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Abstract

This paper deals with the class of uncertain continuous-time linear stochastic hybrid systems with Wiener process. The uncertainties we are considering are of norm bounded type. The robust $\mathscr{H}_\infty$ stochastic stabilization problem is treated. LMIs based conditions are developed to design the state feedback controller that robust stochastically stabilizes the studied class of systems and at the same time rejects the disturbance rejection of a desired level. The minimum disturbance rejection is also determined. A numerical example is provided to show the validness of the proposed results.

Key Words: Stochastic systems, Markovien jumping parameters systems, stabilization, robustness, LMI.

Résumé

Cet article traite de la classe des systèmes incertains à sauts markovien et processus de Wiener. Les incertitudes considérées dans ce travail sont du type borné en norme. La commande $\mathscr{H}_\infty$ est proposée pour stochastiquement stabiliser cette classe de systèmes et en même temps garantir le rejet d'une pertubation externe à énergie finie. Un exemple numérique est donné pour montrer la validté des résultats établis.

1 Introduction

In practice we find some systems that have abrupt changes in their dynamics that results from causes like connections or disconnections of some components, failures in the components etc. Analysis and design of these systems can't be done using the linear invariant system theory since it is unable to model adequately such systems. These systems have stochastic behavior and examples of such systems can be found in manufacturing systems, power systems, telecommunications systems, etc. The occurrence of the abrupt changes is random in more cases. These practical systems have been modelled by the class of linear systems with Markovian jumps that we will term in this paper as stochastic hybrid systems. This class of systems has two components in the state vector. The first component of this state vector takes values in $\mathbb{R}^n$ and evolves continuously in time and it represents the classical state vector that is usually used in the modern control theory. The second one takes values in a finite set and switches in a random manner between a finite number of states (see Mariton (Ref. 1), Boukas and Liu (Ref. 2) and Boukas (Ref. 3) and the references therein. This component is represented by a continuous-time Markov process. Usually the state vector of the class of stochastic hybrid systems is denoted by $(x(t), r_t)$. We have also to notice that used model will never represent adequately the physical system for many know reasons. Therefore uncertainties have to be include to the model, to correct the analysis and the design phases. In this paper, we will assume that the uncertainties of the system are of norm bounded type.

This class of systems has attracted a lot researchers and many problems have been tackled and solved. Among these problems, we quote those of stability, stabilizability, $\mathcal{H}_\infty$ control problem and filtering problem. For more details on what it has been done on this class of systems, we refer the reader to the recent books by Boukas and Liu (Ref. 2) and Boukas (Ref. 3) and the references therein. These two books present a good review of the literature of the subject up to 2004.

The stabilization problem has attracted many researchers from the control community and many results have been reported in the literature. For more details on this subject, we refer the reader to (Refs. 2–11). To the best of our knowledge the case of continuous-time systems with Markovian jumps and multiplicative noise has never studied and our objective in this paper is to study the $\mathscr{H}_\infty$ stabilization of such class of systems.

Our goal in this paper consists of designing a state feedback controller that robust stochastically stabilizes the class of systems we are studying and at the same time reject the disturbance with a desired level $\gamma > 0$. We are also interested by determining the minimum level of the disturbance rejection. In this paper, we will solve these two problems and develop LMI conditions that we can use to determine the state feedback controller that stochastically stabilizes the class of systems of stochastic hybrid systems with multiplicative noise and guarantees the minimum disturbance rejection.

The rest of the paper is organized as follows. In section 2, the problem we are considering is stated and some useful definitions are given. Section 3 gives the main results of the paper. In section 4, some numerical examples are provided to show the usefulness of the proposed results.

2 Problem Statement

Let us consider a dynamical system defined in a fundamental probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and assume that its dynamics is described by the following differential equations:

$$\begin{cases} dx(t) = A(r_t,t)x(t)dt + B(r_t,t)u(t)dt + B_w(r_t)w(t)dt + \mathbb{W}(r_t)x(t)d\omega(t), \\ x(0) = x_0 \\ y(t) = C_y(r_t,t)x(t) + D_y(r_t,t)u(t) + B_y(r_t)w(t), \\ z(t) = C_z(r_t,t)x(t) + D_z(r_t,t)u(t) + B_z(r_t)w(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $x_0 \in \mathbb{R}^n$ is the initial state, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^l$ is the system external disturbance, $\omega(t)$ is a standard Wiener process that is assumed to be independent of the Markov process $\{r_t, t \geq 0\}$ which is a continuous-time Markov process taking values in a finite space $\mathscr{S} = \{1, \ldots, N\}$ and that describes the evolution of the mode at time t, when $r_t = i$ the matrices $A(r_t,t)$, $B(r_t,t)$, $C_y(r_t,t)$, $D_y(r_t,t)$, $C_z(r_t,t)$ and $D_z(r_t,t)$ are given by:

$$\begin{cases} A(r_t,t) = A(r_t) + D_A(r_t)F_A(r_t,t)E_A(r_t), \\ B(r_t,t) = B(r_t) + D_B(r_t)F_B(r_t,t)E_B(r_t), \\ C_y(r_t,t) = C_y(r_t) + D_{C_y}(r_t)F_{C_y}(r_t,t)E_{C_y}(r_t), \\ C_z(r_t,t) = C_z(r_t) + D_{C_z}(r_t)F_{C_z}(r_t,t)E_{C_z}(r_t), \\ D_y(r_t,t) = C_y(r_t) + D_{D_y}(r_t)F_{D_y}(r_t,t)E_{D_y}(r_t), \\ D_z(r_t,t) = D_z(r_t) + D_{D_z}(r_t)F_{D_z}(r_t,t)E_{D_z}(r_t) \end{cases}$$

where the matrices $A(i)$, $B(i)$, $B_w(i)$, $\mathbb{W}(i)$, $C_y(i)$, $D_y(i)$, $B_y(i)$, $C_z(i)$, $D_z(i)$, and $B_z(i)$, are given matrices with appropriate dimension.

The system disturbance, $w(t)$, is assumed to belong to $\mathscr{L}_2[0,\infty)$ which means that the following holds:

$$\mathbb{E}\left[\int_0^\infty w^\top(t)w(t)dt\right] < \infty \tag{2}$$

This implies that the disturbance has finite energy.

The Markov process $\{r_t, t \geq 0\}$ beside taking values in the finite set $\mathscr{S}$, the switching between the different modes is described by the following probability transitions:

$$\mathbb{P}\left[r_{t+h} = j | r_t = i\right] = \begin{cases} \lambda_{ij}h + o(h) & \text{when } r_t \text{ jumps from } i \text{ to } j \\ 1 + \lambda_{ij}h + o(h) & \text{otherwise} \end{cases} \tag{3}$$

where λ_{ij} is the transition rate from mode i to mode j with $\lambda_{ij} \geq 0$ when $i \neq j$ and $\lambda_{ii} = -\sum_{j=1,j\neq i}^{N} \lambda_{ij}$ and $o(h)$ is such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

For system (1), when $u(t) \equiv 0$ and the uncertainties are equal to zero for all $t \geq 0$, we have the following definitions.

Definition 2.1. System (1) is said to be

(i) stochastically stable (SS) if there exists a finite positive constant $T(x_0, r_0)$ such that the following holds for any initial conditions (x_0, r_0):

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt | x_0, r_0\right] \leq T(x_0, r_0); \tag{4}$$

(ii) mean square stable (MSS) if

$$\lim_{t \to \infty} \mathbb{E}\|x(t)\|^2 = 0 \tag{5}$$

holds for any initial condition (x_0, r_0);

(iii) mean exponentially stable (MES) if there exist positive constants α and β such that the following holds for any initial conditions (x_0, r_0):

$$\mathbb{E}\left[\|x(t)\|^2 | x_0, r_0\right] \leq \alpha \|x_0\| e^{-\beta t}. \tag{6}$$

Definition 2.2. System (1) is said to be stabilizable in the SS (MES, MSQS) sense if there exists a controller such that the closed-loop system is SS (MES, MSQS) for every initial conditions (x_0, r_0).

When the system's uncertainties are not equal to zero, the concept of stochastic stability becomes robust stochastic stability and is defined for system (1), as follows.

Definition 2.3. System (1) is said to be

(i) robustly stochastically stable (RSS) if there exists a finite positive constant $T(x_0, r_0)$ such that the condition (4) holds for any initial conditions (x_0, r_0) and for all admissible uncertainties;

(ii) robust mean exponentially stable (RMES) if there exist positive constants α and β such that the condition (6) holds for any initial conditions (x_0, r_0) and for all admissible uncertainties.

Remark 2.1. From these definitions, we can see that the robust mean exponentially stable (RMES) implies the stochastically stable (RSS).

In the rest of this paper we will deal with the design of a state feedback controller that robust stochastically stabilizes the closed-loop systems and guarantees the disturbance rejection with a certain level $\gamma > 0$. Mathematically, we are concerned with the design of a controller that guarantees the following for all $\omega \in \mathscr{L}_2[0, \infty)$:

$$\|z(t)\|_2 < \gamma \left[\|\omega(t)\|_2^2 + M(x_0, r_0)\right]^{\frac{1}{2}}$$

where $\gamma > 0$ is a prescribed level of disturbance rejection to be achieved, x_0 and r_0 are the initial conditions of the state vector and the mode respectively at time $t = 0$, and $M(x_0, r_0)$ is a constant that depends on the initial conditions (x_0, r_0).

Definition 2.4. Let $\gamma > 0$ be a given positive constant. System (1) with $u(t) \equiv 0$ is said to be robust stochastically stable with $\gamma-$disturbance attenuation if there exists a constant $M(x_0, r_0)$ with $M(0, r_0) = 0$, for all $r_0 \in \mathscr{S}$, and for all admissible uncertainties such that the following holds:

$$\|z\|_2 \triangleq \left[\mathbb{E}\int_0^\infty z^\top(t)z(t)dt|(x_0, r_0)\right]^{1/2} \leq \gamma\left[\|w\|_2^2 + M(x_0, r_0)\right]^{\frac{1}{2}}. \tag{7}$$

Definition 2.5. System (1) with $u(t) \equiv 0$ is said to be internally mean square quadratically stable (MSQS) if there exists a set of symmetric and positive-definite matrices $P = (P(1), \ldots, P(N)) > 0$, satisfying the following for every $i \in \mathscr{S}$:

$$A^\top(i)P(i) + P(i)A(i) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) < 0, \tag{8}$$

By virtue of Definition 2.1, it is obvious that internal MSQS means that system (1) is MSQS in case of $w(t) \equiv 0$, i.e., system (1) being free of input disturbance. Likewise, we can give the following definitions:

Definition 2.6. System (1) with $u(t) \equiv 0$ is said to be internally SS (MES) if it is SS (MES) in case of $w(t) \equiv 0$.

Definition 2.7. System (1) is said to be stabilizable with γ-disturbance in the SS (MES, MSQS) sense if there exists a control law such that the closed-loop system under this control law is SS (MES, MSQS) and satisfies (7).

The goal of this paper is to design a state feedback controller that robust stochastically stabilizes the class of stochastic hybrid systems with Wiener process we are considering in this paper and at the same time rejects the effect of the external disturbance $w(t)$ with a desired level $\gamma > 0$. The structure of the controller we will be using here is given by the following expression:

$$u(t) = K(i)x(t) \tag{9}$$

where $K(i)$ is a constant gain that we have to determine when $r_t = i \in \mathscr{S}$.

We are mainly concerned with the design of such controller. LMI based conditions are searched since the design becomes easier and the gain can be obtained by solving the appropriate LMIs using the existing developed algorithms. In the rest of this paper, we will assume the complete access to the mode and to the state vector at time t.

Before closing this section let us give some lemmas that we will use in the rest of the paper.

Lemma 2.1. (Ref. 10) Let H, F and G be real matrices of appropriate dimensions then, for any scalar $\varepsilon > 0$ for all matrices F satisfying $F^T F \leq I$, we have:

$$HFG + G^\top F^\top H^\top \leq \varepsilon HH^\top + \varepsilon^{-1}G^\top G \tag{10}$$

Lemma 2.2. The linear matrix inequality

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0, H - S^\top R^{-1} S > 0$$

where $H = H^\top$, $R = R^\top$ and S is a matrix with appropriate dimension.

3 Main results

Our goal in this paper consists of designing a state feedback controller that robust stochastically stabilizes the class of stochastic hybrid systems with Wiener process we are considering and at the same time rejects the external disturbance with a desired level $\gamma > 0$.

Based on the previous definition, the uncertain system with $u(t), \forall t \geq 0$, will be stochastically stable if the following holds for every $i \in \mathscr{S}$ and for all admissible uncertainties:

$$A^\top(i,t)P(i) + P(i)A(i,t) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij} P(j) < 0$$

This condition is useless since it contains the uncertainties $F_A(i,t)$. Let us now transform it into a useful condition that can be used to check the robust stability.

Now if we use the expression of $A(i,t)$, we get:

$$A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij} P(j) + P(i)D_A(i)F_A(i,t)E_A(i) + E_A^\top(i)F_A^\top(i,t)D_A^\top(i)P(i) < 0,$$

Using now the lemma 2.1, the previous inequality will be satisfied if the following holds:

$$A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij} P(j) + \varepsilon_A(i)E_A^\top(i)E_A(i) + \varepsilon_A^{-1}(i)P(i)D_A(i)D_A^\top(i)P(i) < 0,$$

with $\varepsilon_A(i) > 0$ for all $i \in \mathscr{S}$.

Using Schur complement we get the desired condition:

$$\begin{bmatrix} J_0(i) & P(i)D_A(i) \\ D_A^\top(i)P(i) & -\varepsilon_A(i)\mathbb{I} \end{bmatrix} < 0$$

with

$$J_0(i) = A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + \varepsilon_A(i)E_A^\top(i)E_A(i)$$

The results of this development is summarized by the following theorem.

Theorem 3.1. If there exist a set of symmetric and positive-definite matrices $P = (P(1), \ldots, P(N)) > 0$ and a set of positive scalars $\varepsilon_A = (\varepsilon_A(1), \ldots, \varepsilon_A(N))$ such that the following LMI holds for every $i \in \mathscr{S}$ and for all admissible uncertainties:

$$\begin{bmatrix} J_0(i) & P(i)D_A(i) \\ D_A^\top(i)P(i) & -\varepsilon_A(i)\mathbb{I} \end{bmatrix} < 0$$

then system (1) with $u(t) = 0$ for all $t \geq 0$ is internally mean square quadratically stable.

Theorem 3.2. If system (1) with $u(t) \equiv 0$ is internally mean square stochastically stable for all admissible uncertainties, then it is also robust stochastically stable.

Proof. Let $r_t = i \in \mathscr{S}$. To prove this theorem, let us consider a candidate Lyapunov function be defined as follows:

$$V(x(t), i) = x^\top(t)P(i)x(t)$$

where $P(i) > 0$ is symmetric and positive-definite matrix for every $i \in \mathscr{S}$.

The infinitesimal operator $\mathscr{L}$ is given as follows (Ref. 12):

$$\begin{aligned} \mathscr{L}V(x(t), i) &= \dot{x}^\top(t)P(i)x(t) + x^\top(t)P(i)\dot{x}(t) \\ &\quad + x^\top(t)\mathbb{W}^\top(i)P(i)\mathbb{W}(i)x(t) + \sum_{j=1}^{N} \lambda_{ij}x^\top(t)P(j)x(t) \\ &= x^\top(t)\left[A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j)\right]x(t) \\ &\quad + 2x^\top(t)P(i)D_A(i)F_A(i,t)E_A(i)x(t) + 2x^\top(t)P(i)B_\omega(i)\omega(t) \end{aligned}$$

Using now Lemma 2.1, we get the following:

$$\begin{aligned} 2x^\top(t)P(i)D_A(i)F_A(i,t)E_A(i)x(t) &\leq \varepsilon_A(i)x^\top(t)E_A^\top(i)E_A(i)x(t) \\ &\quad + \varepsilon_A^{-1}(i)x^\top(t)P(i)D_A(i)D_A^\top(i)P(i)x(t) \\ 2x^\top(t)P(i)B_\omega(i)\omega(t) &\leq \varepsilon_w^{-1}(i)x^\top(t)P(i)B_\omega(i)B_\omega^\top(i)P(i)x(t) \\ &\quad + \varepsilon_w(i)\omega^\top(t)\omega(t). \end{aligned}$$

Combining this with the expression of $\mathscr{L}V(x(t),i)$, yields

$$\begin{aligned}
\mathscr{L}V(x(t),i) \leq x^\top(t)\left[A^\top(i)P(i)+P(i)A(i)+\mathbb{W}^\top(i)P(i)\mathbb{W}(i)+\sum_{j=1}^{N}\lambda_{ij}P(j)\right]x(t) \\
+\varepsilon_A(i)x^\top(t)E_A^\top(i)E_A(i)x(t)+\varepsilon_A^{-1}(i)x^\top(t)P(i)D_A(i)D_A^\top(i)P(i)x(t) \\
+\varepsilon_w^{-1}(i)x^\top(t)P(i)B_\omega(i)B_\omega^\top(i)P(i)x(t)+\varepsilon_w(i)\omega^\top(t)\omega(t) \\
= x^\top(t)\left[A^\top(i)P(i)+P(i)A(i)+\mathbb{W}^\top(i)P(i)\mathbb{W}(i)+\sum_{j=1}^{N}\lambda_{ij}P(j)\right]x(t) \\
+\varepsilon_A(i)x^\top(t)E_A^\top(i)E_A(i)x(t)+\varepsilon_A^{-1}(i)x^\top(t)P(i)D_A(i)D_A^\top(i)P(i)x(t) \\
+x^\top(t)\left[\varepsilon_w^{-1}(i)P(i)B_\omega(i)B_\omega^\top(i)P(i)\right]x(t)+\varepsilon_w(i)\omega^\top(t)\omega(t), \\
= x^\top(t)\Upsilon(i)x(t)+\varepsilon(i)\omega^\top(t)\omega(t), \qquad (11)
\end{aligned}$$

with

$$\begin{aligned}
\Upsilon(i) = A^\top(i)P(i)+P(i)A(i)+\mathbb{W}^\top(i)P(i)\mathbb{W}(i)+\sum_{j=1}^{N}\lambda_{ij}P(j) \\
+\varepsilon_A(i)E_A^\top(i)E_A(i)+\varepsilon_A^{-1}(i)P(i)D_A(i)D_A^\top(i)P(i) \\
+\varepsilon_w^{-1}(i)P(i)B_w(i)B_w^\top(i)P(i)
\end{aligned}$$

If $\Upsilon(i)<0$ for each $i\in\mathscr{S}$, we get the following equivalent inequality matrix:

$$\Xi(i)=\begin{bmatrix} J_w(i) & P(i)B_w(i) & P(i)D_A(i) \\ B_w^\top(i)P(i) & -\varepsilon_w(i)\mathbb{I} & 0 \\ D_A^\top(i)P(i) & 0 & -\varepsilon_A(i)\mathbb{I}\end{bmatrix}<0$$

with $J_w(i)=A^\top(i)P(i)+P(i)A(i)+\mathbb{W}^\top(i)P(i)\mathbb{W}(i)+\sum_{j=1}^{N}\lambda_{ij}P(j)+\varepsilon_A(i)E_A^\top(i)E_A(i)$

Based on Dynkin's formula, we get the following:

$$\mathbb{E}[V(x(t),i)]-V(x_0,r_0)]=\mathbb{E}\left[\int_0^t \mathscr{L}V(x(s),r_s)ds|x_0,r_0\right],$$

which combined with (11) yields

$$\begin{aligned}
\mathbb{E}[V(x(t),i)]-V(x_0,r_0)] \leq \mathbb{E}\left[\int_0^t x^\top(s)\Xi(r_s)x(s)ds|x_0,r_0\right] \\
+\varepsilon_w(i)\int_0^t \omega^\top(s)\omega(s)ds. \qquad (12)
\end{aligned}$$

Since $V(x(t),i)$ is non-negative, (12) implies

$$\mathbb{E}[V(x(t),i)] + \mathbb{E}\left[\int_0^t x^\top(s)[-\Xi(r_s)]x(s)ds|x_0,r_0\right] \leq V(x_0,r_0)] + \varepsilon_w(i)\int_0^t \omega^\top(s)\omega(s)ds,$$

which yields

$$\begin{aligned}\min_{i\in\mathscr{S}}\{\lambda_{min}(-\Xi(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds\right] &\leq \mathbb{E}\left[\int_0^t x^\top(s)[-\Xi(r_s)]x(s)ds\right] \\ &\leq V(x_0,r_0) + \varepsilon_w(i)\int_0^\infty \omega^\top(s)\omega(s)ds.\end{aligned}$$

This proves that system (1) is stochastically stable. □

Let us now establish what conditions should we satisfy if we want to get system (1), with $u(t) = 0$ for all $t \geq 0$, stochastically stable and has γ-disturbance rejection. The following theorem gives such conditions.

Theorem 3.3. Let γ be a given positive constant. If there exists a set of symmetric and positive-definite matrices $P = (P(1),\ldots,P(N)) > 0$ such that the following LMI holds for every $i \in \mathscr{S}$

$$\begin{bmatrix} J_u(i) & \begin{bmatrix} C_z^\top(i,t)B_z(i) \\ +P(i)B_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)C_z(i,t) \\ +B_\omega^\top(i)P(i) \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{13}$$

where
$J_u(i) = A^\top(i,t)P(i) + P(i)A(i,t) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) + C_z^\top(i,t)C_z(i,t)$, then system (1) with $u(t) \equiv 0$ is robust stochastically stable and satisfies the following:

$$\|z\|_2 \leq \left[\gamma^2\|w\|_2^2 + x_0^\top P(r_0)x_0\right]^{\frac{1}{2}}, \tag{14}$$

which means that the system with $u(t) = 0$ for all $t \geq 0$ is stochastically stable with γ-disturbance attenuation.

Proof. Let $r_t = i \in \mathscr{S}$. From (13) and using Schur complement, we get the following inequality

$$A^\top(i,t)P(i) + P(i)A(i,t) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) + C_z^\top(i,t)C_z(i,t) < 0.$$

which implies the following since $C_z^\top(i,t)C_z(i,t) > 0$

$$A^\top(i,t)P(i) + P(i)A(i,t) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) < 0.$$

Based on Definition 2.5, this proves that the system under study is internally MSQS. Using now Theorem 3.2, we conclude that system (1) with $u(t) \equiv 0$ is robust stochastically stable.

Let us now prove that (14) is satisfied. To this end, let us define the following performance function:

$$J_T = \mathbb{E}\left[\int_0^T [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t)]dt\right].$$

To prove (14), it suffices to establish that J_∞ is bounded, i.e:

$$J_\infty \leq V(x_0, r_0) = x_0^\top P(r_0)x_0.$$

First of all notice that for $V(x(t), i) = x^\top(t)P(i)x(t)$, we have:

$$\begin{aligned}\mathscr{L}V(x(t), i) = x^\top(t)&\left[A^\top(i,t)P(i) + P(i)A(i,t) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N}\lambda_{ij}P(j)\right]x(t)\\ &+ x^\top(t)P(i)B_\omega(i)\omega(t) + \omega^\top(t)B_\omega^\top(i)P(i)x(t),\end{aligned}$$

and

$$\begin{aligned}&z^\top(t)z(t) - \gamma^2\omega(t)\omega(t)\\ &\qquad = [C_z(i,t)x(t) + B_z(i)\omega(t)]^\top [C_z(i,t)x(t) + B_z(i)\omega(t)] - \gamma^2\omega(t)\omega(t)\\ &\qquad = x^\top(t)C_z^\top(i,t)C_z(i,t)x(t) + x^\top(t)C_z^\top(i,t)B_z(i)\omega(t)\\ &\qquad\quad + \omega^\top(t)B_z^\top(i)C_z(i,t)x(t) + \omega^\top(t)B_z^\top(i)B_z(i)\omega(t) - \gamma^2\omega^\top(t)\omega(t)\end{aligned}$$

which implies the following equality:

$$z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \mathscr{L}V(x(t), i) = \eta^\top(t)\Theta_u(i)\eta(t),$$

with

$$\begin{aligned}\Theta_u(i) &= \begin{bmatrix} J_u(i) & \begin{bmatrix} C_z^\top(i,t)B_z(i) \\ +P(i)B_\omega(i)\end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)C_z(i,t) \\ +B_\omega^\top(i)P(i)\end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I}\end{bmatrix}\\ \eta^\top(t) &= \begin{bmatrix} x^\top(t) & \omega^\top(t)\end{bmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}J_T = \mathbb{E}&\left[\int_0^T [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \mathscr{L}V(x(t), i)]dt\right]\\ &-\mathbb{E}\left[\int_0^T \mathscr{L}V(x(t), i)]dt\right]\end{aligned}$$

Using now Dynkin's formula, i.e:

$$\mathbb{E}\left[\int_0^T \mathscr{L}V(x(t),i)]dt|x_0,r_0\right] = \mathbb{E}[V(x(T),r_T)] - V(x_0,r_0).$$

we get

$$J_T = \mathbb{E}\left[\int_0^T \eta^\top(t)\Theta_u(i)\eta(t)dt\right] - \mathbb{E}[V(x(T),r_T)] + V(x_0,r_0).$$

Since $\Theta_u(i) < 0$ and $\mathbb{E}[V(x(T),r_T)] \geq 0$, (15) implies the following:

$$J_T \leq V(x_0,r_0),$$

which yields $J_\infty \leq V(x_0,r_0)$, i.e., $\|z\|_2^2 - \gamma^2\|\omega\|_2^2 \leq x_0^\top P(r_0)x_0$.

This gives the desired results:

$$\|z\|_2 \leq \left[\gamma^2\|\omega\|_2^2 + x_0^\top P(r_0)x_0\right]^{\frac{1}{2}}$$

This ends the proof of the theorem. □

Let us first of all see how we can design a controller of the form (9). Plugging the expression of the controller in the dynamics (1), we get:

$$\begin{cases} dx(t) = \bar{A}(i,t)x(t)dt + B_w(i)w(t)dt + \mathbb{W}(i)x(t)d\omega(t) \\ z(t) = \bar{C}_z(i,t)x(t) + B_z(i)w(t) \end{cases} \tag{15}$$

where $\bar{A}(i,t) = A(i,t) + B(i,t)K(i)$ and $\bar{C}_z(i,t) = C_z(i,t) + D_z(i,t)K(i)$.

Using now the results of Theorem 3.3, we get the following ones for the stochastic stability and the disturbance rejection of level $\gamma > 0$ for the dynamics of the closed-loop.

Theorem 3.4. Let γ be a given positive constant and $K = (K(1),\ldots,K(N))$ be a set of given gains. If there exists a set of symmetric and positive-definite matrices $P = (P(1),\ldots,P(N)) > 0$ such that the following LMI holds for every $i \in \mathscr{S}$

$$\begin{bmatrix} \bar{J}_0(i,t) & \begin{bmatrix} \bar{C}_z^\top(i,t)B_z(i) \\ +P(i)B_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)\bar{C}_z(i,t) \\ +B_\omega^\top(i)P(i) \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{16}$$

with $\bar{J}_0(i,t) = \bar{A}^\top(i,t)P(i) + P(i)\bar{A}(i,t) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) + \bar{C}_z^\top(i,t)\bar{C}_z(i,t)$, then system (1) is stochastically stable under the controller (9) and satisfies the following

$$\|z\|_2 \leq \left[\gamma^2\|w\|_2^2 + x_0^\top P(r_0)x_0\right]^{\frac{1}{2}}, \tag{17}$$

which means that the system is stochastically stable with γ-disturbance attenuation.

To synthesize the controller gain, let us transform the LMI (16) into a form that can be used easily to compute the gain for every mode $i \in \mathscr{S}$. For this purpose notice that:

$$\left[\begin{array}{cc} \bar{J}_0(i,t) & \left[\begin{array}{c} \bar{C}_z^\top(i,t)B_z(i) \\ +P(i)B_\omega(i) \end{array} \right] \\ \left[\begin{array}{c} B_z^\top(i)\bar{C}_z(i,t) \\ +B_\omega^\top(i)P(i) \end{array} \right] & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{array} \right] = \left[\begin{array}{cc} \bar{J}_1(i,t) & P(i)B_\omega(i) \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} \end{array} \right] + \left[\begin{array}{c} \bar{C}_z^\top(i,t) \\ B_z^\top(i) \end{array} \right] \left[\begin{array}{cc} \bar{C}_z(i,t) & B_z(i) \end{array} \right]$$

with

$$\bar{J}_0(i,t) = \bar{A}^\top(i,t)P(i) + P(i)\bar{A}(i,t) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + \bar{C}_z^\top(i,t)\bar{C}_z(i,t)$$

$$\bar{J}_1(i,t) = \bar{A}^\top(i,t)P(i) + P(i)\bar{A}(i,t) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j)$$

Using now Schur complement we show that (16) is equivalent to the following inequality:

$$\left[\begin{array}{ccc} \bar{J}_1(i,t) & P(i)B_\omega(i) & \bar{C}_z^\top(i,t) \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \bar{C}_z(i,t) & B_z(i) & -\mathbb{I} \end{array} \right] < 0$$

Using now the expression of $\bar{A}(i,t)$ and $\bar{C}_z(i,t)$ and the expressions of their components, we obtain the following inequality:

$$\begin{aligned} &\left[\begin{array}{ccc} J_1(i) & P(i)B_\omega(i) & \left[\begin{array}{c} C_z^\top(i) \\ +K^\top(i)D_z^\top(i) \end{array} \right] \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \left[\begin{array}{c} D_z(i)K(i) \\ +C_z(i) \end{array} \right] & B_z(i) & -\mathbb{I} \end{array} \right] \\ &+ \left[\begin{array}{ccc} E_A^\top(i)F_A^\top(i,t)D_A^\top(i)P(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{ccc} P(i)D_A(i)F_A(i,t)E_A(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \\ &+ \left[\begin{array}{ccc} P(i)D_B(i)F_B(i,t)E_B(i)K(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{ccc} K^\top(i)E_B^\top(i)F_B^\top(i,t)D_B^\top(i)P(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \\ &+ \left[\begin{array}{ccc} 0 & 0 & K^\top(i)E_{D_z}^\top(i)F_{D_z}^\top(i,t)D_{D_z}^\top(i) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_{D_z}(i)F_{D_z}(i,t)E_{D_z}(i)K(i) & 0 & 0 \end{array} \right] \\ &+ \left[\begin{array}{ccc} 0 & 0 & E_{C_z}^\top(i)F_{C_z}^\top(i,t)D_{C_z}^\top(i) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_{C_z}(i)F_{C_z}(i,t)E_{C_z}(i) & 0 & 0 \end{array} \right] < 0 \end{aligned}$$

with

$$J_1(i) = A^\top(i)P(i) + P(i)A(i) + K^\top(i)B^\top(i)P(i) + P(i)B(i)K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j)$$

Based on Lemma 2.1, we get:

$$\begin{bmatrix} J_1(i) & P(i)B_\omega(i) & C_z^\top(i) + K^\top(i)D_z^\top(i) \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ D_z(i)K(i) + C_z(i) & B_z(i) & -\mathbb{I} \end{bmatrix}$$
$$+ \begin{bmatrix} \varepsilon_A(i)P(i)D_A(i)D_A^\top(i)P(i) + \varepsilon_A^{-1}(i)E_A^\top(i)E_A(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \begin{bmatrix} \varepsilon_B(i)P(i)D_B(i)D_B^\top(i)P(i) \\ +\varepsilon_B^{-1}K^\top(i)E_B^\top(i)E_B(i)K(i) \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \varepsilon_{D_z}^{-1}(i)K^\top(i)E_{D_z}^\top(i)E_{D_z}(i)K(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{D_z}(i)D_{D_z}(i)D_{D_z}^\top(i) \end{bmatrix}$$
$$+ \begin{bmatrix} \varepsilon_{C_z}^{-1}(i)K^\top(i)E_{C_z}^\top(i)E_{C_z}(i,t)K(i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{C_z}(i)D_{C_z}(i)D_{C_z}^\top(i) \end{bmatrix} < 0$$

with

$$J_1(i) = A^\top(i)P(i) + P(i)A(i) + K^\top(i)B^\top(i)P(i) + P(i)B(i)K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j)$$

Let $J_2(i)$, $\mathcal{W}(i)$ and $\mathcal{T}(i)$ be defined as:

$$\begin{aligned} J_2(i) &= J_1(i) + \varepsilon_A^{-1}(i)E^\top(i)E_A(i) + \varepsilon_B^{-1}(i)K^\top(i)E_B^\top(i)E_B(i)K(i) \\ \mathcal{W}(i) &= \text{diag}[\varepsilon_A^{-1}(i)\mathbb{I}, \varepsilon_B^{-1}(i)\mathbb{I}, \varepsilon_{C_z}(i)\mathbb{I}, \varepsilon_{D_z}(i)\mathbb{I}] \\ \mathcal{T}(i) &= \Big(P(i)D_A(i), P(i)D_B(i), K^\top(i)E_{C_z}^\top(i), K^\top(i)E_{D_z}^\top(i)\Big) \end{aligned}$$

and using Schur complement we get the equivalent inequality:

$$\begin{bmatrix} J_2(i) & P(i)B_\omega(i) & \begin{bmatrix} C_z^\top(i) \\ +K^\top(i)D_z^\top(i) \end{bmatrix} & \mathcal{T}(i) \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 \\ \begin{bmatrix} D_z(i)K(i) \\ +C_z(i) \end{bmatrix} & B_z(i) & -\mathcal{U}(i) & 0 \\ \mathcal{T}^\top(i) & 0 & 0 & -\mathcal{W}(i) \end{bmatrix} < 0$$

with $\mathcal{U}(i) = \mathbb{I} - \varepsilon_{D_z}(i) D_{D_z}(i) D_{D_z}^\top(i) - \varepsilon_{C_z}(i) D_{C_z}(i) D_{C_z}^\top(i)$

This inequality is nonlinear in $K(i)$ and $P(i)$ and therefore it can not solved using existing linear algorithms. To transform it to an LMI, let $X(i) = P^{-1}(i)$. Pre- and post-multiply this inequality by diag$[X(i), \mathbb{I}, \mathbb{I}]$, where $\mathbb{I}$ is an appropriate identity matrix, gives:

$$\begin{bmatrix} J_3(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +X(i)K^\top(i)D_z^\top(i) \end{bmatrix} & X(i)\mathcal{T}(i) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 \\ \begin{bmatrix} D_z(i)K(i)X(i) \\ +C_z(i)X(i) \end{bmatrix} & B_z(i) & -\mathcal{U}(i) & 0 \\ \mathcal{T}^\top(i)X(i) & 0 & 0 & -\mathcal{W}(i) \end{bmatrix} < 0$$

with

$$\begin{aligned} J_3(i) = {} & X(i)A^\top(i) + A(i)X(i) + X(i)K^\top(i)B^\top(i) + X(i)\mathbb{W}^\top(i)X^{-1}(i)\mathbb{W}(i)X(i) \\ & + B(i)K(i)X(i) + \varepsilon_A^{-1}(i)X(i)E_A^\top(i)E_A(i)X(i) \\ & + \varepsilon_B^{-1}(i)X(i)K^\top(i)E_B^\top(i)E_B(i)K(i)X(i) \\ & + \sum_{j=1}^{N} \lambda_{ij} X(i)X^{-1}(j)X(i) \end{aligned}$$

Notice that:

$$X(i)\mathcal{T}(i) = \Big(D_A(i), D_B(i), X(i)K^\top(i)E_{C_z}^\top(i), X(i)K^\top(i)E_{D_z}^\top(i)\Big)$$

and

$$\sum_{j=1}^{N} \lambda_{ij} X(i)X^{-1}(j)X(i) = \lambda_{ii}X(i) + \mathcal{S}_i(X)\mathcal{X}^{-1}(X)\mathcal{S}_i^\top(X)$$

Letting $Y(i) = K(i)X(i)$ and using Schur complement we obtain:

$$\begin{bmatrix} \tilde{J}(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i) \end{bmatrix} & X(i)\mathbb{W}^\top(i) & \mathcal{Z}(i) & \mathcal{S}_i(X) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 & 0 & 0 \\ \begin{bmatrix} D_z(i)Y(i) \\ +C_z(i)X(i) \end{bmatrix} & B_z(i) & -\mathcal{U}(i) & 0 & 0 & 0 \\ X(i)\mathbb{W}(i) & 0 & 0 & -X(i) & 0 & 0 \\ \mathcal{Z}^\top(i) & 0 & 0 & 0 & -\mathcal{V}(i) & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0$$

with

$$\begin{aligned}\tilde{J}(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i)\\ &\quad + \varepsilon_A(i)D_A(i)D_A^\top(i) + \varepsilon_B(i)D_B(i)D_B^\top(i)\\ &\quad + \lambda_{ii}X(i)\\ \mathcal{Z}(i) &= \Big(X(i)E_A^\top(i), X(i)E_B^\top(i), Y^\top(i)E_{C_z}^\top(i), Y^\top(i)E_{D_z}^\top(i)\Big)\\ \mathcal{V}(i) &= \mathrm{diag}[\varepsilon_A(i)\mathbb{I}, \varepsilon_B(i)\mathbb{I}, \varepsilon_{C_z}(i)\mathbb{I}, \varepsilon_{D_z}(i)\mathbb{I}]\end{aligned}$$

The following theorem summarizes the results of this development.

Theorem 3.5. Let γ be a positive constant. If there exist a set of symmetric and positive-definite matrices $X = (X(1), \ldots, X(N)) > 0$ and a set of matrices $Y = (Y(1), \ldots, Y(N))$ and sets of positive scalars $\varepsilon_A = (\varepsilon_A(1), \ldots, \varepsilon_A(N))$, $\varepsilon_B = (\varepsilon_B(1), \ldots, \varepsilon_B(N))$, $\varepsilon_{C_z} = (\varepsilon_{C_z}(1), \ldots, \varepsilon_{C_z}(N))$, and $\varepsilon_{D_z} = (\varepsilon_{D_z}(1), \ldots, \varepsilon_{D_z}(N))$ such that the following LMI holds for every $i \in \mathscr{S}$ and for all admissible uncertainties:

$$\begin{bmatrix} \tilde{J}(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i)\end{bmatrix} & X(i)\mathbb{W}^\top(i) & \mathcal{Z}(i) & \mathcal{S}_i(X) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 & 0 & 0 \\ \begin{bmatrix} D_z(i)Y(i) \\ +C_z(i)X(i)\end{bmatrix} & B_z(i) & -\mathcal{U}(i) & 0 & 0 & 0 \\ X(i)\mathbb{W}(i) & 0 & 0 & -X(i) & 0 & 0 \\ \mathcal{Z}^\top(i) & 0 & 0 & 0 & -\mathcal{V}(i) & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0 \tag{18}$$

with

$$\begin{aligned}\tilde{J}(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i)\\ &\quad + \varepsilon_A(i)D_A(i)D_A^\top(i) + \varepsilon_B(i)D_B(i)D_B^\top(i)\\ &\quad + \lambda_{ii}X(i)\\ \mathcal{U}(i) &= \mathbb{I} - \varepsilon_{D_z}(i)D_{D_z}(i)D_{D_z}^\top(i) - \varepsilon_{C_z}(i)D_{C_z}(i)D_{C_z}^\top(i)\\ \mathcal{Z}(i) &= \Big(X(i)E_A^\top(i), X(i)E_B^\top(i), Y^\top(i)E_{C_z}^\top(i), Y^\top(i)E_{D_z}^\top(i)\Big)\\ \mathcal{V}(i) &= \mathrm{diag}[\varepsilon_A(i)\mathbb{I}, \varepsilon_B(i)\mathbb{I}, \varepsilon_{C_z}(i)\mathbb{I}, \varepsilon_{D_z}(i)\mathbb{I}]\end{aligned}$$

then, the system (1) under the controller (9) with $K(i) = Y(i)X^{-1}(i)$ is stochastically stable and moreover the closed-loop system satisfies the disturbance rejection of level γ.

From the practical point of view, the controller that stochastically stabilizes the class of systems and at the same time guarantees the minimum disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

$$
\text{Pu}: \begin{cases}
\min\limits_{\substack{\nu>0, \\ \varepsilon_A=(\varepsilon_A(1),\ldots,\varepsilon_A(N))>0, \\ \varepsilon_B=(\varepsilon_B(1),\ldots,\varepsilon_B(N))>0, \\ \varepsilon_{C_z}=(\varepsilon_{D_z}(1),\ldots,\varepsilon_{C_z}(N))>0, \\ \varepsilon_{D_z}=(\varepsilon_{D_z}(1),\ldots,\varepsilon_{D_z}(N))>0, \\ X=(X(1),\ldots,X(N))>0, \\ Y=(Y(1),\ldots,Y(N))}} \nu \\
s.t: \\
\left[\begin{matrix}
\tilde{J}(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i) \end{bmatrix} \\
B_\omega^\top(i) & -\nu\mathbb{I} & B_z^\top(i) \\
\begin{bmatrix} D_z(i)Y(i) \\ +C_z(i)X(i) \end{bmatrix} & B_z(i) & -\mathcal{U}(i) \\
X(i)\mathbb{W}(i) & 0 & 0 \\
\mathcal{Z}^\top(i) & 0 & 0 \\
\mathcal{S}_i^\top(X) & 0 & 0
\end{matrix}\right. \\
\qquad \left.\begin{matrix}
X(i)\mathbb{W}^\top(i) & \mathcal{Z}(i) & \mathcal{S}_i(X) \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-X(i) & 0 & 0 \\
0 & -\mathcal{V}(i) & 0 \\
0 & 0 & -\mathcal{X}_i(X)
\end{matrix}\right] < 0
\end{cases}
$$

where the LMI in the constraints is obtained from (18) by replacing γ^2 by ν.

The following theorem gives the results on the design of the controller that stochastically stabilizes the system (1) and simultaneously guarantees the smallest disturbance rejection level.

Theorem 3.6. Let $\nu > 0$, $\varepsilon_A = (\varepsilon_A(1), \ldots, \varepsilon_A(N)) > 0$, $\varepsilon_B = (\varepsilon_B(1), \ldots, \varepsilon_B(N)) > 0$, $\varepsilon_{C_z} = (\varepsilon_{C_z}(1), \ldots, \varepsilon_{C_z}(N)) > 0$, $\varepsilon_{D_z} = (\varepsilon_{D_z}(1), \ldots, \varepsilon_{D_z}(N)) > 0$, $X = (X(1), \ldots, X(N)) > 0$ and $Y = (Y(1), \ldots, Y(N))$ be the solution of the optimization problem Pu. Then, the controller (9) with $K(i) = Y(i)X^{-1}(i)$ stochastically stabilizes the class of systems we are considering and moreover the closed-loop system satisfies the disturbance rejection of level $\sqrt{\nu}$.

4 Numerical Example

In the previous section, we developed results that determine the state feedback controller that stochastically stabilizes the class of systems we are treating in this paper and at the same time rejects the disturbance $w(t)$ with the desired level $\gamma > 0$. The conditions we developed are in the LMI form which makes their resolution easy. In the rest of this section we will give a numerical example to show the usefulness of our results.

Example 4.1. Let us consider a system with two modes with the following data:

- transition probability rates matrix:

$$\Lambda = \begin{bmatrix} -2.0 & 2.0 \\ 3.0 & -3.0 \end{bmatrix}$$

- mode 1:

$$A(1) = \begin{bmatrix} -0.5 & 1.0 \\ 0.3 & -2.5 \end{bmatrix}, B(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, B_w(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$C_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, B_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, D_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1 \end{bmatrix}$$

$$D_A(1) = \begin{bmatrix} 0.10 \\ 0.20 \end{bmatrix}, E_A(1) = \begin{bmatrix} 0.20 & 0.10 \end{bmatrix}, D_B(1) = \begin{bmatrix} 0.10 \\ 0.20 \end{bmatrix}$$

$$E_B(1) = \begin{bmatrix} 0.20 & 0.10 \end{bmatrix}, D_{C_z}(1) = \begin{bmatrix} 0.10 \\ 0.20 \end{bmatrix}, E_{C_z}(1) = \begin{bmatrix} 0.20 & 0.10 \end{bmatrix}$$

$$D_{D_z}(1) = \begin{bmatrix} 0.10 \\ 0.20 \end{bmatrix}, E_{D_z}(1) = \begin{bmatrix} 0.20 & 0.10 \end{bmatrix}, \mathbb{W}(1) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}$$

- mode 2:

$$A(2) = \begin{bmatrix} -1.0 & 0.1 \\ 0.2 & -2.0 \end{bmatrix}, B(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, B_w(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$C_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, B_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, D_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$D_A(2) = \begin{bmatrix} 0.13 \\ 0.10 \end{bmatrix}, E_A(2) = \begin{bmatrix} 0.10 & 0.20 \end{bmatrix}, D_B(2) = \begin{bmatrix} 0.13 \\ 0.10 \end{bmatrix}$$

$$E_B(2) = \begin{bmatrix} 0.10 & 0.20 \end{bmatrix}, D_{C_z}(2) = \begin{bmatrix} 0.13 \\ 0.10 \end{bmatrix}, E_{C_z}(2) = \begin{bmatrix} 0.10 & 0.20 \end{bmatrix}$$

$$D_{D_z}(2) = \begin{bmatrix} 0.13 & 0.10 \end{bmatrix}, E_{D_z}(2) = \begin{bmatrix} 0.10 & 0.20 \end{bmatrix}, \mathbb{W}(2) = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.2 \end{bmatrix}$$

The required positive scalars are fixed to the following values:

$$\varepsilon_A(1) = \varepsilon_A(2) = 0.50$$
$$\varepsilon_B(1) = \varepsilon_B(2) = \varepsilon_{C_z}(1) = \varepsilon_{C_z}(2) = \varepsilon_{D_z}(1) = \varepsilon_{D_z}(2) = 0.10$$

Solving the problem Pu gives:

$$X(1) = \begin{bmatrix} 0.0769 & -0.1335 \\ -0.1335 & 0.5794 \end{bmatrix}, Y(1) = \begin{bmatrix} -1.0610 & 0.1437 \\ 0.1392 & -1.5717 \end{bmatrix},$$
$$X(2) = \begin{bmatrix} 0.1433 & -0.1433 \\ -0.1433 & 0.5533 \end{bmatrix}, Y(2) = \begin{bmatrix} -1.1449 & 0.1518 \\ 0.1478 & -1.5304 \end{bmatrix},$$

which gives the following gains:

$$K(1) = \begin{bmatrix} -22.2878 & -4.8864 \\ -4.8327 & -3.8260 \end{bmatrix}, K(2) = \begin{bmatrix} -10.4069 & -2.4204 \\ -2.3394 & -3.3718 \end{bmatrix}$$

Using the results of Theorem 3.6, it results that the system of this example is stochastically stable under the state feedback controller with the computed constant gain and assure the disturbance rejection of level $\gamma = 1.01$.

5 Conclusion

This paper dealt with the class of uncertain hybrid stochastic systems. The uncertainties we considered in this paper are of norm bounded type. Both the stability and the stabilizability problems are treated. A state feedback controller is proposed to robustly stochastically stabilize the class of hybrid systems and at the same time rejection the disturbance with a desired level $\gamma > 0$. The conditions we developed are in LMI form which makes the resolution easier using the existing tools.

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