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Abstract

This paper deals with the class of continuous-time linear stochastic hybrid systems with Wiener process. The $\mathscr{H}_\infty$ stochastic stabilization problem is treated. LMIs based conditions are developed to design the state feedback controller that stochastically stabilizes the studied class of systems and at the same time reject the disturbance rejection of a desired level. The minimum disturbance rejection is also determined. Numerical examples are given to show the usefulness of the proposed results.

Résumé

Cet article traite de la classe des systèmes linéaires à sauts markoviens avec bruit Brownien. La commande $\mathscr{H}_\infty$ est utilisée pour stabiliser cette classe de systèmes. Des conditions LMI sont développées pour le design des gains du correcteur par retour d'état qui stabilise et en même temps rejette la perturbation avec un taux désiré $\gamma > 0$.

1 Introduction

In practice we find some systems that have abrupt changes in their dynamics that results from causes like connections or disconnections of some components, failures in the components etc. Analysis and design of these systems can't be done using the linear invariant system theory since it is unable to model adequately such systems. These systems have stochastic behavior and examples of such systems can be found in manufacturing systems, power systems, telecommunications systems, etc. The occurrence of the abrupt changes is random in more cases. These practical systems have been modelled by the class of linear systems with Markovian jumps that we will term in this paper as stochastic hybrid systems. This class of systems has two components in the state vector. The first component of this state vector takes values in $\mathbb{R}^n$ and evolves continuously in time and it represents the classical state vector that is usually used in the modern control theory. The second one takes values in a finite set and switches in a random manner between a finite number of states (see Mariton [9], Boukas and Liu [5] and Boukas [3] and the references therein. This component is represented by a continuous-time Markov process. Usually the state vector of the class of stochastic hybrid systems is denoted by $(x(t), r_t)$.

This class of systems has attracted a lot researchers and many problems have been tackled and solved. Among these problems, we quote those of stability, stabilizability, $\mathcal{H}_\infty$ control problem and filtering problem. For more details on what it has been done on this class of systems, we refer the reader to the recent books by Boukas and Liu [5] and Boukas [3] and the references therein. These two books present a good review of the literature of the subject up to 2004.

The stabilization problem of the class of linear systems with Markovian jumping parameters has attracted a lot of researchers, and many contributions have been reported in the literature. For more details on this topics and the contributions on the subject, we refer the reader to ([5, 3, 12, 4, 6, 11, 8, 7]) and the references therein. To the best of our knowledge the case of continuous-time systems with Markovian jumps and multiplicative noise has never studied and our objective in this paper is to study the $\mathscr{H}_\infty$ stabilization of such class of systems.

Our goal in this paper consists of designing a state feedback controller that stochastically stabilizes the class of systems we are studying and at the same time reject the disturbance with a desired level $\gamma > 0$. We are also interested by determining the minimum level of the disturbance rejection. In this paper, we will solve these two problems and develop LMI conditions that we can use to determine the state feedback controller that stochastically stabilizes the class of systems of stochastic hybrid systems with multiplicative noise and guarantees the minimum disturbance rejection.

The rest of the paper is organized as follows. In Section 2, the problem we are considering is stated and some useful definitions are given. Section 3 gives the main results of the paper. In Section 4, some numerical examples are provided to show the usefulness of the proposed results.

2 Problem Statement

Let us consider a dynamical system defined in a fundamental probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and assume that its dynamics is described by the following differential equations:

$$\begin{cases} dx(t) = A(r_t)x(t)dt + B(r_t)u(t)dt + B_w(r_t)w(t)dt + \mathbb{W}(r_t)x(t)d\omega(t), x(0) = x_0 \\ y(t) = C_y(r_t)x(t) + D_y(r_t)u(t) + B_y(r_t)w(t), \\ z(t) = C_z(r_t)x(t) + D_z(r_t)u(t) + B_z(r_t)w(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $x_0 \in \mathbb{R}^n$ is the initial state, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^l$ is the system external disturbance, $\omega(t)$ is a standard Wiener process that is assumed to be independent of the Markov process $\{r_t, t \geq 0\}$ which is a continuous-time Markov process taking values in a finite space $\mathscr{S} = \{1, \cdots, N\}$ and that describes the evolution of the mode at time t, when $r_t = i$ the matrices $A(i)$, $B(i)$, $B_w(i)$, $\mathbb{W}(i)$, $C_y(i)$, $D_y(i)$, $B_y(i)$, $C_z(i)$, $D_z(i)$, and $B_z(i)$, are given matrices with appropriate dimension.

The system disturbance, $w(t)$, is assumed to belong to $\mathscr{L}_2[0, \infty)$ which means that the following holds:

$$\mathbb{E}\left[\int_0^\infty w^\top(t)w(t)dt\right] < \infty \tag{2}$$

This implies that the disturbance has finite energy.

The Markov process $\{r_t, t \geq 0\}$ beside taking values in the finite set $\mathscr{S}$, the switching between the different modes is described by the following probability transitions:

$$\mathbb{P}\left[r_{t+h} = j | r_t = i\right] = \begin{cases} \lambda_{ij}h + o(h) & \text{when } r_t \text{ jumps from } i \text{ to } j \\ 1 + \lambda_{ij}h + o(h) & \text{otherwise} \end{cases} \tag{3}$$

where λ_{ij} is the transition rate from mode i to mode j with $\lambda_{ij} \geq 0$ when $i \neq j$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^N \lambda_{ij}$ and $o(h)$ is such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

For system (1), when $u(t) \equiv 0$ for all $t \geq 0$, we have the following definitions.

Definition 2.1 *System (1) is said to be*

(i) stochastically stable (SS) if there exists a finite positive constant $T(x_0, r_0)$ such that the following holds for any initial conditions (x_0, r_0):

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt | x_0, r_0\right] \leq T(x_0, r_0); \tag{4}$$

(ii) mean square stable (MSS) if

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0 \tag{5}$$

holds for any initial condition (x_0, r_0);

(iii) mean exponentially stable (MES) if there exist positive constants α *and* β *such that the following holds for any initial conditions* (x_0, r_0)*:*

$$\mathbb{E}\left[\|x(t)\|^2 | x_0, r_0\right] \leq \alpha \|x_0\| e^{-\beta t}. \tag{6}$$

Definition 2.2 *System (1) is said to be stabilizable in the SS (MES, MSQS) sense if there exists a controller such that the closed-loop system is SS (MES, MSQS) for every initial conditions* (x_0, r_0)*.*

In the rest of this paper we will deal with the design of controllers that stochastically stabilizes the closed-loop systems and guarantees the disturbance rejection with a certain level $\gamma > 0$. Mathematically, we are concerned with the design of a controller that guarantees the following for all $\omega \in \mathscr{L}_2[0, \infty)$:

$$\|z(t)\|_2 < \gamma \left[\|\omega(t)\|_2^2 + M(x_0, r_0)\right]^{\frac{1}{2}}$$

where $\gamma > 0$ is a prescribed level of disturbance rejection to be achieved, x_0 and r_0 are the initial conditions of the state vector and the mode respectively at time $t = 0$, and $M(x_0, r_0)$ is a constant that depends on the initial conditions (x_0, r_0).

Definition 2.3 *Let* $\gamma > 0$ *be a given positive constant. System (1) with* $u(t) \equiv 0$ *is said to be stochastically stable with* $\gamma-$*disturbance attenuation if there exists a constant* $M(x_0, r_0)$ *with* $M(0, r_0) = 0$*, for all* $r_0 \in \mathscr{S}$*, such that the following holds:*

$$\|z\|_2 \stackrel{\Delta}{=} \left[\mathbb{E}\int_0^\infty z^\top(t) z(t) dt | (x_0, r_0)\right]^{1/2} \leq \gamma \left[\|w\|_2^2 + M(x_0, r_0)\right]^{\frac{1}{2}}. \tag{7}$$

Definition 2.4 *System (1) with* $u(t) \equiv 0$ *is said to be internally mean square quadratically stable (MSQS) if there exists a set of symmetric and positive-definite matrices* $P = (P(1), \cdots, P(N)) > 0$*, satisfying the following for every* $i \in \mathscr{S}$*:*

$$A^\top(i)P(i) + P(i)A(i) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij} P(j) < 0, \tag{8}$$

By virtue of Definition 2.1, it is obvious that internal MSQS means that system (1) is MSQS in case of $w(t) \equiv 0$, i.e., system (1) being free of input disturbance. Likewise, we can give the following definitions:

Definition 2.5 *System (1) with* $u(t) \equiv 0$ *is said to be internally SS (MES) if it is SS (MES) in case of* $w(t) \equiv 0$*.*

Definition 2.6 *System (1) is said to be stabilizable with* γ*-disturbance in the SS (MES, MSQS) sense if there exists a control law such that the closed-loop system under this control law is SS (MES, MSQS) and satisfies (7).*

The goal of this paper is to design a state feedback controller that stochastically stabilizes the class of stochastic hybrid systems with Wiener process we are considering in this paper and at the same time rejects the effect of the external disturbance $w(t)$ with a desired level $\gamma > 0$. The structure of the controller we will be using here is given by the following expression:

$$u(t) = K(i)x(t) \tag{9}$$

where $K(i)$ is a constant gain that we have to determine when $r_t = i \in \mathscr{S}$.

We are mainly concerned with the design of such controller. LMI based conditions are searched since the design becomes easier and the gain can be obtained by solving the appropriate LMIs using the existing developed algorithms. In the rest of this paper, we will assume the complete access to the mode and to the state vector at time t.

Before closing this section let us give some lemmas that we will use in the rest of the paper.

Lemma 2.1 *[10] Let H, F and G be real matrices of appropriate dimensions then, for any scalar $\varepsilon > 0$ for all matrices F satisfying $F^T F \leq I$, we have:*

$$HFG + G^\top F^\top H^\top \leq \varepsilon H H^\top + \varepsilon^{-1} G^\top G \tag{10}$$

Lemma 2.2 *The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0, H - S^\top R^{-1} S > 0$$

where $H = H^\top$, $R = R^\top$ and S is a matrix with appropriate dimension.

3 Main results

Our goal in this paper consists of designing a state feedback controller that stochastically stabilizes the class of stochastic hybrid systems with Wiener process we are considering and at the same time rejects the external disturbance with a desired level $\gamma > 0$.

Theorem 3.1 *If system (1) with $u(t) \equiv 0$ is internally MSQS, then it is stochastically stable.*

Proof: Let $r_t = i \in \mathscr{S}$. To prove this theorem, let us consider a candidate Lyapunov function be defined as follows:

$$V(x(t), i) = x^\top(t) P(i) x(t)$$

where $P(i) > 0$ is symmetric and positive-definite matrix for every $i \in \mathscr{S}$.

The infinitesimal operator $\mathscr{L}$ is given as follows:

$$\begin{aligned}\mathscr{L}V(x(t),i) &= \dot{x}^\top(t)P(i)x(t) + x^\top(t)P(i)\dot{x}(t) \\ &\quad + x^\top(t)\mathbb{W}^\top(i)P(i)\mathbb{W}(i)x(t) + \sum_{j=1}^{N}\lambda_{ij}x^\top(t)P(j)x(t) \\ &= x^\top(t)\left[A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N}\lambda_{ij}P(j)\right]x(t) \\ &\quad + 2x^\top(t)P(i)B_\omega(i)\omega(t)\end{aligned}$$

Using now Lemma 2, we get the following for any $\varepsilon_w(i) > 0$

$$\begin{aligned}2x^\top(t)P(i)B_\omega(i)\omega(t) \quad &\le \quad \varepsilon_w(i)x^\top(t)P(i)B_\omega(i)B_\omega^\top(i)P(i)x(t) \\ &\quad + \varepsilon_w^{-1}(i)\omega^\top(t)\omega(t).\end{aligned}$$

Combining this with the expression of $\mathscr{L}V(x(t),i)$, yields

$$\begin{aligned}\mathscr{L}V(x(t),i) &\le x^\top(t)\left[A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N}\lambda_{ij}P(j)\right]x(t) \\ &\quad + \varepsilon_w(i)x^\top(t)P(i)B_\omega(i)B_\omega^\top(i)P(i)x(t) + \varepsilon_w^{-1}(i)\omega^\top(t)\omega(t) \\ &= x^\top(t)\left[A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N}\lambda_{ij}P(j)\right]x(t) \\ &\quad + x^\top(t)\left[\varepsilon_w(i)P(i)B_\omega(i)B_\omega^\top(i)P(i)\right]x(t) + \varepsilon_w^{-1}(i)\omega^\top(t)\omega(t), \\ &= x^\top(t)\Xi(i)x(t) + \varepsilon^{-1}(i)\omega^\top(t)\omega(t), \qquad (11)\end{aligned}$$

with

$$\begin{aligned}\Xi(i) = A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N}\lambda_{ij}P(j) \\ + \varepsilon_w(i)P(i)B_\omega(i)B_\omega^\top(i)P(i)\end{aligned}$$

Based on Dynkin's formula, we get the following:

$$\mathbb{E}[V(x(t),i)] - V(x_0,r_0)] = \mathbb{E}\left[\int_0^t \mathscr{L}V(x(s),r_s)ds|x_0,r_0\right],$$

which combined with (11) yields

$$\mathbb{E}[V(x(t),i)] - V(x_0,r_0)] \le \mathbb{E}\left[\int_0^t x^\top(s)\Xi(r_s)x(s)ds|x_0,r_0\right] + \varepsilon_w^{-1}(i)\int_0^t \omega^\top(s)\omega(s)ds. \quad (12)$$

Since $V(x(t),i)$ is non-negative, (12) implies

$$\mathbb{E}[V(x(t),i)] + \mathbb{E}\left[\int_0^t x^\top(s)[-\Xi(r_s)]x(s)ds|x_0,r_0\right] \le V(x_0,r_0)] + \varepsilon_w^{-1}(i)\int_0^t \omega^\top(s)\omega(s)ds,$$

which yields

$$\min_{i\in\mathscr{S}}\{\lambda_{min}(-\Xi(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds\right] \le \mathbb{E}\left[\int_0^t x^\top(s)[-\Xi(r_s)]x(s)ds\right] \le V(x_0,r_0) + \varepsilon_w^{-1}(i)\int_0^\infty \omega^\top(s)\omega(s)ds.$$

This proves that system (1) is stochastically stable. □

Let us now establish what conditions should we satisfy if we want to get system (1), with $u(t) = 0$ for all $t \ge 0$, stochastically stable and has γ-disturbance rejection. The following theorem gives such conditions.

Theorem 3.2 *Let γ be a given positive constant. If there exists a set of symmetric and positive-definite matrices $P = (P(1),\cdots,P(N)) > 0$ such that the following LMI holds for every $i \in \mathscr{S}$*

$$\begin{bmatrix} J_0(i) & \begin{bmatrix} C_z^\top(i)B_z(i) \\ +P(i)B_\omega(i)\end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)C_z(i) \\ +B_\omega^\top(i)P(i)\end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I}\end{bmatrix} < 0, \quad (13)$$

where $J_0(i) = A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) + C_z^\top(i)C_z(i)$, then system (1) with $u(t) \equiv 0$ is stochastically stable and satisfies the following:

$$\|z\|_2 \le \left[\gamma^2\|w\|_2^2 + x_0^\top P(r_0)x_0\right]^{\frac{1}{2}}, \quad (14)$$

which means that the system with $u(t) = 0$ for all $t \ge 0$ is stochastically stable with γ-disturbance attenuation.

Proof: Let $r_t = i \in \mathscr{S}$. From (13) and using Schur complement, we get the following inequality

$$A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + C_z^\top(i)C_z(i) < 0.$$

which implies the following since $C_z^\top(i)C_z(i) > 0$

$$A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) < 0.$$

Based on Definition 2.4, this proves that the system under study is internally MSQS. Using now Theorem 3.1, we conclude that system (1) with $u(t) \equiv 0$ is stochastically stable.

Let us now prove that (14) is satisfied. To this end, let us define the following performance function:

$$J_T = \mathbb{E}\left[\int_0^T [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t)]dt\right].$$

To prove (14), it suffices to establish that J_∞ is bounded, i.e:

$$J_\infty \leq V(x_0, r_0) = x_0^\top P(r_0)x_0.$$

First of all notice that for $V(x(t), i) = x^\top(t)P(i)x(t)$, we have:

$$\begin{aligned}\mathscr{L}V(x(t), i) = x^\top(t)\left[A^\top(i)P(i) + P(i)A(i) + \mathbb{W}^\top(i)P(i)\mathbb{W}(i) + \sum_{j=1}^{N} \lambda_{ij}P(j)\right]x(t) \\ +x^\top(t)P(i)B_\omega(i)\omega(t) + \omega^\top(t)B_\omega^\top(i)P(i)x(t),\end{aligned}$$

and

$$\begin{aligned}z^\top(t)z(t) - \gamma^2\omega(t)\omega(t) \\ = [C_z(i)x(t) + B_z(i)\omega(t)]^\top[C_z(i)x(t) + B_z(i)\omega(t)] - \gamma^2\omega(t)\omega(t) \\ = x^\top(t)C_z^\top(i)C_z(i)x(t) + x^\top(t)C_z^\top(i)B_z(i)\omega(t) \\ +\omega^\top(t)B_z^\top(i)C_z(i)x(t) + \omega^\top(t)B_z^\top(i)B_z(i)\omega(t) - \gamma^2\omega^\top(t)\omega(t)\end{aligned}$$

which implies the following equality:

$$z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \mathscr{L}V(x(t), i) = \eta^\top(t)\Theta(i)\eta(t),$$

with

$$\Theta(i) = \begin{bmatrix} J_0(i) & \begin{bmatrix} C_z^\top(i)B_z(i) \\ +P(i)B_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)C_z(i) \\ +B_\omega^\top(i)P(i) \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix}$$
$$\eta^\top(t) = \begin{bmatrix} x^\top(t) & \omega^\top(t) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} J_T &= \mathbb{E}\left[\int_0^T [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \mathscr{L}V(x(t),i)]dt\right] \\ &\quad -\mathbb{E}\left[\int_0^T \mathscr{L}V(x(t),i)]dt\right] \end{aligned}$$

Using now Dynkin's formula, i.e:

$$\mathbb{E}\left[\int_0^T \mathscr{L}V(x(t),i)]dt|x_0,r_0\right] = \mathbb{E}[V(x(T),r_T)] - V(x_0,r_0).$$

we get

$$J_T = \mathbb{E}\left[\int_0^T \eta^\top(t)\Theta(i)\eta(t)dt\right] - \mathbb{E}[V(x(T),r_T)] + V(x_0,r_0).$$

Since $\Theta(i) < 0$ and $\mathbb{E}[V(x(T),r_T)] \geq 0$, (15) implies the following:

$$J_T \leq V(x_0,r_0),$$

which yields $J_\infty \leq V(x_0,r_0)$, i.e., $\|z\|_2^2 - \gamma^2\|\omega\|_2^2 \leq x_0^\top P(r_0)x_0$.

This gives the desired results:

$$\|z\|_2 \leq \left[\gamma^2\|\omega\|_2^2 + x_0^\top P(r_0)x_0\right]^{\frac{1}{2}}$$

This ends the proof of the theorem. □

Let us first of all see how we can design a controller of the form (9). Plugging the expression of the controller in the dynamics (1), we get:

$$\begin{cases} dx(t) = \bar{A}(i)x(t)dt + B_w(i)w(t)dt + \mathbb{W}(i)x(t)d\omega(t) \\ z(t) = \bar{C}_z(i)x(t) + B_z(i)w(t) \end{cases} \tag{15}$$

where $\bar{A}(i) = A(i) + B(i)K(i)$ and $\bar{C}_z(i) = C_z(i) + D_z(i)K(i)$.

Using now the results of Theorem 3.2, we get the following ones for the stochastic stability and the disturbance rejection of level $\gamma > 0$ for the dynamics of the closed-loop.

Theorem 3.3 *Let γ be a given positive constant and $K = (K(1), \cdots, K(N))$ be a set of given gains. If there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N)) > 0$ such that the following LMI holds for every $i \in \mathscr{S}$*

$$\begin{bmatrix} \bar{J}_0(i) & \begin{bmatrix} \bar{C}_z^\top(i)B_z(i) \\ +P(i)B_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)\bar{C}_z(i) \\ +B_\omega^\top(i)P(i) \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{16}$$

with $\bar{J}_0(i) = \bar{A}^\top(i)P(i) + P(i)\bar{A}(i) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) + \bar{C}_z^\top(i)\bar{C}_z(i)$, then system (1) is stochastically stable under the controller (9) and satisfies the following

$$\|z\|_2 \leq \left[\gamma^2\|w\|_2^2 + x_0^\top P(r_0)x_0\right]^{\frac{1}{2}}, \tag{17}$$

which means that the system is stochastically stable with γ-disturbance attenuation.

To synthesize the controller gain, let us transform the LMI (16) into a form that can be used easily to compute the gain for every mode $i \in \mathscr{S}$. For this purpose notice that:

$$\begin{bmatrix} \bar{J}_0(i) & \begin{bmatrix} \bar{C}_z^\top(i)B_z(i) \\ +P(i)B_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)\bar{C}_z(i) \\ +B_\omega^\top(i)P(i) \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} = \begin{bmatrix} \bar{J}_1(i) & P(i)B_\omega(i) \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} \end{bmatrix} + \begin{bmatrix} \bar{C}_z^\top(i) \\ B_z^\top(i) \end{bmatrix} \begin{bmatrix} \bar{C}_z(i) & B_z(i) \end{bmatrix}$$

with

$$\bar{J}_0(i) = \bar{A}^\top(i)P(i) + P(i)\bar{A}(i) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j) + \bar{C}_z^\top(i)\bar{C}_z(i)$$

$$\bar{J}_1(i) = \bar{A}^\top(i)P(i) + P(i)\bar{A}(i) + \mathbb{W}(i)P(i)\mathbb{W}(i) + \sum_{j=1}^N \lambda_{ij}P(j)$$

Using now Schur complement we show that (16) is equivalent to the following inequality:

$$\begin{bmatrix} \bar{J}_1(i) & P(i)B_\omega(i) & \bar{C}_z^\top(i) \\ B_\omega^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \bar{C}_z(i) & B_z(i) & -\mathbb{I} \end{bmatrix} < 0$$

Since $\bar{A}(i)$ is nonlinear in $K(i)$ and $P(i)$ the previous inequality is then nonlinear and therefore it can not solved using existing linear algorithms. To transform it to an LMI, let $X(i) = P^{-1}(i)$. Pre- and post-multiply this inequality by $\text{diag}[X(i), \mathbb{I}, \mathbb{I}]$, where $\mathbb{I}$ is an appropriate identity matrix, gives:

$$\begin{bmatrix} \bar{J}_X(i) & B_\omega(i) & X(i)\bar{C}_z^\top(i) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \bar{C}_z(i)X(i) & B_z(i) & -\mathbb{I} \end{bmatrix} < 0$$

with $\bar{J}_X(i) = X(i)\bar{A}^\top(i) + \bar{A}(i)X(i) + X(i)\mathbb{W}(i)X^{-1}(i)\mathbb{W}(i)X(i) + \sum_{j=1}^N \lambda_{ij}X(i)X^{-1}(j)X(i)$

Notice that

$$\begin{aligned} X(i)\bar{A}^\top(i) + \bar{A}(i)X(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) \\ &\quad + B(i)Y(i) \\ \sum_{j=1}^N \lambda_{ij}X(i)X^{-1}(j)X(i) &= \lambda_{ii}X(i) + \mathcal{S}_i(X)\mathcal{X}_i^{-1}(X)\mathcal{S}_i^\top(X) \\ X(i)\left[C_z(i) + D_z(i)K(i)\right]^\top &= X(i)C_z^\top(i) + Y^\top(i)D_z^\top(i) \end{aligned}$$

where $Y(i) = K(i)X(i)$, and $\mathcal{S}_i(X)$ and $\mathcal{X}_i(X)$ are defined as:

$$\begin{aligned} \mathcal{S}_i(X) &= \Big[\sqrt{\lambda_{i1}}X(i), \cdots, \sqrt{\lambda_{ii-1}}X(i), \sqrt{\lambda_{ii+1}}X(i), \\ &\qquad \cdots, \sqrt{\lambda_{iN}}X(i)\Big] \\ \mathcal{X}_i(X) &= \text{diag}\left[X(1), \ldots, X(i-1), X(i+1), \cdots, Y(N)\right] \end{aligned}$$

Using Schur complement again this implies that the previous inequality is equivalent to the following:

$$\begin{bmatrix} J(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i) \end{bmatrix} & X(i)\mathbb{W}^\top(i) & \mathcal{S}_i(X) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 & 0 \\ \begin{bmatrix} C_z(i)X(i) \\ +D_z(i)Y(i) \end{bmatrix} & B_z(i) & -\mathbb{I} & 0 & 0 \\ \mathbb{W}(i)X(i) & 0 & 0 & -X(i) & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0$$

with $J(i) = X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \lambda_{ii}X(i)$.

From this discussion we get the following theorem.

Theorem 3.4 *Let γ be a positive constant. If there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N)) > 0$ and a set of matrices $Y = (Y(1), \cdots, Y(N))$ such that the following LMI holds for every $i \in \mathscr{S}$:*

$$\begin{bmatrix} J(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i) \end{bmatrix} & X(i)\mathbb{W}^\top(i) & \mathcal{S}_i(X) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 & 0 \\ \begin{bmatrix} C_z(i)X(i) \\ +D_z(i)Y(i) \end{bmatrix} & B_z(i) & -\mathbb{I} & 0 & 0 \\ \mathbb{W}(i)X(i) & 0 & 0 & -X(i) & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0 \qquad (18)$$

with $J(i) = X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \lambda_{ii}X(i)$*, then the system (1) under the controller (9) with* $K(i) = Y(i)X^{-1}(i)$ *is stochastically stable and moreover the closed-loop system satisfies the disturbance rejection of level* γ*.*

From the practical point of view, the controller that stochastically stabilizes the class of systems and at the same time guarantees the minimum disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

$$\mathrm{P}: \begin{cases} \min\limits_{\substack{\nu>0, \\ X=(X(1),\cdots,X(N))>0, \\ Y=(Y(1),\cdots,Y(N))}} \nu \\ s.t: \\ \begin{bmatrix} J(i) & B_\omega(i) & \begin{bmatrix} X(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i) \end{bmatrix} & X(i)\mathbb{W}^\top(i) & \mathcal{S}_i(X) \\ B_\omega^\top(i) & -\nu\mathbb{I} & B_z^\top(i) & 0 & 0 \\ \begin{bmatrix} C_z(i)X(i) \\ +D_z(i)Y(i) \end{bmatrix} & B_z(i) & -\mathbb{I} & 0 & 0 \\ \mathbb{W}(i)X(i) & 0 & 0 & -X(i) & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0 \end{cases}$$

where the LMI in the constraints is obtained from (18) by replacing γ^2 by ν.

The following theorem gives the results on the design of the controller that stochastically stabilizes the system (1) and simultaneously guarantees the smallest disturbance rejection level.

Theorem 3.5 *Let* $\nu > 0$*,* $X = (X(1), \cdots, X(N)) > 0$ *and* $Y = (Y(1), \cdots, Y(N))$ *be the solution of the optimization problem P. Then, the controller (9) with* $K(i) = Y(i)X^{-1}(i)$ *stochastically stabilizes the class of systems we are considering and moreover the closed-loop system satisfies the disturbance rejection of level* $\sqrt{\nu}$*.*

4 Numerical Examples

In the previous section, we developed results that determine the state feedback controller that stochastically stabilizes the class of systems we are treating in this paper and at the same time rejects the disturbance $w(t)$ with the desired level $\gamma > 0$. The conditions we developed are in the LMI form which makes their resolution easy. In the rest of this section we will give some numerical examples to show the usefulness of our results. Two numerical examples are presented.

Example 4.1 *Let us consider a system with two modes with the following data:*

- *transition probability rates matrix:*

$$\Lambda = \begin{bmatrix} -2.0 & 2.0 \\ 3.0 & -3.0 \end{bmatrix}$$

- *mode 1:*

$$A(1) = \begin{bmatrix} 1.0 & -0.5 \\ 0.1 & 1.0 \end{bmatrix}, B(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, B_w(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

$$B_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, W(1) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix} C_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

$$D_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

- *mode 2:*

$$A(2) = \begin{bmatrix} -0.2 & -0.5 \\ 0.5 & -0.25 \end{bmatrix}, B(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, B_w(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

$$B_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, W(2) = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.2 \end{bmatrix} C_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

$$D_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

First of all notice the system is instable in mode 1 and it is stochastically instable. Letting $\gamma = 10$ *and solving the LMI (18), we get:*

$$X(1) = \begin{bmatrix} 35.2579 & 2.3259 \\ 2.3259 & 29.7626 \end{bmatrix}, Y(1) = \begin{bmatrix} -37.3452 & -2.3342 \\ -2.3117 & -31.8415 \end{bmatrix},$$

$$X(2) = \begin{bmatrix} 44.3439 & -1.2064 \\ -1.2064 & 39.3224 \end{bmatrix}, Y(2) = \begin{bmatrix} -45.5124 & 1.4861 \\ 1.4905 & -40.7028 \end{bmatrix},$$

which gives the following gains:

$$K(1) = \begin{bmatrix} -1.0595 & 0.0044 \\ 0.0050 & -1.0702 \end{bmatrix}, K(2) = \begin{bmatrix} -1.0262 & 0.0063 \\ 0.0055 & -1.0349 \end{bmatrix}$$

All the conditions Theorem 3.4 are satisfied and therefore the closed-loop system is stochastically stable under the state feedback controller we designed for this system and also it assures the disturbance rejection of level 10.

Example 4.2 *To design a stabilizing controller that assures the minimum disturbance rejection, let us consider again the system with two modes we considered in the previous example and solve the optimization problem P. The resolution of such system gives:*

$$X(1) = \begin{bmatrix} 1.9289 & -0.0471 \\ -0.0471 & 1.8714 \end{bmatrix}, Y(1) = \begin{bmatrix} -2.9289 & 0.0471 \\ 0.0471 & -2.8714 \end{bmatrix},$$
$$X(2) = \begin{bmatrix} 2.0797 & -0.1916 \\ -0.1916 & 2.1665 \end{bmatrix}, Y(2) = \begin{bmatrix} -3.0797 & 0.1916 \\ 0.1916 & -3.1665 \end{bmatrix},$$

which gives the following gains:

$$K(1) = \begin{bmatrix} -1.5188 & -0.0131 \\ -0.0131 & -1.5347 \end{bmatrix}, K(2) = \begin{bmatrix} -1.4848 & -0.0429 \\ -0.0429 & -1.4654 \end{bmatrix}$$

Using the results of Theorem 3.5, it results that the system of this example is stochastically stable under the state feedback controller with the computed constant gain and assure the disturbance rejection of level $\gamma = 1.0$.

5 Conclusion

This paper dealt with the class of hybrid stochastic systems. Both the stability and the stabilizability problems are treated. A state feedback controller is proposed to stochastically stabilize the class of hybrid systems and at the same time rejection the disturbance with a desired level $\gamma > 0$. The conditions we developed are in LMI form which makes the resolution easier using the existing tools.

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