***Les Cahiers du GERAD*** ISSN: 0711–2440

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G–2003–69

October 2003

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October, 2003

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**Abstract**

This paper deals with the uncertain class of continuous-time linear systems with Markovian jumping parameters and multiplicative Brownian disturbance. A design method for a non fragile robust controller for this class of systems is proposed when the uncertainties in the system are of norm bounded types. LMI based sufficient condition is developed. The methodology used is mainly based on Lyapunov approach. A numerical example is presented to show the usefulness of the proposed results.

**Key Words**: Jump linear system, Linear matrix inequality, Stochastic stability, state feedback control, Norm bounded uncertainty, Brownian motion.

**Résumé**

Ce papier traite de la classe des systèmes incertains continus à sauts markoviens et sujet à des bruits Brownien. Une méthode de design d'un contrôleur nonfragile et robust pour la classe de systèmes considérée est développée quand les incertitudes sont du type borné norme. Les résultats développés sont en forme de LMI. La méthodologie repose sur la méthode de Lyapunov. Un exemple numérique est présenté pour montrer l'importance des résulats.

# 1 Introduction

There exists in practice many systems that can't be unfortunately modelled by the popular linear model that is widely used in the literature. Among these systems we quote for instance the ones with abrupt changes in their structures that may be caused by many factors like failures, repairs, sudden environmental disturbance, changing subsystem interconnections, abrupt variations of the operating point of a nonlinear system, etc. This class of systems can be modelled by the class of linear system with Markovian jumping parameters which was introduced for the first time by Krasovskii and Lidskii (Ref. [1]). The power of this class of systems to model different practical systems, has been the catalyst of the development of this class of systems. For a recent review on this class of systems and its applications we refer the readers to Boukas and Liu (Ref. [2]) and the references therein on what it has been done on this class of systems. Most of the problems like stability, stabilization, $\mathcal{H}_\infty$ control, filtering and theirs robustness have been tackled and some interesting results already exist in the literature. For example we can refer the readers to Mao (Ref. [3]), Shi and Boukas (Ref. [4]), Shi et al. (Ref. [5]), Wang et al. (Ref. [6]) and to the references therein.

Most of the contributions to the class of systems with Markovian jumping parameters dealt with the design of controllers that cope with the system uncertainties but none of them has addressed the robustness with regard to the controllers uncertainties that may results from different causes like the errors in the electronic components for instance when the controllers are implemented using electronic components. In their study Keel and Bhattacharyya (Ref. [7]) have shown that the controller may be very sensitive or fragile to the errors in the controller parameters even if the design take care of the system uncertainties. To overcome this, the parameters variations should be included in the controller design phase besides the system uncertainties. The goal becomes then how to design a controller that is non fragile in the sense that the closed loop system tolerates a certain changes in the controller parameters and at the same time the system uncertainties that may affect the different matrices.

Our goal in this paper consists of designing a non fragile controller that can cope with norm bounded uncertainties that may affect the class of continuous-time Markovian jumping parameters with Brownian disturbance we are considering in this paper and at the same time tolerate some changes in the controller parameters. To the best of our knowledge this problem has never tackled before for this class of systems. Our choice will be limited to conditions for robust stochastic stabilization in the form of LMI that may be solved easily using the existing convex optimization algorithms. The methodology using in this paper is mainly based on Lyapunov method.

The rest of the paper is organized as follows. In section 2, the stabilization problem is stated. Section 3 gives the main results of the paper. They comprise results on robust stochastic stability and the design method for a non fragile robust controller. In section 4, a numerical example is presented to show the usefulness of the developed results.

The notations used in this paper is standard unless it is mentioned otherwise. For symmetric matrices $X$ and $Y$, $X > Y$ (resp. $X < Y$) means that $X - Y$ is positive-definite (resp. negative-definite). $I$ denotes the identity matrix with the appropriate dimension that may be understood from the context. diag[.] denotes a block diagonal matrix.

# 2 Problem statement

Consider a continuous-time linear Markovian jumping parameters system defined in a fundamental probability space $(\Omega, \mathcal{F}, P)$ with the following dynamics:

$$\begin{cases} dx_t = A(r_t,t)x_t dt + B(r_t)u_t dt + W(r_t)x_t)dw(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state vector at time, $u_t \in \mathbb{R}^p$ is the control at time $t$, $w(t) \in \mathbb{R}^m$ is a standard Wiener process that is independent of the Markov process $\{r_t, t \geq 0\}$, $A(r_t,t)$ is the state matrix that is assumed to contain uncertainties and its expression is given by:

$$A(r_t,t) = A(r_t) + D_A(r_t)F_A(r_t,t)E_A(r_t)$$

with $A(r_t)$, $D_A(r_t)$, $E_A(r_t)$ are known matrices, and $F_A(r_t,t)$ is the uncertainty of the state matrix; $B(r_t)$ is the control matrix that is supposed to be known and $W(r_t)$ is a known real matrix; $\{r_t, t \geq 0\}$ is continuous-time homogeneous Markov process with right continuous trajectories taking values in a finite set $\mathcal{S} = \{1, 2, \ldots, N\}$ with the following stationary transition probabilities:

$$P\left[r_{t+\Delta t} = j | r_t = i\right] = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t) & i \neq j \\ 1 + \lambda_{ii}\Delta t + o(\Delta t) & \text{otherwise} \end{cases} \tag{2}$$

where $\Delta t > 0$, $\lim_{\Delta t \to 0} \frac{o(\Delta t}{\Delta t} = 0$ and $\lambda_{ij} \geq 0$ is the transition probability from the mode $i$ to the mode $j$ at time $t$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$.

The uncertainty in the state matrix is assumed to satisfy the following for every $r_t \in \mathcal{S}$:

$$F_A^\top(r_t,t)F_A(r_t,t) \leq I \tag{3}$$

Let us now define some concepts that will be used in the rest of the paper.

For system (1), when $F_A(r_t,t) \equiv 0$, i.e we drop the system's uncertainties, we have the following definitions.

**Definition 2.1** *System (1), with $F_A(r_t,t) = 0$ for all $t \geq 0$, is said to be*

*(i) stochastically stable (SS) if there exists a finite positive constant $T(x_0, r_0)$ such that the following holds for any initial conditions $(x_0, r_0)$:*

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt | x_0, r_0\right] \leq T(r_0, x_0); \tag{4}$$

*(ii) mean square stable (MSS) if*

$$\lim_{t \to \infty} \mathbb{E}\|x(t)\|^2 = 0 \tag{5}$$

*holds for any initial condition* $(x_0, r_0)$*;*

*(iii) mean exponentially stable (MES) if there exist positive constants* $\alpha$ *and* $\beta$ *such that the following holds for any initial conditions* $(x_0, r_0)$*:*

$$\mathbb{E}\left[\|x(t)\|^2 | x_0, r_0\right] \le \alpha \|x_0\| e^{-\beta t}. \tag{6}$$

**Remark 2.1** From the definitions, we can see that the mean exponentially stable (MES) implies the mean square stable (MSS) and the stochastically stable (SS).

When the system's uncertainties are not equal to zero, the concept of stochastic stability becomes robust stochastic stability and is defined for system (1), as follows.

**Definition 2.2** *System (1) is said to be*

*(i) robustly stochastically stable (RSS) if there exists a finite positive constant* $T(x_0, r_0)$ *such that the condition (4) holds for any initial conditions* $(x_0, r_0)$ *and for all admissible uncertainties;*

*(ii) robust mean exponentially stable (RMES) if there exist positive constants* $\alpha$ *and* $\beta$ *such that the condition (6) holds for any initial conditions* $(x_0, r_0)$ *and for all admissible uncertainties.*

**Remark 2.2** From the definitions, we can see that the robust mean exponentially stable (RMES) implies the stochastically stable (RSS).

**Definition 2.3** *System (1) with* $F_A(r_t, t) = 0$ *for all modes and for* $t \ge 0$*, is said to be stabilizable in the SS (MES, MSQS) sense if there exists a controller such that the closed-loop system is SS (MES, MSQS) for every initial conditions* $(x_0, r_0)$*.*

When the uncertainties are not equal to zero, the previous definition is replaced by the following one:

**Definition 2.4** *System (1) is said to be robustly stabilizable in the stochastic sense if there exists a controller such that the closed-loop system is stochastically stable for every initial conditions* $(x_0, r_0)$ *and for all admissible uncertainties.*

The problem we are facing in this paper consists of designing a state feedback controller that robustly stabilizes the closed loop of the system.

In general the state feedback control is given by:

$$u(t) = K(r_t)x(t), \text{ for every } r_t \in \mathcal{S} \tag{7}$$

But in practice the implementation is quite different from this expression and there is always uncertainties in the gain controller which means that the gain is given by:

$$K(r_t, t) = K(r_t) + \Delta K(r_t, t) \tag{8}$$

with $\Delta K(r_t, t)$ is given by:

$$\Delta K(r_t, t) = \rho(r_t) F_K(r_t, t) K(r_t) \tag{9}$$

where $\rho(r_t)$ is an uncertain real parameter indicating the measure of non fragility against controller gain variations and $F_K(r_t, t)$ is the uncertainty that will be supposed to satisfy the following for every $r_t \in \mathcal{S}$:

$$F_K^\top(r_t, t) F_K(r_t, t) \leq I \tag{10}$$

Our goal in this paper is to synthesize the gain for the state feedback controller with the following form for every $r_t \in \mathcal{S}$:

$$K(r_t) = \gamma(r_t) B^\top(r_t) P(r_t) \tag{11}$$

where $\gamma(r_t)$ is a real number and $P(r_t)$ is symmetric and positive-definite matrix for every $r_t \in \mathcal{S}$.

Plugging the controller in the dynamics we get the following closed loop dynamics:

$$\begin{aligned} dx_t &= \left[A(r_t, t) + B(r_t) K(r_t, t)\right] x_t dt + W(r_t) x_t dw_t \\ &= \Big[A(r_t) + D_A(r_t) F_A(r_t, t) E_A(r_t) + B(r_t) \Big[\gamma(r_t) B^\top(r_t) P(r_t) \\ &\qquad + \rho(r_t) F_K(r_t, t) \gamma(r_t) B^\top(r_t) P(r_t)\Big]\Big] x_t dt + W(r_t) x_t dw_t \end{aligned} \tag{12}$$

In the rest of this we will propose an LMI design approach to compute the controller gain, $P(r_r)$ and $\gamma(r_t)$ for each $r_t \in \mathcal{S}$.

The following lemmas will be used in the rest of the paper. For their proofs, we refer the reader to Boukas and Liu (Ref. [2]).

**Lemma 2.1** *Let $D$, $F$ and $E$ be real constant matrices of compatible dimensions with $F^\top F \leq I$, then, the following:*

$$DFE + E^\top F^\top D^\top \leq \varepsilon D D^\top + \frac{1}{\varepsilon} E^\top E$$

*holds for any $\varepsilon > 0$.*

**Lemma 2.2** *(Schur Complement) Let the symmetric matrix $M$ be partitioned as*

$$M = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix}$$

*with $X, Z$ being symmetric matrices. We have* *(i ) $M$ is nonnegative-definite if and only if either*

$$\begin{cases} Z \geq 0 \\ Y = L_1 Z \\ X - L_1 Z L_1^\top \geq 0 \end{cases} \tag{13}$$

*or*

$$\begin{cases} X \geq 0 \\ Y = X L_2 \\ Z - L_2^\top X L_2 \geq 0 \end{cases} \tag{14}$$

*hold, where $L_1, L_2$ are some (non unique) matrices of compatible dimensions.*

*(ii) $M$ is positive-definite if and only if either*

$$\begin{cases} Z > 0 \\ X - Y Z^{-1} Y^\top > 0 \end{cases} \tag{15}$$

*or*

$$\begin{cases} X > 0 \\ Z - Y^\top X^{-1} Y > 0 \end{cases} \tag{16}$$

*Matrices $X - YZ^{-1}Y^\top$ is called the Schur complement $X(Z)$ in $M$.*

**Lemma 2.3** *Let $V(x(t), r_t)$ be a function from $\mathbb{R}^n \times \mathcal{S}$ into $\mathbb{R}$ such that $V(x(t), r_t)$ and $V_x(x(t), r_t)$ are continuous in $x$ for any $r \in \mathcal{S}$ and such that $|V(x(t), r_t)| < \gamma (1 + \|x\|)$ for a constant $\gamma$, the generator $\mathbb{L}_u$ of $(x(t), r_t)$ under an admissible control law $u$, for $x(t)$ solution of (1) and $r_t$ a continuous time Markov process taking values in $\mathcal{S}$ with transition rates matrix $\Lambda$, is given by:*

$$\begin{aligned} \mathbb{L}_u V(x(t), r_t) &= \left[A(r_t)x(t) + B(r_t)u(t)\right]^\top V_x(x(t), r_t) \\ &\quad + \frac{1}{2} trace\left(x^\top(t) W^\top(r_t) V_{xx}(x(t), r_t) W(r_t) x(t)\right) \end{aligned} \tag{17}$$

# 3 Main results

In this section we will develop the main results of this paper that are related to the robust stability and the robust stabilization problems for the class of systems we are considering. All the results are LMI based which make them easily solvable using existing convex optimization algorithms.

Let us now study the stability problem. For this purpose let the control $u_t = 0$ for $t \geq 0$. The following theorem give the result on robust stochastic stability. The following theorem summarizes the result in this case.

**Theorem 3.1** *If there exist symmetric and positive-definite matrices* $P = (P(1), \ldots, P(N))$ *and positive scalar* $\varepsilon_A$ *such the following LMI holds for every* $r_t \in \mathcal{S}$*:*

$$\begin{bmatrix} J_w(r_t) & P(r_t)D_A(r_t) \\ D_A^\top(r_t)P(r_t) & -\varepsilon_A I \end{bmatrix} < 0 \tag{18}$$

*with* $J_w(r_t) = A^\top(r_t)P(r_t) + P(r_t)A(r_t) + W^\top(r_t)P(r_t)W(r_t) + \varepsilon_A E_A^\top(r_t)E_A(r_t) + \sum_{j=1}^N \lambda_{r_t j} P(j)$*, then the system is stochastically stable.*

Proof: Let $P(r_t)$, $r_t \in \mathcal{S}$ be a symmetric and positive-definite that represents a solution of the LMI (18), then a Lyapunov function candidate can be given by the following expression:

$$V(x_t, r_t) = x^\top(t)P(r_t)x(t) \tag{19}$$

Using the results of lemma 2.3, the infinitesimal generator of the Markov process $(x(t), r_t)$ is given by:

$$\begin{aligned} \mathbb{L}V(x(t), r_t) &= [A(r_t, t)x(t)]^\top V_x(x(t)) + \sum_{j=1}^N \lambda_{r_t j} V(x(t), j) \\ &\quad + \frac{1}{2}\text{trace}\left[x^\top(t)W^\top(r_t)V_{xx}(x(t), r_t)W(r_t)x(t)\right] \end{aligned} \tag{20}$$

Using the following expressions of $V_x(x(t), r_t)$ and $V_{xx}(x(t), r_t)$:

$$V_x(x(t), r_t) = 2P(r_t)x(t) \tag{21}$$

$$V_{xx}(x(t), r_t) = 2P(r_t) \tag{22}$$

we obtain:

$$\begin{aligned} \mathbb{L}V(x(t), r_t) &= 2x^\top(t)A^\top(r_t, t)P(r_t)x(t) \\ &\quad + \sum_{j=1}^N \lambda_{r_t j} x^\top(t)P(j)x(t) + x^\top(t)W^\top(r_t)P(r_t)W(r_t)x(t) \\ &= x^\top(t)\Big[A^\top(r_t)P(r_t) + P(r_t)A(r_t) + E_A^\top(r_t)F_A^\top(r_t, t)D_A^\top(r_t)P(r_t) \\ &\quad + P(r_t)D_A(r_t)F_A(r_t, t)E_A(r_t) + W^\top(r_t)P(r_t)W(r_t) + \sum_{j=1}^N \lambda_{r_t j} P(j)\Big]x(t) \end{aligned}$$

Based on the results of Lemma 2.1, we have:

$$\begin{aligned} 2x^\top(t)P(r_t)D_A(r_t)F_A(r_t, t)E_A(r_t)x(t) &\leq \frac{1}{\varepsilon_A} x^\top(t)P(r_t)D_A(r_t)D^\top(r_t)P(r_t)x(t) \\ &\quad + \varepsilon_A x^\top(t)E_A^\top(r_t)E_A(r_t)x^\top(t) \end{aligned} \tag{23}$$

Considering this, the previous relation becomes:

$$\begin{aligned}\mathbb{L}V(x(t),r_t) &= x^\top(t)\Big[A^\top(r_t)P(r_t)+P(r_t)A(r_t)+\frac{1}{\varepsilon_A}P(r_t)D_A(r_t)D^\top(r_t)P(r_t)\\ &\quad +\varepsilon_A E_A^\top(r_t)E_A(r_t)+W^\top(r_t)P(r_t)W(r_t)+\sum_{j=1}^{N}\lambda_{r_tj}P(j)\Big]x(t)\\ &\le x^\top(t)\Gamma_u(r_t)x(t) \end{aligned} \tag{24}$$

with

$$\begin{aligned}\Gamma_u(r_t) &= A^\top(r_t)P(r_t)+P(r_t)A(r_t)+\frac{1}{\varepsilon_A}P(r_t)D_A(r_t)D^\top(r_t)P(r_t)\\ &\quad +\varepsilon_A E_A^\top(r_t)E_A(r_t)+W^\top(r_t)P(r_t)W(r_t)+\sum_{j=1}^{N}\lambda_{r_tj}P(j)\end{aligned} \tag{25}$$

Using the condition (18) and Schur complement, we get:

$$\mathbb{L}V(x(t),r_t) \le -min_{i\in\mathcal{S}}x^\top(t)\lambda_{min}\left(-\Gamma_u(i)\right) \tag{26}$$

Combining this again with Dynkin's formula yields

$$\begin{aligned}\mathbb{E}\left[V(x(t),r_t)\right]-\mathbb{E}\left[V(x(0),r_0)\right] &= \mathbb{E}\left[\int_0^t \mathbb{L}V(x(s),r_s)ds|(x_0,r_0)\right]\\ &\le -\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Gamma(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right],\end{aligned}$$

implying, in turn,

$$\begin{aligned}&\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Gamma_u(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right]\\ &\quad\le \mathbb{E}\left[V(x(0),r_0)\right]-\mathbb{E}\left[V(x(t),r_t)\right]\\ &\quad\le \mathbb{E}\left[V(x(0),r_0)\right].\end{aligned}$$

This yields that

$$\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds|(x_0,r_0)\right] \le \frac{\mathbb{E}\left[V(x(0),r_0)\right]}{\min_{i\in\mathbf{S}}\{\lambda_{min}(-\Gamma_u(i))\}}$$

holds for any $t>0$. This proves Theorem 3.1. □

**Remark 3.1** The condition we give in this theorem is a sufficient one which means that if we are not able to find a set $P=(P(1),\ldots,P(N))$ of symmetric and positive-definite matrices that satisfies the condition (18), this doesn't imply that the dynamical system is not robustly stochastically stable.

Let us now return to the initial problem and see how we can design a robust controller that can handle the uncertainties in the system matrices and the controller gains.

Based on Theorem 3.1, the closed-loop system will be stable if the following holds for every $r_t \in \mathcal{S}$:

$$\begin{aligned}&\left[A(r_t,t) + B(r_t)\left[K(r_t) + \rho(r_t)F_K(r_t,t)K(r_t)\right]\right]^\top P(r_t)\\&\quad + P(r_t)\left[A(r_t,t) + B(r_t)\left[K(r_t) + \rho(r_t)F_K(r_t,t)K(r_t)\right]\right]\\&\quad + W^\top(r_t)P(r_t)W(r_t) + \sum_{j=1}^{N}\lambda_{r_tj}P(j) < 0 \end{aligned} \tag{27}$$

Using now the fact that $K(r_t) = \gamma(r_t)B^\top(r_t)P(r_t)$ and Lemma 2.1, we get:

$$\begin{aligned}2x_t^\top P(r_t)D_A(r_t)F_A(r_t,t)E_A(r_t)x_t &\leq \varepsilon_A(r_t)x_t^\top P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)x(t)\\&\quad + \varepsilon_A^{-1}x^\top(t)(r_t)E_A^\top(r_t)E_A(r_t)x_t\end{aligned}$$

$$\begin{aligned}&2\rho(r_t)\gamma(r_t)x_t^\top P(r_t)B(r_t)F_K(r_t,t)B^\top(r_t)P(r_t)x_t\\&\quad\leq \varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t\\&\qquad + \varepsilon_K(r_t)\rho(r_t)x_t^\top P(r_t)B(r_t)F_K^\top(r_t,t)F_K(r_t,t)B^\top(r_t)P(r_t)x_t\\&\quad\leq \varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t\\&\qquad + \varepsilon_K(r_t)\rho(r_t)x_t^\top P(r_t)B(r_t)B^\top(r_t)P(r_t)x_t\end{aligned}$$

Based on this, the previous inequality becomes:

$$\begin{aligned}&A^\top(r_t)P(r_t) + P(r_t)A(r_t) + \varepsilon_A(r_t)P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t)\\&+\varepsilon_A^{-1}(r_t)E_A^\top(r_t)E_A(r_t) + \sum_{j=1}^{N}\lambda_{r_tj}P(j)\\&+\varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)P(r_t)B(r_t)B^\top(r_t)P(r_t) + \varepsilon_K(r_t)\rho(r_t)P(r_t)B(r_t)B^\top(r_t)P(r_t)\\&+2\gamma(r_t)P(r_t)B(r_t)B^\top(r_t)P(r_t) + W^\top(r_t)P(r_t)W(r_t)\end{aligned} \tag{28}$$

This inequality is nonlinear in the design parameters $\gamma(r_t)$ and $P(r_t)$ for every $r_t \in \mathcal{S}$ to cast it into in an LMI, let us put $X(r_t) = P^{-1}(r_t)$ for each $r_t \in \mathcal{S}$. Let us now pre- and post-multiplying (28) by $X(r_t)$ we get:

$$\begin{aligned}X(r_t)\Lambda(r_t)X(r_t) \;=\;& X(r_t)A^\top(r_t) + A(r_t)X(r_t) + \varepsilon_A(r_t)D_A(r_t)D_A^\top(r_t)\\&+\varepsilon_A^{-1}(r_t)X(r_t)E_A^\top(r_t)E_A(r_t)X(r_t) + \sum_{j=1}^{N}\lambda_{r_tj}X(r_t)X^{-1}(j)X(r_t)\\&+\varepsilon_K^{-1}(r_t)\rho(r_t)\gamma^2(r_t)B(r_t)B^\top(r_t) + \varepsilon_K(r_t)\rho(r_t)B(r_t)B^\top(r_t)\\&+2\gamma(r_t)B(r_t)B^\top(r_t) + X(r_t)W^\top(r_t)X^{-1}(r_t)W(r_t)X(r_t)\end{aligned} \tag{29}$$

Letting $S_{r_t}(X)$ and $\mathcal{X}_{r_t}(X)$ be defined as follows:

$$\begin{cases} S_{r_t}(X) = \left[\sqrt{\lambda_{r_t 1}}X(r_t), \ldots \sqrt{\lambda_{r_t r_t - 1}}X(r_t), \sqrt{\lambda_{r_t r_t + 1}}X(r_t), \ldots, \sqrt{\lambda_{r_t N}}X(r_t)\right] \\ \\ \mathcal{X}_{r_t}(X) = \mathrm{diag}\left[X(1), \ldots, X(r_t - 1), X(r_t + 1), \ldots, X(N)\right] \end{cases} \tag{30}$$

the term $X(r_t)\left[\sum_{j=1}^{N} \lambda_{r_t j} X^{-1}(j)\right] X(r_t)$ can be rewritten as follows:

$$X(r_t)\left[\sum_{j=1}^{N} \lambda_{r_t j} X^{-1}(j)\right] X(r_t) = \lambda_{r_t r_t} X(r_t) + S_{r_t}(X)\mathcal{X}_{r_t}^{-1}(X)\mathcal{S}_{r_t}^{\top}(X)$$

Using now (29) and Schur complement we get:

$$\begin{bmatrix} J(r_t) & X(r_t)E_A^{\top}(r_t) & \gamma(r_t)B(r_t) & X(r_t)W^{\top}(r_t) & \mathcal{S}_{r_t}(X) \\ E_A(r_t)X(r_t) & -\varepsilon_A(r_t)I & 0 & 0 & 0 \\ \gamma(r_t)B^{\top}(r_t) & 0 & -\frac{\varepsilon_K(r_t)}{\rho(r_t)}I & 0 & 0 \\ W(r_t)X(r_t) & 0 & 0 & -X(r_t) & 0 \\ \mathcal{S}_{r_t}^{\top}(X) & 0 & 0 & 0 & -\mathcal{X}_{r_t}(X) \end{bmatrix} < 0 \tag{31}$$

where

$$\begin{aligned} J(r_t) = {} & X(r_t)A^{\top}(r_t) + A(r_t)X(r_t) + \varepsilon_A(r_t)D_A(r_t)D_A^{\top}(r_t) + 2\gamma(r_t)B(r_t)B^{\top}(r_t) \\ & + \lambda_{r_t r_t}X(r_t) + \varepsilon_K(r_t)\rho(r_t)B(r_t)B^{\top}(r_t) \end{aligned} \tag{32}$$

The following theorem gives a result in the LMI framework that can be used to design a non fragile robust controller for the class of system we are considering.

**Theorem 3.2** *If there exist a set of symmetric and positive-definite matrix $X = (X(1), \ldots, X(N))$ and positive scalars $\varepsilon_A(r_t)$, $\mu(r_t)$, $\nu(r_t)$ and a scalar $\gamma(r_t)$ satisfying the following LMI for every $r_t \in \mathcal{S}$ and for all admissible uncertainties:*

$$\begin{bmatrix} J(r_t) & X(r_t)E_A^{\top}(r_t) & \gamma(r_t)B(r_t) & X(r_t)W^{\top}(r_t) & \mathcal{S}_{r_t}(X) \\ E_A(r_t)X(r_t) & -\varepsilon_A I & 0 & 0 & 0 \\ \gamma(r_t)B^{\top}(r_t) & 0 & -\mu(r_t)I & 0 & 0 \\ W(r_t)X(r_t) & 0 & 0 & -X(r_t) & 0 \\ \mathcal{S}_{r_t}^{\top}(X) & 0 & 0 & 0 & -\mathcal{X}_{r_t}(X) \end{bmatrix} < 0 \tag{33}$$

*where*

$$\begin{aligned} J(r_t) &= X(r_t)A^{\top}(r_t) + A(r_t)X(r_t) + \varepsilon_A D_A(r_t)D_A^{\top}(r_t) + 2\gamma(r_t)B(r_t)B^{\top}(r_t) \\ &\quad + \lambda_{r_t r_t}X(r_t) + \nu(r_t)B(r_t)B^{\top}(r_t) \\ \mu(r_t) &= \frac{\varepsilon_K(r_t)}{\rho(r_t)} \\ \nu(r_t) &= \varepsilon_K(r_t)\rho(r_t) \end{aligned}$$

*then closed-loop system is robustly stochastically stable with non fragility* $\rho(r_t)$ *under the controller (7) with the gain* $K(r_t) = \gamma(r_t)B^\top(r_t)X^{-1}(r_t)$.

When the controller gains don't have uncertainties the previous result becomes easier and it is summarized in the following theorem:

**Theorem 3.3** *If there exist a set of symmetric and positive-definite matrix* $X = (X(1), \ldots, X(N))$ *and positive scalar* $\varepsilon_A(r_t)$, *and* $\gamma(r_t)$ *satisfying the following LMI for every* $r_t \in \mathcal{S}$ *and all admissible uncertainties:*

$$\begin{bmatrix} J(r_t) & X(r_t)E_A^\top(r_t) & \mathcal{S}_{r_t}(X) \\ E_A(r_t)X(r_t) & -\varepsilon_A I & 0 \\ \mathcal{S}_{r_t}^\top(X) & 0 & -\mathcal{X}_{r_t}(X) \end{bmatrix} < 0 \tag{34}$$

*where*

$$J(r_t) = X(r_t)A^\top(r_t) + A(r_t)X(r_t) + \varepsilon_A D_A(r_t)D_A^\top(r_t) + 2\gamma(r_t)B(r_t)B^\top(r_t) + \lambda_{r_t r_t}X(r_t)$$

*then closed loop system is robustly stochastically stable under the controller (7) with the gain* $K(r_t) = \gamma(r_t)B^\top(r_t)X^{-1}(r_t)$.

Proof: The details of the proof is similar to the one of the previous theorem and follows the same steps. □

# 4 Numerical example

In this section we will show the usefulness of the proposed results in this paper. For this purpose let us consider a system with two modes and two components in the state vector. Let the data in each mode be given by:

- mode 1:

$$A(1) = \begin{bmatrix} 1.0 & -0.5 \\ 0.1 & 1.0 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$D_A(1) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$E_A(1) = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}$$

$$W(1) = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.2 \end{bmatrix}$$

- mode 2:

$$A(2) = \begin{bmatrix} -0.2 & 0.5 \\ 0.0 & -0.25 \end{bmatrix}$$

$$B(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$D_A(2) = \begin{bmatrix} 0.13 \\ 0.1 \end{bmatrix}$$

$$E_A(2) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}$$

$$W(2) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}$$

Let the transition probability matrix between these two modes be given by:

$$\Lambda = \begin{bmatrix} -2.0 & 2.0 \\ 3.0 & -3.0 \end{bmatrix}$$

Letting $\varepsilon_A(1) = \varepsilon_A(2) = 0.5$, $\varepsilon_K(1) = \varepsilon_K(2) = 0.1$, and $\rho(1) = 0.5$, $\rho(2) = 0.6$ and solving the LMI (33), we get:

$$\begin{aligned} X(1) &= \begin{bmatrix} 0.0546 & 0.0044 \\ 0.0044 & 0.0484 \end{bmatrix} \\ X(2) &= \begin{bmatrix} 1.6675 & -0.1072 \\ -0.1072 & 0.8727 \end{bmatrix} \\ \gamma(1) &= -0.1966 \\ \gamma(2) &= -0.1590 \end{aligned}$$

which gives the following controller gains:

$$\begin{aligned} K(1) &= \begin{bmatrix} -3.6290 & 0.3330 \\ 0.3330 & -4.0964 \end{bmatrix} \\ K(2) &= \begin{bmatrix} -0.0961 & -0.0118 \\ -0.0118 & -0.1836 \end{bmatrix}. \end{aligned}$$

## 5 Conclusion

This paper deals with the class of uncertain continuous-time Markovian jump linear systems. The uncertainties we considered in this paper were of norm bounded type. A design LMI method was developed to synthesize a state feedback controller that robustly stochastically stabilizes the class under study. The condition we established is easily solvable using existing convex optimization algorithms.

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