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Komei Fukuda
Bohdan Kaluzny

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Komei Fukuda
Institute for Operations Research and
Institute of Theoretical Computer Science
ETH Zentrum, Zurich, and
Institute of Mathematics / ROSO, EPFL
Lausanne, Switzerland
fukuda@ifor.math.ethz.ch

Bohdan Kaluzny
School of Computer Science
McGill University, Montreal, Canada
beezer@cs.mcgill.ca

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Abstract

Using deformed products of arrangements, we construct a family of linear programs with n inequalities in $\Re^d$ on which, in the worst-case, the least-index criss-cross method requires $\Omega(n^d)$ (for fixed d) pivots to reach optimality.

Résumé

En utilisant les produits déformés des polytopes, nous construisons une famille de programmes linéaires avec n contraintes en $\Re^d$ sur lesquels la méthode entrecroisée du plus-petit-indice prend $\Omega(n^d)$ (pour d fixé) pivots pour atteindre le sommet optimal.

Communicated by Professor David Avis, GERAD and School of Computer Science, McGill University.

1 Introduction

What is the maximal number of pivots, $C(d, n)$, (along a path) taken by the *least-index criss-cross method* to solve a linear program in dimension d with n inequality constraints? This question was posed by D. Avis in 2002 (private communication). Clearly

$$C(d, n) \leq \binom{n}{d} - 1 \leq n^d \quad \text{for } n \geq d \tag{1}$$

since the maximal number of vertices of an arrangement is $\binom{n}{d}$. In 1978, even before the birth of the least-index criss-cross method, Avis and Chvátal [1] unknowingly proved its exponential worst-case behaviour by exhibiting an example where the number of pivots taken by the simplex method with Bland's rule on a completely degenerate polytope is bounded from below by the $(2d)^{th}$ fibonacci number which is of the order $(1.618\cdots)^{2d}$.

$$C(d, n) \geq \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2d} - \left(\frac{1-\sqrt{5}}{2}\right)^{2d}\right) - 1 \quad \text{for } n \geq 2d. \tag{2}$$

The result follows from the observation that the least-index criss-cross method and the simplex method with Bland's rule follow the same pivot path on a completely degenerate polytope. In 1990, Roos [13] marginally improved the lower bound by constructing the first example of a nondegenerate polytope where the criss-cross method requires an exponential number of pivots. In his example, the criss-cross method starts at a vertex of the polytope and the inequalities are ordered so that the criss-cross method path mimics the behaviour of the simplex method with Bland's rule and thus always stays on the boundary of the polytope. He showed that

$$C(d, n) \geq 2^{2d} - 1 \quad \text{for } n \geq 2d. \tag{3}$$

Both of these constructions are variants of the *Klee-Minty* examples [10]. To date, all worst-case constructions for the criss-cross method provide a lower bound on $C(d, n)$ which, asymptotically, leaves a significant gap with the upper bound:

$$C(d, n) \text{ is } \Omega(2^{2d}) \text{ and } O(n^d) \text{ for } n \geq 2d \text{ where } d \text{ is fixed.} \tag{4}$$

In fact, it remained unclear whether the criss-cross method could take a path of length longer than the maximal number of vertices, $M(d, n)$, that a d-polytope with n facets can have.

$$C(d, n) \overset{?}{\geq} M(d, n) \ (= \theta(n^{\lfloor \frac{d}{2} \rfloor}) \quad \text{for fixed } d) \tag{5}$$

In the present paper we show how to construct a family of examples which not only answer this question affirmatively, but also show that $C(d, n)$ is $\Omega(n^d)$ for fixed d, implying that nearly every vertex of the arrangement can be visited.

Theorem 1 (Main Theorem) *For fixed d, the function $C(d,n)$ grows like a polynomial of degree d in n:*

$$C(d,n) \text{ is } \theta(n^d) \text{ for } n \geq 2d \tag{6}$$

Our construction uses the powerful tool of a *deformed product of arrangements*, an extension of a *deformed product of polytopes* as defined recently by Amenta and Ziegler [2]. The reader familiar with polytopes, linear programming, and deformed products of polytopes can breeze through section 2. Section 3 examines the behaviour of the least-index criss-cross method on a deformed product of polytopes, and by section 4 we are ready to construct a family of worst-case examples.

2 Preliminaries

2.1 Polyhedra

A *d-polyhedron* P is the intersection of n closed halfspaces (inequalities) in $\Re^d$ or equivalently the convex hull of a finite set of points in $\Re^d$ [8, p. 31]. We write these representations as

$$(\textbf{H}\text{alfspace-rep.}) \quad P = \{x \in \Re^d : a_i^t x \leq b_i \quad \text{for } 1 \leq i \leq n\} \text{ where } a_i \in \Re^d, \text{ and } b_i \in \Re, \tag{7}$$

and

$$(\textbf{V}\text{ertex-rep.}) \quad P = conv\{p_1, \ldots, p_m\} \text{ for points } p_i \in \Re^d. \tag{8}$$

The *faces* of P are all the subsets of the form $F = \{x \in P : \alpha^t x = \beta\}$ for some $\alpha \in \Re^d$, and $\beta \in \Re$, where $\alpha^t x \leq \beta$ is a *valid* inequality for P; meaning that $\alpha^t x \leq \beta$ is satisfied for all $x \in P$. The faces of P are themselves polyhedra and the faces of dimensions $0, 1, d-2, d-1$, and k are called *vertices*, *edges*, *ridges*, *facets,* and *k-faces* of P. A bounded polyhedron is called a *polytope*. The interested reader is referred to the textbooks by Grünbaum [8] and Ziegler [19] for a comprehensive study of convex polytopes and their properties.

Definition 1 (Simple Polytopes) *A d-polytope is simple if every vertex lies on exactly d facets.*

Definition 2 (Combinatorially Equivalent Polytopes) *Two polytopes P and Q are combinatorially equivalent if there is a bijection between their vertices, $vert(P) = \{p_1, \ldots, p_m\}$ and $vert(Q) = \{q_1, \ldots, q_m\}$, such that for any subset $I \subseteq \{1, \ldots, m\}$, the convex hull $conv\{p_i : i \in I\}$ is a face of P if and only if $conv\{q_i : i \in I\}$ is a face of Q.*

Definition 3 (Normally Equivalent Polytopes) *Two polytopes P and Q are normally equivalent if they are combinatorially equivalent and each facet $conv\{p_i : i \in I\}$ of P is parallel to the corresponding facet $conv\{q_i : i \in I\}$ of Q.*

Theorem 2 (Upper Bound Theorem, McMullen [11]) *A d-dimensional polytope with n facets has no more than*

$$\binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor} \tag{9}$$

vertices, where equality is attained only by the polars of neighborly polytopes (for example, by the polars of cyclic polytopes).

This upper bound is a polynomial in n of degree $\lfloor \frac{d}{2} \rfloor$ in the case of fixed dimension:

$$M(d,n) = \theta(n^{\lfloor \frac{d}{2} \rfloor}) \quad \text{for fixed } d. \tag{10}$$

2.2 Hyperplane Arrangements

A *hyperplane* is a set $h = \{x \in \Re^d : \alpha^t x = \beta\}$ for some nonzero $\alpha \in \Re^d$, and $\beta \in \Re$. A finite set of hyperplanes H in $\Re^d$ induces a decomposition of $\Re^d$ into connected open cells called an *arrangement* A_H. The $0, 1, 2, (d-1)$, and k-dimensional cells of A_H are termed vertices, edges, faces, facets, and k-cells. Two vertices of A_H are *adjacent* if they share $d-1$ hyperplanes, in other words they share an edge. Given a hyperplane arrangement, $A_H \subseteq \Re^d$, and a linear functional $\varphi : \Re^d \to \Re$ such that $\pi_{\min}$ and $\pi_{\max}$ are the vertices of A_H that minimize resp. maximize φ with $0 \le \varphi(\pi_{\min}) \le \varphi(\pi_{\max}) \le 1$, then we write $\varphi(vert(A_H)) \subseteq [0,1]$. We will assume that the hyperplanes are labeled as $H = \{h_1, h_2, \ldots, h_n\}$, and that hyperplanes are *oriented*: each has a positive side and negative side that are given by $\{x \in \Re^d : \alpha^t x > \beta\}$ and $\{x \in \Re^d : \alpha^t x < \beta\}$. An arrangement of oriented hyperplanes is an example of an *oriented matroid*, and shares all of its features (see [12]). In particular, each cell of the arrangement is represented as a *signed vector* in $\{+, 0, -\}^n$ indicating the position of the cell with respect to the hyperplanes of H.

Proposition 1 *A polytope $P \subseteq \Re^d$ induces an arrangement of oriented hyperplanes A_P.*

Proof. P is defined by the intersection of a finite number of halfspaces, and a halfspace is an oriented hyperplane. Thus P induces an arrangement of oriented hyperplanes A_P where the feasible region P is simply the interesection of the nonnegative sides of A_P. ■

Definition 4 (Combinatorially Equivalent Arrangements) *Two arrangements are combinatorially equivalent if the two sets of sign vectors of cells of the arrangements are exactly the same (i.e. the underlying oriented matroids are exactly the same).*

Definition 5 (Normally Equivalent Arrangements) *Two hyperplane arrangements are said to be normally equivalent if they are combinatorially equivalent and the corresponding unit hyperplane normals coincide.*

For an introduction to arrangements we refer to Halperin [9], and for a study on their combinatorial structure see [12] and [17].

2.3 Linear Programming

Linear Programming is the problem of maximizing a linear functional φ over a polyhedron P:

$$\max \varphi(x) : x \in P \subseteq \Re^d. \tag{11}$$

For the remainder of our discussion, we will assume that the linear program is *feasible* (P is non-empty), P is simple, and that φ is bounded by P.

Starting at a vertex defined by the set H of d intersecting hyperplanes, a *pivot* is the operation of exchanging a hyperplane $h_i \in H$ for a hyperplane $h_j \notin H$ that intersects the edge $H \backslash \{h_i\}$. This results in a second vertex defined by $H' := H \backslash \{h_i\} \cup \{h_j\}$ (henceforth abbreviated by $H - h_i + h_j$). *Pivot methods* attempt to solve a linear program, (P, φ), by moving along the edges of A_P from vertex to adjacent vertex, and *pivot rules* determine which adjacent vertex to travel to. The sequence of vertices visited is called the *pivot path*, and when a pivot rule guarantees convergence to optimality then we say it is *finite*. *Simplex (pivot) methods* try to solve a linear program by pivoting along the boundary edges of P from vertex $v_i \in P$ to vertex $v_j \in P$ such that $\varphi(v_i) < \varphi(v_j)$. For example, *Bland's least-index rule* [3], is a finite pivot rule for the simplex method where the edge chosen at v_i leaves the hyperplane that is indexed smaller than the other hyperplanes at v_i leaving which would offer to increase φ. Chvátal's classic book [4], provides an expository introduction to linear programming. See Terlaky and Zhang [15] for a survey on pivot rules.

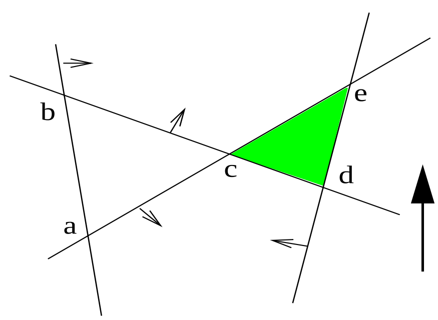


Figure 2.3

Definition 6 (Increasing Edges) *For a linear functional $\alpha : \Re^d \to \Re$, an edge $[p', p'']$ is α-increasing if $\alpha(p'') > \alpha(p')$.*

In Figure 2.3, $[a, b], [a, e]$, and $[d, b]$ are examples of increasing edges.

Definition 7 (Increasing Rays) *Given a start point $p \in \Re^d$ and a vector $\overrightarrow{v} \in \Re^d$, a ray $r = (p, \overrightarrow{v})$ is the set of all points of the form $p + \lambda\overrightarrow{v}$ for all scalar $\lambda \geq 0$. For a linear functional $\alpha : \Re^d \to \Re$, a ray is α-increasing if and only if $\alpha(\overrightarrow{v}) > 0$.*

Definition 8 (Primal Infeasible Vertex) *A vertex ξ_i of the arrangement induced by a linear program (P, φ) is primal infeasible if ξ_i is not a vertex of the polytope P. In other words, ξ_i violates at least one inequality of P.*

In Figure 2.3, a and b are primal infeasible.

Definition 9 (Dual Infeasible Vertex) *A vertex ξ_i, defined by the set H of d intersecting hyperplanes of the arrangement induced by a linear program (P, φ), is dual infeasible if there is at least one ray starting at ξ_i that is φ-increasing and satisfies all $h \in H$ (every point on the ray lies on the nonnegative side of all hyperplanes $h \in H$).*

In Figure 2.3, a, b, c, and d are dual infeasible.

The *optimal* vertex which maximizes φ is a vertex that is both primal and dual feasible.

2.4 Deformed Products of Polytopes

Recently Amenta and Ziegler [2] presented a construction of deformed products of polytopes that generalized all known constructions of linear programs with "many simplex pivots."

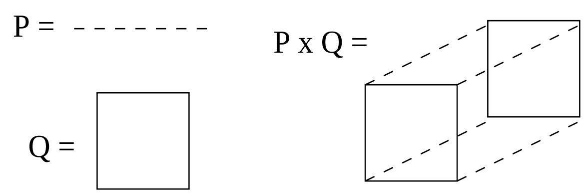
Definition 10 (Product of Polytopes) *The product of two polytopes $P \subseteq \Re^d$ and $Q \subseteq \Re^e$ is given by*

$$P \times Q = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} : \begin{array}{c} x \in P \\ u \in Q \end{array} \right\}. \tag{12}$$

The vertices of the product are given by

$$vert(P \times Q) = \left\{ \begin{pmatrix} p_i \\ q_j \end{pmatrix} : \begin{array}{c} p_i \in vert(P) \\ q_j \in vert(Q) \end{array} \right\}, \tag{13}$$

and the facet-defining inequalities for $P \times Q$ are the inequalities of P together with the inequalities of Q. Thus taking the product of two polytopes multiplies the number of vertices and sums the number of facets.



Definition 11 (Deformed Products of Polytopes) *Let $P \subseteq \Re^d$ be a convex polytope, and $\varphi : P \to \Re$ a linear functional with $\varphi(P) \subseteq [0, 1]$. Let $V,\ W \subseteq \Re^e$ be convex polytopes. Then the deformed product of (P, φ) and of (V, W) is*

$$(P, \varphi) \bowtie (V, W) := \left\{ \begin{pmatrix} x \\ v + \varphi(x)(w - v) \end{pmatrix} : \begin{array}{c} x \in P \\ v \in V, w \in W \end{array} \right\} \subseteq \Re^{d+e}. \tag{14}$$

To help visualize the deformed product, it is helpful to observe that when $V = W$, then the deformed product is the standard product:

$$(P, \varphi) \bowtie (V, V) = P \times V, \tag{15}$$

and if we examine a cross-section of the deformed product at some $\varphi(x) = \lambda$, then we will observe the Minkowski sum of λW with $(1 - \lambda)V$.

We now state some facts about deformed products that are proved in [2].

Lemma 1 *Deformed products are convex.*

Theorem 3 *Let $P \subseteq \Re^d$ be a d-polytope, $\varphi : \Re^d \to \Re$ a linear function with $\varphi(P) \subseteq [0,1]$, and $V, W \subseteq \Re^e$ be normally equivalent e-polytopes, then:*

- If P has m vertices and s facets, and if V and W have n vertices and t facets, then $Q := (P,\varphi) \bowtie (V,W)$ is a $(d+e)$-polytope with mn vertices and $s+t$ facets.
- For $P = conv\{p_1, \dots, p_m\}$, $V = conv\{v_1, \dots, v_n\}$, and $W = conv\{w_1, \dots, w_n\}$, the deformed product is a polytope given by

$$Q = (P,\varphi) \bowtie (V,W) = conv\{q(i,j) : 1 \le i \le m, 1 \le j \le n\}. \tag{16}$$

where the vertices of Q, which we denote $q(i,j)$, are defined as:

$$q(i,j) = \begin{pmatrix} p_i \\ v_j + \varphi(p_i)(w_j - v_j) \end{pmatrix} : \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n. \end{matrix} \tag{17}$$

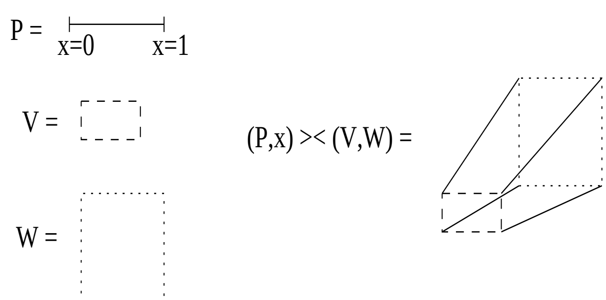
- If P, V and W are given by

$$\begin{aligned} P &= \{x \in \Re^d : \quad a_k x \le \xi_k \text{ for } 1 \le k \le s\}, \\ V &= \{u \in \Re^e : \quad b_l u \le \beta_l \text{ for } 1 \le l \le t\}, \text{ and} \\ W &= \{u \in \Re^e : \quad b_l u \le \beta'_l \text{ for } 1 \le l \le t\}, \end{aligned} \tag{18}$$

then the deformed product is given by

$$(P,\varphi) \bowtie (V,W) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \Re^{d+e} : \quad \begin{matrix} a_k x \le \xi_k \text{ for } 1 \le k \le s \\ (\beta_l - \beta'_l)\varphi(x) + b_l u \le \beta_l \text{ for } 1 \le l \le t \end{matrix} \right\}. \tag{19}$$

- $Q = (P,\varphi) \bowtie (V,W)$ is combinatorially equivalent to $P \times V$.



Note that Q has two types of edges:

- *P-edges* of the form $[q(i',j), q(i'',j)]$ for $1 \le j \le n$ and for any $1 \le i', i'' \le m$ such that $p_{i'}$ and $p_{i''}$ are adjacent vertices of P, and
- *(V,W)-edges* of the form $[q(i,j'), q(i,j'')]$ for $1 \le i \le m$ and for any $1 \le j', j'' \le n$ such that $v_{j'}$ and $v_{j''}$ are adjacent vertices of V (or, equivalently, $w_{j'}$ and $w_{j''}$ are adjacent vertices of W).

Definition 12 (Deformed Product Programs) *Let*

$$\max \varphi(x): \quad x \in P \tag{20}$$

be a linear program in $\Re^d$ *and let*

$$\max \alpha(u) : u \in V \quad \text{and} \quad \max \alpha(u) : u \in W \tag{21}$$

be two normally equivalent linear programs. Define the deformed product program as

$$\max \widehat{\alpha}\binom{x}{u} = \alpha(u): \quad \binom{x}{u} \in Q = (P, \varphi) \bowtie (V, W). \tag{22}$$

The resulting linear program is the deformed product polytope Q *with objective function* $\max \alpha(u)$.

Now let's examine the edges of a deformed product program as defined in (22).

Proposition 2 *A* P*-edge* $[q(i', j), q(i'', j)]$ *is* $\widehat{\alpha}$*-increasing if and only if either* $[p', p'']$ *is* φ*-increasing and* $\alpha(w_j) > \alpha(v_j)$*, or* $[p', p'']$ *is* φ*-decreasing and* $\alpha(w_j) < \alpha(v_j)$.

Proposition 3 *A* (V, W)*-edge* $[q(i, j'), q(i, j'')]$ *is* $\widehat{\alpha}$*-increasing if and only if* $[v_{j'}, v_{j''}]$ *is* α*-increasing.*

3 Criss-Cross Methods

Criss-cross methods are pivot methods for solving a linear program (P, φ) whose pivot path can leave the boundary of P. The first criss-cross method was baptized by Zoints [18], and the first finite criss-cross method, the *least-index criss-cross method*, was discovered independently by Terlaky [14], and Wang [16]. The reader is invited to learn about the properties, history, and recent developments pertaining to criss-cross methods by looking at a survey by Fukuda and Terlaky [5].

3.1 The Least-Index Criss-Cross Method

As the name suggests, criss-cross methods have two types of pivots (with respect to an objective function φ): *admissible type I* pivots and *admissible type II* pivots.

Definition 13 (Admissible Type I Pivot) *For every primal infeasible vertex* ξ *defined by the set* H *of* d *intersecting hyperplanes, there exists an oriented hyperplane* $h_j \notin H$ *that is violated at* ξ*. A pivot from* ξ *to vertex* ξ'*, defined by* $H' := H - h_i + h_j$*, is an admissible type I pivot if* ξ' *lies on the nonnegative side of* h_i.

If h_j is selected such that j is minimized, followed by selecting h_i to minimize i, then the pivot is a *least-index admissible type I* pivot. In Figure 2.3, a pivot from a to c is of *type I*.

Definition 14 (Admissible Type II Pivot) *For every dual infeasible vertex ξ defined by the set H of d intersecting hyperplanes, there exists a φ-increasing ray $(\xi, \overrightarrow{v})$ that lies on an edge $H \backslash \{h_i\}$ that is on the nonnegative side of $h_i \in H$. A pivot from ξ to vertex ξ', defined by $H' := H - h_i + h_j$, is an admissible type II pivot if there exists a point on $(\xi, \overrightarrow{v})$ that lies on the nonpositive side of h_j.*

If h_i is selected such that i is minimized, followed by selecting h_j to minimize j, then the pivot is a *least-index admissible type II* pivot. In Figure 2.3, a pivot from c to e is of *type II*, as is a pivot from b to a.

We will denote a pivot, exchanging h_i for h_j, by *pivot*(i, j). Using these notions, we provide the geometric interpretation of the least-index criss-cross method:

Algorithm 1 (The Least-Index Criss-Cross Method) *Given a linear program $(P, \varphi) \subseteq \Re^d$, a linear ordering of the inequalities of P, and a vertex ξ of A_P:*

Criss-Cross:

If ξ is optimal (both primal feasible and dual feasible) then ***Stop;***

If ξ is primal feasible then let $p := +\infty$. Otherwise let $p := j$ such that pivot(i, j) is the least-index admissible type I pivot from ξ to ξ'. If no pivot exists then the linear program is primal inconsistent, ***Stop;***

If ξ is dual feasible then let $q := +\infty$. Otherwise let $q := i'$ such that pivot(i', j') is the least-index admissible type II pivot from ξ to ξ''. If no pivot exists then the linear program is dual inconsistent, ***Stop;***

If $p < q$, then $\overline{\xi} := \xi'$. Otherwise $\overline{\xi} := \xi''$.

Pivot from ξ to $\overline{\xi}$, let $\xi := \overline{\xi}$ and go to ***Criss-Cross;***

Theorem 4 *The least-index criss-cross method is finite*

Theorem 5 *The least-index criss-cross method solves a linear program.*

See [7] for simple proofs. From this point forward the criss-cross method will refer to the least-index criss-cross method.

3.2 Deformed Product of Arrangements

Our goal is to construct a family of deformed product programs on which the criss-cross method visits almost all vertices of the arrangement. We begin by analyzing the behaviour of the criss-cross method on the arrangement of hyperplanes of a deformed product program. Hence, we define a *deformed product of arrangements* to be the induced hyperplane arrangement of a deformed product of polytopes, and we extend Theorem 3 to express the properties of the induced hyperplane arrangement of $(Q = (P, \varphi) \bowtie (V, W), \widehat{\alpha})$. We facilitate understanding by providing an example (see Figure 3.2) which the reader is encouraged to refer to in order to verify the theorem's statements.

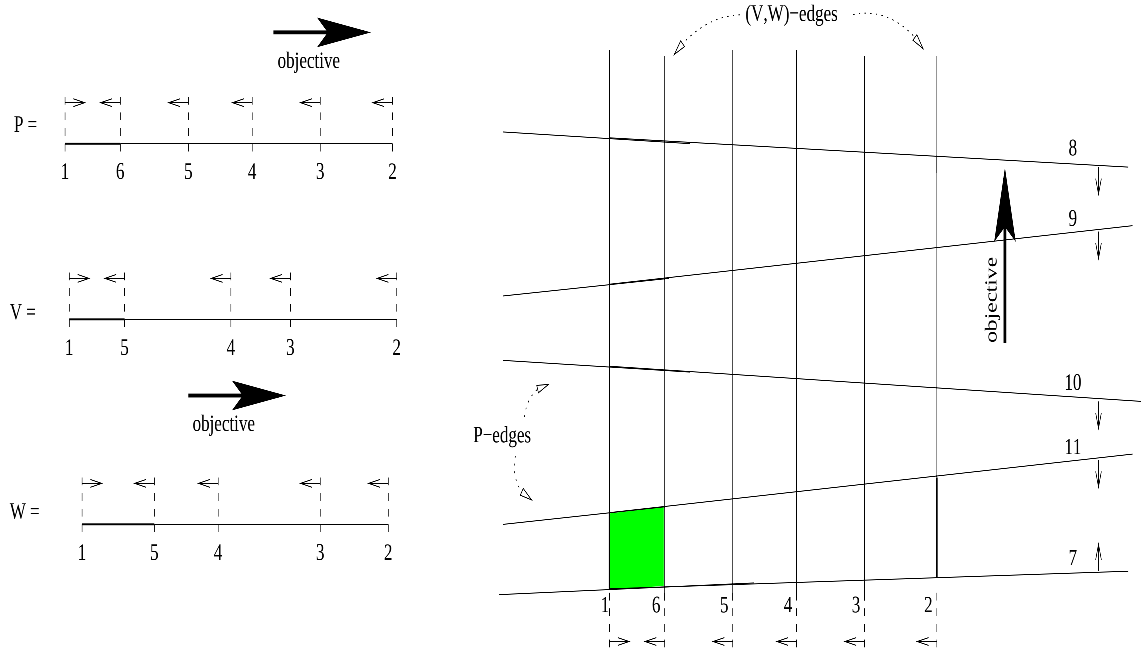


Figure 3.2

Theorem 6 *Let $P \subseteq \Re^d$ be a d-polyhedron, A_P be the underlying hyperplane arrangement of P, $\varphi : \Re^d \to \Re$ a linear function such that $\varphi(vert(A_P)) \subseteq [0,1]$, and $V, W \subseteq \Re^e$ be normally equivalent e-polyhedra inducing normally equivalent hyperplane arrangements A_V and A_W, then:*

- If A_P has $\overline{m}$ vertices and s hyperplanes, and if V and W have $\overline{n}$ vertices and t hyperplanes each, then $Q := (P, \varphi) \bowtie (V, W)$ is a $(d+e)$-polytope whose underlying arrangement A_Q has at least $\overline{m} \cdot \overline{n}$ vertices and exactly $s + t$ hyperplanes.
- Specifically, if $\{\pi_1 \ldots \pi_m\}$ are vertices of A_P, $\{\upsilon_1 \ldots \upsilon_n\}$ and $\{\omega_1 \ldots \omega_n\}$ the vertices of A_V resp. A_W, then we can define $\overline{m} \cdot \overline{n}$ of the vertices of Q, denoted $\gamma(i,j)$, as:

$$\gamma(i,j) = \begin{pmatrix} \pi_i \\ \upsilon_j + \varphi(\pi_i)(\omega_j - \upsilon_j) \end{pmatrix} : \quad \begin{array}{l} 1 \leq i \leq m \\ 1 \leq j \leq n. \end{array} \tag{23}$$

- If A_P, A_V, and A_W are given by

$$\begin{aligned} A_P &= \{a_k x \leq \xi_k \text{ for } 1 \leq k \leq s\}, \\ A_V &= \{b_l u \leq \beta_l \text{ for } 1 \leq l \leq t\}, \text{ and} \\ A_W &= \{b_l u \leq \beta'_l \text{ for } 1 \leq l \leq t\}, \end{aligned} \tag{24}$$

  then the arrangement of hyperplanes of the deformed product Q is given by

$$A_Q = \left\{ \begin{array}{c} a_k x \leq \xi_k \text{ for } 1 \leq k \leq s \\ (\beta_l - \beta'_l)\varphi(x) + b_l u \leq \beta_l \text{ for } 1 \leq l \leq t \end{array} \right\}. \tag{25}$$

- A cell C_Q^i of A_Q is the deformed product of some cell C_P^j with (C_V^k, C_W^k): $C_Q^i = (C_P^j, \varphi) \bowtie (C_V^k, C_W^k)$.

Corollary 1 *A cell C_Q^i of the hyperplane arrangement of the deformed product is convex if $0 \leq \varphi(y) \leq 1$ for $y \in C_Q^i$.*

Proof. This follows from Lemma 1 which states that the deformed product of a polytope is convex if $\varphi(P) \subseteq [0, 1]$. ■

Let's examine the edges of a hyperplane arrangement underlying a deformed product program. Note that A_Q has two types of edges:

- A_P*-edges* of the form $[\gamma(i', j), \gamma(i'', j)]$ for $1 \leq j \leq n$ and for any $1 \leq i', i'' \leq m$ such that $\pi_{i'}$ and $\pi_{i''}$ are adjacent vertices of A_P and
- $A_{(V,W)}$*-edges* of the form $[\gamma(i, j'), \gamma(i, j'')]$ for $1 \leq i \leq m$ and for any $1 \leq j', j'' \leq n$ such that $\upsilon_{j'}$ and $\upsilon_{j''}$ are adjacent vertices of A_V (equivalently, $\omega_{j'}$ and $\omega_{j''}$ are adjacent vertices of A_W).

Proposition 4 *Given a deformed product program (22), an A_P-edge $[\gamma(i', j), \gamma(i'', j)]$ is $\widehat{\alpha}$-increasing if and only if either $[\pi_{i'}, \pi_{i''}]$ is φ-increasing and $\alpha(\omega_j) > \alpha(\upsilon_j)$, or $[\pi', \pi'']$ is φ-decreasing and $\alpha(\omega_j) < \alpha(\upsilon_j)$.*

Proposition 5 *A (V, W)-edge $[\gamma(i, j''), \gamma(i, j'')]$ is $\widehat{\alpha}$-increasing if and only if $[\upsilon_{j'}, \upsilon_{j''}]$ is α-increasing.*

The proofs of the preceding statements follow naturally from the proofs given in [2] for deformed products of polytopes. We are now ready to analyze the behaviour of the criss-cross method on deformed product programs.

Corollary 2 (The Criss-Cross Method on Deformed Product Programs) *Let π_1 and π_l be the vertices of P that minimize respectively maximize φ with $0 \leq \varphi(\pi_1) \leq \varphi(\pi_l) \leq 1$. Construct the deformed product program Q, as defined in (22). If we number the inequalities of Q such that the inequalities of P get smaller indices than the inequalities corresponding to (V, W), then the criss-cross method prefers to pivot along A_P-edges rather than $A_{(V,W)}$-edges.*

The result is that if the criss-cross method on (P, φ) for the objective function φ takes a path of length l from the π_1 to π_l, and for $-\varphi$ takes a path of length l' from π_l to π_1, then for $(Q, \widehat{\alpha})$ the criss-cross method will follow a path of length l from $\gamma(1, j)$ to $\gamma(l, j)$ if $\alpha(\upsilon_j) < \alpha(\omega_j)$, and a path of length l' from $\gamma(l, j)$ to $\gamma(1, j)$ if $\alpha(\upsilon_j) > \alpha(\omega_j)$.

4 The Construction

We construct the worst-case example by first building low dimensional examples where the criss-cross method takes many pivots. We then show how to take deformed products of these base cases to construct polyhedra in any dimension where the criss-cross method behaves badly.

Definition 15 *Let $C(d,n)$ be the maximal number of pivots taken by the least-index criss-cross method for some linear objective function α on a d-dimensional polyhedron with at most n facets.*

Definition 16 *Starting at the vertex of P that minimizes α, let $H(d,n)$ be the maximal number of vertices visited along a path taken by the least-index criss-cross method for some linear objective function max α on the arrangement of hyperplanes induced by a d-dimensional polyhedron with at most n facets.*

Clearly $C(d,n) \geq H(d,n) - 1$.

Lemma 2 *For $n \geq 2$, $H(1,n) = n$*

Proof. Consider the following linear program defined by $max\ \varphi = x$ and polytope, given by the inequalities (indexed in order of appearance): $x \geq 0$ and $x \leq (n-i)\lambda$ for $1 \leq i \leq n-1$ and for some constant $\lambda > 0$.

For example, for $n = 6$:

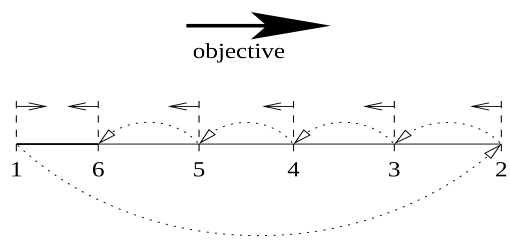


Figure 4a

Let v_i be the vertex defined by hyperplane h_i for $1 \leq i \leq n$. The criss-cross method takes a path of length n from vertex v_1 to vertex v_n when $\varphi = x$. The criss-cross method takes a path of length one from vertex v_n to vertex v_1 when $\varphi = -x$. ■

Note that the construction has $n - 2$ redundant constraints.

Lemma 3 *There exists a pair of normally equivalent 1-polytopes, (V, W), defined by k inequalities each (hence k vertices), and a linear functional α, such that $\alpha(v_i) > \alpha(w_i)$ if i is even and $\alpha(v_i) < \alpha(w_i)$ if i is odd.*

Proof. Construct V as in Lemma 2. To build W, for each hyperplane h_i of V, construct h'_i of W by translating h_i in the positive x-direction by (some suitably small) $\epsilon > 0$ if i is odd and by $-\epsilon$ if i is even. The case when $k = 5$ in is illustrated in Figure 3.2. ■

Note that $\alpha(v_k) < \alpha(w_k)$ when k is odd and $\alpha(v_k) > \alpha(w_k)$ if k is even. The following example illustrates the construction of a deformed product and the path that the criss-cross method takes on the underlying arrangement.

Example 1 (See Figure 4b) Construct P (6 inequalities, variable x_1, $\lambda = 0.1$) and V (5 inequalities, variable x_2) as in Lemma 2, and W (5 inequalities, variable x_2) as in Lemma 3. Let $Q = (P, x) \bowtie (V, W)$ and order the inequalities of Q so that the inequalities coming from P are indexed smaller than those from (V, W). Consider the path that the criss-cross method takes on the deformed product program $(Q, \widehat{\alpha} = x_2)$.

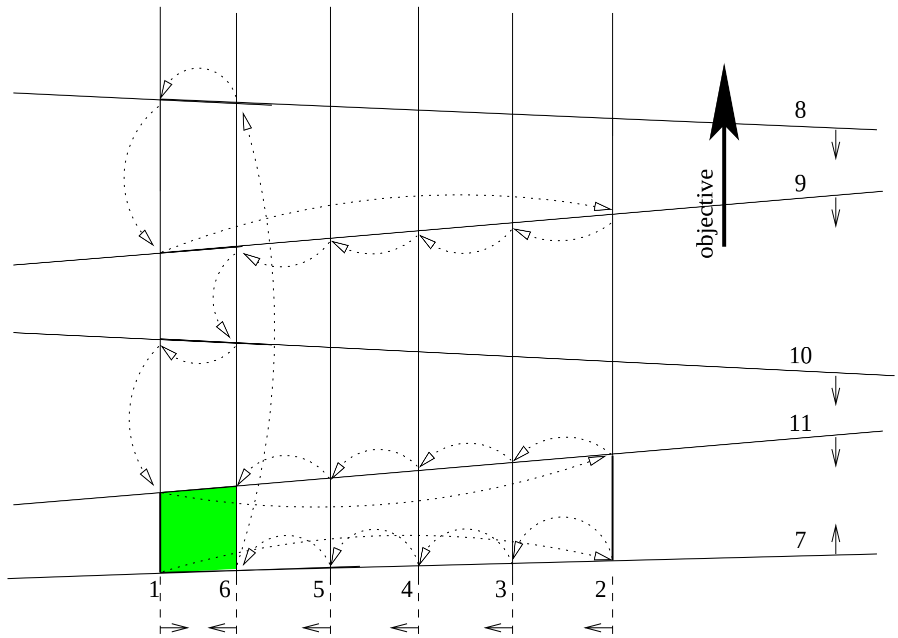


Figure 4b

Theorem 7 *For $k \geq 2$ and $n > d \geq 0$,*

$$H(d+1, n+k) \geq \left\lceil \frac{k}{2} \right\rceil \cdot H(d,n) \tag{26}$$

Proof. Take a polytope $P \subseteq \Re^d$ with n inequalities for which the least-index criss-cross method for a linear functional $\varphi(x)$ (rescaled such that $\varphi(vert(A_P)) \subseteq [0,1]$) follows a criss-cross method path of length $l = H(d,n)$ starting at vertex p_1 and ending at vertex p_l. Now construct the deformed product program

$$\begin{aligned} &\max \ \alpha \\ &\text{st .: } Q = (P, \varphi) \bowtie (V, W), \end{aligned}$$

where $V, W \subseteq \Re$ and α are defined according to Lemma 3. By Corollary 2 we get that the criss-cross method applied to (Q, α) first follows a P-path with l vertices from $\gamma(1,1)$ to $\gamma(l,1)$, then after one (V,W)-pivot it follows a P-path of length one from $\gamma(l,2)$ to $\gamma(1,2)$, then after one (V,W)-pivot it follows a P-path with l vertices from $\gamma(1,3)$ to $\gamma(l,3)$, then after one (V,W)-pivot it follows a P-path of length one from $\gamma(l,4)$ to $\gamma(1,4)$, etc. The complete path will thus visit $\lceil \frac{k}{2} \rceil \, l + \lfloor \frac{k}{2} \rfloor$ vertices arriving at $\gamma(1,k)$ or $\gamma(l,k)$, depending on whether k is even or odd. ∎

Remark 1 We could use this result to construct examples, by induction, where $C(d,n)$ is $\Omega(n^d)$ asymptotically for fixed d. However we choose to postpone this analysis since iterative deformed products with the 1-dimensional construction would contain a large number of redundant constraints, in fact $n - 2d$ of them.

Lemma 4 *For $n \geq 3$, $H(2,n)$ is $\Omega(n^2)$.*

Proof. Consider the following construction: let the i^{th} inequality of P be defined as

$$-2(i-1)x_1 - (2(n-i)-1)x_2 \leq -2(i-1)(2(n-i)-1). \tag{27}$$

This construction ensures that the x_1 intercept of the i^{th} inequality is greater than the x_1 intercept of the $(i-1)^{th}$ while the x_2 intercept of the i^{th} inequality is less than that of the $(i-1)^{th}$ (see Figure 4c).

The least-index criss-cross method on the linear program

$$\begin{aligned} &\text{Maximize } -x_2 \\ &\text{s.t.: } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in P, \end{aligned} \tag{28}$$

starting at the vertex defined by the intersection of hyperplanes 1 and n (which we denote $(1,n)$), will take $n-1$ *type I* pivots $(1,n) \rightarrow (2,n) \rightarrow \cdots \rightarrow (n-1,n)$, and then from $(n-1,n)$ take one *type II* pivot to $(1,n-1)$, and then take $n-2$ *type I* pivots to $(n-2,n-1)$, and then one *type II* pivot to $(1,n-2)$, and then take $n-3$ *type I* pivots to $(1,n-3)$, etc. and ending with one *type II* pivot from $(2,3)$ to $(1,2)$ visiting a total of $\frac{n(n-1)}{2} = \binom{n}{2}$ vertices. ∎

For example, when $n = 7$:

	Maximize $-x_2$
1:	$0x_1 - 11x_2 \leq 0$
2:	$-2x_1 - 9x_2 \leq -18$
3:	$-4x_1 - 7x_2 \leq -28$
4:	$-6x_1 - 5x_2 \leq -30$
5:	$-8x_1 - 3x_2 \leq -24$
6:	$-10x_1 - 1x_2 \leq -10$
7:	$-12x_1 - 0x_2 \leq 0$

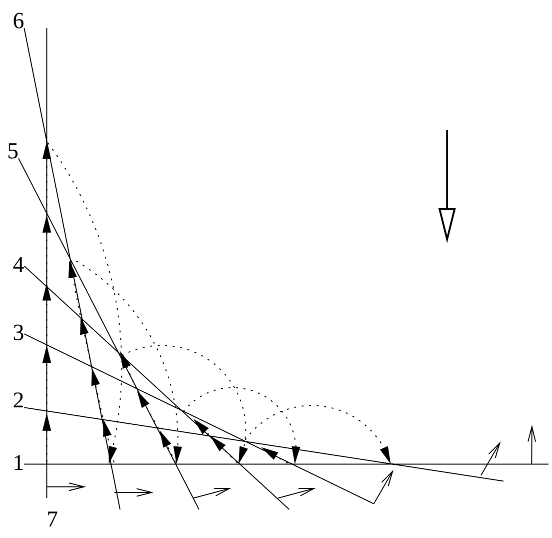


Figure 4c

We offer the following remarks about the construction of Lemma 4:

Remark 2 There are $n-2$ *type II* pivots, and $\frac{(n-1)(n-2)}{2}$ *type I* pivots.

Remark 3 For every *type I* pivot from intersection (i, j) to (g, j), if i is odd then g is even, and if i is even then g is odd. Every *type II* pivot has the form (i, j) to $(1, i)$ where $j = i + 1$.

Lemma 5 *There exist normally equivalent 2-dimensional polyhedra V and W with k facets ($k \geq 4$), and objective function min α for which the criss-cross method takes $\theta(k^2)$ pivots such that corresponding vertices υ of V and ω of W, defined by the intersection of hyperplanes h_i and h_j for $i < j$, have the following property:*

$$\alpha(\upsilon) > \alpha(\omega) \text{ when } i \text{ is odd, and } \alpha(\upsilon) < \alpha(\omega) \text{ when } i \text{ is even.}$$

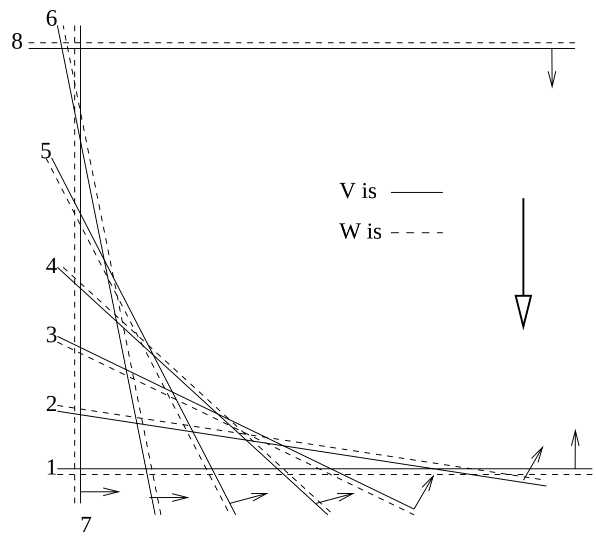


Figure 4d

Proof. Let V be a 2-polyhedron with $k - 1$ inequalities as defined in (27) and define the k^{th} inequality as

$$0x_1 + x_2 \leq 2(k-2) + C \quad \text{for } C > \frac{k}{2}.$$

This additional inequality ensures that V bounds both α and $-\alpha$. C is chosen such that the x_2 intercept of the k^{th} inequality is greater than that of the $(k-2)^{th}$. Build W as follows: for $1 \leq i \leq k-1$ take the i^{th} inequality of V, $b_i x \leq \beta$, and define the i^{th} inequality of W to be $b_i x \leq \beta'$ where $\beta' = \beta - \epsilon$ if i is odd and $\beta' = \beta + \epsilon$ if i is even (see Figure 4d). ϵ is chosen to be positive and suitably small. Let the k^{th} inequality of W be $b_k x \leq \beta'$ where $\beta' = \beta + \epsilon$. Now lets examine corresponding vertices of V and W, υ and ω, defined by the intersection of hyperplanes h_i and h_j for $i < j$:

Case 1: i is odd and j odd. This case is illustrated in Figure 4e. n_i and n_j represent the normals of h_i and h_j respectively, or if you wish the direction of translation by ϵ: $|n_i| = |n_j| = \epsilon$. Let $\delta = |d|, \theta_1 = angle(A)$, and $\theta_2 = angle(B)$. By construction, $0° \leq \theta_1 < \theta_2 \leq 90°$, and $\delta = \epsilon \sin\theta_1$ where $\sin\theta_1 \geq 0°$. Now $\alpha(w) < \alpha(\upsilon - \delta) \leq \alpha(\upsilon)$ which implies $\alpha(\upsilon) > \alpha(\omega)$.

Case 2: i is even and j even. This case is symmetric to *case 1*, hence $\alpha(\upsilon) < \alpha(\omega)$.

Case 3: i is odd and j even. This case is illustrated in Figure 4f. n_i and n_j represent the normals of h_i and h_j respectively, the direction of translation by ϵ: $|n_i| = |n_j| = \epsilon$. Let $\delta = |d|, \theta_1 = angle(A)$, and $\theta_2 = angle(B)$. By construction, $0° \leq \theta_1 < \theta_2 \leq 90°$, and $\delta = \epsilon \sin(90° - \theta_2)$ where $\sin(90° - \theta_2) > 0$. Now $\alpha(w) \leq \alpha(v - \delta) < \alpha(v)$ which implies $\alpha(\upsilon) > \alpha(\omega)$.

Case 4: i is even and j odd. This case is symmetric to *case 3*, hence $\alpha(\upsilon) < \alpha(\omega)$.

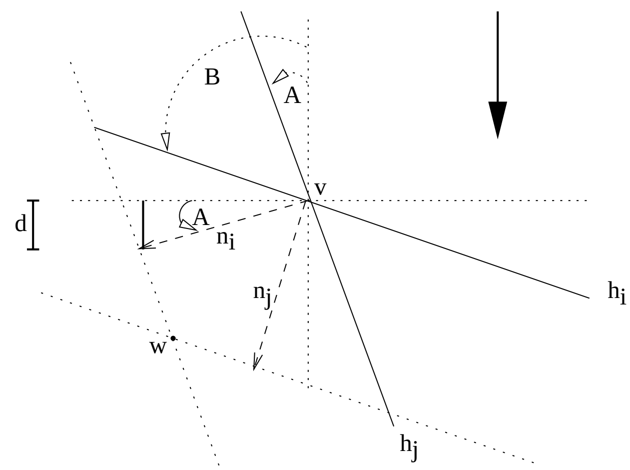


Figure 4e

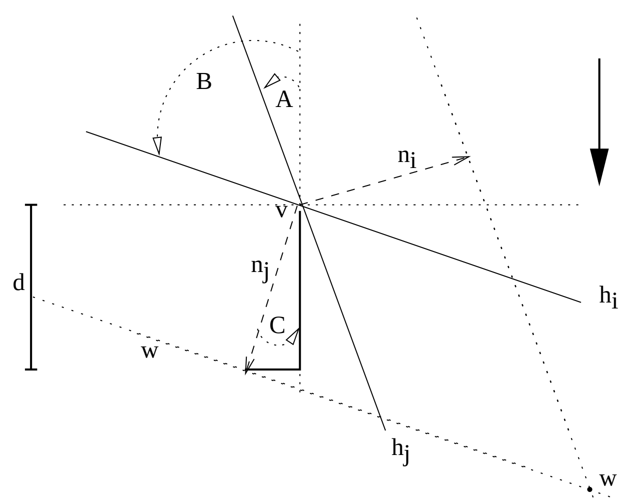


Figure 4f

■

We offer the following remarks about the construction of Lemma 5:

Remark 4 Starting at $(n-2, n)$ the criss-cross method on V (or W) will take one *type II* pivot to $(1, n)$ and then follow the path described in Lemma 4.

Remark 5 There are $k-2$ *type II* pivots, and $\frac{(k-2)(k-3)}{2}$ *type I* pivots (see Remark 2 setting $n = k-1$ and adding one additional *type II* pivot from $(k-2, k)$ to $(1, k)$).

Definition 17 (Switch Pivot) *Given two normally equivalent polyhedra V and W, and a linear objective function α we define a switch pivot to be a pivot from v_i to v_j (w_i to w_j) such that if $\alpha(v_i) > \alpha(w_i)$ then $\alpha(v_j) < \alpha(w_j)$, otherwise if $\alpha(v_i) < \alpha(w_i)$ then $\alpha(v_j) > \alpha(w_j)$.*

Lemma 6 *Let (V, W) be as defined in Lemma 5. Starting at the intersection of h_{k-1} and h_k the least-index criss cross method takes a $\theta(k^2)$ path to the intersection of h_1 and h_2 on which there are $\theta(k^2)$ switch pivots.*

Proof. The least-index criss-cross method on (V, W) takes a $\theta(k^2)$ length path (see Lemma 4). There are five types of pivots with respect to indices of the intersecting hyperplanes of the first vertex v_1 defined by h_i and h_j, and the second vertex v_2 defined by $h_{i'}$ and $h_{j'}$ ($j' = j$ for *type I* pivots):

Pivot	**from (i, j) where $i < j$**		**to (i', j') where $i' < j'$**	**Switch Pivot?**
Type I	i is *odd*	$\rightarrow$	i' is *even*	yes
	i is *even*	$\rightarrow$	i' is *odd*	yes
Type II	i is *odd*	$\rightarrow$	$i' = 1$ is *odd*	no
	i is *even*	$\rightarrow$	$i' = 1$ is *odd*	yes
	$(k-2, k)$	$\rightarrow$	$(1, k)$	yes only if k is even

Thus every *type I* pivot is a switch pivot, and every second *type II* pivot is a switch pivot,

$$\# \text{ of switch pivots} = \begin{cases} \frac{(k-2)(k-3)}{2} + \frac{k-3}{2} & \text{if } k \text{ is odd,} \\ \frac{(k-2)(k-3)}{2} + \frac{k-4}{2} + 1 & \text{if } k \text{ is even,} \end{cases} \tag{29}$$

$$\geq \frac{k^2}{C} \quad \text{for some constant } C > 1 \text{ and all } k \geq 3. \tag{30}$$

The number of switch pivots is $\theta(k^2)$. ■

Theorem 8 *For $k \geq 3$, $n > d \geq 0$, and some constant $C > 1$,*

$$H(d+2, n+k) \geq \frac{k^2}{2C} \cdot H(d, n). \tag{31}$$

Proof. Take a polytope $P \subseteq \Re^d$ with n inequalities for which the least-index criss-cross method for a functional $\varphi(x)$ (rescaled such that $\varphi(vert(A_P)) \subseteq [0,1]$) follows a criss-cross method path of length $l = H(d,n)$ starting at vertex p_1 and ending at vertex p_l. Let l' be the length of the criss-cross method path from p_l to p_1 for $-\varphi$. Now construct the deformed product program

$$\max \alpha \tag{32}$$
$$\text{s.t.: } Q = (P, \varphi) \bowtie (V, W),$$

where V, W and α are defined according to Lemma 5. Let v_{opt} be the optimal vertex of (V, W). By Corollary 2, we get that the criss-cross method applied to (Q, α) first follows a P-path with l vertices from $\gamma(1,1)$ to $\gamma(l,1)$, and then after a (V,W)-*switch pivot* it follows a P-path of length l', and then after a (V,W)-*switch pivot* it follows a P-path of length l, and then after a (V,W)-*switch pivot* it follows a P-path of length l', etc. The complete path will visit at least $\frac{1}{2}\frac{k^2}{C}l + \frac{1}{2}\frac{k^2}{C}l'$ vertices ending at (l, opt). ■

Corollary 3 *For $n \geq 2d \geq 2$ and some constant $C > 1$, $C(d,n) = \Omega((\frac{n}{d})^d)$. More specifically:*

$$C(d,n) \geq \left\lfloor \frac{2n}{d\sqrt{2C}} \right\rfloor^d \qquad \text{if } d \text{ is even,} \tag{33}$$

and

$$C(d,n) \geq \left\lfloor \frac{2n}{(d+1)\sqrt{2C}} \right\rfloor^d \qquad \text{if } d \text{ is odd.} \tag{34}$$

Proof. *(By induction)* Lets begin with the even case, when $d = 2m$ for all $m \geq 0$, let $n = km$:

$$\begin{aligned} H(2m, km) &\geq \frac{k^2}{2C} H(2(m-1), k(m-1)) \\ &\vdots \qquad m \text{ times} \\ &\geq \left(\frac{k^2}{2C}\right)^{m-1} H(2,k) \\ &= \frac{k^{2m}}{(2C)^m} \qquad \text{by Lemma 4.} \end{aligned}$$

Substituting for $m = \frac{d}{2}$ and $k = \lfloor \frac{2n}{d} \rfloor$, we get

$$H(d,n) \geq \left\lfloor \frac{2n}{d\sqrt{2C}} \right\rfloor^d. \tag{35}$$

For the odd case, when $d = 2m + 1$ for all $m \geq 0$, let $n = k(m + 1)$:

$$\begin{aligned} H(2m+1, k(m+1)) &\geq \frac{k^2}{2C} H(2(m-1)+1, km) \\ &\vdots \quad m \text{ times} \\ &\geq \left(\frac{k^2}{2C}\right)^m H(1,k) \\ &= \frac{k^{2m+1}}{(2C)^m} \qquad \text{by Lemma 2.} \end{aligned}$$

Substituting for $m = \frac{d-1}{2}$ and $k = \left\lfloor \frac{2n}{d+1} \right\rfloor$, we get

$$H(d,n) \geq \left\lfloor \frac{2n}{(d+1)\sqrt{2C}} \right\rfloor^d . \tag{36}$$

The condition $n \geq 2d$ guarantees $k \geq 4$. ∎

Remark 6 The construction has no redundant constraints when d is even, and $\left\lfloor \frac{2n}{d+1} \right\rfloor - 2$ redundant constraints when d is odd.

Corollary 4 (Main Theorem) *For fixed dimension d, the function $C(d,n)$ grows like a polynomial of degree d in n:*

$$C(d,n) \text{ is } \theta(n^d) \text{ for } n \geq 2d \text{ where } d \text{ is fixed.} \tag{37}$$

5 Conclusion

Using a construction of deformed product programs, we proved that the worst-case path length that the least-index criss-cross method for solving a linear program can take is $\Omega(n^d)$ for a d-polyhedron defined by n halfspaces (when d is fixed). This result provides a tighter lower bound that assymptotically achieves the upperbound, and also shows that the least-index criss-cross method is worse than simplex methods in the worst case. Despite this negative result, criss-cross methods remain perhaps the best hope of finding a strongly polynomial algorithm for linear programming (see [6]).

References

[1] D. Avis and V. Chvátal, Notes on Bland's rule, *Mathematical Programming Study* **8** (1978) 24–34.

[2] N. Amenta and G. Ziegler, Deformed products and maximal shadows of polytopes, *Contemporary Mathematics* **223** (1999) 57–90.

[3] R.G. Bland, New finite pivot rules for the simplex method, *Mathematics of Operations Research* **2** (1977) 103–107.

[4] V. Chvátal, *Linear Programming*, Freeman, 1980.

[5] K. Fukuda and T.Terlaky, Criss-cross methods: A fresh view on pivot algorithms, *Mathematical Programming* **79** (1997) 369–395.

[6] K. Fukuda and T.Terlaky, On the existence of a short admissible pivot sequence for feasibility and linear optimization problems, *Pure Mathematics and Applications, Mathematics of Optimization*, **10** (4) (2000) 431–447.

[7] K. Fukuda and T. Matsui, On the finiteness of the criss–cross method, *European Journal of Operations Research* **52** (1991) 119–124.

[8] B. Grünbaum, *Convex Polytopes*, Springer Verlag; 2nd edition, June 2003.

[9] D. Halperin, Arrangements. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry,* 389–412*,* CRC Press, New York, 1997.

[10] V. Klee and G. J. Minty, How good is the simplex algorithm? In O. Shisha, editor, *Inequalities–III*, 159–175. Academic Press, New York, 1972.

[11] P. McMullen, The maximum number of faces of a convex polytope. *Mathematika* **17** (1970) 179–184.

[12] J. Richter-Gebert and G. Ziegler, Oriented Matroids. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry,* 111–132*,* CRC Press, New York, 1997.

[13] C. Roos, An exponential example for Terlaky's pivoting rule for the criss-cross simplex method, *Mathematical Programming* **46** (1990) 78–94.

[14] T. Terlaky, A convergent criss-cross method, *Math. Oper. und Stat. ser. Optimization* **16** (5) (1985) 683–690.

[15] T. Terlaky and S. Zhang, A survey on pivot rules for linear programming, *Annals of Operations Research* **46** (1993) 203–233.

[16] Zh. Wang, A conformal elimination free algorithm for oriented matroid programming, *Chinese Annals of Mathematics* **8** (B1) (1987).

[17] T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.*, vol. 1 (1): No 154 MR 50 (1975).

[18] S. Zionts, The criss-cross method for solving linear programming problems, *Management Science* **15** (7) (1969) 426–445.

[19] G. Ziegler, *Lectures on polytopes.* Springer-Verlag, New York, 1995.