Les Cahiers du GERAD ISSN: 0711–2440

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G–2022–20

May 2022

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2022
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Omnichannel fulfillment strategies and sales credit allocation

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May 2022
Les Cahiers du GERAD
G–2022–20

Abstract : Faster delivery, lower shipping costs, and a higher chance of product availability, are some of the benefits offered by an omnichannel business model. Assuming that ordering can only be done before the start of the selling season, we address two questions that are crucial for the success of pooling inventories. First, what is the optimal order-quantity decision for each channel, i.e., the brick-and-mortar store and the online channel, under different omnichannel fulfillment strategies? Second, who should get the sales credit? To answer these questions, we propose a noncooperative game theory approach and characterize the equilibrium order quantities in four scenarios, namely, no cross-fulfillment of customer's demand, ship-to-store, ship-from-store, and hybrid fulfillment. We compare the equilibrium strategies and outcomes, and obtain, among other things, that sales credit allocation is a key factor in determining whether the firm will benefit from the omnichannel strategies. We consider various sales credit allocation rules and provide insights on the best practice for each fulfillment strategy, taking into account customers' shopping preference.

Keywords: Retail operations, omnichannel, ship-to-store, ship-from-store, sales credit allocation

Résumé : Une livraison plus rapide, des frais d'expédition réduits et une plus grande probabilité de disponibilité des produits sont certains des avantages offerts par un modèle commercial omnicanal. En supposant que la commande doit être faite avant le début de la saison de vente, nous abordons deux questions cruciales pour le succès d'une mutualisation des inventaires. Tout d'abord, quelles sont les quantités optimales que le magasin physique et le canal en ligne doivent commander? Deuxièmement, quel canal devrait obtenir le crédit d'une vente? Pour répondre à ces questions, nous proposons un jeu non-coopératif pour caractériser les quantités d'équilibre dans quatre scénarios, à savoir, les deux canaux n'effectuent pas de livraisons croisées, expédier la commande du dépôt au magasin, livrer du magasin une commande placée en ligne, et implémenter les deux dernières possibilités (option hybride). Nous comparerons les stratégies d'équilibre et obtenons, entre autres, que l'option hybride conduit à de meilleurs résultats collectifs et individuels. Nous considérons diverses règles d'allocation des crédits de vente et déterminons la meilleure façon de procéder pour chacun des scénarios.

Mots clés : Vente au détail, omnicanal, livraison en magasin, expédition depuis le magasin, attribution du crédit de vente

Acknowledgements: Research supported by NSERC, Canada, grant RGPIN–2021–02462.

1 Introduction

Retailers and brands are accelerating their digitalization strategies and making investments to improve their omnichannel retailing capabilities (Bell et al., 2014). In recent years, many retailers, e.g., Best Buy, Nordstrom, Sears, Walmart, and Canadian Tire, adopted omnichannel strategies or are planning to do so in the near future (Forrester-Research, 2022). To provide an integrated shopping experience, an omnichannel retailer uses the full inventory at its disposal to get the order to the customer. For example, when an order is accepted by the online channel, it can be shipped directly from the distribution center (DC) to the customer. If the DC cannot satisfy the order due to a stock out, an SFS fulfillment strategy allows the order to be delivered to the customer from the BM store (Ziobro 2016). Moreover, if a product is no longer available in the store, a ship-to-store (STS) strategy allows the retailer to ship the product to the BM store, at no extra cost, for the customer to pick up.

Faster delivery, lower shipping costs, and a higher chance of product availability are some of the benefits offered by an omnichannel business model. According to GlobalData, the proportion of online sales in the US supported by physical stores rose to 37% during the 2020 holiday season, up from 32% in 2019. Pickup of online orders from stores grew by 103% last year, while retailers shipping orders from stores grew by 80% (www.retaildive.com). In the past, sales crediting frameworks were very straightforward and based on the assumption that online shoppers shop online and offline shoppers shop in-store. With online-offline integration, BM stores have a great impact on a company's e-commerce profitability. There is a significant boost in online sales where stores are available and a proven drop in online sales where stores have closed (Gallino et al., 2017). In addition, in-store pick up, ship from store, and other BM services significantly increase e-commerce profit margins (Gao and Su, 2017a). Although many retailers choose to embrace the omnichannel approach, many have yet to resolve a critical piece of the puzzle: omnichannel sales attribution. For instance, if a customer buys a product online but the product is shipped to their home from the store, which channel should get credit for the sale: the online channel where demand originated or the BM store where the order was fulfilled?

With omnichannel fulfillment strategies, sales attribution can be extremely tricky because, when the stores are leveraging the fulfillment of online orders, measuring store success based purely on in-person purchases is not realistic. Despite the fact that many retailers have already implemented a fully functioning omnichannel, yet according to Forrester Consulting report, only 16% of retailers allocate revenues between channels, while 31% and 21%, respectively, attribute revenue from such sales exclusively to either the online or store channel (Forrester-Research (2014), www.emarketer.com). Interestingly, eMarketer found that over 40% of merchants still assign credit to a single touchpoint (Benes). According to the 2016 Customer Experience/Unified Commerce Survey conducted by Boston Retail Partners (BRP), many retailers (55%) do not (unfortunately) offer store employees incentives for fulfilling omnichannel orders. In practice, many retailers have yet to create clear and logical practices for attribution of sales and revenues. For example, some furniture retailers have chosen a geographic compensation model in which all online orders are attributed to the store in closest proximity to the sale. So far, much of the conversation on attribution has focused on marketing and advertising budget allocation. Whereas a first-touch (last-touch) attribution model assigns full credit to the channel with which the customer had the first (last), engagement, the linear attribution model assigns credit to all touchpoints equally, without identifying which customer engagement most influenced the sale.

These attribution methods have obvious limitations. For instance, if the BM store gets credit for an STS sale, but the DC manager has to take on the expense of processing the order, and shipping the goods from the DC's inventory, the mismatched reward system could make tensions rise. If retailers fail to allocate sales to the appropriate channel or combination of channels, they risk creating disharmony in the goals and motives of the online and in-store management teams, which may be damaging in an omnichannel environment. The challenges involved in assessing the proper credit for an omnichannel sale when it is partly influenced by offline or online, may lead to under ordering by channel managers, and in some cases, to one channel sabotaging the other's sales.

In this research, we analyze the decision on a sales-credit allocation policy in omnichannel retailing. Sales attribution has only been studied in the marketing literature, and to the best of our knowledge, our research is the first to analyze this problem from an operations management perspective. We specifically seek to answer the following interconnected questions:

1. When an order is fulfilled using omnichannel strategies, e.g., STS or SFS, which channel should get credit for that sale?
   Is it the store where the customer picks up the order purchased online? Or is it the online channel that provides the information and processes the transaction? And what about the store that delivers an online order to the customer's address?
2. Under each fulfillment strategy, does the optimal sales allocation policy also reflect a fair allocation? What are the conditions under which the retailer's optimal credit allocation is also a fair sales allocation between the channels?
3. Are the various credit attribution policies implemented in practice, e.g., giving full credit to the BM store, giving full credit to the DC, or assigning equal credit to both, part of an optimal policy of a profit- maximizing omnichannel firm? What is the impact of the sales-credit allocation policy on retailer's profit under omnichannel strategies?

To answer these questions, we consider an inventory game between the store and the online channel of an omnichannel brand that sells a product to customers. The two channels are operated by two independent managers, and each channel's inventory decision affects the other channel's profit. We suppose that the market is made of two segments: (i) store-visiting customers who always prefer to visit the store to purchase or pick up the product; (ii) direct-shipping customers who prefer direct delivery to their address. As a benchmark, we first analyze an inventory game without any omnichannel strategy implementation. Next, we consider the STS, SFS, and hybrid fulfillment strategies, respectively, and analyze how the crediting of sales affects the channels' ordering decisions and profits.

Our main contributions are the following:

- Prior work in omnichannel operations management is mostly based on a centralized supply chain, where the retailer decides the order quantities for the online and store channels. We focus on a decentralized system where each channel is managed by an independent team facing an uncertain demand and analyze how the order quantity of each channel is affected by the sales-credit attribution policies. In our work, we do not intend to determine whether a company should adopt STS, SFS, or hybrid strategies, but rather to offer insights for companies that already plan to launch a fulfillment service on how to foster collaboration rather than competition among channels. We show that under the ship-to-store strategy, both channels place higher orders when the DC (where orders are fulfilled)) gets full credit. Whereas, under a ship-from-store strategy, if the store gets full credit for each sale, then both channels order higher quantities.
- Multichannel attribution models in the marketing literature focus on how to allocate the advertising campaign budget to each marketing touchpoint, based on the role they play in the customer journey. These studies do not consider the impact of sales-credit allocation on the channels' decisions. In this research, we use a game theory approach with sales credit attribution between channels to investigate the optimal order-quantity decision of each channel under each omnichannel fulfillment strategy. We provide explanations on why each firm needs a different sales credit allocation based on the implemented fulfillment strategy and the combination of store-visiting and direct shipping customers in the market. An important insight is that the sales credit allocation is a key factor in determining whether the firm will benefit from the omnichannel strategies.

  We find that, when the firm's policy is to allocate equal sales credit, then the hybrid fulfillment strategy is the preferred omnichannel strategy for the store, the online channel, and the firm. From our numerical analysis we, observe that, when the fraction of store-visiting customers is higher in the market, it is more profitable for the firm to implement the STS strategy and allocate the full credit to the demand origin (store) or allocate credit equally between the two channels.

- In our game, we assume that the omnichannel firm first determines the sales credit policy, and next, the two channels make their decisions. Our third contribution is in providing the conditions under which the retailer's optimal sales credit allocation is also a fair allocation policy. We show that, in a market with equal proportions of store-visiting and direct-shipping customers, under all fulfillment strategies, the Nash bargaining solution (NBS) outcome is the most beneficial for the omnichannel company. We note that, under some conditions that will be explored, such an outcome is only implementable for a narrow range of parameter values.

The remainder of this paper is organized as follows. In Section 2, we review the most related literature. In Section 3, we provide preliminaries and introduce our key assumptions. In Section 4, we characterize the Nash equilibrium quantities under the STS, SFS, and hybrid fulfillment strategies, respectively, and analyze the NBS outcomes. In Section 5, we derive the optimal sales crediting policy and illustrate our results using realistic parameter values that represent a typical omnichannel setting. Concluding remarks are provided in Section 6.

2 Literature review

This study is connected to the literature in omnichannel operations, sales compensation and attribution, and inventory transshipment. While inventory transshipment has received extensive study in the past decade, omnichannel operations is an emerging research stream. Below, we review each research stream, position our work, and discuss our contributions to the literature.

2.1 Omnichannel retailing

Omnichannel retailing is a relatively new area in operations management, and it has gained traction in recent years; see, e.g., (Bell et al., 2016; Ishfaq and Bajwa, 2019; Davis-Sramek et al., 2020; Verhoef et al., 2015; Bijmolt et al., 2021) for comprehensive reviews of the topic. A part of this stream empirically explores the effects of implementing different omnichannel fulfillment strategies observed in practice such as BOPS (buy-online-pickup-in-store) (Bell et al., 2014; Gallino and Moreno, 2014), STS (ship-to-store) (Gallino et al., 2017; Ertekin et al., 2021) and offline showrooms (Bell et al., 2018). Akturk et al. (2018) empirically find that, for high-value items, some customers switched from the online channel to the BM channel after the implementation of STS, which led to increasing BM store sales and lower online sales. Cao et al. (2016) show that in-store pickup provides customers with a new purchasing option, leading to an increase in sales. Strategically allocating and using inventory across both online and offline channels is a very limited area in the omnichannel literature. Hu et al. (2022) analytically study the positive pooling effect and negative depooling effect of BOPS as well as its potential margin-loss and market-expansion effects. The majority of previous studies examined the potential impact of implementing omnichannel fulfillment strategies on market expansion and customers' switching behavior by considering customer utility. Unlike previous studies, we do not aim to compare different fulfillment strategies profitability or to determine whether a company should adopt one. We study an inventory game between channels to determine the best way to allocate sales credit. Thus, considering customer valuation would not affect our results.

A series of papers focused on integrating online demand with a network of physical stores (Jalilipour-Alishah et al., 2015; Govindarajan et al., 2021). Other analytical studies investigate how retailers' operational decisions and profitability are impacted by omnichannel strategies, through various aspects such as the effect of information provided to customers (Gao and Su, 2017b), the impact of implementing BOPS on customers' channel choice (Gao and Su, 2017a), retailer's optimal BOPS service area (Jin et al., 2018), and retailers' return policy strategies (Jin et al., 2020; Mandal et al., 2021). Gao et al. (2021) investigate how the adoption of three omnichannel retailing strategies, namely, showrooms, flexible returns, and fulfillment flexibility, influence a retailer's decisions regarding the number and size of physical stores.

Little attention has been paid to the SFS option in the operations management literature. Li (2020) studies the effects of the ship-from-store-to-store option on the retailer's multiperiod inventory decisions. He et al. (2021) study the impacts of opening physical stores and the SFS option on consumers' shopping behaviors. Bayram and Cesaret (2021) investigates dynamic SFS fulfillment decisions from which store to fulfill an online order when it arrives and finds that it is not practical to implement ship-from-store when a large number of stores are included in the network.

Most of the studies in the literature are based on a centralized supply chain, where the retailer decides the order quantity for online and offline channels. In this paper, we are interested in a case where each channel is managed by separate and independent teams. By analyzing the functionality of two omnichannel fulfillment options, namely, STS and SFS, we add to the literature by considering how inventory decisions are made under different sales allocation policies.

None of the reviewed works have focused on sales credit allocation between the store and online channel once the omnichannel strategies are implemented. To the best of our knowledge, only Gao and Su (2017a) briefly investigate how to allocate BOPS revenue between channels. They assume that the store obtains a fraction of the BOPS revenue and they compare the store's inventory under centralized and decentralized systems. They find that the store is usually either overstocked or understocked relative to the centralized benchmark. They show that, with BOPIS fulfillment, a simple revenue-sharing contract between channels can coordinate the decentralized system and align the incentives of the BM store with the entire organization. They do not consider inventory allocation for the online channel. Our study provides a different perspective to the stream of work on omnichannel inventory operation by considering a game between the BM store management team and the DC's management team. Our study is different from Gao and Su (2017a) in two ways. First, when investigating profit allocation, we consider an inventory game between the store and the online channel, where each player's inventory decision affects the other channel's profit. Second, we consider the implementation of different omnichannel strategies (STS and SFS) and their related operating costs for the store and the online channel.

There are also some similarities between our work and salesforce compensation studies in the marketing literature, where the amount of effort made is the only factor affecting sales. Although there have been some discussions about the interactions between inventory and sales effort in centralized settings, inventory rarely is a focus in this literature. Recently, a few papers have considered limited inventory (Dai et al., 2021), however the inventory decision is mostly set exogenously before the sales agent makes the effort decision. Li et al. (2020) study the problem of designing compensation contracts to incentivize the retail store managers, who next choose the effort level and order quantity. When all else is equal, they find that store managers exert more effort under target contracts than under profit sharing.

2.2 Inventory transshipment

Our paper also draws on and contributes to the literature on lateral transshipment, which allows inventory to be moved between multiple locations in order to better meet demand (Axsäter, 1990). In omnichannel retailing, the fact that demands from either market can be fulfilled by either stock of inventory, is analogous to a reactive transshipment setting. Here, we are particularly interested in studies that examine transshipment between independent, decentralized players and their optimal stocking and sharing decisions, in a game theoretical framework (Rudi et al., 2001; Çömez et al., 2012; Zhang et al., 2020; Zhao and Atkins, 2009; Shao et al., 2011; Silbermayr, 2020). A major part of the literature considers a two-symmetric-retailer system, with identical cost parameters, or symmetric market demands (Archibald et al., 2009; Paterson et al., 2012; Hezarkhani and Kubiak, 2010). Our research is closer to transshipment studies in a dual-channel retailer setting, where the inventory pooling benefits or transshipment operating costs are not symmetric. Yang and Qin (2007) analyze a case where online demand can be fulfilled from a network of BM stores and called this "virtual lateral transshipment,"that is, the demand is redirected to another location without inventory being transferred between the locations. Zhao et al. (2016) determine the optimal order quantities of different channels under a lateral inventory transshipment strategy where a manufacturer forwards its online orders to an offline retailer. He et al. (2014) focus on a manufacturer's dual channel and propose a quantity-discount contract to achieve coordination. Seifert et al. (2006) consider a supply chain with multiple independent retailers and a single "virtual" store.

Most of the studies on transshipment in a dual channel assume that a retailer owns both channel, and that decisions are centralized to maximize the total profit. As mentioned before, here, we assume that the channels are managed by separate teams with distinct profit functions. Transfer price (i.e., a fee charged by one store for transshipping products to another) has been well studied as a coordinating mechanism (Rudi et al., 2001; Hezarkhani and Kubiak, 2010; He et al., 2014). Hu et al. (2007) show that, in a two-location inventory model with asymmetric locations, coordinating prices may exist for only a narrow region in the parameter space. They also find that transshipment costs have the most direct effect on the existence of coordinating prices. Li and Li (2018) discuss the impact of bargaining power in a two-echelon supply chain consisting of one manufacturer and two symmetric retailers with bidirectional transshipment between them. Katok and Villa (2021) use behavioral laboratory experiments to investigate how transfer prices should be set centrally when ordering decisions are made by two symmetric local retailers. They find a positive relationship between transfer prices and ordering decisions and show that having retailers negotiate transfer prices performs quite well. Li and Chen (2020) empirically study transfer price from a behavioral perspective and examine how the decisions of decentralized retailers can be affected by commitments in transfer price and in sharing. They show that, when order quantities are set after the transfer price, the ordering decisions are influenced by the transfer price. As far as we know, the only study that considers inventory transshipment in an omnichannel environment is Derhami et al. (2021). They propose a data-driven model to estimate product availability in a network of interconnected retailers within the customer's acceptable time frame.

Our analysis is distinct from the aforementioned literature due to modeling features specific to the omnichannel setting, such as asymmetric demands, fulfillment process, and costs for the store and online channels. In addition, one of the main assumptions in the above literature is that all of the unsatisfied demand will be fulfilled with the transshipped inventory. Whereas, in the omnichannel setting, due to the difference in fulfillment waiting times, only a portion of customers would agree to order with the cross-channel fulfillment strategies in case of stock out. Clearly, this gap in the literature, especially in light of the the extensive offerings of omnichannel services in the marketplace, signals a significant research need. None of the reviewed works have addressed the issue of revenue allocation between the store and online channel after implementing omnichannel strategies. The novel aspects of our model enable us to generate insights into the decisions of an omnichannel firm, because these seemingly beneficial fulfillment strategies may undermine the firm's overall profit when the sales credit is not allocated correctly between channels.

3 Setup and assumptions

We consider a manufacturer/brand/firm that sells a product to customers through online (e-commerce site) and offline (BM store) channels at price p. We assume that the total number of potential consumers in the market (the market size) is a uniformly distributed random variable between 0 and M, i.e., $D \sim U[0, M]$. Consumers are of two types, namely, store-visiting customers, representing a proportion ϕ of the total, and the remaining $1 - \phi$ being direct-shipping customers. Store-visiting customers always choose to visit the BM store or order online and pick up the order in-store. Direct-shipping customers always prefer to have their orders shipped to their address. A store-visiting customer typically lives near a BM store, prefers traditional shopping, or simply needs the product immediately. This segmentation is consistent with empirical studies (Skrovan, 2017; Nageswaran et al., 2020). Consequently, the in-store and online demands are given by $D_s \sim U[0, \phi M]$ and $D_o \sim U[0, (1 - \phi)M]$, respectively. Denote by p the fixed retail price.

The store and the online channel are operated by two separate management teams, each maximizing its own profit. The firm may hold an inventory in the store and in a DC. Orders q_s and q_o for the two channels are placed before the selling season starts. Throughout the paper, the subscript o stands for the $o$nline channel (distribution center) and the subscript s refers the $s$tore. We consider four scenarios for fulfilling consumers' orders, that is, benchmark, STS, SFS, and a hybrid scenario.

Benchmark scenario: Each channel fulfills its own demand and faces a newsvendor-type problem where any excess demand is lost. This scenario allows us to measure the benefit of the three others.

STS scenario: When in-store customers encounter a stock out, although they are given the STS option, not all are willing to wait the shipping time for product delivery. We assume that only a fraction α of in-store customers switch to STS in case of an in-store stock out. The STS sales are given by

$$T_t = \mathbb{E} \min[\alpha(D_s - q_s)^+, (q_o - D_o)^+],$$

where the subscript t stands for ship to store. We suppose that there are no back orders and that lack of available inventory equals lost sales.

SFS scenario: The store serves all in-store customers, and the excess stock, if any, may be used to fulfill unsatisfied online orders by shipping products directly to the customer's address. The SFS sales are given by the minimum between the store's excess inventory and the online channel's excess demand, that is,

$$T_f = \mathbb{E} \min[(D_o - q_o)^+, (q_s - D_s)^+],$$

where the subscript f stands for ship $f$rom store. Our implicit (reasonable) assumption is that online customers do not care from where the product is shipped from. Also, as they are not sensitive to the delivery waiting time, even if shipping from the BM takes more time than shipping from the DC, we suppose that all unsatisfied online customers choose SFS in case of stock-out.

Hybrid scenario: Both STS and SFS options are available. We refer to this model by h for $h$ybrid.

Due to the differences in fulfillment processes, the handling costs vary across the two channels. Let c_s and c_o denote in-store procurement cost and the online channel's direct shipping fulfillment cost (shipping cost for last mile delivery), respectively. If the BM store fulfills online orders with SFS, a cost c_f is incurred for each unit delivered to the customers. With SFS, not only are items shipped twice, proper structure and store associates are required to pick, pack, and ship single items from stores to the customer's address. Compared to DCs, most BM stores are poorly positioned for picking and packing orders for delivery. Therefore, we assume that $c_f \geq c_o$. If the online channel fulfills an order via STS, a cost c_t is incurred for each unit sent to store. From the retailer standpoint, one can verify that the retailer's logistics cost for last mile deliveries c_o is much higher than selling through the STS channel, i.e., $c_t < c_o$. To avoid trivial cases, we assume that $p > \max\{c_s, c_o\}$, which implies that p is also larger than c_f and c_t.

To save on notation, let $m_o = p - w - c_o$ denote the online profit margin. Further, denote by η the store's holding cost for unused inventory. Due to the lower rent and storage fees, we assume that the DC's holding is negligible and set equal to zero.

In the STS, SFS, and hybrid scenarios, the profit of the BM store and the DC will depend on the sales credit share, that is, the compensation that a channel gets for fulfilling a customer's order received by the other channel. Denote by k_j this sales credit share, with $k_j \in [0, p]$ for $j = f, h, t$. The actual value of k_j can be determined by some ad-hoc rules, e.g., the full credit goes to where the demand originated or is split half-and-half. Here, we assume that k_j is exogenously determined by the retailer and then by implementing the Nash bargaining solution, we examine whether the optimal sales allocation policy also reflect a fair allocation.

Table 1 summarizes our notation.

Table 1: Notation

Symbol	Definition
w	Wholesale price
M	Market size
p	Retail price
c_s	In-store fulfillment cost (wholesale price + store's operating cost)
c_o	DC's direct shipping fulfillment cost, e.g., packing and delivery cost
c_t	STS fulfillment cost
c_f	SFS fulfillment cost
k_j	Sales credit share, for $j = f, h, t$.
ϕ	Fraction of customers choosing to visit or pick up in-store
α	Fraction of the store demand switched to STS in case of stock out
η	Holding cost

4 Model analysis

In this section, we characterize the Nash equilibrium order quantities in the different fulfillment scenarios, and determine the sales credit using the NBS in the relevant scenarios. In all, but the benchmark, the expressions of the equilibrium quantities are very large and are not shown. Instead, their properties are highlighted and they are illustrated numerically.

4.1 Benchmark

To evaluate the benefits of the omnichannel strategies, we start by examining a benchmark system in which the firm's online and offline channels are independently managed, and cross-channel sales are not possible. The expected profit are given by

$$\Pi_s^b(q_s) = p\mathbb{E}\min(D_s, q_s) - c_s q_s - \eta\mathbb{E}[q_s - D_s]^+, \tag{1}$$

$$\Pi_o^b(q_o) = (p - c_o)\mathbb{E}\min(D_o, q_o) - wq_o, \tag{2}$$

where Π_s^b and Π_o^b represent the store's and the online channel's expected profit, respectively. The first two terms in (1) represent the expected profit from selling the product in the store, while the third term measures the expected inventory holding cost for leftover inventory. To have $q_o^b > 0$, we make the intuitive assumption that $p > w + c_o$. In (2), the first term represents the revenues and the second one the purchasing cost.

From the first-order optimality conditions, it is easy to verify that the optimal order quantities are given by

$$q_s^b = \phi M \left(\frac{p - c_s}{p + \eta}\right) > 0,$$

$$q_o^b = (1 - \phi)M \left(\frac{m_o}{p - c_o}\right) > 0.$$

Without any surprise, q_s^b decreases with the cost c_s, and q_o^b decreases with w and c_o. Also, as expected, both order quantities increase with the retail price. Finally, q_s^b increases with the fraction of customers who choose to visit or pick up in-store, whereas q_o^b decreases with this proportion. Inserting the equilibrium quantities in the profit functions, we obtain

$$\Pi_s^b = \frac{\phi M(p - c_s)^2}{2(p + \eta)},$$

$$\Pi_o^b = \frac{(1 - \phi)\, Mm_o^2}{2\,(p - c_o)}.$$

4.2 Ship to Store (STS)

To handle in-store stock outs, STS is an option retailers can use to provide in-store pickup services. If the store can get the items transferred from the DC, the customer is given the option of coming back later to pick up their order. Although it is not as good as using in-store inventory to fulfill the order, STS is an improvement over not providing the service at all or losing sales.

The number of orders fulfilled by STS is defined by

$$T_t = \mathbb{E}\min[\alpha(D_s - q_s)^+, (q_o - D_o)^+].$$

Let k_t be the credit allocation rule under STS. Then, the store and the online channel earn $(p - k_t)$ and $(k_t - c_t)$ per STS sale, respectively. The expected profits of the store and the online channel under the STS strategy are given by

$$\Pi_s^t(q_s) = p\mathbb{E}\min(D_s, q_s) - c_s q_s - \eta\mathbb{E}[q_s - D_s]^+ + (p - k_t)T_t, \tag{3}$$

$$\Pi_o^t(q_o) = (p - c_o)\mathbb{E}\min(D_o, q_o) - wq_o + (k_t - c_t)T_t, \tag{4}$$

where

$$T_t = \begin{cases} \alpha(D_s - q_s), & \text{if } (q_o - D_o) > \alpha(D_s - q_s) > 0, \\ q_o - D_o, & \text{if } \alpha(D_s - q_s) > (q_o - D_o) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

In (3) and (4), the terms $(p - k_t)T_t$ and $(k_t - c_t)T_t$ represent the expected profit from the STS sales for the store and online channel, respectively. We assume that $k_t \in [c_t, p]$ to guarantee that the online channel is willing to fulfill STS orders.

Given our assumption that the two channels are managed independently, we seek a Nash equilibrium.

Proposition 1. *If $\alpha(k_t - c_t) \leq (p - c_o)$, then there exists a unique Nash-equilibrium order quantities under STS fulfillment.*

Proof. See Appendix. □

Proposition 1 shows that a unique STS Nash equilibrium exists when the online channel's profit margin from direct delivery sales $(p - c_o)$ is higher than the profit margin of fulfilling STS orders $\alpha(k - c_t)$. This condition ensures that it is not always beneficial to the DC to sell through the STS strategy. The best response functions of the two players are given by

$$q_s^t(q_o^t) = \frac{2\phi(1-\phi)(p - c_s)M^2 - (p - k_t)\left(q_o^t\right)^2}{2(1-\phi)M(p+\eta)}, \tag{5}$$

$$q_o^t(q_s^t) = \frac{2\phi(1-\phi)(p - w - c_s)M^2 + \alpha(k_t - c_t)(\phi M - q_s^t)^2}{2\phi M(p - c_o)}. \tag{6}$$

Remark 1. The equilibrium quantities are obtained by solving the above system, which requires finding the roots of a fourth-degree polynomial. As the resulting expressions are very long and do not give any qualitative insight, we do not show them. In the numerical examples, we always obtain only one root that leads to positive quantities and demands. The same situation happens in all omnichannel scenarios.

To characterize the strategic interaction between the two players, we compute the derivatives of the reaction functions to obtain

$$\frac{dq_s^t(q_o^t)}{dq_o^t} = -\frac{(p - k_t)q_o^t}{(1-\phi)M(p+\eta)} < 0,$$

$$\frac{dq_o^t(q_s^t)}{dq_s^t} = -\frac{\alpha(k_t - c_t)(\phi M - q_s^t)}{\phi M(p - c_o)} < 0,$$

which implies strategic substitution, that is, each manager's order quantity is decreasing in the other manager's order.[1]

Solving the nonlinear response functions (5)–(6) for some special cases leads to the following observations:

1. If the firm gives the full credit $k_t = p$ to where the demand is fulfilled (to the DC/online channel), then solving (5)–(6), we obtain

$$q_s^t = q_s^b = \phi M \left(\frac{p - c_s}{p + \eta}\right), \tag{7}$$

$$q_o^t = (1-\phi)M\left(\frac{m_o}{p - c_o}\right) + \frac{\alpha\phi M(p - c_t)(c_s + \eta)^2}{2(p - c_o)(p + \eta)^2} > q_o^b, \tag{8}$$

   that is, the store orders the same quantity as in the benchmark, while the online channel increases its order with respect to the benchmark. Note that the upper bound of the BM store's equilibrium order quantity is equal to the optimal order quantity in the benchmark model when $k = p$; otherwise, the store always orders less than the benchmark's optimal order quantity $q_s^t \leq q_s^b$.

2. If the firm allocates the credit to the store ($k_t = c_t$), then the equilibrium would be

$$q_s^t = \phi M\left(\frac{p - c_s}{p + \eta}\right) - \frac{(1-\phi)M(p - c_t)m_o^2}{2(p + \eta)(p - c_o)^2} < q_s^b, \tag{9}$$

$$q_o^t = q_o^b = (1-\phi)M\left(\frac{m_o}{p - c_o}\right). \tag{10}$$

   In this scenario, the online channel orders the same quantity as in the benchmark, while the store orders less, implying that the total order is lower than in the benchmark. The intuition behind this result is that the store is somehow incentivized to free ride the online channel.

3. In the intermediate case where $c_t < k_t < p$, the online channel (store) always orders more (less) than in the benchmark scenario.

Given the very large expressions of the equilibrium quantities, any further analysis of how they vary with the credit allocation k_t must be done numerically. We use the following realistic parameter values that represent a typical omnichannel setting (Petersen, 2017), and satisfy the assumptions made previously:

$$w = 6, \quad M = 600, \quad p = 40, \quad c_s = 10, \quad c_o = 20, \quad c_t = 3, \quad c_f = 23, \quad \phi = 0.5, \quad \alpha = 0.6, \quad \eta = 7. \tag{11}$$

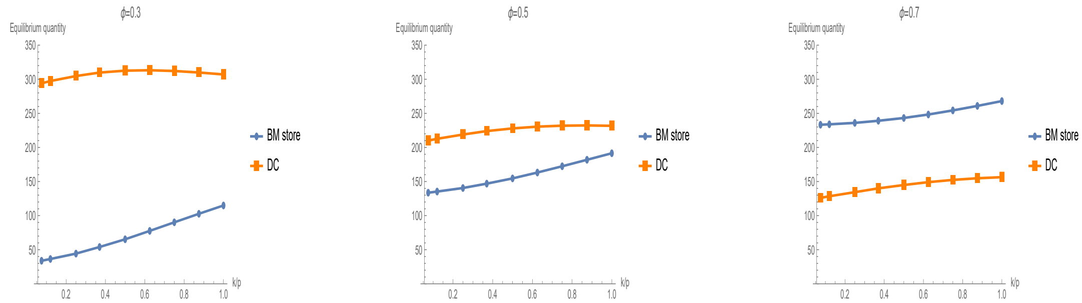
Figure 1 shows the equilibrium orders as a function of the sales-credit allocation ratio k_t/p for three different values of ϕ. We observe that the store's order quantities are increasing in this ratio, whereas the DC's orders are almost constant in k_t/p.

4.2.1 Nash bargaining solution under STS strategy

Suppose that the retailer uses the Nash bargaining solution (NBS) to determine k_t. The rationale for doing so is the fairness property of the NBS, that is, both parties improve their outcomes equally with respect to the status quo point, which gives the payoffs the channels would obtain if there was no agreement. We let the players' benchmark profits to be the status quo.

The determination of equilibrium quantities and the sales credit allocation involves the following two steps:

[1] We recall that when the strategy of each player decreases (increases, independent) in the other player's strategy, then we have strategic substitution (complementarity, no interaction).



(a) Online-dominant market $\phi = 0.3$

(b) Equal fraction of customers $\phi = 0.5$

(c) Store-dominant market $\phi = 0.7$

Figure 1: Comparison of Nash equilibrium order quantities under the STS strategy for various fractions of store-visiting customers

Operational stage: For any sales credit rule k_t set by the retailer, the store and the online channel simultaneously choose their order quantities q_s and q_o by solving the optimization problems

$$\max_{q_s} \Pi_s\left(q_s, q_o, k_t(q_s, q_o)\right),$$
$$\max_{q_o} \Pi_o\left(q_s, q_o, k_t(q_s, q_o)\right).$$

Denote by q_s^t and q_o^t the resulting optimal values.

Credit allocation stage: To determine k_t using the NBS, the retailer solves the following maximization problem:

$$\max_{k_t} \left[\Pi_s^t(k_t, q_s^t, q_o^t) - \Pi_s^b\right] \left[\Pi_o^t(k_t, q_s^t, q_o^t) - \Pi_o^b\right], \tag{12}$$

that is, the product of the two players' incremental profits with respect to the status quo.

Next, the demands are realized and the two channels receive their sales credits.

Proposition 2. *Under the STS fulfillment strategy, if*

$$\frac{2p - c_s + \alpha c_t - \sqrt{(p - c_s + \alpha c_t) - \alpha^2 c_t p}}{p + 2(p - c_s + \alpha c_t) + \alpha^2 c_t} \leq \phi \leq \frac{2p(p - c_s)}{2p(p - c_s) + c_t(p - c_t)\alpha^2},$$

then there exists a unique NBS credit rule

$$k_t^*(q_s^t, q_o^t) = \frac{p(\mathbb{E}\min(D_s, q_s^t) + T_t) - c_s q_s^t - \eta\mathbb{E}[q_s^t - D_s]^+ - \Pi_s^b}{T_t} - \frac{(p - c_o)\mathbb{E}\min(D_o, q_o^t) - wq_o - c_t T_t - \Pi_o^b}{T_t},$$

where $c_t < k_t^*(q_s^t, q_o^t) < p$.

Proof. See Appendix. □

The proposition shows that the NBS exists only under some restrictions on the parameter values, written here in terms of the fraction of store-visiting customers. Using the values in (11), Figure 2 shows the area where the NBS exists for different values of ϕ and c_t. In particular, if ϕ is low enough, then the NBS no longer exists.

Figure 3 shows how the expected profits of the store, online channel, and firm change with the sales-credit allocation rate $\frac{k_t}{p}$ for different fractions of store-visiting customers $\phi \in \{0.3, 0.5, 0.7\}$. We note that the store's profit Π_s^t is decreasing in k_t, while the DC's profit Π_o^t is increasing in k_t. Figure 3 also exhibits the NBS value when $\phi = 0.5$ and 0.7. At the NBS point, the firm's profit is the highest. Note that the NBS does not exist when the fraction of store-visiting customers is low, i.e., $\phi = 0.3$.

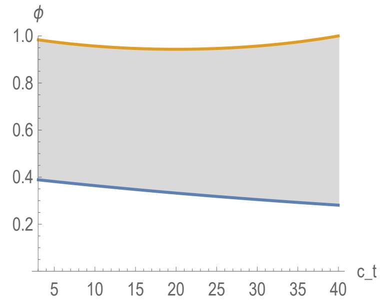
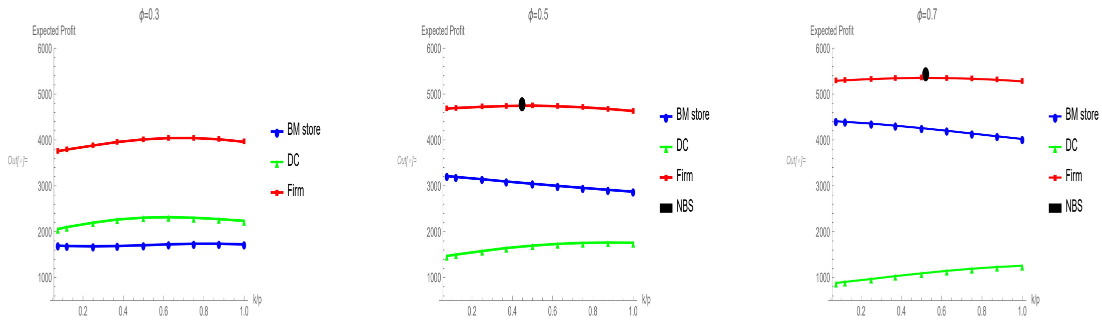


Figure 2: The area where a unique NBS exists under STS (as a function of ϕ and c_t)



(a) Online-dominant market $\phi = 0.3$ **(b) Equal fractions of customers** $\phi = 0.5$ **(c) Store-dominant market** $\phi = 0.7$

Figure 3: Comparison of expected profits under STS strategy for various fractions of store-visiting customers

4.3 Ship-from-Store (SFS)

In an SFS strategy, the BM store can accommodate unmet online orders, after satisfying in-store customers' orders. In this scenario, the store operates as a virtual distribution center and fulfills the online orders by shipping parcels to consumers. For instance, in addition to Walmart and Macy's, recently Zara and GAP, two apparel giants, have converted their physical outlets worldwide to fulfill online orders (Thau, 2018; Yang and Zhang, 2020). Most of the retailers do not deliver themselves, but rely on third party for shipping. Further, when the cost of staff handling SFS orders is factored in, SFS is more expensive than direct delivery from the DC (Reagan, 2017), that is, $c_f > c_o$. To ensure that the BM store is willing to fulfill SFS orders, we also assume that $k_f \in [c_f, p]$, where k_f is the allocation under SFS. The number of orders fulfilled via SFS is defined as

$$T_f = \begin{cases} (D_o - q_o), & \text{if } (q_s - D_s) > (D_o - q_o) > 0, \\ (q_s - D_s), & \text{if } (q_o - D_o) > (D_s - q_s) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The expected profit functions of the store and online channel under SFS are

$$\Pi_s^f(q_s) = p\mathbb{E}\min(D_s, q_s) - c_s q_s + (k_f - c_f)T_f - \eta\mathbb{E}[q_s - D_s - T_f]^+ \tag{13}$$

$$\Pi_o^f(q_o) = (p - c_o)\mathbb{E}\min(D_o, q_o) - wq_o + (p - k_f)T_f \tag{14}$$

The terms $(k_f - c_f)T_f$ and $(p - k_f)T_f$ are the expected profit from the SFS sales for each channel, while the last term in (13) represents one of the benefits of SFS, that is, the lower inventory-holding cost due to leveraging the store's inventory.

Proposition 3. *If $(p - k_f) \leq (p - c_o)$ and $k_f - c_f + \eta \geq 0$, then there exists a unique Nash equilibrium order quantities under STS fulfillment.*

Proof. See Appendix. □

The above proposition shows that the existence of a unique Nash equilibrium requires that (i) the online channel's marginal profit from direct delivery to be higher than the marginal profit from an SFS sale ($p - k_f \leq p - c_o$), and (ii) the store's unit inventory holding cost be higher than the marginal cost of fulfilling an SFS order. The two inequalities can be rewritten in the compact form $k_f \in [\min\{c_o, c_f - \eta\}, p]$.

From the first-order equilibrium conditions, we obtain the players' best response functions, that is,

$$q_s^f(q_o^f) = \frac{(k_f + \eta - c_f)((1-\phi)M - q_o^f)^2 + 2\phi(1-\phi)(p - c_s)M^2}{2((1-\phi)(p+\eta)M)} \tag{15}$$

$$q_o^f(q_s^f) = \frac{2\phi(1-\phi)m_o M^2 - (p - k_f)(q_s^f)^2}{2\phi(p - c_o)M} \tag{16}$$

Their derivatives are given by

$$\frac{dq_s^f(q_o^f)}{dq_o^f} = -\frac{(k_f + \eta - c_f)((1-\phi)M - q_o^f)}{(1-\phi)M(p+\eta)} < 0,$$
$$\frac{dq_o^f(q_s^f)}{dq_s^f} = -\frac{(p - k_f)q_s^f}{\phi M(p - c_o)} < 0,$$

which implies strategic substitution between the two decision variables. As in the STS scenario, each manager's order quantity is decreasing in the other manager's order.

Solving (15)–(16) for some special cases leads to the following observations:

1. If the firm gives full credit to where the order was fulfilled (i.e., store), $k_f = p$, then we can observe that the upper bound of the DC's equilibrium order quantity is equal to the benchmark model's optimal order quantity, $q_o^f = q_o^b$. Otherwise, the DC always orders less than in the benchmark case $q_o^f < q_o^b$. The equilibrium quantities are

$$q_s^f = \phi M \left(\frac{p - c_s}{p + \eta}\right) + \frac{(1-\phi)w^2 M(k + \eta - c_f)}{2(p - c_o)(p+\eta)} > q_s^b, \tag{17}$$

$$q_o^f = q_o^b = (1-\phi)M\left(\frac{m_o}{p - c_o}\right). \tag{18}$$

2. If $k_f = c_f - \eta$, that is, the DC gets the SFS sales credit, then the equilibrium quantities become

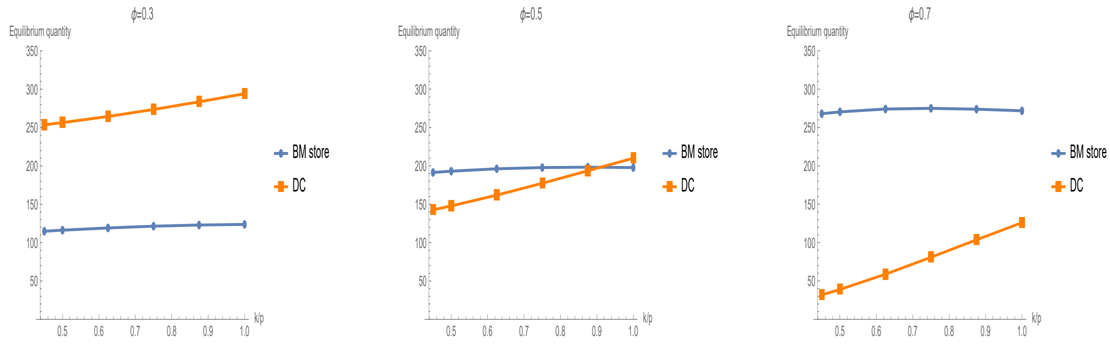
$$q_s^f = q_s^b = \phi M \left(\frac{p - c_s}{p + \eta}\right) \tag{19}$$

$$q_o^f = (1-\phi)M\left(\frac{m_o}{p - c_o}\right) - \frac{\phi M(p + \eta - c_f)(p - c_s)^2}{2(p - c_o)(p+\eta)^2} < q_o^b. \tag{20}$$

   As SFS sales do not yield any marginal profit to the store, it orders the same quantity as in the benchmark. The online channel's order is below the benchmark. The total expected quantity is then lower in an SFS scenario, with $k_f = c_f - \eta$, than in the benchmark.

3. Under an SFS strategy, if $c_f - \eta < k_f < p$, then the BM store (DC) always orders more (less) than the benchmark case.

As illustrated in Figure 4, under an SFS fulfillment strategy, the DC's Nash equilibrium order quantities are increasing in the credit allocation ratio k_f/p, whereas the store's quantities vary little with this ratio.



(a) Online-dominant market $\phi = 0.3$

(b) Equal market share $\phi = 0.5$

(c) Store-dominant market $\phi = 0.7$

Figure 4: Comparison of Nash equilibrium order quantities under SFS strategy for various fractions of store visiting customers

4.3.1 Nash bargaining solution under SFS strategy

Similar to the previous scenario, the NBS allocation rule k_f is given by solving the following optimization problem:

$$\max_{k_f} \left[\Pi_s^f(k_f, q_s^f, q_o^f) - \Pi_s^b\right] \left[\Pi_o^f(k_f, q_s^f, q_o^f) - \Pi_o^b\right].$$

Proposition 4. *Under the SFS fulfillment strategy, if*

$$\frac{(c_f - p)(c_f - \eta)}{(c_f - p)(c_f - \eta) - 2m_o(p - h)} < \phi \leq \frac{m_o + 2(c_f - \eta)}{2m_o + p - c_f + 4(c_f - \eta)} + \frac{\sqrt{(m_o + 2(c_f - \eta))^2 - (c_f - \eta)(2m_o + p - c_f + 4(c_f - \eta))}}{2m_o + p - c_f + 4(c_f - \eta)},$$

then there exists a unique NBS credit rule

$$k_f^*(q_s^f, q_o^f) = \frac{(p - c_o)\mathbb{E}\min(D_o, q_o^f) - wq_o + pT_f - \Pi_o^b}{T_f} - \frac{p\mathbb{E}\min(D_s, q_s^f) - c_s q_s^f - \eta E[q_s^f - D_s - T_f]^+ - c_f T_f - \Pi_s^b}{T_f}$$

Proof. See Appendix. □

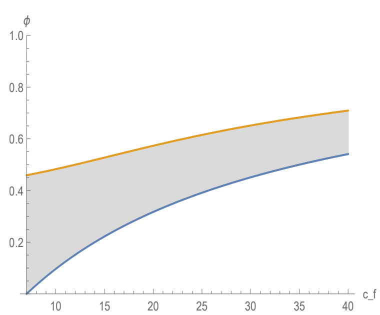
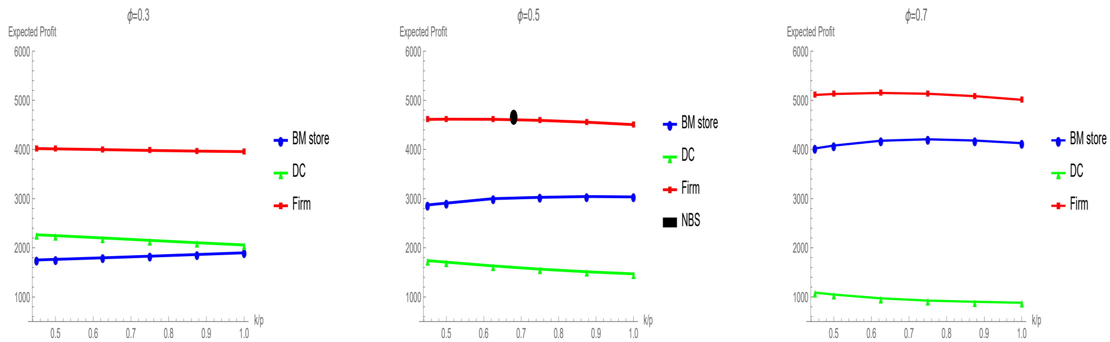


Figure 5: The area where a unique NBS exists under SFS, as a function of the ϕ and operating costs c_f

Figure 5 shows the area where the NBS exists as a function of the fraction of store-visiting customers (ϕ) and the operating costs of SFS fulfillment strategy c_f. We observe that the higher the ϕ, the larger must be the value of c_f for the NBS to exist. Also, no solution exists for too high a value of ϕ.

Figure 6 displays the profits of the DC, store and firm as function of the ratio k_f/p, for three different values of ϕ. Under this SFS strategy, the store's profit Π_s^f increases and the DC's profit Π_o^f decreases as k_f becomes higher. Further, the NBS exists when $\phi = 0,5$, but not if ϕ is equal to 0.3 or $0,7$. The results tend to indicate that the firm would benefit from allocating more credit to the online channel (the demand origin) if ϕ is low (i.e., $\phi = 0.3$), whereas, in a market where store-visiting customers dominate (i.e., $\phi = 0.7$), the firm is better off allocating equal sales credit to both channels.



(a) Online-dominant market $\phi = 0.3$ **(b) Equal market share** $\phi = 0.5$ **(c) Store-dominant market** $\phi = 0.7$

Figure 6: Comparison of expected profits under an SFS strategy for various market share combinations

4.4 Hybrid fulfillment strategy

Consider now a hybrid strategy that consists in implementing both STS and SFS fulfilment options. Recall that the number of orders fulfilled via STF and SFS are defined by $T_t = \mathbb{E}\min[\alpha(D_s - q_s)^+, (q_o - D_o)^+]$ and $T_t = \mathbb{E}\min[\alpha(D_o - q_o)^+, (q_s - D_s)^+]$. To incentivize both channels to fulfill the STS and SFS orders, we assume that $k_h \in [\max\{c_t, c_f\}, p]$, where k_h is the allocation in this scenario. The expected profits for the store and online channel under a hybrid strategy are as follows:

$$\Pi_s^h(q_s) = p\mathbb{E}\min(D_s, q_s) - c_s q_s + (p - k_h)T_t + (k_h - c_f)T_f - \eta\mathbb{E}[q_s - D_s - T_f]^+ \tag{21}$$

$$\Pi_o^h(q_o) = (p - c_o)\mathbb{E}\min(D_o, q_o) - w q_o + (k_h - c_t)T_t + (p - k_h)T_f \tag{22}$$

In (21) and (22), the third and fourth terms represent the expected profit from STS and SFS sales for each channel, respectively.

Proposition 5. *Under the following parameter restrictions,*

$$(p - k_h) \le (k_h + \eta - c_f),$$
$$(p - k_h) \le \alpha(k_h - c_t) \le (p - c_o),$$

there exists a unique Nash equilibrium order quantity for both the store and for the DC.

Proof. See Appendix. □

The first condition in Proposition 5 means that the store gains more from SFS fulfillment than it does from STS fulfillment. The second condition stipulates that the marginal gain of the online channel is higher under STS than under an SFS fulfillment strategy. From the first-order equilibrium conditions, we get the following response functions:

$$q_s^h(q_o) = \frac{2\phi(1-\phi)(p-c_s)M^2 + (k_h+\eta-c_f)(M(1-\phi)-q_o)^2 - (p-k_h)q_o^2}{2M(1-\phi)(p+\eta)},$$
$$q_o^h(q_s) = \frac{2\phi(1-\phi)m_oM^2 + \alpha(k_h-c_t)(M\phi-q_s)^2 - (p-k_h)q_s^2}{2M\phi(p-c_o)}.$$

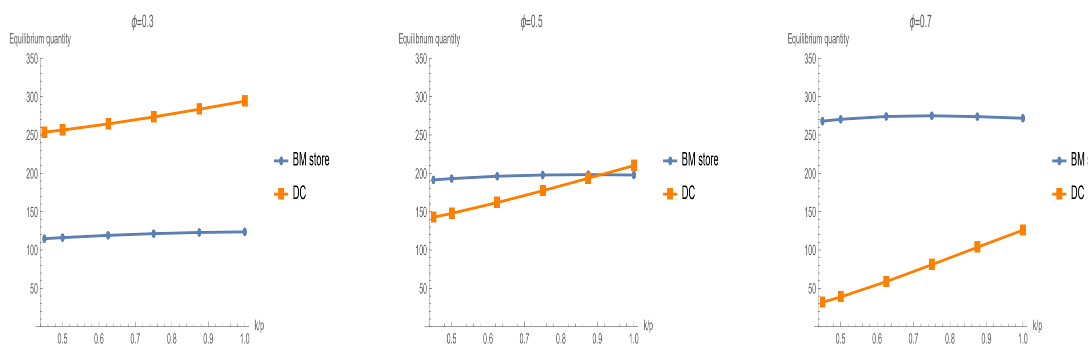
Computing their derivatives, we obtain

$$\frac{dq_s^h(q_o^h)}{dq_o^h} = -\frac{(k+\eta-c_f)((1-\phi)M-q_o^h)+(p-k_h)q_o^h}{(1-\phi)(p+\eta)M} < 0,$$
$$\frac{dq_o^h(q_s^h)}{dq_s^h} = -\frac{\alpha(k-c_t)(\phi M-q_s^h)+(p-k_h)q_s^h}{\phi(p-c_o)M} < 0.$$

As in the two other scenarios, we have strategic substitution between the two decision variables.

If $k = p$, meaning that the channel fulfilling the demand gets full credit, i.e., the store for SFS sales and the online channel for STS sales, then $q_s^h(q_o) = q_s^f(q_o)$ and $q_o^h(q_s) = q_o^t(q_s)$. We conclude that, under a hybrid fulfillment strategy, the store and the online channel always order higher quantities than in the benchmark, that is, $q_s^h > q_s^b$ and $q_o^h > q_o^b$.

As illustrated in Figure 7, under a hybrid fulfillment strategy, the DC's order quantities increase with k_h/p, whereas the store's order quantities vary little with this ratio.



(a) Online-dominant market $\phi = 0.3$ **(b) Equal market share** $\phi = 0.5$ **(c) Store-dominant market** $\phi = 0.7$

Figure 7: Comparison of Nash equilibrium order quantities under hybrid strategy for various fractions of store-visiting customers

4.4.1 Nash bargaining solution under a hybrid strategy

Under a hybrid system, the NBS sales credit is obtained by solving the following optimization problem:

$$\max_{k_h} \left[\Pi_s^h(k_h, q_s^h, q_o^h) - \Pi_s^h\right]\left[\Pi_o^h(k_h, q_s^h, q_o^h) - \Pi_o^b\right].$$

Proposition 6. *Under the hybrid fulfillment strategy, if*

$$\phi \geq \frac{(2p+c_t\alpha-c_s) - \sqrt{(2p+c_t\alpha-c_s)(p+c_t\alpha-c_s) - p(p-c_s+c_t\alpha(1+\alpha))}}{(2p+c_t\alpha-c_s)+(p-c_s+c_t\alpha(1+\alpha))}, \quad (23)$$

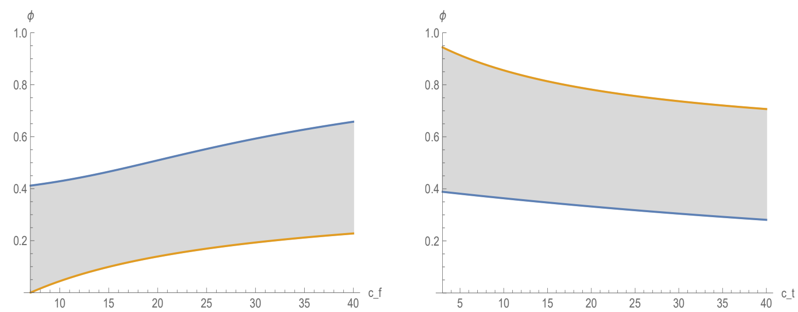
$$\phi < \frac{m_o + 2(c_f-\eta) - \sqrt{(p-c_s-w+2(c_f-\eta))^2 - (c_f-\eta)(2(m_o+c_f-\eta)+p+c_f-\eta)}}{2(p-c_o-w+c_f-\eta)+p+c_f-\eta}, \quad (24)$$

then there exists a unique NBS allocation rule given by

$$k_h^*(q_s^h, q_o^h) = \frac{p\mathbb{E}\min(D_s, q_s^h) - c_s q_s^h - \eta E[q_s^h - D_s - T_f]^+ + pT_t - c_f T_f - \Pi_s^b}{T_t - T_f} - \frac{(p - c_o)\mathbb{E}\min(D_o, q_o^h) - wq_o^h + pT_f - c_f T_f - \Pi_o^b}{T_t - T_f}.$$

Proof. See Appendix. □

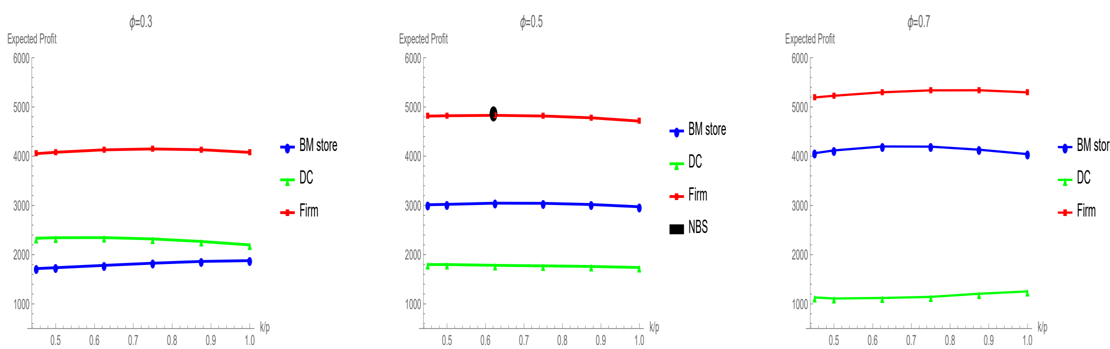
The area where the NBS exists is shown in Figure 8. We note that such a solution does not exist in markets with a very small or very large fraction of store-visiting customers. In such situations, the firm is better off allocating a higher sales credit to the channel fulfilling the request.



(a) c_f **(b)** c_t

Figure 8: The area where a unique NBS exist under SFS, as a function of the ϕ and the operating costs c_f and c_t

As illustrated in Figure 9, under a hybrid fulfillment strategy, the Nash equilibrium order quantities are increasing in the credit allocation k_h/p, albeit only slightly for the BM store.



(a) Online-dominant market $\phi = 0.3$ **(b) Equal market share** $\phi = 0.5$ **(c) Store-dominant market** $\phi = 0.7$

Figure 9: Comparison of expected profits under a hybrid strategy for various fractions of store-visiting customers

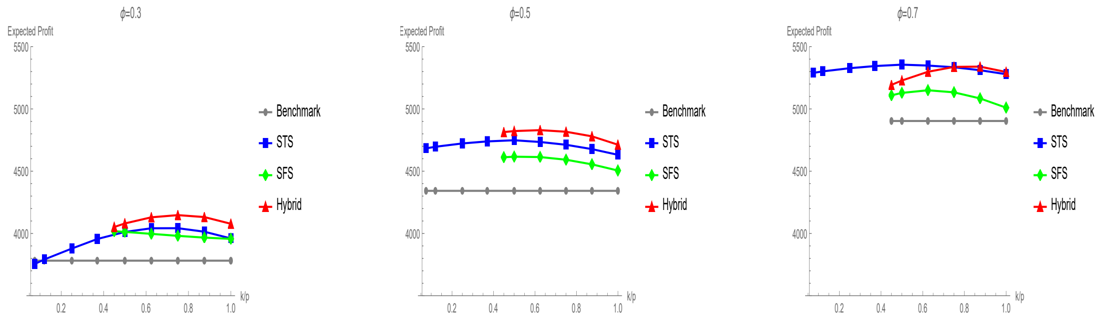
5 Numerical analysis: Performance comparison

To complement our analytical findings, we present a numerical study of the inventory game between channels using the parameter values in (11).

Table 2 compares the equilibrium order quantities and expected profits when the credit allocation is given by the NBS, for $\phi = 0.5$. The total expected profit of the firm from both channels is given by $\Pi_s + \Pi_o$. Figure 10 illustrates how the firm's expected profit changes with respect to the sales credit allocation rate $\frac{k}{p}$ for different fractions of store-visiting customers $\phi \in \{0.3, 0.5, 0.7\}$.

Table 2: Equilibrium quantities, NBS allocation, and profits ($\phi = 0.5$)

Fulfillment	NBS credit allocation	q_s	q_o	Π_s	Π_o	$\Pi_s + \Pi_o$
Benchmark		191.4	210.0	2,872.3	1,470.0	4,342.3
STS	17.9	144.1	243.9	3,083.7	1,681.3	4,765.1
SFS	27.2	220.6	143.0	3,017.7	1,615.4	4,633.1
Hybrid	25.1	181.2	144.8	3,093.3	1,690.9	4,784.2



(a) Online-dominant market $\phi = 0.3$

(b) Equal market share $\phi = 0.5$

(c) Store-dominant market $\phi = 0.7$

Figure 10: Profit comparison of fulfillment strategies under different fractions of store-visiting customers

Our numerical study allows to make the following comments:

1. The fraction of store-visiting customers ϕ plays a critical role in determining the allocation of the cross-channel sales profit, and consequently, whether a retailer will benefit by implementing omnichannel strategies. For instance, if $\phi = 0.5$, then hybrid fulfillment yields the highest total profit. However, it is STS that maximizes profit if $\phi = 0.7$ and the ratio k/p is below a given threshold. Further, Figure 10b shows that, for $\phi = 0.5$, under all fulfillment strategies, the firm is always better off attributing equal sales credit to each channel ($\frac{k}{p} = 0.5$). From Figure 10a, we see that, when direct-shipping customers are dominant in the market, after implementing the STS and hybrid strategies, it is more beneficial for the firm to allocate more credit to where the demand is fulfilled (e.g., to online channel for STS sales). Under hybrid strategy, Figure 10c shows that in a market where store-visiting customers are dominant ($\phi = 0.7$), the firm is better off giving higher sales credit to the channel that fulfills the demand (e.g., to the store for SFS sales).
2. All omnichannel fulfillment strategies lead to a higher total profit than the benchmark. The reason is that the total ordered quantity is highest in the benchmark case, and some items end up remaining unsold.
3. All omnichannel fulfillment strategies are Pareto-optimal with respect to the benchmark. This is a strong argument in favor of implementing an omnichannel approach. Further, a hybrid fulfillment strategy is collectively optimal and Pareto-dominates STS and SFS. Note, however, that the improvement with respect to STS is marginal, at least for the considered parameter constellation.

Recall that direct shipping costs (direct shipping and SFS) are more expensive than store-visiting fulfillment (in-store and STS) costs for the channels. Therefore, store-visiting fulfillment strategies have higher profit margins, and the firm's expected profit in a store-visiting-dominant market is higher.

Result 1. Numerical experiments confirm that with any omnichannel fulfillment strategy, if the proportion of store-visiting and direct-shipping customers is equal, it is beneficial for the firm to attribute equal sales credit $k = \frac{p}{2}$ to each channel.

Remark 2. Under the hybrid fulfillment strategy, whether the fraction of store-visiting customers is higher or lower than that of direct-shipping customers, allocating full credit to where the demand is fulfilled is more beneficial for the omnichannel retailer rather than allocating it to where the demand originates, $k = p$.

Considering Figure 10b, we observe that when the firm's policy is to allocate equal sales credit to use the NBS, then the hybrid fulfillment strategy is the preferred omnichannel strategy for the store, the online channel, and the firm.

The main findings from our numerical analysis are summarized in Figure 11. A firm that would like to implement an STS strategy in a market with higher fraction of direct-shipping customers ($0 < \phi < 0.5$) is unlikely to achieve the NBS. The firm would be better off allocating a higher sales credit to the DC (demand fulfiller). Whereas, if a firm is implementing SFS in the same market combination, allocating more credit to the demand origin leads to a higher profit for the firm. In addition, our results indicate that, after implementing STS and SFS strategies in a market with equal fractions of store-visiting and direct-shipping customers or in a store-visiting-dominant market ($0 \leq \phi < 0.5$), it is beneficial to allocate equal credit to both channels.

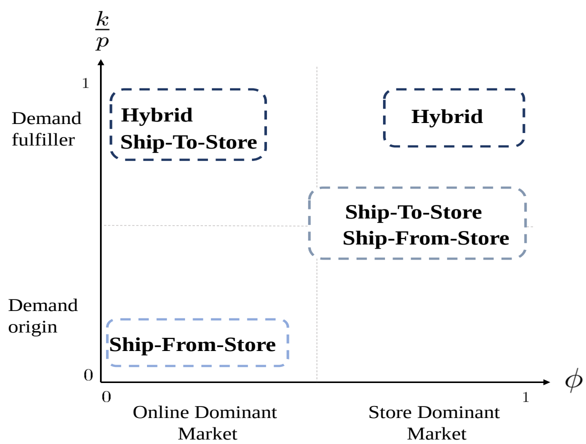


Figure 11: Sales credit allocation policy

6 Conclusion

This paper investigates the problem of designing a proper sales-credit allocation policy in omnichannel retailing. We study the impact of the sales credit allocation policy on the retailer's profit under different omnichannel strategies. We develop an inventory game between the store and the online channel, where each channel is managed by separate and independent teams. We consider four scenarios for fulfilling consumers' orders, that is, a benchmark, ship-to-store, ship-from-store, and hybrid scenarios. We assume that the omnichannel firm first determines on its sale-credit allocation policy, and next, the two channels make their ordering decisions. We characterize the Nash equilibrium of order quantities and find that, under all fulfillment strategies, each channel's order quantity is decreasing in the other's order. For some special cases, we derive bounds on the values of equilibrium order quantities and discuss some insights. Our analysis shows that the fraction of store-visiting customers in the market plays a critical role in determining the allocation of the cross-channel sales profit, and consequently, whether a retailer will benefit from implementing omnichannel strategies. Later we consider the possibility of negotiation between the store and the online channel over the sales credit attribution and investigate a fair sales credit allocation policy across channels. We showed that a fair allocation by NBS is not guaranteed to exist and that it can only be implemented for a narrow range of parameters, such as fulfilling costs.

The results of our numerical analysis indicate that, under all fulfillment strategies, when, market is made up equally of store-visiting and direct-shipping customers, it is more beneficial for the firm to allocate sales credit equally or to use the NBS. Under a hybrid strategy, if the fraction of store-visiting customers is higher or lower than that of direct-shipping customers, in either case, our results suggest that the firm is better off giving higher credit to the channel fulfilling the order. Without taking the market make-up into consideration, under an STS strategy, it is beneficial to allocate equal or higher credit to the demand fulfiller. However, under an SFS strategy, the firm gains in allocating equal or higher credit to the demand origin.

In practice, many companies give annual performance bonuses to employees in various departments of the company (e.g., store operations, e-commerce). The bonus is primarily based on conventional metrics such as the sales and profit generated within the four walls of the store, without considering how that impact extends to other channels. Some sophisticated omnichannel retailers are designing innovative programs for incentivizing store associates. According to the 2016 Customer Experience/Unified Commerce Survey from BRP, here are some of the compensation plans used to reward cross-channel sales: (1) sharing credit for the sale equally across the channels involved; (2) adjusting store labor costs to compensate for orders fulfilled from a store; (3) a commission to the sales associate for all online sales, to the store closest to that customer's location; and (4) hiring separate employees to handle omnichannel fulfillment. The implications of these practices correlate with our results. For example, we verify that, under SFS strategy, when the demand origin (DC) gets credit for the sale, the costs associated to fulfilling SFS orders need to be compensated to the store ($k_f > c_f$). With such policy, the store employees would be encouraged to quickly pick and ship orders that originate from the online channel.

We believe that there are many directions in which to extend this work. Future research may introduce layers of complexity, such as capacity constraints or behavioral effects. Another interesting avenue is to study the inventory game when each channel has asymmetric information on the demand. In addition, in this paper, we only consider credit sharing between a DC and one BM store. It would be interesting to investigate ship-from-store-to-store sales credit allocation between DC and multiple stores (Li, 2020). Finally, empirical analysis could be useful to determine if our results can be confirmed in a much more realistic setting.

A Appendix

Proof for Proposition 1. The expected profit function of the store and the online channel can be rewritten as

$$\begin{aligned}\Pi_s^t(q_s) =&\frac{p}{\phi M}\left(\int_0^{q_s} x dx + \int_{q_s}^{\phi M} q_s dx\right) - c_s q_s \\ &+ \frac{(p-k)}{\phi M(1-\phi)M}\left(\int_0^{q_o}\int_{q_s}^{\frac{(q_o-y)}{\alpha}+q_s} \alpha(x-q_s)dxdy + \int_0^{q_o}\int_{\frac{(q_o-y)}{\alpha}+q_s}^{\phi M}(q_o-y)dxdy\right) \\ &- \frac{\mu}{\phi M}\int_0^{q_s}(q_s-x)dx.\end{aligned}$$

$$\begin{aligned}\Pi_o^t(q_o) =&\frac{p-c_o}{(1-\phi)M}\left(\int_0^{q_o} y dy + \int_{q_o}^{(1-\phi)M} q_o dy\right) - w q_o \\ &+ \frac{k-c_t}{\phi M(1-\phi)M}\left(\int_{q_s}^{\phi M}\int_0^{q_o-\alpha(x-q_s)} \alpha(x-q_s)dydx + \int_{q_s}^{\phi M}\int_{q_o-\alpha(x-q_s)}^{q_o}(q_o-y)dydx\right).\end{aligned}$$

(i) Existence. For $p > c_o$, it is easy to check that $\frac{\partial^2\Pi_s^t}{\partial(q_s)^2} = -\frac{(p+\eta)}{\phi M} < 0$ and $\frac{\partial^2\Pi_o^t}{\partial(q_o)^2} = -\frac{(p-c_o)}{(1-\phi)M} < 0$, which means that Π_s^t is a concave function of q_s for any given q_o and Π_o^t is a concave function of q_o for any given q_s. Therefore, there exists at least one Nash equilibrium in pure strategies.

(ii) Uniqueness. To establish the uniqueness of the Nash equilibrium, it suffices to show that the reaction functions are monotone, and the absolute value of the slope is less than 1 (Cachon and Netessine, 2006). Under the parameter restriction $\alpha(k - c_t) \leq (p - c_o)$, we clearly have

$$\left|\frac{\partial q_s(q_o)}{\partial(q_o)}\right| = \frac{(p-k)}{(p+\eta)}\frac{q_o}{(1-\phi)M} < 1 \quad \text{and} \quad \left|\frac{\partial q_o(q_s)}{\partial(q_s)}\right| = \frac{\alpha(k-c_t)}{(p-c_o)}\frac{(\phi M - q_s)}{\phi M} < 1. \qquad \square$$

Proof for Proposition 2. The store's response functions can be written as

$$q_s^1(q_o) = \frac{-B - \sqrt{B^2 - 4AC}}{C}, \tag{25}$$

$$q_s^2(q_o) = \frac{-B + \sqrt{B^2 - 4AC}}{C}, \tag{26}$$

where

$$\begin{aligned} A &= -(\phi M^2(2(p-c_s)(1-\phi) - c_t\phi\alpha^2) + (2c_t\alpha\phi M - pq_o)q_o), \\ B &= M((p+\eta)(1-\phi) - \alpha^2 c_t\phi) + \alpha c_t q_o, \\ C &= \alpha^2 c_t \end{aligned}$$

- Since $C > 0$, it suffices to prove that $A \leq 0$, then $q_s^1(q_o) \leq 0$ and $q_s^2(q_o) \geq 0$.
- Next, we show that there exist $q_o = q_o^{'}$ such that, if $q_o \leq q_o^{'}$, then $A \leq 0$. By solving $A = 0$ we have

$$q_o^{'} = \frac{\alpha\phi c_t M + \sqrt{(\alpha\phi c_t M)^2 + \phi p M^2(2(1-\phi)(p-c_s) - \phi\alpha^2 c_t)}}{p}$$

  Since q_o is always smaller than the online market share, $(1-\phi)M$, then if we find under which conditions $(1-\phi)M \leq q_o^{'}$ then $q_o(q_s) \leq q_o^{'}$. As long as the market share fraction of store-visiting customers is such that the following condition holds, then, $q_o(q_s) \leq q_o^{'}$.

$$\frac{(2p - c_s + \alpha c_t) - \sqrt{(p - c_s + \alpha c_t)^{-}\alpha^2 c_t p}}{p + 2(p - c_s + \alpha c_t) + \alpha^2 c_t} < \phi < \frac{2p(p-c_s)}{2p(p-c_s) + c_t(p-c_t)\alpha^2}$$

The DC's response functions can be written as

$$q_o^1(q_s) = \frac{-D - \sqrt{D^2 - 4FG}}{G}, \tag{27}$$

$$q_o^2(q_s) = \frac{-D + \sqrt{D^2 - 4FG}}{G}, \tag{28}$$

where

$$\begin{aligned} F &= -\alpha(2M^2\phi(1-\phi)(p-c_o-w) - \alpha c_t(\phi M - q_s)^2) \\ D &= \alpha(pq_s - \phi c_o M) \\ G &= p \end{aligned}$$

- Since $G > 0$, therefore it suffices to prove that $F \leq 0$, then we can show that $q_o^1(q_s) \leq 0$ and $q_o^2(q_s) \geq 0$.
- F is a convex quadratic function of q_s and $F(q_s = 0) < 0$, therefore it is easy to check that if F is also negative in the upper bound of q_s (Equation 7) then, for all values of q_s, $F < 0$. Considering the related parameter restrictions, we can show that, for all values of q_s, $q_o^2(q_s) - q_o^{'} < 0$, which implies that $A \leq 0$ hence, $q_s^1(q_o) \leq 0$ and $q_s^2(q_o) \geq 0$. $\square$

Proof for Proposition 3. Under the SFS fulfillment strategy, the expected profit function of the store and the online channel can be rewritten as

$$\begin{aligned}\Pi_s^f(q_s,q_o) =& p(\frac{1}{\phi M})\left(\int_0^{q_s} x dx + \int_{q_s}^{\phi M} q_s dx\right) - c_s q_s \\ &+ (k-c_f)\left(\frac{1}{\phi M}\right)\left(\frac{1}{(1-\phi)M}\right) \\ &\left(\int_{q_o}^{(1-\phi)M}\int_0^{q_s-(y-q_o)}(y-q_o)dxdy + \int_{q_o}^{(1-\phi)M}\int_{q_s-(y-q_o)}^{q_s}(q_s-x)dxdy\right) \\ &- \mu\left(\frac{1}{\phi M}\right)(\int_0^{q_o}\int_0^{q_s}(q_s-x)dxdy + \int_{q_o}^{(1-\phi)M}\int_0^{q_s-(y-q_o)}((q_s-x)-(y-q_o))dxdy) \\ \Pi_o^f(q_o,q_s) =& (p-c_o)\left(\frac{1}{(1-\phi)M}\right)\left(\int_0^{q_o} y dy + \int_{q_o}^{(1-\phi)M} q_o dy\right) - wq_o \\ &+ (p-k)\left(\frac{1}{\phi M}\right)\left(\frac{1}{(1-\phi)M}\right) \\ &\left(\int_0^{q_s}\int_{q_o}^{(q_o-x)+q_s}(y-q_o)dydx + \int_0^{q_s}\int_{q_o+(q_s-x)}^{(1-\phi)M}(q_s-x)dydx\right)\end{aligned}$$

(i) Existence. For $p > c_o$, it is easy to check that $\frac{\partial^2 \Pi_s^{sfs}(q_s)}{\partial (q_s)^2} = -\frac{p+\eta}{M\phi} < 0$ and $\frac{\partial^2 \Pi_o^{sfs}(q_o)}{\partial (q_o)^2} = -\frac{p-c_o}{M(1-\phi))} < 0$, implying that Π_s^{sfs} is a concave function of q_s for any given q_o and Π_o^{sfs} is a concave function of q_o for any given q_s. Therefore, there exists at least one Nash equilibrium in pure strategies.

(ii) Uniqueness. For $c_o < k < p$, we clearly have

$$\left|\frac{\partial q_s^f(q_o)}{\partial (q_o)}\right| = \frac{(k+\eta-c_{sfs})}{(p+\eta)}\frac{((1-\phi)M-q_o)}{(1-\phi)M} < 1, \tag{29}$$

$$\left|\frac{\partial q_o^f(q_s)}{\partial (q_s)}\right| = \frac{(p-k)}{(p-c_o)}\frac{q_s}{M\phi} < 1. \tag{30}$$

Therefore, the equilibrium is unique. □

Proof for Proposition 4. The BM store's response functions can be written as

$$q_s^1(q_o) = \frac{-B-\sqrt{B^2-4AC}}{C}, \tag{31}$$

$$q_s^2(q_o) = \frac{-B+\sqrt{B^2-4AC}}{C}, \tag{32}$$

where

$$\begin{aligned} A &= -(2\phi(1-\phi)M^2(p-c_s) - (c_f-\eta)(M(1-\phi)-q_o^f)^2), \\ B &= ((p-\eta)q_o^f + M\eta(1-\phi)), \\ C &= p. \end{aligned}$$

Since $B > 0$ and $C > 0$, therefore it suffices to prove that $A \le 0$, then $q_s^1(q_o) \le 0$ and $q_s^2(q_o) \ge 0$.

A is a convex quadratic function of q_o and $A(q_o = 0) < 0$, therefore it is easy to check that if A is also negative in the upper bound of q_o^f, Equation (18), then, for all values of q_o^f, $A \le 0$. Therefore $q_s^2(q_o^f) \ge 0$.

The DC's response functions can be written as

$$q_o^1(q_s) = \frac{-D - \sqrt{D^2 - 4GF}}{G}, \tag{33}$$

$$q_o^2(q_s) = \frac{-D + \sqrt{D^2 - 4GF}}{G}, \tag{34}$$

where

$$\begin{aligned}
F &= -((1-\phi)((2\phi(p-c_o-w)-(c_f-\eta)(1-\phi))M^2 + 2(c_f-\eta)q_s^f M) - p(q_s^f)^2),\\
D &= M((c_f-\eta)(1-\phi)-(p-c_o)\phi) - (q_s^f(c_f-\eta),\\
G &= (c_f-\eta).
\end{aligned}$$

If $G > 0$ and $F \leq 0$, then $q_o^1(q_s) \leq 0$ and $q_o^2(q_s) \geq 0$. This implies the existence of a unique response function that leads to a unique Nash equilibrium.

F is a convex function of q_s^f. We can show that if $\phi > \frac{c_f-\eta}{2(p-c_o-w)+(c_f-\eta)}$, then $F(q_s^f = 0) < 0$. In addition, we can show that there exists $(q_s^f)' > 0$ such that, if $q_s < (q_s^f)'$, then $F < 0$. By solving $F = 0$, we find that if $\phi > \frac{(c_f-p)(c_f-\eta)}{(c_f-p)(c_f-\eta)-2(p-c_o-w)(p-h)}$, then $(q_s^f)' > 0$.

$$(q_s^f)' = \frac{-(c_{sfs}-\eta)(1-\phi)+\sqrt{((c_{sfs}-\eta)(1-\phi))^2 + p(1-\phi)(M^2(2(p-c_o-w)\phi-(c_{sfs}-\eta)(1-\phi))}}{p}.$$

Since q_s^f is always smaller than the store-visiting market share, ϕM, then if we find under which conditions $\phi M \leq (q_s^f)'$, then $q_s^f < (q_s^f)'$. We show that as long as the fraction of store-visiting customers is such that the following condition holds, then $q_s^f < (q_s^f)'$.

$$\phi > \frac{(c_f-p)(c_f-\eta)}{(c_f-p)(c_f-\eta)-2(p-c_o-w)(p-h)}$$

$$\phi \leq$$

$$\frac{(p-c_o-w)+2(c_f-\eta)+\sqrt{((p-c_o-w)+2(c_f-\eta))^2-(c_f-\eta)(2(p-c_o-w)+(p-c_f)+4(c_f-\eta))}}{2(p-c_o-w)+(p-c_f)+4(c_f-\eta)}$$

□

Proof for Proposition 5.

$$\begin{aligned}
\Pi_s^h(q_s,q_o) =& p(\frac{1}{\phi M})\left(\int_0^{q_s} x dx + \int_{q_s}^{\phi M} q_s dx\right) - c_s q_s\\
&+ (p-k)\left(\frac{1}{\phi M}\right)\left(\frac{1}{(1-\phi)M}\right)\\
&\left(\int_0^{q_o}\int_{q_s}^{\frac{(q_o-y)}{\alpha}+q_s} \alpha(x-q_s)dxdy + \int_0^{q_o}\int_{\frac{(q_o-y)}{\alpha}+q_s}^{\phi M}(q_o-y)dxdy\right)\\
&+ (k-c_f)\left(\frac{1}{\phi M}\right)\left(\frac{1}{(1-\phi)M}\right)\\
&\left(\int_{q_o}^{(1-\phi)M}\int_0^{q_s-(y-q_o)}(y-q_o)dxdy + \int_{q_o}^{(1-\phi)M}\int_{q_s-(y-q_o)}^{q_s}(q_s-x)dxdy\right)\\
&- \mu\left(\frac{1}{\phi M}\right)\left(\int_0^{q_o}\int_0^{q_s}(q_s-x)dxdy + \int_{q_o}^{(1-\phi)M}\int_0^{q_s-(y-q_o)}((q_s-x)-(y-q_o))dxdy\right).
\end{aligned}$$

$$\begin{aligned}\Pi_o^h(q_o,q_s) =&(p-c_o)\left(\frac{1}{(1-\phi)M}\right)\left(\int_0^{q_o} ydy+\int_{q_o}^{(1-\phi)M} q_o dy\right)-wq_o\\
&+(p-k)\left(\frac{1}{\phi M}\right)\left(\frac{1}{(1-\phi)M}\right)\\
&\left(\int_0^{q_s}\int_{q_o}^{(q_o-x)+q_s}(y-q_o)dydx+\int_0^{q_s}\int_{q_o+(q_s-x)}^{(1-\phi)M}(q_s-x)dydx\right)\\
&+(k-c_t)\left(\frac{1}{\phi M}\right)\left(\frac{1}{(1-\phi)M}\right)\\
&\left(\int_{q_s}^{\phi M}\int_0^{q_o-\alpha(x-q_s)}\alpha(x-q_s)dydx+\int_{q_s}^{\phi M}\int_{q_o-\alpha(x-q_s)}^{q_o}(q_o-y)dydx\right).\end{aligned}$$

(i) Existence. For $p>c_o$, it is easy to check that $\frac{\partial^2\Pi_s^h(q_s)}{\partial(q_s)^2}=-\frac{p+\eta}{M\phi}<0$ and $\frac{\partial^2\Pi_o^h(q_o)}{\partial(q_o)^2}=-\frac{p-c_o}{M(1-\phi))}<0$, implying that Π_s^h is a concave function of q_s for each given q_o and Π_o^h is a concave function of q_o for each given q_s. Therefore, there exists at least one Nash equilibrium in pure strategies.

(ii) Uniqueness. We need to show that

$$\left|\frac{\partial q_s^h(q_o)}{\partial(q_o)}\right|=\frac{(k+\eta-c_f)}{(p+\eta)}-\frac{(k+\eta-c_f)-(p-k)}{(p+\eta)}\frac{q_o}{M(1-\phi)}<1, \tag{35}$$

$$\left|\frac{\partial q_o^h(q_s)}{\partial(q_s)}\right|=\frac{\alpha(k-c_t)}{(p-c_o)}-\frac{\alpha(k-c_t)-(p-k)}{(p-c_o)}\frac{q_s}{M\phi}<1. \tag{36}$$

Under the restrictions $p\geq k$ and $p>c_f$, we can conclude that, if $(k+\eta-c_f)\geq(p-k)$, then the inequality in (35) holds true. The inequality in (36) follows from $(p-k)\leq\alpha(k-c_t)\leq(p-c_o)$. □

Proof for Proposition 6. The BM store's response functions can be written as

$$q_s^1(q_o)=\frac{-B-\sqrt{B^2-4AC}}{C}, \tag{37}$$

$$q_s^2(q_o)=\frac{-B+\sqrt{B^2-4AC}}{C}, \tag{38}$$

where

$$\begin{aligned}A=&-(2(1-\phi)\phi(p-c_s)-(c_f-\eta)(1-\phi)^2-c_t\alpha^2\phi^2)M^2,\\
&(c_t\alpha\phi+(c_f-\eta)(1-\phi))-2Mq_o+(p+c_f-\eta)q_o^2\\
B=&M(\eta(1-\phi)-\phi\alpha^2c_t)+q_o(p+\alpha c_t)>0,\\
C=&p+\alpha^2c_t.\end{aligned}$$

- Since $B>0$ and $C>0$, therefore it suffices to prove that $A\leq 0$, then $q_s^1(q_o)\leq 0$ and $q_s^2(q_o)\geq 0$.
- A is a convex quadratic function of q_o^h and $A(q_o^h=0)<0$, therefore we can show that if

$$\phi\geq\frac{(p-c_s)+(c_f-\eta)-\sqrt{(p-c_s)^2+\alpha^2c_t(\eta-c_f)}}{(p-c_s)+(c_f-\eta)+(p-c_s)+\alpha^2c_t},$$

then, there exist $(q_o^h)'\geq 0$ such that if $q_o\leq(q_o^h)'$, then $A\leq 0$. By solving $A=0$ we have

$$(q_o^h)'=\frac{M((c_f-\eta)(1-\phi)-c_t\alpha\phi)}{(p+c_f-\eta)}$$
$$+\frac{\sqrt{M^2((c_f-\eta)(1-\phi)-c_t\alpha\phi)^2+M^2(p+c_f-\eta)(2(p-c_s)\phi(1-\phi)-(c_f-\eta)(1-\phi)^2-c_t\alpha^2\phi^2)}}{(p+c_f-\eta)}>0.$$

Since q_o^h is always smaller than the online market share, $(1-\phi)M$, then if we find under which conditions $(1-\phi)M \leq (q_o^h)'$ then $q_o^h \leq q_o^{'}$. We find that, as long as the fraction of store-visiting customers is such that the following condition holds, then, $q_o(q_s) \leq (q_o^h)'$. This implies that $A \leq 0$, and, $q_s^1(q_o) \leq 0$ and $q_s^2(q_o) \geq 0$.

$$\phi \geq \frac{(2p + c_t\alpha - c_s) - \sqrt{(2p + c_t\alpha - c_s)(p + c_t\alpha - c_s) - p(p - c_s + c_t\alpha(1+\alpha))}}{(2p + c_t\alpha - c_s) + (p - c_s + c_t\alpha(1+\alpha))}.$$

The DC's response functions can be written as

$$q_o^1(q_s) = \frac{-D - \sqrt{D^2 - 4FG}}{G}, \tag{39}$$

$$q_o^2(q_s) = \frac{-D + \sqrt{D^2 - 4FG}}{G}, \tag{40}$$

where

$$\begin{aligned}
F =& - (M^2\alpha((1-\phi)(2(p - c_o - w)\phi - (c_{sfs} - \eta)(1-\phi)) - c_{sts}\alpha\phi^2) \\
&+ 2q_sM\alpha((c_{sfs} - \eta)(1-\phi) + \phi c_{sts}\alpha) - \alpha(c_{sts}\alpha + p)q_s^2), \\
D =& - \alpha(M((1-\phi)(c_{sfs} - \eta) + c_o\phi) - (p + c_{sfs} - \eta)q_s), \\
G =& (p + \alpha(c_{sfs} - \eta)).
\end{aligned}$$

- Since $G > 0$, it suffices to prove that $F \leq 0$, then $q_o^1(q_s) \leq 0$ and $q_o^2(q_s) \geq 0$, which implies the existence of a unique response function and a unique Nash equilibrium.
- F is a convex quadratic function of q_s^h and $A(q_s^h = 0) < 0$, therefore we can show that if the following condition holds then there exist $q_s^{'} \geq 0$ such that if $q_s^h \leq q_s^{'}$, then $F \leq 0$. By solving $F = 0$ we have

$$(q_s^h)' = \frac{-M(c_f - \eta)(1-\phi)}{p + \alpha c_f}$$

$$+ \frac{\sqrt{M^2((c_f - \eta)(1-\phi))^2 + M^2(p + \alpha c_f)(2(p - c_o - w)(1-\phi)\phi - (c_f - \eta)(1-\phi)^2 - c_t\alpha\phi^2)}}{p + \alpha c_f} > 0$$

- We find that if

$$\phi < \frac{(p - c_o - w) + 2(c_f - \eta)}{2(p - c_o - w + c_f - \eta) + p + c_f - \eta}$$

$$- \frac{\sqrt{(p - c_s - w + 2(c_f - \eta))^2 - (c_f - \eta)(2(p - c_o - w + c_f - \eta) + p + c_f - \eta)}}{2(p - c_o - w + c_f - \eta) + p + c_f - \eta}$$

then for all values of q_o^h, $q_s^2(q_o) - (q_s^h)^{'} < 0$ (decreasing in q_o), which implies that $F \leq 0$ hence, $q_o^1(q_s) \leq 0$ and $q_o^2(q_s) \geq 0$. □

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