Stability and Stabilizability of Dynamical Systems with Multiple Time-Varying Delays: Delay Dependent Criteria

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Abstract

This paper deals with the class of uncertain systems with multiple time-delays. The stability and stabibizability of this class of systems are considered. Their robustness are also studied when the norm bounded uncertainties are considered. LMIs Delay-dependent sufficient conditions for both stability and stabilizability and their robustness are established to check if a system of this class is stable and/or is stabilizable. Some numerical examples are provided to show the usefulness of the proposed results.

Keywords: Dynamical systems, multiple time-varying delays, stability, robust stability, stabilizability, robust stabilizability.

Résumé

Cet article traite de la classe des systèmes incertains à retard multiples. Les problèmes de stabilité de stabilisabilité et leur robustesse sont considérés. Les incertitudes utilisées dans ce travail sont du type bornées en norme. Des conditions en forme de LMI sont établies pour la stabilité, la stabilisabilité et leur robustesse. Des exemples numériques sont présentés pour montrer la validité des résulats exposés.

1 Introduction

It was shown in different studies that the presence of the time-delay in the systems dynamics is the primary cause of instability and performance degradation. The class of dynamical systems with time-delay has in fact attracted a lot of researchers mainly from the control community. Many results on this class of systems have been reported to the literature. We refer the reader to [1, 2] and the references therein for more information.

In the present literature there exist two techniques that can be used to study the stability and the stabilizability. The first one is based on the Lyapunov-Razumikhin technique and it consists of considering a Lyapunov function of the form, $V(x_t) = x_t^\top P x_t$, with P a symmetric and positive-definite matrix with appropriate dimension and x_t is the state vector of the system, to develop the conditions that can be used to check if the system under study is stable; and/or stabilizable. This technique gives a condition that depends on the maximum value of the delay. The reader can consult the work of [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 2] and the references therein for more information.

The second technique is based on the Lyapunov-Krasovskii approach and it consists of considering a more complicated Lyapunov functional to determine the appropriate delay-dependent condition that in general depends on the upper bound of the first derivative of the delay when it is time-varying. This technique has been extensively used and the large number of references using it confirms this. See for example the works done by [1, 2] and the references therein for more information;

But from the practical point of view we are interested by conditions that depend on both, i.e: the upper bound of the delay and the lower and the upper bounds of the first derivative of the time-varying delay. Since in practice the delay is in fact always time-varying, that can be usually represented by a function $h(t)$, and bounded by a constant $\bar{h}$, it is therefore desirable to have conditions that depend on the upper bound of the time-varying delay and on the lower and the upper bounds of the first derivative of the time-varying delay.

The goal of this paper consists of considering the class of uncertain linear systems with multiple time-varying delays and develop sufficient conditions for stability and stabilizability and their robustness that depend on the upper bounds of the delays and on the lower and upper bounds of the first derivative of the time-varying delays. The Lyapunov-Krasovskii approach will be used in this paper.

The paper is organized as follows. In section 2, the problem is stated and the required assumptions are formulated. Section 3 deals with the stability and the robust stability. Section 4 covers the stabilizability and the robust stabilizability of the class of systems under study. Section 5 presents some numerical examples to show the usefulness of the proposed results.

Notation: In the rest of this paper the notation is standard unless it is specified otherwise. $L > 0$ ($L < 0$) means that the matrix L is symmetric and positive-definite (symmetric and negative-definite) matrix. $\mathbf{Sym}(M) = M + M^\top$

2 Problem statement

Let us consider the following class of systems with multiple time-varying delays:

$$\dot{x}_t = A(t)x_t + \sum_{j=1}^{p} A_{dj}(t)x_{t-h_j(t)} + B(t)u_t \quad (1)$$

where x_t is the state vector, u_t is the control input, $h_j(t)$; $j = 1, 2, \ldots, p$, is the time-varying delay of the system and the matrices $A(t)$, $A_{dj}(t)$ and $B(t)$ are given by:

$$\begin{aligned} A(t) &= A + DF(t)E \\ A_{dj}(t) &= A_{dj} + D_jF_j(t)E_j, \forall j = 1, 2, \ldots, p \\ B(t) &= B + D_bF_b(t)E_b \end{aligned}$$

with A, A_{dj}, $j = 1, 2, \ldots, p$, B, D, E, D_j, E_j; $j = 1, 2, \ldots, p$, D_b and E_b are given matrices with appropriate dimensions and $F(t)$, $F_j(t)$; $j = 1, 2, \ldots, p$ and $F_b(t)$ represent the system uncertainties satisfying the following assumption.

Assumption 2.1 *Let us assume that the following hold:*

$$\begin{aligned} F^\top(t)RF(t) &\leq R \\ F_d^\top(t)R_dF_d(t) &\leq R_d, \\ F_b^\top(t)R_bF_b(t) &\leq R_b \end{aligned} \quad (2)$$

where R_d and $F_d(t)$ are diagonal matrices given by

$$F_d(t) = \begin{bmatrix} F_1(t) & & \\ & \ddots & \\ & & F_p(t) \end{bmatrix} \qquad R_d(t) = \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_p \end{bmatrix} \quad (3)$$

with R, $R_1, \ldots, R_p$ and R_b are given matrices with appropriate dimensions

Remark 2.1 *The uncertainties that satisfy (2) will be referred to as admissible uncertainties. Notice that the uncertainties $F(t)$, $F_j(t)$, $j = 1, 2, \ldots, p$ and $F_b(t)$ can be chosen dependent on the system state and the developed results will remain valid. However, in the present paper we will consider only the case of time-varying uncertainties.*

Assumption 2.2 *For each j, $(j = 1, 2, \ldots, p)$, the time-varying delay $h_j(t)$, is assumed to satisfy the following:*

$$0 \leq h_j(t) \leq \bar{h}_j < \infty \quad (4)$$

$$\underline{l}_j \leq \dot{h}_j(t) \leq \bar{l}_j < 1, \quad (5)$$

where $\bar{h}_j$, $\underline{l}_j$ and $\bar{l}_j$ are given positive constants.

Let us define $\bar{\tau}$ as $\bar{\tau} = max\left(\bar{h}_1, \ldots, \bar{h}_1\right)$ and $\mathbf{x}_t$ as $\mathbf{x}_t(s) = x_{t+s}$, $t - \bar{\tau} \leq s \leq t$. In the rest of the paper we will use $\mathbf{x}_t$ instead of $\mathbf{x}_t(s)$.

Lemma 2.1 *Let Z, E, F, R and Δ be matrices of appropriate dimensions. Assume that Z is symmetric, R is symmetric and positive definite and $\Delta^\top R \Delta \leq R$, then*

$$Z + E\Delta F + F^\top \Delta^\top E^\top < 0$$

if and only if there exists a scalar $\lambda > 0$ satisfying

$$Z + E\Big(\lambda R\Big)E^\top + F^\top \Big(\lambda R\Big)^{-1} F < 0$$

3 Stability and robust stability

The goal of this section consists of establishing what will be the sufficient conditions that can be used to check whether or not the class of systems under study is stable. We are also interested by the robust stability of this class of systems. These two problems will be discussed in the following subsections.

3.1 Stability

Let us now suppose that the control is equal to zero, i.e: $u_t = 0, \ \forall \ t \geq 0$ and that the system doesn't contain uncertainties which gives the following dynamics:

$$\dot{x}_t = Ax_t + \sum_{j=1}^{p} A_{dj} x_{t-h_j(t)} \tag{6}$$

The goal of this subsection consists of developing a condition that can be used to check if the class of systems under study is stable. The condition we are looking for should depend on the upper bound of the delay and on the lower and upper bounds of the first derivative of the time-varying delays given in Assumption 2.2. The following theorem states such result.

Theorem 3.1 *Let assume that the assumption 2.2 is satisfied. If there exist $P > 0$, $Q_j > 0$, $W_j > 0$, X_j, Y_j and Z_j for $j = 1, 2, \ldots, p$ such that the following hold:*

$$\mathcal{Z}_j = \begin{bmatrix} Z_j & Y_j \\ Y_j^\top & X_j \end{bmatrix} \ > \ 0 \tag{7}$$

$$\left(\bar{l}_j - \underline{l}_j\right) X_j + \left(\bar{l}_j - 1\right) W_j < 0, \tag{8}$$

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & \\ 0 & 0 & -\mathcal{W} & \mathcal{W}I\!I^\top \\ P & 0 & I\!I\mathcal{W} & 0 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} [\, A \quad A_d \quad 0 \quad -I \,] \right\} < 0 \tag{9}$$

where

$$\begin{array}{rcl}
\mathbb{I} & = & [\, I \quad \ldots \quad I \,] \\
A_d & = & [\, A_{d1} \quad \ldots \quad A_{dp} \,] \\
\mathcal{W} & = & diag(h_1 W_1, \ \ldots \ \ h_p W_p) \\
\Psi_1 & = & \sum_{j=1}^{p} \left[Q_j + \left(\bar{l}_j - \underline{l}_j\right) \left(\bar{h}_j Z_j + Y_j + Y_j^\top \right) \right] \\
\Psi_3 & = & \left[\ \left(\bar{l}_1 - \underline{l}_1\right) Y_1 \quad \ldots \quad + \left(\bar{l}_p - \underline{l}_p\right) Y_p \ \right] \\
\Psi_2 & = & diag(\left(1 - \bar{l}_1\right) Q_1, \ \ldots \ , \ \left(1 - \bar{l}_p\right) Q_p)
\end{array}$$

then the system under study is asymptotically stable.

In order to the proof of Theorem 3.1, we give the following lemma

Lemma 3.1 *The two statements are equivalent*

a)

$$\begin{bmatrix} A^\top P + PA + \Psi_1 & PA_d - \Psi_3 & A^\top \mathbb{I} \mathcal{W} \\ \left(PA_d - \Psi_3\right)^\top & -\Psi_2 & A_d^\top \mathbb{I} \mathcal{W} \\ \mathcal{W} \mathbb{I}^\top A & \mathcal{W} \mathbb{I}^\top A_d & -\mathcal{W} \end{bmatrix} < 0 \qquad (10)$$

b)

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & \\ 0 & 0 & -\mathcal{W} & \mathcal{W} \mathbb{I}^\top \\ P & 0 & \mathbb{I} \mathcal{W} & 0 \end{bmatrix} + \mathrm{Sym} \left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} [\, A \quad A_d \quad 0 \quad -I \,] \right\} < 0 \qquad (11)$$

Proof of Lemma 3.1 The proof is a direct application of the elimination lemma; that is, notice that

$$\mathcal{N}_{right} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A & A_d & 0 \end{bmatrix}$$

satisfies

$$[\, A \quad A_d \quad 0 \quad -I \,] \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A & A_d & 0 \end{bmatrix} = 0$$

then condition (11) is equivalent to

$$\mathcal{N}_{right}^{\top}\begin{bmatrix}\Psi_1 & -\Psi_3 & 0 & P\\ -\Psi_3^{\top} & -\Psi_2 & 0 & 0\\ 0 & 0 & -\mathcal{W} & \mathcal{W}I\!I^{\top}\\ P & 0 & I\!I\mathcal{W} & 0\end{bmatrix}\mathcal{N}_{right}=\begin{bmatrix}A^{\top}P+PA+\Psi_1 & PA_d-\Psi_3 & A^{\top}I\!I\mathcal{W}\\ \left(PA_d-\Psi_3\right)^{\top} & -\Psi_2 & A_d^{\top}I\!I\mathcal{W}\\ \mathcal{W}I\!I^{\top}A & \mathcal{W}I\!I^{\top}A_d & -\mathcal{W}\end{bmatrix}<0$$

and this ends the proof of Lemma 3.1. $\nabla\nabla\nabla$

Proof of Theorem 3.1 Let the Lyapunov functional be defined by:

$$V(\mathbf{x}_t) = V_1(\mathbf{x}_t) + V_2(\mathbf{x}_t) + V_3(\mathbf{x}_t) + V_4(\mathbf{x}_t)$$

where

$$\begin{aligned}
V_1(\mathbf{x}_t) &= x_t^{\top}Px_t\\
V_2(\mathbf{x}_t) &= \sum_{j=1}^{p}\int_{t-h_j(t)}^{t}\int_{s}^{t}\dot{x}_z^{\top}W_j\dot{x}_z dzds\\
V_3(\mathbf{x}_t) &= \sum_{j=1}^{p}\int_{t-h_j(t)}^{t}x_s^{\top}Q_jx_sds\\
V_4(\mathbf{x}_t) &= \sum_{j=1}^{p}\int_{0}^{t}\left(\bar{l}_j-\dot{h}_j(z)\right)\\
&\quad\int_{z-h_j(z)}^{z}\left[\begin{array}{cc} x_z^{\top} & \dot{x}_s\end{array}\right]\left[\begin{array}{cc} Z_j & Y_j\\ Y_j^{\top} & X_j\end{array}\right]\left[\begin{array}{c} x_z\\ \dot{x}_s\end{array}\right]dsdz
\end{aligned}$$

After taking the derivative of these functionals and some algebraic manipulations we get

$$\dot{V}(\mathbf{x_t}) = \xi_t^{\top}M\xi_t + \sum_{j=1}^{p}\int_{t-h_j(t)}^{t}\dot{x}_s^{\top}\left[\left(\bar{l}_j-\underline{l}_j\right)X_j+\left(\bar{l}_j-1\right)W_j\right]\dot{x}_sds$$

with

$$\begin{aligned}
\xi_t^{\top} &= \left[x_t^{\top}\quad x_{t-h_1(t)}^{\top}\quad \cdots\quad x_{t-h_p(t)}^{\top}\right]\\
M &= \left[\begin{array}{cc} M_{11} & M_{12}\\ M_{12}^{\top} & M_{22}\end{array}\right]
\end{aligned}$$

where M_{11}, M_{12} and M_{22} are given by

$$M_{11} = A^{\top}P + PA + A^{\top}I\!I\ \mathcal{W}\ I\!I^{\top}A + \Psi_1$$

$$M_{12} = PA_d - \Psi_3 + A^\top I\!I\, \mathcal{W}\, I\!I^\top A_d$$

$$M_{22} = A_d^\top I\!I\, \mathcal{W}\, I\!I^\top A_d - \Psi_2$$

Therefore, the system is then asymptotically stable if the following hold:

$$\begin{cases} M < 0 \\ \left[\left(\bar{l}_j - \underline{l}_j\right) X_j + \left(\bar{l}_j - 1\right) W_j\right] < 0, \forall j = 1, \ldots, p \end{cases}$$

Notice that matrix M can be expressed as follows

$$\begin{aligned} M &= \begin{bmatrix} A^\top P + PA + \Psi_1 & PA_d - \Psi_3 \\ \left(PA_d - \Psi_3\right)^\top & -\Psi_2 \end{bmatrix} + \begin{bmatrix} A^\top I\!I\, \mathcal{W}\, I\!I^\top A & A^\top I\!I\, \mathcal{W}\, I\!I^\top A_d \\ \left(A^\top I\!I\, \mathcal{W}\, I\!I^\top A_d\right)^\top & A_d^\top I\!I\, \mathcal{W}\, I\!I^\top A_d \end{bmatrix} \\ &= \begin{bmatrix} A^\top P + PA + \Psi_1 & PA_d - \Psi_3 \\ \left(PA_d - \Psi_3\right)^\top & -\Psi_2 \end{bmatrix} + \begin{bmatrix} A^\top I\!I\, \mathcal{W} \\ A_d^\top I\!I\, \mathcal{W} \end{bmatrix} \left(\mathcal{W}\right)^{-1} \begin{bmatrix} A^\top I\!I\, \mathcal{W} \\ A_d^\top I\!I\, \mathcal{W} \end{bmatrix}^\top \end{aligned}$$

Using Shur complement, we conclude that M is negative definite if and only if

$$\begin{bmatrix} A^\top P + PA + \Psi_1 & PA_d - \Psi_3 & A^\top I\!I \mathcal{W} \\ \left(PA_d - \Psi_3\right)^\top & -\Psi_2 & A_d^\top I\!I \mathcal{W} \\ \mathcal{W} I\!I^\top A & \mathcal{W} I\!I^\top A_d & -\mathcal{W} \end{bmatrix} < 0 \tag{12}$$

is satisfied. Furthermore, since condition (12) is equivalent to (9) according to Lemma 3.1 and since (9) is verified by assumption as well as conditions (8) and (7) then the system under study is asymptotically stable. This ends the proof of the theorem. ∇∇∇

Remark 3.1 *The results of Theorem 3.1 are only sufficient and therefore if these conditions are not verified we can't claim that the system under study is not stable.*

3.2 Robust stability

Let us now assume that the control is still equal to zero for all time and assume that the system has uncertainties on all the matrices, i.e:

$$\dot{x}_t = \left[A + DF(t)E\right] x_t + \sum_{j=1}^{p} \left[A_{dj} + D_j F_j(t) E_j\right] x_{t-h_j(t)} \tag{13}$$

where all the terms keep the same meaning as before.

We introduce the following notations

$$\begin{aligned} \tilde{A} &= A + DF(t)E \\ \tilde{A}_d &= \left[\, A_{d1} + D_1 F_1(t) E_1 \quad \ldots \quad A_{dp} + D_p F_p(t) E_p \,\right] = A_d + D_d F_d E_d \end{aligned}$$

where E_d and D_d are given by

$$D_d \;=\; [\,D_1 \quad \dots \quad D_p\,] \qquad E_d = diag(E_1,\ \dots E_p)$$

Note that conditions (7) and (8) do not depend on the system matrices so they do not need to be adapted to the uncertain case. Besides, we have to replace A and A_d respectively by $\tilde{A}$ and $\tilde{A}_d$ in condition (9) to get a condition for the robust case which is stated by Theorem 3.2.

Theorem 3.2 *Let assume that the Assumptions 2.1-2.2 are satisfied. If there exist F_1, F_2, F_3, F_4, $P > 0$, $Q_j > 0$, $W_j > 0$, X_j, Y_j, Z_j for $j = 1, 2, \dots, p$ and λ such that conditions (7), (8) and*

$$\begin{bmatrix} \alpha_{11} & * & * & * & * & * \\ A_d^T F_1^T + F_2 A - \Psi_3^\top & \alpha_{22} & * & * & * & * \\ F_3 A & F_3 A_d & -\mathcal{W} & * & * & * \\ F_4 A + P - F_1^\top & F_4 A_d - F_2^\top & I\!I\mathcal{W} - F_3^\top & -F_4 - F_4^\top & * & * \\ D^\top F_1^\top & D^\top F_2^\top & D^\top F_3^\top & D^\top F_4^\top & -\lambda R & * \\ D_d^\top F_1^\top & D_d^\top F_2^\top & D_d^\top F_3^\top & D_d^\top F_4^\top & 0 & -\lambda R_d \end{bmatrix} < 0 \quad (14)$$

hold with

$$\begin{aligned} \alpha_{11} &= \Psi_1 + A^T F_1^T + F_1 A + \lambda E^\top R E \\ \alpha_{22} &= A_d^T F_2^T + F_2 A_d - \Psi_2 + \lambda E_d^\top R_d E_d \end{aligned}$$

then the uncertain system under study is asymptotically stable for all admissible uncertainties.

Proof of Theorem 3.2 As we said before the robust stability is achieved, according to Theorem 3.1, if conditions (7), (8) and

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & \\ 0 & 0 & -\mathcal{W} & \mathcal{W}I\!I^\top \\ P & 0 & I\!I\mathcal{W} & 0 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} [\,\tilde{A} \quad \tilde{A}_d \quad 0 \quad -I\,] \right\} < 0 \qquad (15)$$

are satisfied. And since conditions (7), (8) remains unchanged in the presence of uncertainty, we have to work out only condition 15.

First notice that in condition (15) the second term of the left side can be split into two part to yield

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}I\!I^\top \\ P & 0 & I\!I\mathcal{W} & 0 \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} [\,A \quad A_d \quad 0 \quad -I\,] \right\}$$

$$+ \quad \mathsf{Sym}\left\{ \begin{bmatrix} F_1D & F_1D_d \\ F_2D & F_4D_d \\ F_3D & F_3D_d \\ F_4D & F_4D_d \end{bmatrix} \begin{bmatrix} F(t) & \\ & F_d(t) \end{bmatrix} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 \end{bmatrix} \right\} < 0$$

And according to Lemma 2.1, the previous inequality is equivalent to

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathcal{W} & 0 \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} [\, A \quad A_d \quad 0 \quad -I \,] \right\}$$
$$+ \quad \begin{bmatrix} F_1D & F_1D_d \\ F_2D & F_4D_d \\ F_3D & F_3D_d \\ F_4D & F_4D_d \end{bmatrix} \begin{bmatrix} \lambda R & \\ & \lambda R_d \end{bmatrix}^{-1} \begin{bmatrix} F_1D & F_1D_d \\ F_2D & F_4D_d \\ F_3D & F_3D_d \\ F_4D & F_4D_d \end{bmatrix}^\top$$
$$+ \quad \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} \lambda R & \\ & \lambda R_d \end{bmatrix} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 \end{bmatrix} < 0$$

Using a Schur complement operation yields, for a scalar λ, the following condition

$$\begin{bmatrix} f_{11} & * & * & * & * & * \\ -\Psi_3^\top & f_{22} & * & * & * & * \\ 0 & 0 & -\mathcal{W} & * & * & * \\ P & 0 & \mathbb{I}\mathcal{W} & 0 & * & * \\ 0 & 0 & 0 & 0 & -\lambda R & * \\ 0 & 0 & 0 & 0 & 0 & -\lambda R_d \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 0 \\ 0 \end{bmatrix} [\, A \;\; A_d \;\; 0 \;\; -I \;\; D \;\; D_d] \right\} < 0 \qquad (16)$$

with

$$\begin{aligned} f_{11} &= \Psi_1 + \lambda E^\top R E \\ f_{22} &= -\Psi_2 + \lambda E_d^\top R_d E_d \end{aligned}$$

It is worth noting that (16) can be rewritten as condition (14) and this ends the proof. This condition combined with (7) and (8) represent the sufficient conditions for robust stability of the class of systems under consideration. $\nabla\nabla\nabla$

The next section will deal with the stabilizability and the robust stabilizability of the class of systems under study.

3.3 Stabilizability

This section deals with the stabilizability problem, and we will try to design a controller that stabilizes the closed-loop system. We will restrict our self to the class of memoryless state feedback controller.

Thus the state feedback controller is of the form:

$$u(t) = Kx(t) \tag{17}$$

Substituting (17) in the plant model and taking $A^{cl} = (A+BK)$ we get the closed-loop dynamics:

$$\dot{x}_t = A^{cl}x_t + \sum_{j=1}^{p} A_{dj}(t)x_{t-h_j(t)} \tag{18}$$

We note that only condition (9) must be adapted to the stabilizability case. We replace A by A^{cl} in (9) to get

$$\mathcal{M}^{cl} = \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathcal{W} & 0 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A^{cl} & A_d & 0 & -I \end{bmatrix} \right\} < 0$$

The problem of robust stabilization by state feedback is stated by Theorem 3.3.

Theorem 3.3 *Let assume that the Assumption 2.2 is satisfied. If there exist F_1, F_2, F_3, F_4, $P > 0$, $Q_i > 0$, $W_i > 0$, X_i, Y_i, Z_i for $i = 1, \ldots, p$ and L and G such that the following hold for $i = 1, \ldots, p$*

$$\begin{bmatrix} \bar{Z}_i & \bar{Y}_i \\ \bar{Y}_i^\top & \bar{X}_i \end{bmatrix} > 0 \tag{19}$$

$$\left(\bar{l}_i - \underline{l}_i\right)\bar{X}_i + \left(\bar{l}_i - 1\right)\bar{W}_i < 0 \tag{20}$$

$$\begin{bmatrix} \Psi_1 + A_o^T F_1^T + F_1 A_o & * & * & * & * \\ A_{d_o}^T F_1^T + F_2 A_o - \Psi_3^\top & A_{d_o}^T F_2^T + F_2 A_{d_o} - \Psi_2 & * & * & * \\ F_3 A_o & F_3 A_{d_o} & -\mathcal{W} & * & * \\ F_4 A_o + P - F_1^\top & F_4 A_{d_o} - F_2^\top & \mathbb{I}\mathcal{W} - F_3^\top & -F_4 - F_4^\top & * \\ B^T F_1^T + L - GK_o & B^T F_2^T - GK_{d_o} & B^T F_3^T & B^T F_4^T & -G - G^T \end{bmatrix} < 0 \tag{21}$$

for given gains K_o and K_{d_o} that make the matrices $A_o = (A+BK_o)$ and $A_{d_o} = (A_d+BK_{d_o})$ stable then the closed loop system is asymptotically stable with the stabilizing feedback gain given by

$$K = G^{-1}L$$

Proof of Theorem 3.3 The closed loop matrix A^{cl} can also be rewritten as

$$\begin{array}{rcl} A^{cl} & = & A + BK = A + BK_o + B\,(K - K_o) = A_o + BS_o \\ A_d & = & A_d + BK_{d_o} - BK_{d_o} = A_{d_o} + BS_{d_o} \end{array}$$

where the gain K_o and K_{d_o} are chosen in such a way that $A + BK_o$ and $A_d + BK_{d_o}$ are stable. This allows us to rewrite $\mathcal{M}^{cl}$ as

$$\mathcal{M}^{cl} = \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}I\!I^\top \\ P & 0 & I\!I\mathcal{W} & 0 \end{bmatrix}$$
$$+\mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} [\,A_o \quad A_{d_o} \quad 0 \quad -I\,] \right\} + \mathrm{Sym}\left\{ \begin{bmatrix} F_1B \\ F_2B \\ F_3B \\ F_4B \end{bmatrix} [\,S_o \quad S_{d_o} \quad 0 \quad 0\,] \right\} < 0$$

and using similar arguments as in the proof of lemma 3.1, we introduce a new variable G to get the following condition.

$$\mathcal{M}_{SF} = \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P & 0 \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}I\!I^\top & 0 \\ P & 0 & I\!I\mathcal{W} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 0 \end{bmatrix} [\,A_o \quad A_{d_o} \quad 0 \quad -I \quad B\,] \right\} + \mathrm{Sym}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} G\,[\,S_o \quad S_{d_o} \quad 0 \quad 0 \quad -I\,] \right\} < 0$$

which is in fact condition (21) where we have introduced the change of variable $L = GK$.

Indeed, notice that

$$\begin{bmatrix} I & 0 & 0 & 0 & -S_o^\top \\ 0 & I & 0 & 0 & -S_{d_o}^\top \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \mathcal{M}_{SF} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ S_o & S_{d_o} & 0 & 0 \end{bmatrix} = \mathcal{M}^{cl} < 0$$

The introduction of the new variable G allows us to perform a decoupling between the matrices B and S_o and hence between B and the state feedback gain K. $\nabla\nabla\nabla$

3.4 Robust stabilizability

In this subsection, we are concerned by robust stabilisability of the uncertain system under the control law (17). The closed loop system is then given by

$$\dot{x}_t = [A + BK + DF(t)E + D_b F_b(t) E_b K]\, x_t + \sum_{j=1}^{p} [A_{dj} + D_j F_j(t) E_j]\, x_{t-h_j(t)} \tag{22}$$

where all the terms keep the same meaning as previously. Taking account of the uncertainties in (21), we get

$$\begin{aligned}
\tilde{\mathcal{M}}_{SF} &= \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P & 0 \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top & 0 \\ P & 0 & \mathbb{I}\mathcal{W} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} G\, [\, S_o \quad S_{d_o} \quad 0 \quad 0 \quad -I\,] \right\} \\
&\quad + \mathsf{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 0 \end{bmatrix} [\, \tilde{A}^{cl} \quad \tilde{A}_d^{cl} \quad 0 \quad -I \quad \tilde{B}\,] \right\} \\
&= \mathcal{M}_{SF_o} + \mathsf{Sym}\left\{ \begin{bmatrix} F_1 D & F_1 D_b & F_1 D_d & F_1 D_b & F_1 D_b \\ F_2 D & F_2 D_b & F_2 D_d & F_2 D_b & F_2 D_b \\ F_3 D & F_3 D_b & F_3 D_d & F_3 D_b & F_3 D_b \\ F_4 D & F_4 D_b & F_4 D_d & F_4 D_b & F_4 D_b \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right. \\
&\quad \left. \times \mathcal{F}(t) \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ E_b K_o & 0 & 0 & 0 & 0 \\ 0 & E_b K_{d_o} & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_b \end{bmatrix} \right\} < 0
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{F}(t) &= \mathsf{diag}\Big[\, F(t) \quad F_b(t) \quad F_d(t) \quad F_b(t) \quad F_b(t) \,\Big] \\
\tilde{A}^{cl} &= \tilde{A} + \tilde{B} K_o \\
\tilde{A}_d^{cl} &= \tilde{A}_d + \tilde{B} K_{d_o}
\end{aligned}$$

and $\mathcal{M}_{SF_o}$ the part of $\tilde{\mathcal{M}}_{SF}$ that contains only the non uncertain terms and using Lemma 2.1 as previously we get

$$
\begin{aligned}
\mathcal{M}_{SFR} = \mathsf{Sym} & \left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left[A_o \quad A_{d_o} \quad 0 \quad -I \quad B \quad D \quad D_b \quad D_d \quad D_b \quad D_b \right] \right\} \\
+ & \begin{bmatrix}
\beta_{11} & * & * & * & * & * & * & * & * & * \\
-\Psi_3^\top & \beta_{22} & * & * & * & * & * & * & * & * \\
0 & 0 & -\mathcal{W} & * & * & * & * & * & * & * \\
P & 0 & \mathbb{I}\mathcal{W} & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & \lambda E_b^\top R_b E_b & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\lambda R & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_d & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b
\end{bmatrix} \\
+\mathrm{Sym} & \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} G \left[S_o \quad S_{d_o} \quad 0 \quad 0 \quad -I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right] \right\} < 0
\end{aligned}
\tag{23}
$$

with

$$
\begin{aligned}
\beta_{11} &= \Psi_1 + \lambda \Big(E^\top R E + K_o^\top E_b^\top R_b E_b K_o \Big) \\
\beta_{22} &= -\Psi_2 + \lambda \Big(E_d^\top R_d E_d + K_{d_o}^\top E_b^\top R_b E_b K_{d_o} \Big)
\end{aligned}
$$

In condition (23), we have to proceed to the change of variable $L = GK$ and then we get an LMI problem from which when it is feasible we get the feedback gain as

$$K = G^{-1}L \tag{24}$$

The following theorem summarizes the result of robust stability.

Theorem 3.4 *Assume that the assumptions 2.1-2.2 are satisfied. If there exist F_1, F_2, F_3, F_4, $P > 0$, $Q_i > 0$, $W_i > 0$, X_i, Y_i, Z_i for $i = 1, \ldots, p$ and L, G and λ such that the LMI problem constituted by the three conditions (19), (20)and (23) is feasible, the robust stabilizing state feedback gain is given by (24) and the uncertain closed loop system under study is asymptotically stable for the set of admissible uncertainties.*

4 Example

To show the usefulness of our results, let us consider some numerical examples.

Example 4.1 *In this example, we consider that the system under study has one time-delay and try to apply the results of Theorem 3.2. Let us assume that the dynamics is described by the following matrices:*

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \qquad A_d = \begin{bmatrix} -0.2 & 0.1 \\ -0.3 & -0.1 \end{bmatrix}$$

$$D = D_d = 0.2I \qquad E = E_d = I \qquad R = R_d = I$$

	Maximal $\bar{h}$	
$\dot{h}(t)$	*[15]*	*Theorem 3.2*
≤ 0.9	*0.1621*	*0.225*
≤ 0.8	*0.3802*	*0.49*
≤ 0.6	*1.0662*	*1.425*
≤ 0.4	*7.1784*	∞

Table 1

In this example we proceed to a comparison with the result given in [15] and we show that in the robust case, Theorem 3.2 of the present paper provides better results than [15, Theorem 3.2]. It is worth noting that for the nominal system we get similar bounds.

Example 4.2 In this example, we consider the robust stabilizability problem. For this purpose let us consider the following data:

$$A = \begin{bmatrix} 2.0 & 0.0 \\ 1.0 & 3.0 \end{bmatrix} \qquad D = 0.2I \qquad E = I, \qquad R = I$$

$$B = \begin{bmatrix} 1.0 & 2.0 \\ 1.0 & 0.0 \end{bmatrix} \qquad D_b = 0.2I \qquad E_b = I, \qquad R_b = I$$

$$A_d = \begin{bmatrix} -0.1 & 0.0 \\ -0.8 & -1.0 \end{bmatrix} \qquad D_d = 0.2I \qquad E_d = I, \qquad R_d = I$$

The characteristics of the first derivative of the delay are as follows

$$\underline{l} = 0. \qquad \bar{l} = 0.825$$

The application of Theorem 3.4 leads to the following results

$$X = \begin{bmatrix} 30.1136 & -3.9990 \\ -3.9990 & 0.5311 \end{bmatrix}, \qquad Y = \begin{bmatrix} -3.3436 & 0.4440 \\ 4.4596 & -0.5922 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0.3712 & -0.4950 \\ -0.4950 & 0.6626 \end{bmatrix}, \qquad W = \begin{bmatrix} 141.9641 & -18.8526 \\ -18.8526 & 2.5036 \end{bmatrix}$$

$$P = \begin{bmatrix} 8999.5211 & -28.5684 \\ -28.5684 & 37679.0852 \end{bmatrix}, \qquad Q = \begin{bmatrix} 7942.9421 & 17142.9757 \\ 17142.9757 & 303352.7590 \end{bmatrix}$$

$$F1 = \begin{bmatrix} 8946.4080 & 152.8018 \\ -88.7054 & 37661.7437 \end{bmatrix}, \qquad F2 = \begin{bmatrix} -100.1079 & 85.6633 \\ -1447.3920 & 38.6734 \end{bmatrix}$$

$$F3 = \begin{bmatrix} 350.8231 & -40.7311 \\ -46.5886 & 5.4090 \end{bmatrix}, \qquad F4 = \begin{bmatrix} 3449.8341 & -375.8000 \\ -500.6067 & 161.2343 \end{bmatrix}$$

$$L = \begin{bmatrix} -223754.6705 & 19190.1742 \\ -258925.2826 & 120109.2273 \end{bmatrix}, \qquad G = \begin{bmatrix} 37765.6548 & 53117.4530 \\ 42764.2230 & 62293.1025 \end{bmatrix}$$

$$K_o = \begin{bmatrix} -0.5000 & -6.2506 \\ -3.8114 & 6.2506 \end{bmatrix}, \qquad K_{d_o} = \begin{bmatrix} 0.4000 & 0.4697 \\ -0.3486 & -0.4697 \end{bmatrix}$$

$$\lambda = 816.2566$$

The stabilizing state feedback gain is then

$$K = \begin{bmatrix} -2.2826 & -63.9955 \\ -2.5896 & 45.8611 \end{bmatrix}$$

These result were obtained for

$$\bar{h} = 26.9650$$

The parameter $\bar{h}$ has been found by search and it is worth noting that this value does not correspond to the maximal value and one can improve this result by choosing adequately the parameters K_o and K_{d_o}. Based on the results of the previous theorem, we conclude that the system under study in this example is robustly stable for all admissible uncertainties.

5 Conclusion

This paper dealt with a class of dynamical linear uncertain systems with multiple time-varying delays in the state. delay-dependent sufficient conditions have been developed to check if a system of this class of systems is stable and/or stabilizable. A memoryless state feedback controller with consequent parameters has been used to stabilize the system. The LMI technique is used in all the development.

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