On Minimizing Some Merit Functions for Complementarity Problems under H-Differentiability

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Abstract

In this paper, we describe H-differentials of some well known NCP functions and their merit functions. We show how, under appropriate conditions on an H-differential of f, minimizing a merit function corresponding to f leads to a solution of the nonlinear complementarity problem. Our results give a unified treatment of such results for C^1-functions, semismooth-functions, and for locally Lipschitzian functions. Illustrations are given to show the usefulness of our results.

Key Words. H-Differentiability, semismooth-functions, locally Lipschitzian, generalized Jacobian, nonlinear complementarity problem, NCP function, merit function, regularity conditions.

Résumé

Cet article décrit les H–différentiels associés à des problèmes de complémentarité non-linéaire et à leurs fonctions mérites. Sous des conditions appropriées sur le H–différentiel de f, la minimisation de la fonction mérite correspondant à f conduit à la solution du problème de complémentarité non-linéaire. Ces résultats donnent un traitement unifié de résultats analogues dans le cas de fonctions C^1, "*semismooth*" et localement Lipschitz. Nous illustrons l'utilité de ces résultats par plusieurs exemples.

1 Introduction

The concepts of H-differentiability and H-differential were introduced in [12] to study the injectivity of nonsmooth functions. It has been shown in [12] that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [2], the Bouligand subdifferential of a semismooth function [19], [21], [23], and the C-differential of a C-differentiable function [22] are examples of H-differentials. Characterizations of $\mathbf{P}_0$- and $\mathbf{P}$- properties of a function were studied in [26] and some applications of H-differentiability to optimization, complementarity, and variational inequalities are treated in [28], [29]. The inverse and implicit function theorems of H-differentiability have been proven in [11]. It were observed in [10] that H-differentials are related to an approximate Jacobian [14] in that the closure of an H-differentials an approximate Jacobian.

In this article, we consider a nonlinear complementarity problem NCP(f) corresponding to an H-differentiable function $f : R^n \to R^n$: Find $\bar{x} \in R^n$ such that

$$\bar{x} \geq 0\,, \quad f(\bar{x}) \geq 0 \;\; \text{and} \; \langle f(\bar{x}), \bar{x}\rangle = 0.$$

By considering an NCP function $\Phi : R^n \to R^n$ associated with NCP(f) so that

$$\Phi(\bar{x}) = 0 \Leftrightarrow \; \bar{x} \text{ solves NCP}(f),$$

and the corresponding merit function

$$\Psi(x) := \sum_{i=1}^{n} \Phi_i(x). \tag{1}$$

In this paper, we describe H-differentials of some well known NCP functions and their merit functions. Also, we show how, under appropriate $\mathbf{P}_0(\mathbf{P}$, regularity), positive definite-conditions on an H-differential of f, finding local/global minimum of Ψ (or a 'stationary point' of Ψ) leads to a solution of the given nonlinear complementarity problem. Our results unify/extend various similar results proved in the literature for C^1, locally Lipschitzian, and semismooth functions [4], [5], [6], [13], [15], [16], [18], [25], [30], [31], [32].

2 Preliminaries

Throughout this paper, we regard vectors in R^n as column vectors. We denote the inner-product between two vectors x and y in R^n by either $x^T y$ or $\langle x, y\rangle$. Vector inequalities are interpreted componentwise. For a matrix A, A_i denotes the ith row of A. For a differentiable function $f : R^n \to R^m$, $\nabla f(\bar{x})$ denotes the Jacobian matrix of f at $\bar{x}$.

Definition 1 *A function $\phi : R^2 \to R$ is called an NCP function if*

$$\phi(a, b) = 0 \Leftrightarrow ab = 0, a \geq 0,\, b \geq 0.$$

For the problem NCP(f), we define

$$\Phi(x) = \begin{bmatrix} \phi(x_1, f_1(x)) \\ \vdots \\ \phi(x_i, f_i(x)) \\ \vdots \\ \phi(x_n, f_n(x)) \end{bmatrix} \quad (2)$$

and, call $\Phi(x)$ an NCP function for NCP(f).

We need the following definitions from [3], [20].

Definition 2 *A matrix $A \in R^{n\times n}$ is called*
(a) $\mathbf{P_0^+}$ $(\mathbf{P}^+)$*-matrix if*

$$\forall x \in R^n_+, x \neq 0, \quad \text{there exists } i \text{ such that } \; x_i \neq 0 \text{ and } x_i\,(Ax)_i \geq 0 \quad (>0).$$

(b) *semimonotone ($\mathbf{E_0}$) [strictly semimonotone ($\mathbf{E}$)]-matrix if*

$$\forall x \in R^n_+, \; x \neq 0, \quad \text{there exists } i \text{ such that } \; x_i\,(Ax)_i \geq 0 \quad [>0].$$

Definition 3 *For a function $f : R^n \to R^n$, we say that f is a*
(i) *monotone if*

$$\langle f(x) - f(y), x - y\rangle \geq 0 \quad \text{for all } x, y \in R^n.$$

(ii) $\mathbf{P_0}(\mathbf{P})$*-function if, for any $x \neq y$ in R^n,*

$$\max_{\{i: x_i \neq y_i\}} (x-y)_i [f(x) - f(y)]_i \geq 0 \; (>0). \quad (3)$$

A matrix $A \in R^{n\times n}$ is said to be a $\mathbf{P}_0(\mathbf{P})$-matrix if the function $f(x) = Ax$ is a $\mathbf{P_0}(\mathbf{P})$-function or equivalently, every principle minor of A is nonnegative (respectively, positive).

We note that every monotone (strictly monotone) function is a $\mathbf{P}_0(\mathbf{P})$-function.

The following result is from [20] and [26].

Theorem 1 *Under each the following conditions, $f : R^n \to R^n$ is a $\mathbf{P}_0(\mathbf{P})$-function.*

(a) f is Fréchet differentiable on R^n and for every $x \in R^n$, the Jacobian matrix $\nabla f(x)$ is a $\mathbf{P}_0(\mathbf{P})$-matrix.

(b) f is locally Lipschitzian on R^n and for every $x \in R^n$, the generalized Jacobian $\partial f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices.

(c) f is semismooth on R^n (in particular, piecewise affine or piecewise smooth) and for every $x \in R^n$, the Bouligand subdifferential $\partial_B f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices.

(d) *f is H-differentiable on R^n and for every $x \in R^n$, an H-differential $T_f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices.*

We now recall the following definition and examples from Gowda and Ravindran [12].

Definition 4 *Given a function $F : \Omega \subseteq R^n \to R^m$ where Ω is an open set in R^n and $x^* \in \Omega$, we say that a nonempty subset $T(x^*)$ (also denoted by $T_F(x^*)$) of $R^{m\times n}$ is an H-differential of F at x^* if for every sequence $\{x^k\} \subseteq \Omega$ converging to x^*, there exist a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(x^*)$ such that*

$$F(x^{k_j}) - F(x^*) - A(x^{k_j} - x^*) = o(||x_j^k - x^*||). \tag{4}$$

We say that F is H-differentiable at x^* if F has an H-differential at x^*.

Remarks

As noted in [12], any superset of an H-differential is an H-differential, H-differentiability implies continuity, and H-differentials enjoy simple sum, product and chain rules.

As noted in [29], it is easily seen that if a function $F : \Omega \subseteq R^n \to R^m$ is H-differentiable at a point $\bar{x}$, then there exist a constant $L > 0$ and a neighbourhood $B(\bar{x}, \delta)$ of $\bar{x}$ with

$$||F(x) - F(\bar{x})|| \leq L||x - \bar{x}||, \quad \forall x \in B(\bar{x}, \delta). \tag{5}$$

Conversely, if condition (5) holds, then $T(\bar{x}) := R^{m\times n}$ can be taken as an H-differential of F at $\bar{x}$. We thus have, in (5), an alternate description of H-differentiability. But, as we see in the sequel, it is the identification of an appropriate H-differential that becomes important and relevant.

Clearly any function locally Lipschitzian at $\bar{x}$ will satisfy (5). For real valued functions, condition (5) is known as the 'calmness' of F at $\bar{x}$. This concept has been well studied in the literature of nonsmooth analysis (see [24], Chapter 8).

Example 1 Let $F : R^n \to R^m$ be Fréchet differentiable at $x^* \in R^n$ with Fréchet derivative matrix (= Jacobian matrix derivative) $\{\nabla F(x^*)\}$ such that

$$F(x) - F(x^*) - \nabla F(x^*)(x - x^*) = o(||x - x^*||).$$

Then F is H-differentiable with $\{\nabla F(x^*)\}$ as an H-differential.

Example 2 Let $F : \Omega \subseteq R^n \to R^m$ be locally Lipschitzian at each point of an open set Ω. For $x^* \in \Omega$, define the Bouligand subdifferential of F at x^* by

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}$$

where Ω_F is the set of all points in Ω where F is Fréchet differentiable. Then, the (Clarke) generalized Jacobian [2]

$$\partial F(x^*) = co\partial_B F(x^*)$$

is an H-differential of F at x^*.

Example 3 Consider a locally Lipschitzian function $F : \Omega \subseteq R^n \to R^m$ that is semismooth at $x^* \in \Omega$ [19], [21], [23]. This means for any sequence $x^k \to x^*$, and for $V_k \in \partial F(x^k)$,

$$F(x^k) - F(x^*) - V_k(x^k - x^*) = o(||x^k - x^*||).$$

Then the Bouligand subdifferential

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}.$$

is an H-differential of F at x^*. In particular, this holds if F is piecewise smooth, i.e., there exist continuously differentiable functions $F_j : R^n \to R^m$ such that

$$F(x) \in \{F_1(x), F_2(x), \ldots, F_J(x)\} \quad \forall\, x \in R^n.$$

Example 4 Let $F : R^n \to R^n$ be C-differentiable [22] in a neighborhood D of x^*. This means that there is a compact upper semicontinuous multivalued mapping $x \mapsto T(x)$ with $x \in D$ and $T(x) \subset R^{n\times n}$ satisfying the following condition at any $a \in D$: For $V \in T(x)$,

$$F(x) - F(a) - V(x - a) = o(||x - a||).$$

Then, F is H-differentiable at x^* with $T(x^*)$ as an H-differential.

Remark While the Fréchet derivative of a differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [2], the Bouligand differential of a semismooth function [21], and the C-differential of a C-differentiable function [22] are particular instances of H-differential, the following simple example, is taken from [10], shows that an H-differentiable function need not be locally Lipschitzian nor directionally differentiable. Consider on R,

$$F(x) = x \sin(\frac{1}{x}) \text{ for } x \neq 0 \ \text{ and } F(0) = 0.$$

Then F is H-differentiable on R with

$$T(0) = [-1, 1] \ \text{ and } \ T(c) = \{\sin(\frac{1}{c}) - \frac{1}{c} cos(\frac{1}{c})\} \text{ for } c \neq 0.$$

We note that F is not locally Lipschitzian around zero. We also see that F is neither Fréchet differentiable nor directionally differentiable.

3 The H-differentiability of the merit function

In this section, we consider an NCP function Φ corresponding to NCP(f) and its merit function $\Psi := \sum_{i=1}^n \Phi_i$.

Theorem 2 *Suppose Φ is H-differentiable at $\bar{x}$ with $T_\Phi(\bar{x})$ as an H-differential. Then $\Psi := \sum_{i=1}^n \Phi_i$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{e^T B : B \in T_\Phi(\bar{x})\}.$$

Proof. To describe an H-differential of Ψ, let $\theta(x) = x_1 + \cdots + x_n$. Then $\Psi = \theta \circ \Phi$ so that by the chain rule for H-differentiability, we have $T_\Psi(\bar{x}) = (T_\theta \circ T_\Phi)(\bar{x})$ as an H-differential of Ψ at $\bar{x}$. Since $T_\theta(\bar{x}) = \{e^T\}$ where e is the vector of ones in R^n, we have

$$T_\Psi(\bar{x}) = \{e^T B : B \in T_\Phi(\bar{x})\}.$$

This completes the proof. □

4 H-differentials of some NCP/merit functions associated with H-differentiable functions

In this section, we describe the H-differentials of some well known NCP functions and their merit functions.

Example 5 In [18], Mangasarian and Solodov introduced the so-called implicit Lagrangian function for solving NCP(f). For $\alpha > 1$, let

$$\phi(a,b) := a\,b + \frac{1}{2\alpha}\left[\max{}^2\{0, a - \alpha b\} + \max{}^2\{0, b - \alpha a\} - a^2 - b^2\right].$$

Then the implicit Lagrangian at $\bar{x}$ is

$$\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x})$$

where

$$\begin{aligned}\Phi_i(x) = \phi(x_i, f_i(x)) := x_i f_i(x) + \tfrac{1}{2\alpha}\big[&\max{}^2\{0, x_i - \alpha f_i(x)\} \\ &+\max{}^2\{0, f_i(x) - \alpha x_i\} - x_i^2 - f_i(x)^2\big].\end{aligned} \tag{6}$$

Suppose that f is H-differentiable at $\bar{x}$ with $T(\bar{x})$ as an H-differential. We claim that $\Psi(\bar{x})$ is H-differentiable with an H-differential $T_\Psi(\bar{x})$ consisting of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are columns vectors with entries defined by

$$\begin{aligned} v_i &= \bar{x}_i + \tfrac{1}{\alpha}\left[-\alpha \max\{0, \bar{x}_i - \alpha f_i(\bar{x})\} + \max\{0, f_i(\bar{x}) - \alpha \bar{x}_i\} - f_i(\bar{x})\right], \\ w_i &= f_i(\bar{x}) + \tfrac{1}{\alpha}\left[\max\{0, \bar{x}_i - \alpha f_i(\bar{x})\} - \bar{x}_i - \alpha \max\{0, f_i(\bar{x}) - \alpha \bar{x}_i\}\right]. \end{aligned} \tag{7}$$

First we show that an H-differential of

$$\begin{aligned}\Phi(x) := x * f(x) + \tfrac{1}{2\alpha}\big[\max{}^2\{0, x - \alpha f(x)\} + \max{}^2\{0, f(x) - \alpha x\} \\ -x^2 - f(x)^2\big]\end{aligned} \tag{8}$$

is given by

$$T_\Phi(\bar{x}) = \{B = VA + W : \ A \in T(\bar{x}),\ V = diag(v_i) \text{ and } W = diag(w_i) \\ \text{where } v_i, w_i \text{ satisfy (7)}\}.$$

Let $g(x) = \max\{0, x - \alpha f(x)\}$, $h(x) = \max\{0, f(x) - \alpha x\}$. For each $A \in T(\bar{x})$, let A' and A'' be matrices such that for $i = 1, \dots, n$,

$$A'_i \in \begin{cases} \{e_i - \alpha A_i\} & \text{if } \ \bar{x}_i - \alpha f_i(\bar{x}) > 0 \\ \{0, e_i - \alpha A_i\} & \text{if } \ \bar{x}_i - \alpha f_i(\bar{x}) = 0 \\ \{0\} & \text{if } \ \bar{x}_i - \alpha f_i(\bar{x}) < 0, \end{cases} \tag{9}$$

and

$$A''_i \in \begin{cases} \{A_i - \alpha e_i\} & \text{if } \ f_i(\bar{x}) - \alpha \bar{x}_i > 0 \\ \{0, A_i - \alpha e_i\} & \text{if } \ f_i(\bar{x}) - \alpha \bar{x}_i = 0 \\ \{0\} & \text{if } \ f_i(\bar{x}) - \alpha \bar{x}_i < 0. \end{cases} \tag{10}$$

Then it can be easily verified that $T_g(\bar{x}) = \{A' | A \in T(\bar{x})\}$ and $T_h(\bar{x}) = \{A'' | A \in T(\bar{x})\}$ are H-differentials of g and h, respectively. Now simple calculations show that $T_\Phi(\bar{x})$ consists of matrices of the form

$$B = [diag(\bar{x})\, A + diag(f(\bar{x}))] + \tfrac{1}{2\alpha}\,[2\,diag(g(\bar{x}))\, A' + 2\,diag(h(\bar{x}))\, A'' \\ -2\,diag(\bar{x}) - 2\,diag(f(\bar{x}))] \tag{11}$$

where A' and A'' corresponding $A \in T(\bar{x})$ are defined by (9) and (10), respectively.

Since $g_i(x) = 0$ when $x_i - \alpha f_i(x) \le 0$, we have $diag(g(\bar{x}))\, A' = diag(g(\bar{x}))(I - \alpha A)$. Similarly, $diag(h(\bar{x}))\, A'' = diag(h(\bar{x}))(A - \alpha I)$. Therefore, (11) becomes

$$\begin{aligned} B = \ & \left[diag(\bar{x}) + \tfrac{1}{\alpha}\,[-\alpha\, diag(\max\{0, \bar{x} - \alpha f(\bar{x})\}) + diag(\max\{0, f(\bar{x}) - \alpha\bar{x}\}) \right. \\ & \left. -diag(f(\bar{x}))\right]\, A + \left[diag(f(\bar{x})) + \tfrac{1}{\alpha}\,[diag(\max\{0, \bar{x} - \alpha f(\bar{x})\}) \right. \\ & -\alpha\, diag(\max\{0, f(\bar{x}) - \alpha\bar{x}\})]] = VA + W \end{aligned} \tag{12}$$

where V and W are diagonal matrices with diagonal entries given by (7). By Theorem 2, we have

$$T_\Psi(\bar{x}) = \{e^T(VA + W) = v^T A + w^T : A \in T(\bar{x}), v \text{ and } w \text{ are vectors in } R^n \\ \text{with components defined by (7)}\}. \tag{13}$$

Example 6 The following NCP function is proposed independently by Fukushima [8] and Auchmuty [1] and its merit function is called the regularized gap function. For $\alpha > 0$, let

$$\phi(a,b) := a\,b + (1/2\alpha)\left[\max{}^2\{0, a-\alpha b\} - a^2\right].$$

Then the regularized gap function associated to NCP function at $\bar{x}$ is

$$\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x})$$

where

$$\Phi_i(x) = \phi(x_i, f_i(x)) := x_i f_i(x) + (1/2\alpha)\left[\max{}^2\{0, x_i - \alpha f_i(x)\} - x_i^2\right]. \tag{14}$$

In previous example, we describe the H-differential of implicit Lagrangian. A similar analysis can be carried out for NCP function $\Phi(\bar{x})$ in (14) and its merit function $\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x})$.

$\Psi(\bar{x})$ is H-differentiable with an H-differential $T_\Psi(\bar{x})$ consisting of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are columns vectors with entries defined by

$$\begin{aligned} v_i &= \bar{x}_i + \max\{0, \bar{x}_i - \alpha f_i(\bar{x})\} \\ w_i &= f_i(\bar{x}) + (1/\alpha)\left[\max\{0, \bar{x}_i - \alpha f_i(\bar{x})\} - \bar{x}_i\right]. \end{aligned} \tag{15}$$

Example 7 The following NCP function was proposed by Solodov [25]

$$\phi(a,b) := a \max{}^2\{0, b\} - \max{}^2\{0, -b\}.$$

Then the merit function associated to NCP function at $\bar{x}$ is

$$\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x})$$

where

$$\Phi_i(x) = \phi(x_i, f_i(x)) := x_i \max{}^2\{0, f_i(x)\} + \max{}^2\{0, -f_i(x)\}. \tag{16}$$

A straightforward calculation shows that $\Psi(\bar{x})$ is H-differentiable with an H-differential $T_\Psi(\bar{x})$ consisting of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are columns vectors with entries defined by

$$\begin{aligned} v_i &= 2\bar{x}_i \max\{0, f_i(x)\} - 2\max\{0, -f_i(x)\} \\ w_i &= \max{}^2\{0, f_i(x)\}. \end{aligned} \tag{17}$$

Example 8 Suppose $f : R^n \to R^n$ has an H-differential $T(\bar{x})$ at $\bar{x} \in R^n$. Consider the associated NCP function [31]

$$\phi(a,b) := \max\{0,a\} \max^3\{0,b\} + (1/2)[a+b-\sqrt{a^2+b^2}]^2.$$

Then the merit function associated to NCP function at $\bar{x}$ is

$$\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x})$$

where

$$\begin{aligned} \Phi_i(x) &= \phi(x_i, f_i(x)) \\ &:= \max\{0,x_i\} \max^3\{0,f_i(x)\} + (1/2)\left[x_i + f_i(x) - \sqrt{x_i^2 + f_i(x)^2}\right]^2. \end{aligned} \tag{18}$$

Let

$$J(\bar{x}) = \{i : f_i(\bar{x}) = 0 = \bar{x}_i\} \quad \text{and} \quad K(\bar{x}) = \{i : \bar{x}_i > 0,\ f_i(\bar{x}) > 0\}.$$

We can describe the H-differential of Φ in a way similar to the calculation and analysis of Examples 5-7 in [29]. The H-differential of Φ is given by

$$T_\Phi(\bar{x}) = \{VA + W :\ (A,V,W,d) \in \Gamma\},$$

where Γ is the set of all quadruples (A,V,W,d) with $A \in T(\bar{x})$, $||d|| = 1$, $V = diag(v_i)$ and $W = diag(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} \left[\bar{x}_i + f_i(\bar{x}) - \sqrt{\bar{x}^2 + f_i(\bar{x})^2}\right]\left(1 - \frac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + 3\,\bar{x}_i\, f_i(\bar{x})^2 & \text{when } i \in K(\bar{x}) \\ \left[d_i + A_i\, d - \sqrt{{d_i}^2 + (A_i\, d)^2}\right]\left(1 - \frac{A_i d}{\sqrt{d_i^2 + (A_i d)^2}}\right) & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 > 0 \\ \left[\bar{x}_i + f_i(\bar{x}) - \sqrt{\bar{x}^2 + f_i(\bar{x})^2}\right]\left(1 - \frac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 = 0, \end{cases} \tag{19}$$

$$w_i = \begin{cases} \left[\bar{x}_i + f_i(\bar{x}) - \sqrt{\bar{x}^2 + f_i(\bar{x})^2}\right]\left(1 - \frac{\bar{x}_i}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + f_i(\bar{x})^3 & \text{when } i \in K(\bar{x}) \\ \left[d_i + A_i\,d - \sqrt{{d_i}^2 + (A_i\,d)^2}\right]\left(1 - \frac{d_i}{\sqrt{d_i^2 + (A_i d)^2}}\right) & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 > 0 \\ \left[\bar{x}_i + f_i(\bar{x}) - \sqrt{\bar{x}^2 + f_i(\bar{x})^2}\right]\left(1 - \frac{\bar{x}_i}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 = 0. \end{cases}$$

By Theorem 2, the H-differential $T_\Psi(\bar{x})$ of $\Psi(\bar{x})$ consists of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are columns vectors with entries defined by (19).

5 Minimizing the merit function

For a given H-differentiable function $f : R^n \to R^n$, consider the associated NCP function Φ and the corresponding merit function $\Psi := \sum_{i=1}^n \Phi_i$. It should be recalled that

$$\Psi(\bar{x}) = 0 \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves NCP}(f).$$

Assume that Ψ is H-differentiable with an H-differential $T_\Psi(\bar{x})$ and Φ is nonnegative H-differentiable with an H-differential $T_\Phi(\bar{x})$ is given by

$$T_\Phi(\bar{x}) = \{VA + W : \; A \in T(\bar{x}), V = diag(v_i) \text{ and } W = diag(w_i)\} \tag{20}$$

where Φ, V and W satisfy the following properties:

$$\left.\begin{array}{rl} \textbf{(i)} & \bar{x} \text{ solves NCP(f)} \;\Leftrightarrow \Phi(\bar{x}) = 0. \\ \textbf{(ii)} & \text{For } i \in \{1,\ldots,n\},\; v_i\,w_i \geq 0. \\ \textbf{(iii)} & \text{For } i \in \{1,\ldots,n\},\; \Phi_i(\bar{x}) = 0 \Leftrightarrow (v_i, w_i) = (0,0). \\ \textbf{(iv)} & \text{For } i \in \{1,\ldots,n\} \text{ with } \bar{x}_i \geq 0 \text{ and } f(\bar{x}_i) \geq 0, \text{ we have } v_i \geq 0. \\ \textbf{(v)} & \text{If } 0 \in T_\Psi(\bar{x}), \text{then } \Phi(\bar{x}) = 0 \Leftrightarrow v = 0. \end{array}\right\} \tag{21}$$

Remarks We note that the NCP function of Example 5 satisfies the properties (i)-(v) in (21) and is known as unrestricted NCP and its merit function unrestricted implicit Lagrangian function. While the NCP functions in Examples 6-8 are called restricted NCP function because they are nonnegative and satisfy properties (i)-(v) in (21) over the non-negative orthant R^n_+, i.e., for restricted NCP function, the properties in (21) will be

$$\left.\begin{array}{rl} \textbf{(i)} & \bar{x} \text{ solves NCP(f) } \Leftrightarrow \Phi(\bar{x}) = 0. \\ \textbf{(ii)} & \text{For } i \in \{1,\ldots,n\},\ v_i\, w_i \geq 0 \text{ for all } \bar{x}_i \geq 0. \\ \textbf{(iii)} & \text{For } i \in \{1,\ldots,n\},\ \Phi_i(\bar{x}) = 0 \Leftrightarrow (v_i, w_i) = (0,0) \text{ with } \bar{x}_i \geq 0. \\ \textbf{(iv)} & \text{For } i \in \{1,\ldots,n\} \text{ with } \bar{x}_i \geq 0 \text{ and } f(\bar{x}_i) \geq 0, \text{ we have } v_i \geq 0. \\ \textbf{(v)} & \text{If } 0 \in T_\Psi(\bar{x}), \text{then } \Phi(\bar{x}) = 0 \Leftrightarrow v = 0 \text{ with } \bar{x}_i \geq 0. \end{array}\right\} \quad (22)$$

In the following subsections, starting with an H-differentiable function f, we show that under appropriate conditions, a vector $\bar{x}$ is a solution of the NCP(f) if and only if zero belongs $T_\Psi(\bar{x})$.

5.1 Minimizing the merit function under P_0-conditions

Theorem 3 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is an NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (iii) and (v) in (21), and $v_i\, w_i > 0$ whenever $\Phi_i(\bar{x}) \neq 0$.

Further suppose that $T(\bar{x})$ consists of $\mathbf{P}_0$-matrices. Then

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. Suppose $\Phi(\bar{x}) = 0$. Then by property (iii) in (21) and the description of $T_\Psi(\bar{x})$, we have $T_\Psi(\bar{x}) = \{0\}$. Conversely, suppose that $0 \in T_\Psi(\bar{x})$, so that for some $v^T A + w^T \in T_\Psi(\bar{x})$,

$$0 = v^T A + w^T$$

yielding $A^T v + w = 0$. Note that for any index i, $\Phi_i(\bar{x}) \neq 0 \Leftrightarrow\ v_i \neq 0$ (by property (v) in (21) and $v_i w_i > 0$ when $\Phi_i(\bar{x}) \neq 0$) in which case $v_i(A^T v)_i = -v_i\, w_i < 0$ contradicting the $\mathbf{P}_0$-property of A. We conclude that $\Phi(\bar{x}) = 0$. □

Remarks Theorem 3 is applicable to the following NCP functions:

- $\Phi(x) = \Phi_F(x) = x + f(x) - \sqrt{x^2 + f(x)^2}$. (Clarification Example 5 in [29])
- $\Phi(x) = x + f(x) - \sqrt{(x - f(x))^2 + \lambda x f(x)}$. (Clarification Example 6 in [29])
- $\Phi(x) = \lambda \Phi_F(x) + (1 - \lambda) x_+\, f(x)_+$. (Clarification Example 7 in [29])

The following are consequences of the above theorems, we state the results for Fischer-Burmeister function for simplicity. However, it is possible to state a general result for any NCP function.

Corollary 1 *Let $f : R^n \to R^n$ be differentiable and $\Phi(x)$ be the Fischer-Burmeister function and $\Psi := \sum_{i=1}^n \Phi_i$. If f is $\mathbf{P_0}$-function, then $\bar{x}$ is a local minimizer to Ψ if and only if $\bar{x}$ solves NCP(f).*

Remarks

When f is C^1 (in which case we can let $T(\bar{x}) = \{\nabla f(\bar{x})\}$), the above result reduces to Prop. 3.4 in [5]. Also in view of Example 3, if f is locally Lipschitzian with $T(\bar{x}) = \partial f(\bar{x})$, the above theorem reduces to a result by Fischer [7]. Moreover, our result extend/generalize a result obtained by Geiger and Kanzow [9] under monotonicity of a C^1 function and by Jiang [15] under uniform **P**- property of a directionally differentiable function .

Corollary 2 *Let $f : R^n \to R^n$ be locally Lipschitzian. Let Φ be the Fischer-Burmeister function and $\Psi := \sum_{i=1}^n \Phi_i$. Further suppose that $\partial f(\bar{x})$ consists of $\mathbf{P_0}$-matrices.*

Then

$$0 \in \partial\Psi(\bar{x}) \Leftrightarrow \Psi(\bar{x}) = 0.$$

Proof. The proof has been established by Fischer [7]. In fact, by taking $T_f(x) = \partial f(x)$ in Theorem 3 and noting $\partial\Psi(x) \subseteq T_\Psi(x)$ for all x, we have the proof. □

5.2 Minimizing the merit function under $\mathbf{P_0^+}$-conditions

Theorem 4 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (iii) and (v) in (21), and

$$\text{for } i \in \{1, \ldots, n\} \text{ with } \bar{x}_i > 0 \text{ and } f(\bar{x}_i) > 0, \text{ we have } v_i > 0,\ w_i > 0.$$

Further suppose that $\bar{x}$ is a strictly feasible point of NCP(f) and $T(\bar{x})$ consists of $\mathbf{P_0^+}$-matrices.Then

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. Suppose $0 \in T_\Psi(\bar{x})$. Then $v^T A + w^T = 0 \Rightarrow A^T v + w = 0$. We claim that $\Phi(\bar{x}) = 0$. Suppose, if possible, $\Phi(\bar{x}) \neq 0$. Then by property (v) in (21), $v \neq 0$. Since $\bar{x}$ is a strictly feasible point to NCP(f), we have $v > 0,\ w > 0$.

Since $T(\bar{x})$ consists of $\mathbf{P}_\mathbf{0}^+$-matrices and $A \in T(\bar{x})$, there exists an index i such that $0 \neq \Phi_i,\ \ 0 \neq v_i > 0$ and $0 \leq v_i(Av)_i$. By the fact , $v_i\, w_i > 0$, we have $0 \leq v_i(Av)_i = -v_i\, w_i < 0$ which is a contradiction. Hence $\Phi(\bar{x}) = 0$. Conversely, suppose $\Phi(\bar{x}) = 0$. Then by property *(iii)* in (21)and the description of $T_\Psi(\bar{x})$, we have $T_\Psi(\bar{x}) = \{0\}$. $\square$

Remarks

- We note that Theorem 4 is applicable to the NCP functions of Examples 7 and 8.
- If we assume the continuous differentiability of f in the above theorem, we get Corollary 3.2 in [25].

A slight modification of the above theorem leads to the following result.

Theorem 5 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (iii), (iv), and (v) in (21).

Further suppose that $\bar{x}$ is a feasible point of NCP(f) and $T(\bar{x})$ consists of $\mathbf{P}^+$-matrices. Then

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. The proof is similar to that of Theorem 4. $\square$

5.3 Minimizing the merit function under P-conditions

Theorem 6 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (ii), (iii), and (v) in (21).

Further suppose that $T(\bar{x})$ consists of $\mathbf{P}$-matrices. Then

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. To see this, suppose $0 \in T_\Psi(\bar{x})$. Then $v^T A + w^T = 0 \Rightarrow A^T v + w = 0$. We claim that $\Phi(\bar{x}) = 0$. Suppose, if possible, $\Phi(\bar{x}) \neq 0$. Then by property (v) in (21), $v \neq 0$. Since $T(\bar{x})$ consists of $\mathbf{P}$-matrices and $A \in T(\bar{x})$, there exists an index i such that $v_i \neq 0$ and

$0 < v_i(Av)_i$. By property (ii) in (21), $v_i\, w_i \geq 0$. But $0 < v_i(Av)_i = -v_i\, w_i \leq 0$ which is a contradiction. Hence $\Phi(\bar{x}) = 0$. Conversely, suppose $\Phi(\bar{x}) = 0$. Then by property (iii) in (21) and the description of $T_\Psi(\bar{x})$, we have $T_\Psi(\bar{x}) = \{0\}$. □

Remark Theorem 6 is applicable to the NCP functions in Examples 5-8.

5.4 Minimizing the merit function under positive-definite-conditions

Theorem 7 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (ii), (iii), and (v) in (21).

Further suppose that $T(\bar{x})$ consists of positive-definite matrices. Then

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. Suppose $\Phi(\bar{x}) = 0$. Then by property (iii) in (21) and the description of $T_\Psi(\bar{x})$, we have $T_\Psi(\bar{x}) = \{0\}$. Conversely, suppose $0 \in T_\Psi(\bar{x})$. Then $v^T A + w^T = 0 \Rightarrow A^T v + w = 0$. We claim that $\Phi(\bar{x}) = 0$. Suppose, if possible, $\Phi(\bar{x}) \neq 0$. Then by property (v) in (21), $v \neq 0$. Since $T(\bar{x})$ consists of positive definite matrices and $A \in T(\bar{x})$,

$0 < \langle v, Av\rangle$. By property (ii) in (21), $\langle v, w\rangle \geq 0$. But $0 < \langle v, Av\rangle = -\langle v, w\rangle \leq 0$ which is a contradiction. Hence $\Phi(\bar{x}) = 0$. □

Remarks

- We note that Theorem 7 is applicable to the NCP function of Examples 5.
- Since every positive definite matrix is also a **P**-matrix, the proof of Theorem 7 follows from Theorem 6. However, we gave a general proof of Theorem 7.

5.5 Minimizing the merit function under strictly semi-monotone (E)-conditions

Theorem 8 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (iii), (iv) and (v) in (21).

Further suppose that $\bar{x}$ is a feasible point of NCP(f) and $T(\bar{x})$ consists of **E***-matrices.Then*

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. Suppose $0 \in T_\Psi(\bar{x})$. Then $v^T A + w^T = 0 \Rightarrow A^T v + w = 0$. We claim that $\Phi(\bar{x}) = 0$. Suppose, if possible, $\Phi(\bar{x}) \neq 0$. Then by property (v) in (21), $v \neq 0$. Since $\bar{x}$ is a feasible point to NCP(f), by property (iv) in (21), we have $v \geq 0$.

Since $T(\bar{x})$ consists of **E**-matrices and $A \in T(\bar{x})$, there exists an index i such that $0 < v_i(Av)_i$. By property (ii) in (21), $v_i\, w_i \geq 0$. But $0 < v_i(Av)_i = -v_i\, w_i \leq 0$ which is a contradiction. Hence $\Phi(\bar{x}) = 0$. Conversely, suppose $\Phi(\bar{x}) = 0$. Then by property (iii) in (21) and the description of $T_\Psi(\bar{x})$, we have $T_\Psi(\bar{x}) = \{0\}$. □

Remark Theorem 8 is applicable to NCP functions of Examples 5-8.

A slight modification of the above theorem leads to the following result.

Theorem 9 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (iii) and (v) in (21), and

$$\text{for } i \in \{1, \ldots, n\} \text{ with } \bar{x}_i > 0 \text{ and } f(\bar{x}_i) > 0, \text{ we have } v_i > 0,\ w_i > .$$

Further suppose that $\bar{x}$ is a strictly feasible point of NCP(f) and $T(\bar{x})$ consists of $\mathbf{E_0}$*-matrices.Then*

$$0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0.$$

Proof. The proof is similar to that of Theorem 8. □

Remark Theorem 9 is applicable to NCP functions of Examples 7 and 8.

5.6 Minimizing the merit function under regularity (strict regularity) conditions

We generalize the concept of a regular (strictly regular) point [4] in order to weaken the hypotheses in the previous Theorems.

For a given H-differentiable function f and $\bar{x} \in R^n$, we define the following index sets:

$$\begin{array}{llll} \mathcal{P}(\bar{x}) & := \{i : v_i > 0\}, & \mathcal{N}(\bar{x}) & := \{i : v_i < 0\}, \\ \mathcal{C}(\bar{x}) & := \{i : v_i = 0\}, & \mathcal{R}(\bar{x}) & := \mathcal{P}(x) \cup \mathcal{N}(x) \end{array}$$

where v_i are the entries of V in (20) (e.g., v_i is defined in Examples 5-8).

Definition 5 *Consider f, Φ, and Ψ as above. A vector $x^* \in R^n$ is called strictly regular if, for every nonzero vector $z \in R^n$ such that*

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0, \tag{23}$$

there exists a vector $s \in R^n$ such that

$$s_{\mathcal{P}} \geq 0, \quad s_{\mathcal{N}} \leq 0, \quad s_{\mathcal{C}} = 0, \quad \textit{and} \tag{24}$$

$$s^T A^T z > 0 \quad \textit{for all } A \in T(x^*). \tag{25}$$

Theorem 10 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (ii), (iii), and (v) in (21).

Then $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a strictly regular point if and only if $\bar{x}$ solves NCP(f).

Proof. Suppose that $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a strictly regular point. Then for some $v^T A + w^T \in T_\Psi(\bar{x})$,

$$0 = v^T A + w^T \Rightarrow A^T v + w = 0. \tag{26}$$

We claim that $\Phi(\bar{x}) = 0$. Assume the contrary that $\bar{x}$ is not a solution of NCP(f). Then by property (v) in (21), we have v as a nonzero vector satisfying $v_{\mathcal{C}} = 0,\ \ v_{\mathcal{P}} > 0,\ \ v_{\mathcal{N}} < 0$. Since $\bar{x}$ is a strictly regular point, and $v_i w_i \geq 0$ by property (ii) in (21), by taking a vector $s \in R^n$ satisfying (24) and (25), we have

$$s^T A^T v > 0 \tag{27}$$

and

$$s^T w = s_{\mathcal{C}}^T w_{\mathcal{C}} + s_{\mathcal{P}}^T w_{\mathcal{P}} + s_{\mathcal{N}}^T w_{\mathcal{N}} \geq 0. \tag{28}$$

Thus we have $s^T(A^T v + w) = s^T A^T v + s^T w > 0$. We reach a contradiction to (26). Hence, $\bar{x}$ is a solution of NCP(f).

The 'if' part of the theorem follows easily from the definitions. □

Remark Another proof of Theorem 7 can be obtained by taking $s = z$ in Definition 5 of a strictly regular point and by using Theorem 10.

Before we state the next theorem, we recall a definition from [27].

Definition 6 *Consider a nonempty set $\mathcal{C}$ in $R^{n\times n}$. We say that a matrix A is a row representative of $\mathcal{C}$ if for each index $i = 1, 2, \ldots, n$, the ith row of A is the ith row of some matrix $C \in \mathcal{C}$. We say that $\mathcal{C}$ has the row-$\mathbf{P}_0$-property (row-$\mathbf{P}$-property) if every row representative of $\mathcal{C}$ is a $\mathbf{P}_0$-matrix ($\mathbf{P}$-matrix). We say that $\mathcal{C}$ has the column-$\mathbf{P}_0$-property (column-$\mathbf{P}$-property) if $\mathcal{C}^T = \{A^T :\ A \in \mathcal{C}\}$ has the row-$\mathbf{P}_0$-property (row-$\mathbf{P}$-property).*

Theorem 11 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Suppose Φ is a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^{n} \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (ii), (iii), and (v) in (21).

Further suppose that $T(\bar{x})$ has the column-$\mathbf{P}$-property. Then

$$0 \in T_\Psi(\bar{x}) \text{ if and only if } \bar{x} \text{ solves NCP}(f).$$

Proof. In view of Theorem 10, it is enough to show $\bar{x}$ is a strictly regular point. To see this, let v be a nonzero vector satisfying (23). Since $T(\bar{x})$ has the column-$\mathbf{P}$-property, by Theorem 2 in [27], there exists an index j such that $v_j \left[A^T v\right]_j > 0 \ \ \forall A \in T(\bar{x})$. Choose $s \in R^n$ so that $s_j = v_j$ and $s_i = 0$ for all $i \neq j$. Then $s^T A^T v = v_j \left[A^T v\right]_j > 0 \ \ \forall A \in T(\bar{x})$. Hence $\bar{x}$ is a strictly regular point. □

As a consequence of the above theorem is the following corollary.

Corollary 3 *Let $f : R^n \to R^n$ be locally Lipschitzian. Let Φ be a nonnegative NCP function of f. Assume that $\Psi := \sum_{i=1}^{n} \Phi_i(\bar{x})$. Further suppose that $\partial_B f(\bar{x})$ has the column-$\mathbf{P_0}$-property. Then*

$$0 \in \partial\Psi(\bar{x}) \Leftrightarrow \Psi(\bar{x}) = 0.$$

Proof. Note that by Corollary 1 in [29], every matrix in $\partial f(\bar{x}) = co\, \partial_B f(\bar{x})$ is a $\mathbf{P}_0$-matrix. Now by Corollary 2, we have the claim.□

Remarks

- Theorem 10 is applicable to the NCP functions of Examples 5-8.
- Corollary 3 might be useful when the function f is piecewise smooth in which case $\partial_B f(\bar{x})$ consists of a finite number of matrices.

Concluding Remarks

In this paper, we described the H-differential of the so called restricted and unrestricted implicit Lagrangian functions. Also, we considered a nonlinear complementarity problem corresponding to an H-differentiable function, with an associated NCP function Φ and a merit function $\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x})$, we described conditions under which every global/local minimum or a stationary point of Ψ is a solution of NCP(f).

Our results recover/extend various well known results stated for continuously differentiable (locally Lipschitzian, semismooth, C-differentiable) functions.

We note here that similar methodologies under H-differentiability can be carried out for other merit functions such as Luo-Tseng function [17]. We can consider the NCP function [17]:

$$\Phi(x) := \phi_0(x^T f(x)) + \sum_{i=1}^{n} \phi_i(-f_i(x), -x_i),$$

where $\phi_0 : R \to [0, \infty)$ and $\phi_1, \cdots, \phi_n : R^2 \to [0, \infty)$ are continuous functions that are zero on the nonpositive orthant only. By defining the merit function

$$\Psi(\bar{x}) := \sum_{i=1}^{n} \Phi_i(\bar{x}) \quad \text{or/and,} \quad \Psi(\bar{x}) :== \frac{1}{2}||\Phi||^2.$$

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