseteq \delta^-_{G[W]}(w)$. Again, by Proposition 1, $\delta^+_{G[W]}(v)$ must be a minimal dicut of G and then w is the unique successor of v in G. If w is a predecessor of a node of $V \setminus W$ in G then, using similar arguments, one can show that condition (ii) is satisfied too.

Suppose that $G^{\star^s}$ has an essential arc f that does not satisfy condition (iii). Let F be the edge set containing f and all arcs of $G^\star$ that are parallel to f. Trivially, F is a minimal dicut of both $G^\star$ and G. As f does not satisfy condition (iii), F is not a trivial dicut of G.

However, by Theorem 2, inequality $x(F) \geq 1$ defines a facet of $DESP(G)$ but it does not appear in system (5), a contradiction.

Assume now that conditions (i), (ii) and (iii) hold and there exists an extreme point $\widetilde{x}$ of $P(G)$ which does not belong to $DESP(G)$. By Lemma 4, $\widetilde{x}$ is $\{0,1\}$-vector. Let $\widetilde{T}$ be the subgraph of G induced by $\{e \in E \ : \ \widetilde{x}(e) = 1\}$. $\widetilde{T}$ is not an equivalent subgraph of G.

At first, note that each minimal dicut of $G^\star$ consists of a set of parallel arcs and corresponds to an essential arc of $G^{\star^s}$. Furthermore, since condition (iii) holds, all minimal dicuts of $G^\star$ are trivial. Therefore, by Proposition 1 and Proposition 2, the edge set $\{e \in E^\star \ : \ \widetilde{x}(e) = 1\}$ induces an equivalent subgraph of $G^\star$. Thus, as $\widetilde{T}$ is not an equivalent subgraph of G, there exists a strongly connected component $G[W]$ of G, $W \subseteq V$, such that the subgraph $\widetilde{T}_W$ of $G[W]$ induced by the edge set $\{e \in E(W) \ : \ \widetilde{x}(e) = 1\}$ is not strongly connected. Let $\widetilde{T}_W[S^i]$ and $\widetilde{T}_W[S^t]$, with $S^i \subset W$ and $S^t \subset W$, be respectively an initial and a terminal strongly connected components of $\widetilde{T}_W$; i.e., no node of S^t (resp. of S^i) is a predecessor (resp. a successor) of a node of $W \setminus S^t$ (resp. $W \setminus S^i$) in $\widetilde{T}_W$.

Note that any node of W has a predecessor and a successor in $\widetilde{T}_W$. Indeed, suppose that a node $w \in W$ has no predecessor (resp. no successor) in $\widetilde{T}_W$. As $\widetilde{x}(\delta^-(w)) \geq 1$ (resp. $\widetilde{x}(\delta^+(w)) \geq 1$), w is a successor (resp. a predecessor) of a node of $V \setminus W$ in $\widetilde{T}$. In this case, condition (ii) implies that w is the unique successor (resp. a predecessor) of a node v of W in G. Hence $\widetilde{x}(\delta^+(v)) < 1$ (resp. $\widetilde{x}(\delta^-(v)) < 1$), a contradiction.

Particularly, any node of S^t (res. S^i) has a successor (resp. a predecessor) in $\widetilde{T}_W$ that must belong to S^t (res. S^i). Consequently, S^i and S^t are both of them of size at least two. Hence $2 \leq |S^t| \leq |W| - 2$ and, as S^t is a terminal component, any node of $W \setminus S^t$ has a predecessor in $\widetilde{T}_W[W \setminus S^t]$ and, trivially, in $G[W \setminus S^t]$ which contradicts condition (i). □

Corollary 1 *A strongly connected digraph G is ESP-trivial if and only if condition (i) of Theorem 5 holds.*

Proof. When the digraph is strongly connected, conditions (ii) and (iii) of Theorem 5 are redundant. □

We denote by $\mathcal{E}$ the class of ESP-trivial digraphs. A first example of ESP-trivial digraphs is that one obtained from a given digraph by subdividing each arc into two arcs in series; one can check easily that the new digraph, with all its arcs being essential, satisfies the three conditions of Theorem 5. In the remainder of this section, we will show that $\mathcal{E}$ is a large class that contains many other digraph classes.

Corollary 2 *If $G = (V, E)$ is a strongly connected ESP-trivial digraph and $G' = (V, E')$ is a strongly connected subgraph of G then G' is ESP-trivial.*

Proof. Since G' is strongly connected, no arc of $E \setminus E'$ is essential for G and by Theorem 1, $\forall e \in E$, inequality $x(e) \geq 0$ defines a facet of $DESP(G)$. Hence $DESP(G')$ is a face of $DESP(G)$, i.e., $DESP(G') = DESP(G) \cap \{x \in R^E : x(e) = 0,\ \forall e \in E \setminus E'\}$. □

Corollary 3 *Any strongly connected digraph that does not contain two node disjoint circuits is ESP-trivial.*

Proof. If a strongly connected digraph $G = (V, E)$ is not ESP-trivial then, by Corollary 1, there exists a strongly connected subgraph $G[S]$ of G such that $S \subset V$, $2 \leq |S| \leq |V| - 2$ and any node of $V \setminus S$ has a predecessor in $G[V \setminus S]$. Hence both $G[S]$ and $G[V \setminus S]$ contain circuits. □

Proposition 3 *Any 2-connected graph can be oriented such that the resulting digraph is strongly connected and ESP-trivial.*

Proof. Let $G = (V, E)$ be a 2-connected graph. We shall show that one can orient G and get a strongly connected digraph without node disjoint circuits. Let $C \cup P_1 \cup ... \cup P_t$ be an ear-decomposition of G where C is a cycle and $P_1, ..., P_t$ are paths (the ears of the decomposition). Since G is 2-connected, the endnodes of every path P_i are distinct. Assume (w.l.o.g.) that C has an edge, with ends u and v, which does not belong to any path P_i, $i = 1, ..., t$. Let $G^d = (V, E^d)$ be the digraph obtained by orienting the edges of G in the following way :

1- Orient the edges of C such that $G^d[C]$ is a circuit and $e = (u, v) \in E^d$;

2- For $i = 1, ..., t$, let u_i and v_i be the endnodes of the path P_i. Assume that there exists no dipath from v_i to u_i in the acyclic digraph $G^d[(C \setminus \{e\}) \cup P_1 \cup ... \cup P_{i-1}]$. Orient the edges of P_i such that we get a new dipath from u_i to v_i.

One can check easily that G^d is strongly connected and does not contain two node disjoint circuits since $G^d \setminus \{e\}$ is acyclic. Hence, by Corollary 3, G^d is a strongly connected ESP-trivial digraph. □

Note that the orientation used in the proof of the above proposition is not unique. Corollary 4, lists some graphs for which any strongly connected orientation leads to an ESP-trivial digraph.

A wheel W_k, $k \geq 3$, is a graph that consists of a cycle of size k and a center node w_0 adjacent to all the nodes of the cycle.

Corollary 4 *Digraphs obtained by considering any strongly connected orientation of graphs with five or fewer nodes, wheels and complete bipartite graphs $K_{k,l}$, with $k = 2, 3$ and $l \geq 2$, are ESP-trivial.*

Proof. The corollary is a trivial consequence of Corollary 3. □

Observe however that some graphs can not be oriented so that the resulting digraphs are acyclic and ESP-trivial. For instance, consider the simple complete graph on four nodes K_4 and let K_4^d be the digraph relative to an arbitrary acyclic orientation of K_4. Let v_1, v_2, v_3, v_4 be the topological order of the nodes of K_4^d. The arc (v_2, v_3) is essential in K_4^d while $\delta^+_{K_4^d}(v_2) = \{(v_2, v_3), (v_2, v_4)\}$ and $\delta^-_{K_4^d}(v_3) = \{(v_1, v_3), (v_2, v_3)\}$; which means that $\{(v_2, v_3)\}$ is a non trivial minimal dicut of K_4^d and then K_4^d is not ESP-trivial.

Let us now list some operations that extend the class $\mathcal{E}$. Given a digraph $G = (V, E)$ and let e and v be respectively an arc and a node of G. Using Theorem 5 and Lemma 3, it is easy to see that the following operations leave any ESP-trivial digraph in $\mathcal{E}$:

- Arc duplication : consists of adding a new arc parallel to e.
- Arc subdivision : consists of subdividing e into two arcs in series.
- Node splitting : consists of replacing v by two nodes v' and v'' such that v'' is the unique successor of v', all predecessors of v in G become predecessors of v' and all successors of v in G become successors of v'' (the other nodes and arcs are kept unchanged).

For ESP-trivial digraphs, the equivalent subgraph problem can be solved as a polynomial size linear program. Furthermore, it can be reduced to the maximum cost 1-capacitated b-matching problem in bipartite graphs. Indeed, consider the bipartite graph $\widehat{G}$ defined in Lemma 4. Hence an equivalent subgraph of G corresponds to an edge covering of $\widehat{G}$. The complement of an edge covering of $\widehat{G}$ is a 1-capacitated b-matching of $\widehat{G}$, where $b(v) = d_{\widehat{G}}(v) - 1$ for each node v of $\widehat{G}$ and $d_{\widehat{G}}(v)$ is the degree of v in $\widehat{G}$. Thus the minimum cost equivalent subgraph of G corresponds to the complement of a maximum cost 1-capacitated b-matching of $\widehat{G}$ (the edge costs remain unchanged).

4 Digraphs with 3-cuts

Given a connected digraph $G = (V, E)$. Let $S \subset V$ such that $2 \leq |S| \leq n - 2$ and assume that the cut $\delta(S)$ is minimal. Consider the digraph $G_1 = (V_1, E_1)$ (resp. $G_2 = (V_2, E_2)$) obtained from G by shrinking S (resp. $\overline{S} = V \setminus S$) to a single node and removing the resulting loops. Thus $E_1 \cap E_2 = \delta(S)$ and $E_1 \cup E_2 = E$.

We will say that the digraph G decomposes into G_1 and G_2 by the operation Φ and we denote it by $G = G_1 \Phi_{\delta(S)} G_2$. In this section, we aim at deriving the description of $DESP(G)$ from those of $DESP(G_1)$ and $DESP(G_2)$.

When $\delta(S)$ is a dicut then digraphs G, G_1 and G_2 are not strongly connected and, using Propositions 1 and 2, we can deduce easily that $DESP(G)$ is described by the union of the inequality systems defining $DESP(G_1)$ and $DESP(G_2)$. Thus we will restrict ourselves to strongly connected digraphs.

Observe also that if $|\delta^+(S)| = |\delta^-(S)| = 1$ then the two arcs of $\delta(S)$ are essential in G, G_1 and G_2 and obviously, in this case, the union of the inequality systems defining $DESP(G_1)$ and $DESP(G_2)$ is sufficient to describe $DESP(G)$.

In what follows, we will specialize operation Φ to 3-cuts and we will derive a description of $DESP(G)$ for strongly connected digraphs for which if we apply recursively operation Φ we are left at the end with a set of ESP-trivial digraphs. These digraphs constitute a large class containing the directed Halin graphs. This decomposition scheme has been studied by Cornuéjols, Naddef and Pulleyblank [4] for the symmetric traveling salesman problem and by Barahona and Mahjoub [1] for the 2-connected and the 2-edge-connected subgraph problems. The following results give, in a similar fashion, the polyhedral consequences of operation Φ.

In Theorems 6 and 7 and Corollaries 5 and 6, $G = (V, E)$ will be a strongly connected digraph such that $G = G_1\Phi_{\{e,f,g\}}G_2$; where $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $S \subset V$, $2 \leq |S| \leq n-2$, $\delta^+(S) = \{e, f\}$ and $\delta^-(S) = \{g\}$.

Theorem 6 *The union of the two systems of inequalities defining $DESP(G_1)$ and $DESP(G_2)$ is sufficient to define $DESP(G)$.*

Proof. Let P be the polyhedron defined by the union of the two inequality systems defining $DESP(G_1)$ and $DESP(G_2)$. We shall prove that $DESP(G) = P$. Trivially, if T is an equivalent subgraph of $G_1\Phi_{\delta(S)}G_2$ then the restrictions of T to E_1 and E_2 are equivalent subgraphs of G_1 and G_2. Thus $DESP(G) \subseteq P$.

Suppose that P has an extreme point $\overline{x}$ that does not correspond to an equivalent subgraph of G. Let $\overline{x}^1$ and $\overline{x}^2$ be the projections of $\overline{x}$ on R^{E_1} and R^{E_2} respectively. For $k = 1, 2$, $\overline{x}^k \in DESP(G_k)$, so there exist t_k extreme points $\{x_i^k\}$ of $DESP(G_k)$, l_k unit vectors $\{\xi_j^k\}$ of $\{0,1\}^{E_k}$, t_k nonnegative scalars $\{\beta_i^k\}$ and l_k nonnegative scalars $\{\gamma_j^k\}$ such that

$$\overline{x}^k = \sum_{i=1}^{t_k} \beta_i^k x_i^k + \sum_{j=1}^{l_k} \gamma_j^k \xi_j^k \qquad \text{and} \qquad \sum_{i=1}^{t_k} \beta_i^k = 1.$$

Denote by $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ the smallest faces of P, $DESP(G_1)$ and $DESP(G_2)$ containing respectively $\overline{x}$, $\overline{x}^1$ and $\overline{x}^2$. Since $\overline{x}$ is an extreme point, $\mathcal{F} = \{\overline{x}\}$. As well, $\{x_i^k : \beta_i^k > 0,\ i = 1, ..., t_k\}$ are extreme points of $\mathcal{F}_k$, $k = 1, 2$.

First of all, as $g \in \Psi(G) \cap \Psi(G_1) \cap \Psi(G_2)$ and by extremality, $\overline{x}(g) = 1$ and $x_i^k(g) = 1$ for $i = 1, ..., t_k$ and $k = 1, 2$.

Note that, for $k = 1, 2$, $i = 1, ..., t_k$ and $j = 1, ..., l_k$,

$$(\overline{x}_i^k(e), \overline{x}_i^k(f)) \in \{(0,1), (1,0), (1,1)\}$$

$$(\xi_j^k(e), \xi_j^k(f)) \in \{(0,0), (0,1), (1,0)\}.$$

For $k = 1, 2$ and $r, s \in \{0, 1\}$, let

$$I^k_{rs} = \{i : 1 \le i \le t_k,\ \beta^k_i > 0,\ x^k_i(e) = r,\ x^k_i(f) = s\}$$

$$J^k_{rs} = \{j : 1 \le j \le l_k,\ \gamma^k_j > 0,\ \xi^k_j(e) = r,\ \xi^k_j(f) = s\}$$

$$\alpha^k_{rs} = \sum_{i \in I^k_{rs}} \beta^k_i \quad , \quad \lambda^k_{rs} = \sum_{j \in J^k_{rs}} \gamma^k_i .$$

Thus, for $k = 1, 2$, we have the following system

$$\left\{ \begin{array}{lllllllll} \overline{x}(e) & = & \overline{x}^k(e) & = & & \alpha^k_{101} & + \ \alpha^k_{111} & + \ \lambda^k_{100} & \\ \overline{x}(f) & = & \overline{x}^k(f) & = & \alpha^k_{011} & & + \ \alpha^k_{111} & & + \ \lambda^k_{010} \\ & & 1 & = & \alpha^k_{011} & + \ \alpha^k_{101} & + \ \alpha^k_{111} & & \end{array} \right. \tag{7}$$

Observe that the following equations hold,

$$\alpha^1_{rs}.\alpha^2_{rs} = 0, \quad \forall (r, s) \in \{(0, 1), (1, 0), (1, 1)\}. \tag{8}$$

Indeed, let $i_1 \in I^1_{rs}$ and $i_2 \in I^2_{rs}$. Consider the vector $\widetilde{x}$ obtained by matching $x^1_{i_1}$ and $x^2_{i_2}$. Hence $\widetilde{x}$ is a vector of $\mathcal{F}$ that corresponds to an equivalent subgraph of G. But this is impossible since $\mathcal{F} = \{\overline{x}\}$.

On the other hand, α^1_{11} must be zero. In fact, suppose that I^1_{11} is not empty and let $i \in I^1_{11}$. Thus $\mathcal{F}_2$ must contain no equivalent subgraph that possess both arcs e and f, since otherwise one can match it with x^1_i to get an equivalent subgraph of G which belongs to $\mathcal{F}$ and then contradict the fact that $\mathcal{F} = \{\overline{x}\}$. Therefore we have $\alpha^2_{11} = 0$ and, as the set of equivalent subgraph of a given digraph is closed under supersets, $\alpha^2_{10}.\lambda^2_{01} = 0$ and $\alpha^2_{01}.\lambda^2_{10} = 0$. Moreover, because $\alpha^2_{10} + \alpha^2_{01} = 1$, one can see readily that α^2_{10} and α^2_{01} have to be positive (then $\lambda^2_{10} = \lambda^2_{01} = 0$). According to equations (8), $\alpha^1_{10} = \alpha^1_{01} = 0$. Hence $\alpha^1_{11} = 1$, $\alpha^2_{10} > 1$ and $\alpha^2_{01} > 1$, a contradiction.

With similar arguments, we prove that $\alpha^2_{11} = 0$.

So, by equations (8), one of the following cases arises; either $\alpha^1_{10} = \alpha^2_{01} = 1$, $\lambda^1_{01} > 0$ and $\lambda^2_{10} > 0$ or $\alpha^1_{01} = \alpha^2_{10} = 1$, $\lambda^1_{10} > 0$ and $\lambda^2_{01} > 0$. But again, in both cases, this means that each one of $\mathcal{F}_1$ and $\mathcal{F}_2$ contains an equivalent subgraph with both arcs e and f. Thus $\mathcal{F}$ contains an equivalent subgraph of G, a contradiction. □

Corollary 5 *The union of the two systems of inequalities defining $ESP(G_1)$ and $ESP(G_2)$ is sufficient to define $ESP(G)$.*

Proof. By Theorem 4 and Theorem 6. □

Corollary 6 *The inequality $x(e) + x(f) \geq 1$ defines a facet of $ESP(G)$ (resp. of $DESP(G)$) if and only if it does so for $ESP(G_1)$ and $ESP(G_2)$ (resp. for $DESP(G_1)$ and $DESP(G_2)$).*

Proof. Observe that he dicut $\{e, f\}$ is minimal in G if and only if it is minimal in both digraphs G_1 and G_2. Hence the corollary follows immediately from Theorems 2 and 3. □

Theorem 7 *Assume that* $\{e, f\} \cap [\Psi(G_1) \cup \Psi(G_2)] = \emptyset$. *Let*

$$ax \geq \alpha \tag{9}$$

be an inequality such that $a(h) = 0,\ \forall\ h \in E_2 \setminus E_1$.

Inequality (9) defines a nontrivial facet of $ESP(G)$ *(resp.* $DESP(G)$*) if and only if it does so for* $ESP(G_1)$ *(resp.* $DESP(G_1)$*).*

Proof. Obviously, inequality (9) is valid for $ESP(G)$ if and only if it is valid for $ESP(G_1)$.

Since $\{e, f\} \cap [\Psi(G_1) \cup \Psi(G_2)] = \emptyset$, the dicut $\{e, f\}$ is minimal in G_1, G_2 and G. Thus, by Theorem 3, the inequality $x(e)+x(f) \geq 1$ is facet defining for $ESP(G_1)$, $ESP(G_2)$ and $ESP(G)$. Therefore, we will assume that inequality (9) induces a nontrivial facet different from that defined by $x(e) + x(f) \geq 1$.

Suppose that inequality (9) defines a facet $\mathcal{F}_1$ of $ESP(G_1)$. Let $\mathcal{F}$ be the face of $ESP(G)$ induced by inequality (9). For $i = 1, 2$, consider the following sets of extreme points of $ESP(G_i)$:

$$\begin{aligned} C_1^i &= \{x \ : \ x(e) = 0 \text{ and } x(f) = 1\} \\ C_2^i &= \{x \ : \ x(e) = 1 \text{ and } x(f) = 0\} \\ C_3^i &= \{x \ : \ x(e) = 1 \text{ and } x(f) = 1\} \end{aligned}$$

Note that $\mathcal{F}_1 \cap C_j^1 \neq \emptyset$ for $j = 1, 2, 3$; otherwise, as e and f are not essential in G_1, $\mathcal{F}_1$ would be contained in one of the facets of $ESP(G_1)$ induced by $x(e) \leq 1$, $x(f) \leq 1$ or $x(e) + x(f) \geq 1$. As well, since $\{e, f\} \cap \Psi(G_2) = \emptyset$, $ESP(G_2) \cap C_j^2 \neq \emptyset$ for $j = 1, 2, 3$.

Let $D_1 = \{x^1, ..., x^{t_1}\}$ and $D_2 = \{y^1, ..., y^{t_2}\}$ be the sets of extreme points of respectively $\mathcal{F}_1$ and $ESP(G_2)$. As $\mathcal{F}_1 \cap C_j^1 \neq \emptyset$ and $ESP(G_2) \cap C_j^2 \neq \emptyset$, $j = 1, 2, 3$, one can match each vector of D_1 with an appropriate vector of D_2 and vice versa (vectors of C_j^1 are matched with vectors of $C_{j'}^2$ if and only if $j = j'$) and get vectors $z^1, ..., z^{t_1+t_2}$ which belong to $\mathcal{F}$.

Consider the vector $\overline{z} = \frac{1}{t_1+t_2} \sum_{l=1}^{t_1+t_2} z^l$ and let $\overline{x}$ and $\overline{y}$ its projections on R^{E_1} and R^{E_2} respectively. $\overline{x}$ is a convex combination of all extreme points of $\mathcal{F}_1$ and, hence, it satisfies (9) as equality and all the other inequalities, of the system defining $ESP(G_1)$, as strict inequalities. On the other hand, the vector $\overline{y}$, which is a convex combination of all extreme points of $ESP(G_2)$, satisfies all the inequalities defining $ESP(G_2)$ as strict inequalities. Consequently, vector $\overline{z}$ satisfies (9) as equality and all the other inequalities, of the union of the two systems defining $ESP(G_1)$ and $ESP(G_2)$, as strict inequalities. This implies that inequality (9) defines a facet of $ESP(G)$.

Suppose now that inequality (9) defines a facet $\mathcal{F}$ of $ESP(G)$. Let $D = \{z^1, ..., z^t\}$ be a set of linearly (and affinely) independent vectors of $\mathcal{F}$, where $t = |E \setminus \Psi(G)|$. Let $D_1 = \{z_1^1, ..., z_1^t\}$ be the set of restrictions of the vectors of D to E_1. Note that vectors of D_1 belong to $ESP(G_1)$ and satisfy (9) as equality. As $\Psi(G_1) = \Psi(G) \cap E_1$, one can check easily that D_1 contains $|E_1 \setminus \Psi(G_1)|$ linearly independent vectors. Hence, inequality (9) induces a facet of $ESP(G_1)$.

With similar arguments one can prove that the result holds for the dominant. □

These results enable us to obtain a minimal complete linear description of $ESP(G)$ and $DESP(G)$ for reducible digraphs by the operation Φ (restricted at 3-cuts).

Let $\mathcal{E}'$ be the class of strongly connected ESP-trivial digraphs. Denote by $\mathcal{G}(\mathcal{E}')$ the class of strongly connected digraphs which can be decomposed by repeated applications of the operation Φ such that the irreducible digraphs are ESP-trivial. Thus $\mathcal{E}'$ is a subclass of $\mathcal{G}(\mathcal{E}')$.

Theorem 8 *Given a strongly connected digraph $G = (V, E)$, let*

$$\mathcal{S}_3 = \{S \subset V : 2 \leq |S| \leq n-2,\ |\delta(S)| = 3\}.$$

If $G \in \mathcal{G}(\mathcal{E}')$ then

$$DESP(G) = \begin{cases} x(\delta^+(v)) \geq 1 & \forall\ v \in V \\ x(\delta^-(v)) \geq 1 & \\ x(\delta^+(S)) \geq 1 & \forall\ S \in \mathcal{S}_3 \\ x(\delta^-(S)) \geq 1 & \\ x(e) \geq 0 & \forall\ e \in E \setminus \Psi(G). \end{cases} \tag{10}$$

Proof. The theorem holds if $G \in \mathcal{E}'$. If $G \in \mathcal{G}(\mathcal{E}') \setminus \mathcal{E}'$ then there exist a nontrivial 3-cut $\{e, f, g\}$ of G such that $G = G_1 \Phi_{\{e,f,g\}} G_2$ and G_1 and G_2 are digraphs of $\mathcal{G}(\mathcal{E}')$. Therefore, by induction, Theorem 8 follows from Theorem 6. □

For this class of digraphs, the only minimal dicuts are trivial dicuts and those induced by the node subsets of $\mathcal{S}_3$ (defined in Theorem 8). Let us consider now the dicut polyhedron. Let $DCP(G)$ denote the dicut polytope associated with a digraph $G = (V, E)$; i.e., the convex hull of the incidence vectors of all dicuts of G, where the incidence vector of a dicut is a $\{0,1\}$-vector $y \in R^E$ such that $y(e) = 1$ if and only if e belongs to the dicut. We denote by $DDCP(G)$ the dominant of $DCP(G)$. The extreme points of $DDCP(G)$ are in bijection with the minimal dicuts of G. Using blocking polyhedra theory (see Chopra [3]), the following corollary is a direct consequence of Theorem 8.

Corollary 7 *Given a strongly connected digraph $G = (V, E)$. Let $\mathcal{T}$ be the set of minimal equivalent subgraphs of G. If $G \in \mathcal{G}(\mathcal{E}')$ then*

$$DDCP(G) = \begin{cases} y(T) \geq 1 & \forall\ T \in \mathcal{T} \\ y(e) \geq 0 & \forall\ e \in E. \end{cases} \tag{11}$$

A well known subclass of $\mathcal{G}(\mathcal{E}')$ is that one of strongly connected directed Halin graphs. A *Halin graph* $H = (V, A \cup C)$ is a planar graph which consists of a tree A having no node of degree two and a cycle C whose nodes are the leaves of A. These graphs were introduced by Halin [8] as an example of planar minimally 3-connected graphs. They can be recognized in polynomial time. Note that wheels are Halin graphs for which the tree A is a star.

Let $\mathcal{H}$ be the class of strongly connected directed Halin graphs. Let $H = (V, A \cup C)$ be a digraph of $\mathcal{H}$. Any arc of A belongs to a unique 3-cut containing two arcs of C. By considering this type of 3-cuts, one can decompose H by repeated applications of the operation Φ such that the irreducible digraphs are directed wheels. Thus, $\mathcal{H} = \mathcal{G}(\mathcal{W}) \subset \mathcal{G}(\mathcal{E}')$, where $\mathcal{W}$ denotes the class of strongly connected wheels which are, by Corollary 4, ESP-trivial. Hence, by Theorem 8 and Corollary 7, we get the following corollary.

Corollary 8 *If H is a strongly connected directed Halin graph then $DESP(H)$ and $DDCP(H)$ are given respectively by system (10) and system (11).*

By considering the above 3-cut decomposition, Cornuéjols, Naddef and Pulleyblank [4] gave a description of the symmetric traveling salesman polytope for a large class of graphs (similar to $\mathcal{G}$) that contains Halin graphs. Using the same decomposition technique, Barahona and Mahjoub [1] described the 2-connected and the 2-edge-connected subgraph polytopes for Halin graphs and Coullard, Rais, Rardin and Wagner [5] proposed a linear-time algorithm for the 2-connected Steiner subgraph problem on these graphs.

Finally observe that System (10) has a polynomial number of inequalities, so the equivalent subgraph problem can be solved in polynomial time on $\mathcal{G}(\mathcal{E}')$ by linear programming. Furthermore, using a classical approach, the equivalent subgraph problem on digraphs of $\mathcal{G}(\mathcal{E}')$ can be reduced to a linear number of equivalent subgraph problems on ESP-trivial digraphs. Indeed, let $G = (V, E)$ be a digraph of $\mathcal{G}(\mathcal{E}')$ such that $G = G_1 \Phi_{\{e,f,g\}} G_2$, where $\{e, f\}$ and $\{g\}$ are dicuts of G, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let c be a cost vector associated with E. Assume (w.l.o.g.) that G_2 is ESP-trivial. For $(i, j) \in \{(0, 1), (1, 0), (1, 1)\}$, let

$$\Gamma_{ij} = Min\{cx : x \in ESP(G_2),\ x(e) = i,\ x(f) = j\} - [ic(e) + jc(f) + c(g)]$$

if $\{x \in ESP(G_2),\ x(e) = i,\ x(f) = j\}$ is not empty, or an arbitrary big value M otherwise.

The following system

$$\begin{cases} \alpha & & + & \lambda & = & \Gamma_{10} \\ & \beta & + & \lambda & = & \Gamma_{01} \\ \alpha & + \beta & + & \lambda & = & \Gamma_{11} \end{cases}$$

has the unique solution

$$\begin{aligned} \alpha &= \Gamma_{11} - \Gamma_{01} \\ \beta &= \Gamma_{11} - \Gamma_{10} \\ \lambda &= \Gamma_{10} + \Gamma_{01} - \Gamma_{11} \end{aligned}$$

Let c' be a vector of arc costs for G_1 defined by

$$c'(h) = \begin{cases} c(e) + \alpha & if \quad h = e \\ c(f) + \beta & if \quad h = f \\ c(g) + \lambda & if \quad h = g \\ c(h) & if \quad h \in E_1 \setminus \{e, f, g\} \end{cases}$$

Hence, the cost of an optimal solution of the equivalent subgraph problem on G_1 with respect to c' is equal to the cost of an optimal solution of the equivalent subgraph problem on G with respect to c. Also, any optimal equivalent subgraph of G_1 with respect to c' can be extended to an optimal equivalent subgraph of G by considering an appropriate optimal equivalent subgraph of G_2.

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