Control of Stochastic Systems with Time-Varying Multiple Time-Delays: An LMI Approach

E.K. Boukas

GERAD and Mechanical Engineering Department
École Polytechnique de Montréal
boukas@meca.polymtl.ca

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Abstract

This paper deals with the class of continuous-time linear systems with Markovian jumps and multiple time-delays. The systems we are treating are assumed to have time-varying delays in their dynamics that can be different and also have uncertainties in the systems parameters. The time-varying structure bounded uncertainties are considered. Delay-dependent conditions for stochastic stability and stochastic stabilizability and their robustness are considered. A design algorithm for a stabilizing memoryless controller is proposed. All the results are given in the LMI formalism.

Key Words: Jump linear system, Linear matrix inequality, Stochastic stability, Stochastic stabilizability, Norm bounded uncertainty.

Résumé

Cet article traite de la commande des systèmes à sauts markoviens avec retard sur le vecteur d'état. Sous l'hypothèse que les incertitudes du système sont bornées en norme, des conditions suffisantes pour la stabilité et la stabilisabilité sont établies. Les résultats sont donnés en forme d'inégalités matricielles linéaires souvent appelées LMI dans littérature. Enfin, un algorithme de design d'un controleur sans mémoire stabilisant de manière robuste la classe de systèmes étudiée est proposé.

1 Introduction

Practical systems like manufacturing systems, chemical processes, transmission lines and rolling mill systems have delays in their dynamics. The reader is referred to Boukas and Liu (Ref. 1), Mahmoud (Ref. 2), and the references therein for more examples of systems with time-delays. The presence of these delays is in general the source of many problems. It is well known in the literature that the delay is a great source of systems instability and poor performance. It is also known that the control of such class of systems is in general a hard control problem. During the last decades, we have seen an increasing interests for the control of this class of systems and many results have been reported in the literature. For a recent review of literature on the subject of systems with time-delay we refer the reader to Boukas and Liu (Ref. 1) and the references therein.

Among the results that have been reported in the literature, we have the ones on stability and stabilizability and their robustness. This includes delay-dependent and delay-independent results. It is well known that the delay-dependent conditions present less conservatism compared to delay-independent conditions and it is all the time preferable to use delay-dependent to deal with the control of the class of systems we are dealing with.

In practice some systems may have in their dynamics besides the time-delay random parameters that makes the study more complicated. Among the results on stochastic systems with time-delays we quote the work of Boukas and his coauthors (Ref. 3), Mao (Ref. 4), Cao and Lam (Ref. 5).

The goal of this paper is to consider dynamical class of systems with Markovian jumps and multiple time-varying time-delays and develop results on stochastic stability and its robustness. Our objective is also to develop a design algorithm using the LMI framework to compute the control that will stabilize and robustly stabilize the class of systems we are considering.

The rest of this paper is organized as follows. In Section 2, the problem is stated and some preliminary results are given. In Section 3, the main results are given and they include results on stochastic stability and its robustness and the stochastic stabilizability and its robustness. A memoryless controller is used in this paper. A design algorithm is proposed to synthesize the controller we are using.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^\mathsf{T}$" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrices with compatible dimensions. $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respective to some probability measure $\mathcal{P}$. L_2 is the space of integral vector over $[0,\infty)$. $\|\cdot\|$ will refer to the Euclidean vector norm whereas $\|\cdot\|$ denotes the L_2-norm over $[0,\infty)$ defined as $\|f\|^2 = \int_0^\infty f^T(t)f(t)\ dt$.

2 Problem statement

Consider a hybrid system with N modes, i.e., $\mathcal{S} = \{1, 2, \cdots, N\}$. The mode switching is governed by a continuous-time Markov process $\{r_t, t \geq 0\}$ taking values in the state space $\mathcal{S}$ and having the following infinitesimal generator

$$\Lambda = (\lambda_{ij}), i, j \in \mathcal{S},$$

where $\lambda_{ij} \geq 0, \forall j \neq i, \lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$.

The mode transition probabilities are described as follows:

$$P[r_{t+\Delta} = j | r_t = i] = \begin{cases} \lambda_{ij}\Delta + o(\Delta), & j \neq i \\ 1 + \lambda_{ii}\Delta + o(\Delta), & j = i \end{cases} \tag{1}$$

where $\lim_{\Delta \to 0} o(\Delta)/\Delta = 0$.

Let $x(t) \in \mathbb{R}^n$ be the physical state of the system, which satisfies the following dynamics:

$$\begin{cases} \dot{x}(t) = A_0(r_t, t)x(t) + \sum_{k=1}^{l} A_k(r_t, t)x(t - h_k(t)) + B(r_t, t)u(t), \\ x(s) = \phi(s), -\tau \leq s \leq 0, \end{cases} \tag{2}$$

where $u(t) \in \mathbb{R}^m$ is the control system, $A_k(r_t, t) = A_k(r_t) + D_k(r_t)F_k(r_t, t)E_k(r_t) \in \mathbb{R}^{n \times n}, k = 0, 1, \ldots, k$, and $B(r_t, t) = B(r_t) + D_b(r_t)F_b(r_t, t)E_b(r_t) \in \mathbb{R}^{n \times m}$ with $A_k(r_t)$, $D_k(r_t)$, and $E_k(r_t)$, $k = 0, 1, \ldots, k$, $B(r_t)$, $D_b(r_t)$ and $E_b(r_t)$ are known real matrices with appropriate dimensions for each $r_t \in \mathcal{S}$, and $F_k(r_t)$, $k = 0, 1, \ldots, k$ and $F_b(r_t, t)$ are unknown real time-varying matrices with appropriate dimensions satisfying the following:

$$F_k^\top(r_t, t)F_k(r_t, t) \leq I, k = 0, 1, \ldots, l, \forall r_t \in \mathcal{S}, \tag{3}$$

$$F_b^\top(r_t, t)F_b(r_t, t) \leq I, \forall r_t \in \mathcal{S}. \tag{4}$$

$h_k(t) > 0, k = 1, 2, \ldots, l$ represents the delay of the system, that satisfies $0 \leq h_k(t) \leq \tau$ and $\dot{h}_k(t) \leq \tau_k < 1$, $\tau = \max\{h_1(t), \ldots, h_l(t)\}$, and $\psi(t)$ is a smooth vector-valued initial function in $[-\tau, 0]$.

The initial condition of the system is specified as $(r_0, \phi(\cdot))$ with $x(s) = \phi(s) \in L_2[-\tau, 0] \triangleq \{f(\cdot) | \int_0^\infty f^\top(t)f(t)dt < \infty\}$.

Remark 2.1 *The uncertainties that satisfies the conditions (3)-(4) are referred to be admissible. The uncertainties we are considering here are time and mode system dependent only. The results we will develop here are applied for systems with uncertainties that may be dependent on time, mode and state systems.*

For system (2) with $u(.) \triangleq 0$ for $t \geq 0$, we have the following definitions:

Definition 2.1 *System (2) with $u(.) \triangleq 0, \forall t \geq 0$ and all the uncertainties equal to zero, is said to be* *(i) stochastically stable (SS) if there exists a constant* $T(r_0, \phi(\cdot))$ *such that*

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt \middle| r_0, x(s) = \phi(s), s \in [-\tau, 0]\right] \leq T(r_0, \phi(\cdot)); \tag{5}$$

(ii) mean square stable (MSS) if

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0$$

holds for any initial condition $(r_0, \phi(\cdot))$*;*

(iii) mean exponentially stable (MES) if there exist constants $\alpha(r_0, \phi(\cdot)) > 0, \beta > 0$ *such that*

$$\mathbb{E}[\|x(t)\|^2] \leq \alpha(r_0, \phi(\cdot))e^{-\beta t}.$$

Obviously, MES implies MSS and SS.

Definition 2.2 *System (2) with* $u(.) \stackrel{\Delta}{=} 0$ *for* $t \geq 0$*, is said to be*

(i) robustly stochastically stable (RSS) if there exists a constant $T(r_0, \phi(\cdot))$ *such that*

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt \middle| r_0, x(s) = \phi(s), s \in [-\tau, 0]\right] \leq T(r_0, \phi(\cdot)); \tag{6}$$

holds for all admissible uncertainties

(ii) robustly mean exponentially stable (RMES) if there exist constants $\alpha(r_0, \phi(\cdot)) > 0, \beta > 0$ *such that*

$$\mathbb{E}[\|x(t)\|^2] \leq \alpha(r_0, \phi(\cdot))e^{-\beta t}.$$

holds for all admissible uncertainties.

Obviously, we can show that RMES implies RSS.

Definition 2.3 *System (2) with all the uncertainties equal to zero, is said to be stabilizable in the stochastic sense if there exists a state feedback controller:*

$$u(t) = K(r_t)x(t) \tag{7}$$

such that the closed-loop system is stochastically stable, where $K(i)$*,* $i \in \mathcal{S}$ *are constant gain matrices.*

Definition 2.4 *System (2) is said to be robustly stabilizable in the stochastic sense if there exists a state feedback controller of the form (7) such that the closed-loop system is robustly stochastically stable for all admissible uncertainties, where* $K(i)$*,* $i \in \mathcal{S}$ *are constant gain matrices.*

Let us now give the following lemma that we will use extensively in proving our results in the rest of this paper. The proofs of the results of this lemma can be found in Boukas and Liu (Ref. 1).

Lemma 2.1 *Let Y be a symmetric matrix, and H, E be given matrices of appropriate dimensions, and F satisfying $F^\top F \leq I$. Then, we have:*

(i) for any $\varepsilon > 0$, $Y + HFE + E^\top F^\top H^\top \leq \varepsilon H^\top H + \frac{1}{\varepsilon} E^\top E$

(ii) $Y + HFE + E^\top F^\top H^\top < 0$ holds if and only if there exists a scalar $\varepsilon > 0$ such that $Y + \varepsilon H H^\top + \frac{1}{\varepsilon} E^\top E < 0$

This paper studies the stochastic stability and the stochastic stabilizability of the class of systems (2) and their robustness. Our goal in this paper is to establish delay-dependent sufficient conditions that guarantee the relative stability and its robustness, the stabilizability and its robustness for the class of systems we are considering. In the rest of this paper, we will assume that all the required assumptions are satisfied. Sometimes, we will use $P(i)$ and P_i to denote $P(r_t)$ when $r_t = i$, $i \in \mathcal{S}$ and the meaning can be understood from the context.

3 Main results

The main goal of this paper is to develop sufficient conditions that guarantee the relative stochastic stability and its robustness, the stochastic stabilizability and its robustness of the class of systems we are dealing with.

3.1 Stability and stabilizability

Let us now consider the free system (2) (i.e. $u(.) \stackrel{\Delta}{=} 0, \forall t \geq 0$) and assume that the uncertainties are equal zero, i.e. $F_k(r_t, t) = 0, k = 0, 1, 2, \ldots, k$ and $F_b(r_t, t) = 0$, $\forall r_t \in \mathcal{S}$, $\forall t \geq 0$. The following theorem states the first result on stability of the class of systems we are considering in this paper. The conditions are in LMI formalism and are delay-dependent.

Theorem 3.1 *If there exist symmetric and positive-definite matrices $P = (P_1, \ldots, P_N) > 0$ and $R = (R_1, \ldots, R_l) > 0$ satisfying the following LMIs for every $r_t \in \mathcal{S}$:*

$$\begin{aligned} &\left[A_0(r_t) + \alpha I\right]^\top P(r_t) + P(r_t)\left[A_0(r_t) + \alpha I\right] + \sum_{k=1}^{l} R_k + \frac{l(l+1)}{2} P(r_t) \\ &\quad + \sum_{j=1}^{N} \lambda_{r_t j} P(j) \stackrel{\Delta}{=} \Phi(r_t) < 0 \end{aligned} \tag{8}$$

$$A_k^\top(r_t) P(r_t) A_k(r_t) \leq (1 - \tau_k) e^{-2\alpha\tau_k} R_k, k = 1, 2, \ldots, l \tag{9}$$

then, system (2) is stochastically stable with a degree α.

Proof: Let $\alpha > 0$ be a given stability degree. Then, via the following state transformation $z(t) = e^{\alpha t}x(t)$, the dynamics (2) becomes:

$$\begin{aligned} \dot{z}(t) &= \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t) \qquad (10)\\ &= \left[A_0(r_t) + \alpha I + D_0(r_t)F_0(r_t,t)E_0(r_t)\right]z(t) \\ &\quad + \sum_{k=1}^{l} e^{\alpha h_k(t)}\left[A_k(r_t) + D_k(r_t)F_k(r_t,t)E_k(r_t)\right]z(t-h_k(t)) \qquad (11) \end{aligned}$$

Let $\mathbb{C}[-\tau, 0]$ be a space of continuous functions on the interval $[-\tau, 0]$ and for any $\mathbf{z} \in \mathbb{C}[-\tau, 0]$, define $\|\mathbf{z}\| = \mathbf{sup}_{-\tau\leq \mathbf{s}\leq \mathbf{0}}\|(\mathbf{s})\|$. Obviously, the evolution of $z(t)$ depends on $z(s), t-\tau \leq s \leq t$, which means that $\{(z(t), r_t), t \geq 0\}$ is not a Markov process. To cast our model into the framework of Markov system, let us define a process $\mathbf{z}(t)$ taking values in $\mathbb{C}[-\tau, 0]$ by

$$\mathbf{z}_s(t) = z(s+t), t-\tau \leq s \leq t \qquad (12)$$

then, $\{(\mathbf{z}(t), r_t), t \geq 0\}$ is a strong Markov process. Consider the Lyapunov functional candidate with the following form:

$$V(\mathbf{z}(t), r_t) = z^\top(t)P(r_t)z(t) + \sum_{k=1}^{l}\int_{t-h_k(t)}^{t} z^\top(\theta)R_k z(\theta)d\theta \qquad (13)$$

Let $\mathcal{A}$ be the infinitesimal generator of the process $\{(\mathbf{z}(t), r_t), t \geq 0\}$. Then, we get:

$$\begin{aligned} \mathcal{A}V(\mathbf{z}(t), r_t) &= \dot{z}^\top(t)P(r_t)z(t) + z^\top(t)P(r_t)\dot{z}(t) + z^\top(t)\left[\sum_{j=1}^{N}\lambda_{r_t j}P(j)\right]z(t) \\ &\quad + \sum_{k=1}^{l} z^\top(t)R_k z(t) - \sum_{k=1}^{l}(1-\dot{h}_k(t))z^\top(t-h_k(t))R_k z(t-h_k(t)) \\ &= z^\top(t)\Bigg[\left(A_0(r_t)+\alpha I\right)^\top P(r_t) + P(r_t)\left(A_0(r_t)+\alpha I\right) + \sum_{k=1}^{l} R_k \\ &\qquad + \sum_{j=1}^{N}\lambda_{r_t j}P(j)\Bigg]z(t) \\ &\quad + 2z^\top(t)P(r_t)\sum_{k=1}^{l} e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t)) \\ &\quad - \sum_{k=1}^{l}(1-\dot{h}_k(t))z^\top(t-h_k(t))R_k z(t-h_k(t)) \end{aligned}$$

Using the fact that

$$
\begin{aligned}
&2\sum_{k=1}^{l} z^\top(t)P(r_t)e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t)) = \\
&\quad -\sum_{k=1}^{l}\left[z(t)-e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t))\right]^\top P(r_t) \\
&\quad \left[z(t)-e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t))\right]+\sum_{k=1}^{l} z^\top(t)P(r_t)z(t) \\
&\quad +\sum_{k=1}^{l}\left(e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t))\right)^\top P(r_t)e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t)) \qquad (14)
\end{aligned}
$$

we get:

$$
\begin{aligned}
\mathcal{A}V(\mathbf{z}(t),r_t) \;=\;& z^\top(t)\Big[(A_0(r_t)+\alpha I)^\top P(r_t)+P(r_t)\left(A_0(r_t)+\alpha I\right) \\
&+\sum_{l=1}^{l} R_k+\frac{l(l+1)}{2}P(r_t)+\sum_{j=1}^{N}\lambda_{r_tj}P(j)\Big]z(t) \\
&-\sum_{k=1}^{l}\left[z(t)-e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t))\right]^\top \\
&\qquad P(r_t)\left[z(t)-e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t))\right] \\
&+\sum_{k=1}^{l}\left(e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t))\right)^\top P(r_t)e^{\alpha h_k(t)}A_k(r_t)z(t-h_k(t)) \\
&-\sum_{k=1}^{l}(1-\dot{h}_k(t))z^\top(t-h_k(t))R_k z(t-h_k(t))
\end{aligned}
$$

This gives the following:

$$
\begin{aligned}
\mathcal{A}V(\mathbf{z}(t),r_t) \;\le\;& z^\top(t)\Big[(A_0(r_t)+\alpha I)^\top P(r_t)+P(r_t)\left(A_0(r_t)+\alpha I\right) \\
&+\sum_{k=1}^{l} R_k+\frac{l(l+1)}{2}P(r_t)+\sum_{j=1}^{N}\lambda_{r_tj}P(j)\Big]z(t) \\
&-\sum_{k=1}^{l} z^\top(t-h_k(t))\left[(1-\tau_k)R_k-e^{2\alpha\tau_k}A_k^\top(r_t)P(r_t)A_k(r_t)\right] \\
&\qquad z(t-h_k(t)) \\
\le\;& z^\top(t)\Phi(r_t)z(t)
\end{aligned}
$$

Therefore, we obtain:

$$\mathcal{A}V(\mathbf{z}(t), r_t) \quad \leq \quad -\min_{j\in\mathcal{S}} \{\lambda_{min}(-\Phi(j))\} z^\top(t) z(t)$$

Combining this with Dynkin formula, we get:

$$\begin{aligned} \mathbb{E}\left[V(\mathbf{z}(t), r_t)\right] - \mathbb{E}\left[V(\mathbf{z}(0), r_0)\right] &= \mathbb{E}\left[\int_0^t \mathcal{A}V(\mathbf{z}(s), r_s) ds | (r_0, \Phi(.))\right] \\ &\leq -min_{j\in\mathcal{S}}\{\lambda_{min}(-\Phi(j))\}\mathbb{E}\left[\int_0^t z^\top(s) z(s) ds | (r_0, \Phi(.))\right] \end{aligned} \tag{15}$$

which gives in turn:

$$min_{j\in\mathcal{S}}\{\lambda_{min}(-\Phi(j))\}\mathbb{E}\left[\int_0^t z^\top(s) z(s) ds | (r_0, \Phi(.))\right] \leq \mathbb{E}\left[V(\mathbf{z}(0), r_0)\right] \tag{16}$$

This implies in turn that the following relation holds for all $t \geq 0$:

$$\mathbb{E}\left[\int_0^t z^\top(s) z(s) ds | (r_0, \Phi(.))\right] \leq \frac{\mathbb{E}\left[V(\mathbf{z}(0), r_0)\right]}{min_{j\in\mathcal{S}}\{\lambda_{min}(-\Phi(j))\}} \tag{17}$$

This proves Theorem 3.1. □

From the practical point of view, for a given system it is of great importance to know what is the maximum stability degree? This question can be answered by solving the following optimization problem:

$$\begin{cases} \max_{(P_1,\ldots,P_N),(R_1,\ldots,R_k)} \alpha \\ \text{s.t.} \quad (8) - (9) \end{cases} \tag{18}$$

This optimization problem is nonlinear and therefore, it can't be solved using the LMI toolboxes that are available in the market. To cast it into a solvable problem we can use the fact that $e^{-2\alpha\tau} < 1$ and therefore the sufficient condition (9) becomes

$$A_k^\top(r_t) P(r_t) A_k(r_t) \leq (1 - \tau_k) R_k, k = 1, 2, \ldots, l, \forall r_t \in \mathcal{S}$$

which is a bilinear matrix inequality that can be used easily.

Notice also, that we can compute the maximum time-delay that the system can have and remain stochastically stable or to design the memoryless controller (7) that stochastically stable. For more details on this matter, we refer the reader to Boukas and Liu (Ref. 1) for equivalent studies.

We are now in a position to synthesize a memoryless state feedback controller (7) that stabilizes the certain system (2) in the SS sense, i.e. $F_k(r_t, t) \stackrel{\Delta}{=} 0, k = 0, 1, 2, \ldots, l$ and $F_b(r_t, t) \stackrel{\Delta}{=} 0$ $\forall t \geq 0$ and for $r_t \in \mathcal{S}$.

Theorem 3.2 *If there exist symmetric and positive-definite matrices* $X = (X_1, \cdots, X_N) > 0$ *and* $U = (U_1, \ldots, U_l) > 0$ *satisfying*

$$\begin{pmatrix} \# & Z_i(X) & S_i(X) \\ Z_i^\top(X) & -\mathcal{Z}_i & \mathbf{0} \\ S_i^\top(X) & \mathbf{0} & -\mathcal{X}_i \end{pmatrix} < 0 \tag{19}$$

$$\begin{pmatrix} (1-\tau_k)e^{-2\tau_k\alpha}U_k & U_k A_k^\top(i) \\ A_k(i)U_k & X_i \end{pmatrix} > 0, \tag{20}$$

where $\# = [A_0(i) + \alpha I]\, X_i + B(i)Y_i + X_i\, [A_0(i) + \alpha I]^\top + Y_i^\top B^\top(i) + \lambda_{ii}X_i + \frac{l(l+1)}{2}X_i$, *then controller (7) with* $K(i) = Y_i X_i^{-1}$ *stabilizes system (2) in the SS sense,*

$$S_i(X) = \left(\sqrt{\lambda_{i1}}X_i \;\cdots\; \sqrt{\lambda_{ii-1}}X_i \; \sqrt{\lambda_{ii+1}}X_i \;\cdots\; \sqrt{\lambda_{iN}}X_i\right) \tag{21}$$

$$\mathcal{X}_i = \mathrm{diag}\{X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_N\}, \tag{22}$$

$$Z_i(X) = (X_i, \ldots, X_i) \tag{23}$$

$$\mathcal{Z}_i = \mathrm{diag}\{U_1, \cdots, U_l\} \tag{24}$$

Proof: Substituting (7) into (2) yields the dynamics of the closed-loop system described as follows:

$$\dot{x}(t) = \bar{A}_0(r_t)x(t) + \sum_{k=1}^{l} A_k(r_t)x(t - h_k(t)).$$

where $\bar{A}_0(r_t) = A_0(r_t) + B(r_t)K(r_t)$.

Using Theorem 3.1, to prove that the controller (7) stabilizes system (2) in the stochastic sense it suffices to show that there exist symmetric and positive-definite matrices $P = (P_1, \cdots, P_N) > 0$ and $R = (R_1, \ldots, R_l) > 0$ satisfying

$$\left[\bar{A}_0(r_t) + \alpha I\right]^\top P(r_t) + P(r_t)\left[+\bar{A}_0(r_t) + \alpha I\right] + \sum_{k=1}^{l} R_k + \frac{l(l+1)}{2}P(r_t)$$
$$+ \sum_{j=1}^{N} \lambda_{r_t j} P(j) < 0 \tag{25}$$

$$A_k^\top(r_t)P(r_t)A_k(r_t) \le (1-\tau_k)e^{-2\alpha\tau_k}R_k, k = 1, 2, \ldots, l. \tag{26}$$

Let $X_i = P^{-1}(i)$ and $U_k = R_k^{-1}$, $k = 1, 2, \ldots, l$, when $r_t = i$. Pre- and post-multiplying (25) by X_i yields

$$X_i\left[\bar{A}_0^\top(i)+\alpha I\right]+\left[\bar{A}_0(i)+\alpha I\right]X_i+\sum_{k=1}^{l}X_iU_k^{-1}X_i+\lambda_{ii}X_i$$
$$+X_i\left[\sum_{j\neq i}\lambda_{ij}X_j^{-1}\right]X_i+\frac{l(l+1)}{2}X_i<0.$$

Note that

$$X_i\left[\sum_{j\neq i}\lambda_{ij}X_j^{-1}\right]X_i=S_i(X)\mathcal{X}_i^{-1}S_i^\top(X),$$

and

$$X_i\left[\sum_{k=1}^{l}U_k^{-1}\right]X_i=Z_i(X)\mathcal{Z}_i^{-1}Z_i^\top(X),$$

with $Z_i=(X_i,\ldots,X_i)$ and $\mathcal{Z}_i=\mathrm{diag}(U_1,\ldots,U_l)$.

Letting now $Y_i=K_iX_i$, the above inequality becomes:

$$X_i\left[A_0(i)+\alpha I\right]^\top+\left[A_0(i)+\alpha I\right]X_i+Y_i^\top B_i^\top+B_iY_i+Z_i(X)\mathcal{Z}_i^{-1}Z_i^\top(X)+\lambda_{ii}X_i$$
$$+S_i(X)\mathcal{X}_i^{-1}S_i^\top(X)+\frac{l(l+1)}{2}X_i<0.$$

which is equivalent to (19) after using the Schur complement. Likewise, (34) is equivalent to

$$\begin{pmatrix}(1-\tau_k)e^{-2\tau_k\alpha}U_k^{-1} & A_k^\top(i)\\ A_k(i) & X_i\end{pmatrix}>0.$$

Pre- and post-multiplying both sides of the above inequality by $\mathrm{diag}\{U_k,I\}$ yields (20). From the above derivation, we conclude that if there exist matrices $X=(X_1,\cdots,X_N)>0$, $Y=(Y_1,\cdots,Y_N)$, and $U=(U_1,\ldots,U_l)>0$ satisfying (19) and (20), then $P(i)=X_i^{-1},K(i)=Y_iX_i^{-1}$ for every $i\in\mathcal{S}$, and $R_k=U_k^{-1}$, $k=1,2,\ldots,l$ satisfy (25)-(34). This completes the proof of Theorem 3.2. □

3.2 Robust stability and robust stabilizability

Let us now return to the original problem and see how the previous stochastic stability conditions will be changed when the uncertainties are not zero. To make the results and the LMIs simpler, we will assume that the following hold for every $r_t \in \mathcal{S}$:

$$
\begin{aligned}
D_0(r_t) &= D_b(r_t), \\
F_0(r_t, t) &= F_b(r_t, t), \forall t.
\end{aligned}
$$

The next theorem states the stability conditions for the uncertain free system (2). These conditions are directly obtained from Theorem 3.1.

Theorem 3.3 *If there exist symmetric and positive-definite matrices $P = (P_1, \cdots, P_N) > 0$, and $R = (R_1, \ldots, R_k) > 0$ such that the following hold for any $r_t \in \mathcal{S}$ and for all admissible uncertainties:*

$$
\begin{aligned}
&[A_0(r_t,t) + \alpha I]^\top P(r_t) + P(r_t)\left[A_0(r_t,t) + \alpha I\right] + \sum_{k=1}^{l} R_k \\
&\quad + \sum_{j=1}^{N} \lambda_{r_t j} P(j) + \frac{k(k+1)}{2} P(r_t) \triangleq \Xi_0(r_t,t) < 0 \qquad (27) \\
&A_k^\top(r_t,t) P(r_t) A_k(r_t,t) < e^{-2\alpha\tau_k}(1-\tau_k) R_k, \qquad (28)
\end{aligned}
$$

then system (2) with $u(t) \equiv 0$ is robust SS.

Proof: The proof of this theorem follows the same steps as the one of Theorem 3.1 and the details is omitted. □

Notice that the conditions of Theorem 3.3 depend of the uncertainties and therefore it can't be solved. The following theorem provides an LMI-based sufficient conditions for the system under study to be robustly SS.

Theorem 3.4 *If there exist symmetric and positive-definite matrices $P = (P_1, \cdots, P_N) > 0$, $R = (R_1, \ldots, R_k) > 0$, and positive scalars $\varepsilon_i, \gamma_i, i \in \mathcal{S}$, satisfying for every $i \in \mathcal{S}$ the LMIs*

$$
\begin{pmatrix} J(i) + \varepsilon_i E_0^\top(i) E_0(i) & P(i) D_0(i) \\ D_0^\top(i) P(i) & -\varepsilon_i I \end{pmatrix} < 0 \qquad (29)
$$

$$
\begin{pmatrix} -(1-\tau_k) e^{-2\alpha\tau_k} R_k + \gamma_i E_k^\top(i) E_k(i) & A_k^\top(i) P(i) & \mathbf{0} \\ P(i) A_k(i) & -P(i) & P(i) D_k(i) \\ \mathbf{0} & D_k^\top(i) P(i) & -\gamma_i I \end{pmatrix} < 0, k = 1, \ldots, l, \qquad (30)
$$

then system (2) with $u(t) \equiv 0$ is SS, where $J(i) = [A_0(i) + \alpha I]^\top P(i) + P(i)[A_0(i) + \alpha I] + \sum_{k=1}^{l} R_k + \sum_{j=1}^{N} \lambda_{ij} P(j) + \frac{l(l+1)}{2} P(i)$.

Proof: To prove this theorem, it suffices to prove that (27) is equivalent to (29), and (28) is equivalent to (30). In fact, noticing that $\Xi_0(r_t, t)$ can be rewritten as

$$\begin{aligned}\Xi_0(r_t, t) = J(r_t) + P(r_t)D_0(r_t)F_0(r_t, t)E_0(r_t) \\ + E_0^\top(r_t)F_0^\top(r_t, t)D_0^\top(r_t)P(r_t),\end{aligned}$$

we find that (27) holds for all admissible uncertainties if and only if there exist $\varepsilon_i > 0$ satisfying

$$J(i) + \varepsilon_i E_0^\top(i)E_0(i) + \frac{1}{\varepsilon_i}P(i)D_0(i)D_0^\top(i)P(i) < 0.$$

Using the Schur complement yields that the above inequality is equivalent to (29). Furthermore, using the Schur complement we find that (28) is equivalent to

$$\begin{pmatrix} -(1-\tau_k)e^{-2\tau_k\alpha}R_k & A_k^\top(r_t, t)P(r_t) \\ P(r_t)A_k(r_t, t) & -P(r_t) \end{pmatrix} < 0, \; k = 1, 2, \ldots, l, \tag{31}$$

Note that the left-hand side of the above inequality can be rewritten as

$$\begin{aligned}\begin{pmatrix} -(1-\tau_k)e^{-2\tau_k\alpha}R_k & A_k^\top(r_t)P(r_t) \\ P(r_t)A_k(r_t) & -P(r_t) \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ P(r_t)D_k(r_t) \end{pmatrix} F_k(r_t, t) \left(E_k(r_t) \; \mathbf{0}\right) \\ + \begin{pmatrix} \mathbf{0} \\ E_k^\top(r_t) \end{pmatrix} F_k^\top(r_t, t) \Big(\mathbf{0} \; D_k^\top(i)P(i)\Big).\end{aligned}$$

Likewise, using Lemma 2.1 we obtain that (31) holds if and only if there exist scalars $\gamma_i > 0$ such that

$$\begin{aligned}\begin{pmatrix} -(1-\tau_k)e^{-2\tau_k\alpha}R_k & A_k^\top(i)P(i) \\ P(i)A_0(i) & -P(i) \end{pmatrix} + \gamma_i \begin{pmatrix} E_k^\top(i)E_k(i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ + \frac{1}{\gamma_i}\begin{pmatrix} \mathbf{0} \\ P(i)D_k(i) \end{pmatrix}\Big(\mathbf{0} \; D_k^\top(i)P(i)\Big) < 0.\end{aligned}$$

Using Schur complement, the above inequality is equivalent to (30). This concludes the proof of Theorem 3.4. □

Theorem 3.4 can be used to design a state feedback controller in the form of (7) that stabilizes system (2) in the SS sense. Substituting (7) into (2) yields the following dynamics for the closed-loop system:

$$\dot{x}(t) = \bar{A}_0(r_t, t)x(t) + \sum_{k=1}^{l} A_k(r_t, t)x(t - h_k(t)),$$

where $\bar{A}_0(r_t, t) = \bar{A}_0(r_t) + D_0(r_t)F_0(r_t, t)\bar{E}_0(r_t)$ with $\bar{A}_0(r_t) = A_0(r_t) + B(r_t)K(r_t), \bar{E}_0(r_t) = E_0(r_t) + E_b(r_t)K(r_t)$.

For a given controller (7), using Theorem 3.4, we find that the closed-loop system is robust SS if there exist symmetric, positive-definite matrices $P = (P(1), \cdots, P(N)) > 0$, $R = (R_1, \ldots, R_l) > 0$, and scalars $\varepsilon_i > 0$, $\gamma_i > 0$, $i \in \mathcal{S}$ such that the following inequalities hold for every $i \in \mathcal{S}$:

$$\begin{pmatrix} \bar{J}(i) + \varepsilon_i \bar{E}_0^\top(i)\bar{E}_0(i) & P(i)D_0(i) \\ D_0^\top(i)P(i) & -\varepsilon_i I \end{pmatrix} < 0 \tag{32}$$

$$\begin{pmatrix} -(1-\tau_k)e^{-2\alpha\tau_k} + \gamma_i E_k^\top(i)E_k(i) & A_k^\top(i)P(i) & \mathbf{0} \\ P(i)A_k(i) & -P(i) & P(i)D_0(i) \\ \mathbf{0} & D_0^\top(i)P(i) & -\gamma_i I \end{pmatrix} < 0, \tag{33}$$

where $\bar{J}(i)$ is obtained from $J(i)$ by replacing $A(i)$ and $E_0(i)$ by $\bar{A}(i)$ and $\bar{E}_0(i)$, respectively.

Using the Schur complement twice yields that (32) is equivalent to

$$\begin{pmatrix} \bar{J}(i) + \frac{1}{\varepsilon_i}P(i)D_0(i)D_0^\top(i)P(i) & \bar{E}_0^\top(i) \\ \bar{E}_0(i) & -\frac{1}{\varepsilon_i}I \end{pmatrix} < 0.$$

Let $X_i = P^{-1}(i)$ and $U_k = R_k^{-1}$. Pre- and post-multiplying both sides of the above inequality by $\mathrm{diag}\{X_i, I\}$ yields

$$\begin{pmatrix} X_i\bar{J}(i)X_i + \frac{1}{\varepsilon_i}D_0(i)D_0^\top(i) & X_i\bar{E}_0^\top(i) \\ \bar{E}_0(i)X_i & -\frac{1}{\varepsilon_i}I \end{pmatrix} < 0. \tag{34}$$

Note that

$$\begin{aligned} X_i\bar{J}(i)X_i &= \left[\bar{A}(i) + \alpha I\right]X_i + X_i\left[\bar{A}(i) + \alpha I\right]^\top + \lambda_{ii}X_i + \frac{l(l+1)}{2}X_i \\ &\quad + \sum_{k=1}^{l} X_i\, U_k^{-1}X_i + X_i\left[\sum_{j\neq i}\lambda_{ij}X_j^{-1}\right]X_i. \end{aligned}$$

Letting $\rho_{1i} = 1/\varepsilon_i, Y_i = K(i)X_i$, and using the Schur complement, we obtain that (34) is equivalent to

$$\begin{pmatrix} J_1(i) & X_iE_a^\top(i) + Y_i^\top E_b^\top(i) & Z_i(X) & S_i(X) \\ \star & -\rho_{1i}I & \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} & -\mathcal{Z}_i & \mathbf{0} \\ \star & \mathbf{0} & \mathbf{0} & -\mathcal{X}_i \end{pmatrix} < 0, \tag{35}$$

where $J_1(i) = \left[A_0(i) + \alpha I\right]X_i + B(i)Y_i + X_i\left[A_0(i) + \alpha I\right]^\top + Y_i^\top B^\top(i) + \lambda_{ii}X_i + \frac{l(l+1)}{2}X_i + \rho_{1i}D_0(i)D_0^\top(i)$, and $S_i(X)$, $\mathcal{X}_i$, $Z_i(X)$ and $\mathcal{Z}_i$ are defined by (21), (22), (23) and (24).

Likewise, using Schur complement twice gives that (33) is equivalent to

$$\begin{pmatrix} -(1-\tau_k)e^{-2\alpha\tau_k}R_k & A_k^\top(i)P(i) & E_k^\top(i) \\ P(i)A_k(i) & -P(i)+\frac{1}{\gamma_i}P(i)D_k(i)D_k^\top(i)P(i) & \mathbf{0} \\ E_k(i) & \mathbf{0} & -\frac{1}{\gamma_i}I \end{pmatrix} < 0.$$

Pre- and post-multiplying both sides of the above inequality by diag$\{U_k,\ X_i,\ I\}$ and letting $\rho_{2i} = 1/\gamma_i$ leads to

$$\begin{pmatrix} -U_k & U_kA_k^\top(i) & U_kE_k^\top(i) \\ A_k(i)U_k & -X_i+\rho_{2i}D_0(i)D_0^\top(i) & \mathbf{0} \\ E_k(i)U_k & \mathbf{0} & -\rho_{2i}I \end{pmatrix} < 0. \tag{36}$$

Therefore, from the above derivation we get a controller design algorithm, which is given by the following theorem.

Theorem 3.5 *If there exist symmetric and positive-definite matrices* $X = (X_1, \ldots, X_N) > 0$, $U = (U_1, \ldots, U_l) > 0$, *and scalars* $\rho_{1i} > 0, \rho_{2i} > 0$ *satisfying LMIs (35) and (36) for every* $i \in \mathcal{S}$, *then controller (7) with* $K(i) = Y_iX_i^{-1}, i \in \mathcal{S}$, *robustly stabilizes system (2).*

This theorem provides an algorithm to design a memoryless state feedback controller of form (7) that stabilizes system (2) in the robust SS sense. To show its validity, let us give a numerical example.

4 Conclusion

This paper dealt with the class of continuous-time linear systems with Markovian jumps and multiple time-varying time-delays in the state vector. Results on stochastic stability and its robustness, and stochastic stabilizability and its robustness are developed. The LMI framework is used to stablish the different results on stability and stabilizability. The conditions we developed are all delay-dependent and therefore, are less conservative than the one developed before. The results we developed can easily be solved using any LMI toolbox like the one of Matlab or the one of Scilab.

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