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B. Crettez, N. Hayek, G. Martín-Herrán

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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Capacity games in Cournot's duopoly model of complements

Bertrand Crettez [a]

Naila Hayek [a]

Guiomar Martín-Herrán [c, b]

[a] *Université Panthéon-Assas, Paris II, CRED, EA 7321, 75005 Paris, France*

[b] *GERAD, Montréal (Qc), Canada, H3T 1J4*

[c] *IMUVA and Departamento Matemática Aplicada, Universidad de Valladolid, 47011 Valladolid, Spain*

bertrand.crettez@u-paris2.fr
naila.hayek@u-paris2.fr
guiomar.martin-herran@uva.es

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Abstract : We analyze a capital accumulation game in a dynamic version of Cournot duopoly model of complements. In this game, firms' instant profits are discontinuous and therefore one cannot use standard optimal control approaches to study dynamic equilibria. We find that in contrast with Reynolds' (1987) seminal analysis of capacity games in Cournot quantity duopoly model, an open-loop Nash equilibrium generally does not exist. Existence of equilibrium only holds when firms cannot disinvest and have the same initial production capacity. These conclusions diverge from the finding that a Nash equilibrium always exists in the static version of the Cournot model of complement with capacity constraints.

Keywords: Cournot model of complements, open-loop Nash equilibrium

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1 Introduction

While capital accumulation games have received a certain attention for Cournot quantity games (see, e.g., Lambertini (2018), 3.3), to the best of our knowledge, they have not been considered in the second Cournot oligopoly model, namely Cournot model of complements. This papers aims at filing this gap.

In the Cournot model of complements, consumers have a downward-sloping demand for a final product which is made out of n different components, each of which is made by a monopoly supplier. These n components are perfect complements in the sense that each of them is necessary to make the product (one unit of the final good requiring one unit of each of the complement goods). Therefore, when consumers demand one unit of the final good, they actually demand one unit of the complement goods and as a consequence the price of this final good is the sum of the prices of the complement goods.[1]

The Cournot's complementary monopoly model has been used to address issues like corruption in government services, patents and innovation policy, merger theory, competition policy, and in the economics of property rights (a quick review of these applications can be found in Amir and Gama (2019)). Amir and Gama (*ibid*) present a general approach of the model and tackle the existence and uniqueness of equilibrium, the effects of an increase in the number of complements, as well as the interesting property that integrating the n-different monopoly suppliers into a super-monopoly would be Pareto-improving (prices would be lower, and profits higher).[2]

Attempts at analyzing Cournot's complementary model from a dynamic view point are scarce. Casadesus-Masanell and Yoffie (2007) study the dynamics of competition between two complementary firms, the complementors, dubbed as Intel and Microsoft, which both make a final product, the PC. They show that when the two firms have different decision horizons, natural conflicts occur over pricing and the size of initial investments in complementary R&D. Laussel and Van Long (2012) analyze a version of the model where a monopolistic downstream firm (an assembler) looks for the best way to separate from its upstream subsidiaries across time.

This paper contributes to the dynamic analysis of the Cournot model of complements by focusing on capacity accumulation across time.[3] It also contributes to the literature on capacity games.[4] In these games, each firm incurs costly investments to increase capacity. Further, capacity and output are the same (the rate of capacity utilization is always 100%). Firms produce and sell more under closed-loop rules than under open-loop ones: that is, the richer the information used in devising accumulation strategy, the larger the investment and production (see Reynolds (1987, 1991), and Lambertini, 2018).

We concentrate on the duopoly case, where firms face a linear-demand curve and production is costless. Such setting is close to that used by Reynolds (1987). In both cases, the models are linear-quadratic differential games. Yet our model displays a major difference with Reynolds' in that firms' instant profits are discontinuous. This discontinuity stems from the assumption that firms' products are perfect complements. Simply said, it does not pay to produce more than other producers. Moreover, when firms have the capital stocks, any of them can increase its profit by slightly decreasing its capacity (that brings about a rise in the firm's price, that more than compensates the drop in its production). Such discontinuity, however, leads to a major technical difficulty in that one cannot apply standard optimal control approaches to study the dynamic equilibria.

The discontinuity in firms' instant profits explains why we obtain different results from Reynolds (1987). We first show that capacity games for complementors do not have open-loop equilibria gen-

[1] Sonnenschein (1968) made an interesting connection between the two Cournot models. Under certain conditions, notably that production is costless, the equilibrium quantities in one model are equal to the equilibrium prices in the other model.

[2] Existence of the equilibrium is also addressed in Babaioff et al. (2017) who focus on a discretized version of the model in which demand changes only finitely many times.

[3] Casadesus-Masanell and Yoffie (2007) only consider dynamic price competition. Investment is indeed realized at the initial date.

[4] We here concentrate on what Lambertini (2018) calls the Solow-Swan game.

erally. This situation arises whether firms can disinvest or not, under the symmetry assumption that firms can either both disinvest, or else that they are both unable to do so. More precisely, when disinvestment is not possible, open-loop equilibria only exist when the initial values of the complementors' capital are identical.

The general result that open-loop equilibria generally do not exist in our dynamic version of the Cournot dupoly model of complements also holds when only one firm can disinvest. That is, the non-existence result does not depend on the assumption that firms are symmetric with regard to the possibility of disinvestment.

These results illustrate the interest of a dynamic approach to the Cournot model of complements. Firstly, the differences between Reynolds' results and ours are another instance where the duality between the quantity Cournot duopoly and the Cournot model of complements breaks down (see Amir and Gama, 2019, for an analysis of the conditions under which the duality breaks down in a static setting).[5] Secondly, the fact that the static Cournot model of complements model with capacity constraints has an equilibrium whereas the dynamic does not have an open-loop Nash equilibrium illustrates the stark difference between the two approaches.

The remaining of the paper unfolds as follows. In the next section we present the static version of the duopoly Cournot model of complements with capacity constraints (to the best of our knowledge, the study of this model is new). We also lay out the dynamic version of this duopoly. Section 3 studies the existence of open-loop equilibrium when firms can disinvest. Section 4 addresses the existence issue when investment is irreversible. Section 5 concentrates on the case where only one firm can disinvest. Section 6 offers some concluding remarks.

2 Model

2.1 Static price game with capacity constraints

We consider a duopoly with perfect complements where direct demand $D(z)$ is given by $D(z) = \frac{a-z}{b}$, with $a > 0$, $b > 0$, and where z is the addition of the individual prices, that is, $z = p_1 + p_2$. There is zero production cost but firms face capacity constraints as in Reynolds (1987). That is, firm i solves

$$\max_{p_i} \quad p_i \left(\frac{a - (p_i + p_j)}{b} \right) \tag{1}$$

$$\text{s.t.:} \quad \frac{a - (p_i + p_j)}{b} \leq K_i \tag{2}$$

where p_j is the price of the rival's firm and K_i denotes firm i's capacity.

Proposition 1.

The equilibrium values of firm i's profit $R^i(K_i, K_j)$ and the Nash equilibria of the static price game are as follows.

1. *If $K_i < \frac{a}{3b}$, $K_i < K_j$, then $R^i(K_i, K_j) = (a - 2bK_i)K_i$, $p_i = a - 2bK_i$, $p_j = bK_i$.*
2. *If $K_j < \frac{a}{3b}$, $K_j < K_i$, then $R^i(K_i, K_j) = bK_j^2$, $p_i = bK_j$, $p_j = a - 2bK_j$.*
3. *If $K_i = K_j = K < \frac{a}{3b}$, then $R^i(K, K) = \frac{a-bK}{2}K$, $p_i = p_j = \frac{a-bK}{2}$.*
4. *If $\frac{a}{3b} \leq \min\{K_i, K_j\}$, then $R^i(K_i, K_j) = \frac{a^2}{9b}$, $p_i = p_j = \frac{a}{3}$.*

Proof. See the appendix. □

[5] Actually, as we shall see in the following section, the duality breaks down in the static games where firms have capacity constraints, even when production costs are nil.

Observe that the equilibria of the duopoly with perfect complements and capacity constraints are different from those of the standard Cournot duopoly with capacity constraints. Indeed, an important feature of the Cournot duopoly with perfect complements is that producers sell the *same* quantity. An implication of this feature is that there are always unused production capacities if they are different. It also follows straightforwardly that the duality identified by Sonnenschein (1968) between the two Cournot oligopoly models breaks down when firms have capacity constraints (even though they have zero production costs).[6]

Notice that when $K_i = K_j < \frac{a}{3b}$, there are actually many Nash equilibria (p_1, p_2), all of which satisfy the condition $a-(p_i+p_j) = bK$. In particular, the two following pairs of strategies are included in the Nash equilibria: $(p_1, p_2) = (a-2bK, bK)$ and $(p_1, p_2) = (bK, a-2bK)$. Notice, however, that these strategies are not symmetric, whereas firms' capacities are equal. Notice also that the instant profit functions are not continuous on the set of capital stocks K_i and K_j that satisfy $K_i = K_j < \frac{a}{3b}$.

2.2 Dynamic model

We now consider a dynamic version of Cournot model of complements where firms accumulate capital and at each date compete by setting prices simultaneously. More precisely, at each date the result of price competition is the Nash equilibrium studied in the preceding subsection and at each date, firms also make their decisions on their investment rate in order to maximize the discounted sum of instant profits.

From the preceding subsection, firm i's instant revenues at any date read

$$R^i(K_i, K_j) = \begin{cases} (a-2bK_i)K_i, \ K_i \le \frac{a}{3b}, \ K_i<K_j, \\ bK_j^2, \ K_j \le \frac{a}{3b}, \ K_j<K_i, \\ \frac{a-bK_i}{2}K_i, \ K_i = K_j < \frac{a}{3b}, \\ \frac{a^2}{9b}, \ \frac{a}{3b} \le \min\{K_i, K_j\}. \end{cases} \tag{3}$$

Interestingly, firm i's profit function does not depend upon its capital stock K_i when $K_j \le K_i$ and is concave respect to K_i when $K_i \le K_j$.

The dynamic problem faced by each firm is as follows

$$\begin{aligned} \max_{I_i(\cdot)} \quad & \int_0^\infty e^{-rt}\left[R^i(K_i(t), K_j(t)) - C(I_i(t))\right] dt \\ \text{s.t.:} \quad & \dot{K}_i(t) = I_i(t) - \delta K_i(t), \ K_i(0) = K_{i0}, \ K_{i0} \le \frac{a}{3b}, \ i = 1, 2, \end{aligned}$$

where

$$C(I_i) = \alpha I_i + \beta \frac{I_i^2}{2}, \ \alpha > 0, \ \beta > 0.$$

In this problem, firm i maximizes the flows of discounted instant profits $R^i(K_i, K_j) - C(I_i)$ with respect to investment $I_i(t)$. Instant profit at any date t is the difference between the firm's revenues $R_i(K_i(t), K_j(t))$ and instantaneous investment costs C_i (we continue to suppose that marginal production cost is zero). These costs are convex increasing in the investment rate and are identical for both firms. Capital depreciates at the instantaneous rate δ. The discount factor is also the same for both firms and is equal to r, with r strictly positive. We shall, furthermore, assume that $\alpha < \frac{a}{r+\delta}$. This assumption guarantees that there is a positive steady-state capacity if there is an equilibrium. This assumption means that the maximum market price (a) is large enough to finance small investment expenditures. Otherwise, there is, of course, no point in building a long-term capacity.

[6] This conclusion complements that of Amir and Gama (2019), according to whom the duality between the two Cournot models no longer holds when firms have production costs.

We consider two versions of the problem above. In the first version, as in Reynolds (1987), firms can disinvest (capital, however, is always non-negative). The problem is then similar to that considered by Reynolds, with the major exception that the revenue function is not continuous. The second version also departs from Reynolds' setting by assuming that investment is irreversible. Formally, we add the constraint $I(t) \geq 0$ to the problem above.

We next study open-loop Nash equilibria when firms can disinvest.

3 Open-loop Nash equilibrium when investment is reversible

In this section, we suppose that there is no sign condition on $I(t)$, but we must make sure that $K_i(t) \geq 0$ for all t. The major result of this section is as follows.

Theorem 1. *There is no open-loop Nash equilibrium when investment is reversible.*

Notice that this Theorem is in stark contrast with Reynolds (1987)' result where both open-loop and feedback Nash equilibria exist (under some mild conditions, however). The Theorem will be deduced from other results which are of independent interest. The two first of these results (Propositions 2 and 3) assert that firms' capacities cannot be equal in equilibrium. The other Propositions (Propositions 4 and 5) deal with the cases where firms' capacities are always ordered in the same way, and where the capacity paths double-cross at least once.

Proposition 2. *Assume that the initial capital stocks of the two firms are identical and such that $K_{i0} = K_{j0} < \frac{a}{3b}$. Then there is no symmetric open-loop Nash equilibrium such that $K_h(t) > 0$ on an interval $[\underline{t}, \bar{t}]$, with $\underline{t} < \bar{t}$, and $K_h(t) < \frac{a}{3b}$, $h = i, j$.*

Proof. By way of contradiction, suppose that there is an open-loop equilibrium as described in the statement of the Proposition. Denote by I_h^* the equilibrium decision of firm h, $h = i, j$, and by K_h^* the corresponding equilibrium time-path for the capital. Without loss of generality, we can assume that $\underline{t} = 0$.[7] By assumption, the equilibrium time-path for firm j's capacity K_j^* satisfies $K_j^*(t) > 0$ for all $t \in [0, \bar{t}]$. Consider the following deviation strategy for firm i. Let $\epsilon > 0$ and $t_\epsilon > 0$ with $t_\epsilon < \bar{t}$ such that for all t in $[0, t_\epsilon]$, $I_i^\epsilon(t) = I_i^*(t) - \epsilon = I_j^*(t) - \epsilon$ (since by assumption $I_i^* = I_j^*$), and $I_i^\epsilon(t) = I_j^*(t)$ elsewhere. Thus $K_i^\epsilon(t) = \int_0^t e^{\delta(s-t)}(I_j^*(s) - \epsilon)ds + e^{-\delta t}K_j(0) > 0$.[8] Recall that for $t \geq t_\epsilon$, $\dot{K}_i^\epsilon(t) = I_j^*(t) - \delta K_i^\epsilon(t)$. Then for $t \leq t_\epsilon$ we have

$$
\begin{aligned}
K_i^\epsilon(t) = & \int_0^t e^{\delta(s-t)} I_i^\epsilon(s)ds + e^{-\delta t}K_{j0} = \int_0^t e^{\delta(s-t)} I_j^*(s)ds - \int_0^t \epsilon e^{\delta(s-t)}ds + e^{-\delta t}K_{j0} \\
= & K_j^*(t) - \epsilon \frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1].
\end{aligned}
$$

And for $t \geq t_\epsilon$ it holds that

$$
\begin{aligned}
K_i^\epsilon(t) = & \int_0^t e^{\delta(s-t)} I_i^\epsilon(s)ds + e^{-\delta t}K_{j0} = \int_0^t e^{\delta(s-t)} I_j^*(s)ds - \int_0^{t_\epsilon} \epsilon e^{\delta(s-t)}ds + e^{-\delta t}K_{j0} \\
= & K_j^*(t) - \epsilon \frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1].
\end{aligned}
$$

Now, since $K_i^\epsilon(t) < K_j^*(t)$ for all t, it follows that firm i's revenue is written $(a - 2bK_i^\epsilon(t))K_i^\epsilon(t)$ instead of $\frac{(a - bK_j^*(t))}{2}K_j^*(t)$ when $I_i^*(t) = I_j^*(t)$ for all t. Let us now compare the profit obtained by firm i when $\epsilon > 0$ with its value when $\epsilon = 0$ (in that last case, both firms use the same strategy). Deviating from firm j's strategy is profitable whenever

$$
\int_0^\infty \left((a - 2bK_i^\epsilon(t))K_i^\epsilon(t) - \frac{(a - bK_j^*(t))}{2} K_j^*(t) \right) e^{-rt}dt
$$

[7] That is because, in a symmetric equilibrium we would have $K_i^*(\underline{t}) = K_j^*(\underline{t})$ and we can focus on the dynamics from date $\underline{t}$ on.

[8] The fact that $K_j^*(t) > 0$ on $[0, t_\epsilon]$ allows us to choose such an ϵ.

$$-\int_0^{t_\epsilon}\left(\alpha I_i^\epsilon(t)+\frac{\beta}{2}(I_i^\epsilon(t))^2-\alpha I_i^*(t)-\frac{\beta}{2}(I_i^*(t))^2\right)e^{-rt}dt>0$$

for some $\epsilon > 0$.

Let us then compute $aK_i^\epsilon(t) - b(K_i^\epsilon(t))^2$. For $t \le t_\epsilon$, we have:

$$\begin{aligned} aK_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 =& aK_j^*(t) - a\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] \\ & - 2b\left[(K_j^*(t))^2 - 2\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] + \epsilon^2\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t} - 1]^2\right] \end{aligned}$$

and for $t \ge t_\epsilon$ we get

$$\begin{aligned} aK_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 =& aK_j^*(t) - a\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1] \\ & - 2b\left[(K_j^*(t))^2 - 2\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{t_\epsilon} - 1] + \epsilon^2\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t_\epsilon} - 1]^2\right]. \end{aligned}$$

For $t \le t_\epsilon$, we then get

$$\begin{aligned} aK_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 - \frac{(a - bK_j^*(t))}{2}K_j^*(t) =& \frac{a}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2 - a\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] \\ & + 4b\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] - 2b\epsilon^2\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t} - 1]^2, \end{aligned}$$

whereas for $t \ge t_\epsilon$ we have

$$\begin{aligned} aK_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 - \frac{(a - bK_j^*(t))}{2}K_j^*(t) =& \frac{a}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2 - a\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1] \\ & + 4b\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1] - 2b\epsilon^2\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t_\epsilon} - 1]^2. \end{aligned}$$

Now, observe that for all t, $t \le t_\epsilon$, it holds that

$$\begin{aligned} \alpha I_i^\epsilon(t) + \frac{\beta}{2}(I_i^\epsilon(t))^2 - \left(\alpha I_j^*(t) + \frac{\beta}{2}(I_j^*(t))^2\right) &= \alpha(I_j^*(t) - \epsilon) + \frac{\beta}{2}(I_j^*(t) - \epsilon)^2 - \left(\alpha I_j^*(t) + \frac{\beta}{2}(I_j^*(t))^2\right) \\ &= -\alpha\epsilon + \frac{\beta}{2}\left(-2\epsilon I_j^*(t) + \epsilon^2\right). \end{aligned}$$

Now, using the above results, we obtain

$$\begin{aligned} &\int_0^\infty [(a - 2bK_i^\epsilon(t))K_i^\epsilon(t)) - \frac{(a - bK_j^*(t))}{2}K_j^*(t)]e^{-rt}dt \\ &\quad - \int_0^{t_\epsilon}\left(\alpha I_i^\epsilon(t) + \frac{\beta}{2}(I_i^\epsilon(t))^2 - \alpha I_i^*(t) - \frac{\beta}{2}(I_i^*(t))^2\right)e^{-rt}dt \\ &\quad = \int_0^\infty e^{-rt}\left(\frac{a}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2\right)dt \\ &\quad\quad - \epsilon\int_0^{t_\epsilon} e^{-rt}\left(a\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] - 4bK_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] + 2b\epsilon\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t} - 1]^2]\right)dt \\ &\quad\quad - \epsilon\int_{t_\epsilon}^\infty e^{-rt}\left(a\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1] - 4bK_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1] + 2b\epsilon\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t_\epsilon} - 1]^2]\right)dt \\ &\quad\quad + \epsilon\int_0^{t_\epsilon} e^{-rt}\left(\alpha + \beta I_j^*(t) - \frac{\beta}{2}\epsilon\right)dt. \end{aligned} \tag{4}$$

Now since $K_j^*(t) < a/(3b)$ by assumption, we have

$$\int_0^\infty \left(\frac{a}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2\right)e^{-rt}dt > 0.$$

Moreover, we can always choose ϵ small enough so that the expression (4) is positive. Therefore, we have found a profitable deviation. The result follows. □

The intuition of the result is as follows. By slightly decreasing investment expenditures over a non-negligible time interval, a firm obviously decreases its production capacity. Since this policy departs from a candidate symmetric equilibrium and capital depreciates at a rate δ, this means that this firm's capital will always be lower than its rival's. Firm i's profit will be enhanced by a higher price but will also be negatively affected by a lower volume of production. Because of the discontinuity in the revenue function, however, it turns out that for a small decrease in investment (that also increases instant profits), the positive effect more than compensates the negative one and thus brings about a profitable deviation.

The next Proposition addresses the case where firms' capacities are always equal to $a/(3b)$.

Proposition 3. *There is no open-loop Nash equilibrium where $K_h(t) = \frac{a}{3b}$, for all t, and $h = i, j$.*

Proof. Suppose that such an equilibrium exists. Consider the following deviation strategy for firm i. Set $I_i(t) = 0$, as long as $K_i(t)$ is not equal to $\frac{a}{4b}$, and $I_i(t) = \delta \frac{a}{4b}$ afterwards. From Proposition 1, we see that along this alternative path, firm i's revenues are equal to $(a - 2bK)K$. These revenues reach their maximum values at $K = \frac{a}{4b}$ and we have $(a - 2bK)K = \frac{a-bK}{2}K$ when $K = \frac{a}{3b}$. Therefore, along the deviation path, firm i's revenues are always higher than the equilibrium ones. Moreover, the investment expenditures are also always lower than the equilibrium ones, since they are nil until the date at which $K_i(t) = \frac{a}{4b}$ and equal to $\delta \frac{a}{4b}$ afterwards, which is lower than $\delta \frac{a}{3b}$. The Proposition follows. □

The proof directly gives the intuition of the result. Choosing to always maintain the same relatively large capacity does not pay in equilibrium as a firm can increase instant profits by diminishing its production capacity. This increase in instant profits stems from an increase in the firm's revenues (the increase in the sale price compensates the decrease in the quantity sold), and from a decrease in investment expenditures.

Proposition 4. *Let assume that $K_{i0} < K_{j0}$, $K_{j0} \leq \frac{a}{3b}$. Then there is no open-loop Nash equilibrium such that $K_i^*(t) < K_j^*(t)$ for all t.*

Proof. Consider firm j's problem. Under our assumption that $K_i^*(t) < K_j^*(t)$ for all t, it follows from Proposition 1 that firm j's equilibrium payoff reads

$$\int_0^\infty e^{-rt}\big(b(K_i^*(t))^2 - C(I_j^*(t)\big)dt.$$

That is because, the sale proceeds only depend on firm i's capital. Therefore, the equilibrium investment decision $I_j^*(\cdot)$ solves the following problem

$$\begin{array}{rcl} \max\limits_{I_j(\cdot)} & & \displaystyle\int_0^\infty e^{-rt}\big(b(K_i^*(t))^2 - C(I_j(t)\big)dt \\ \text{s.t.:} \quad \dot{K}_j(t) & = & I_j(t) - \delta K_j(t), \quad K_j(0) = K_{j0}, \\ K_i^*(t) & < & K_j(t), \ \forall t. \end{array}$$

Neglecting the sale proceeds, we can write the Hamiltonian associated with the problem above as follows

$$-e^{-rt}\left(\alpha I_j(t) + \frac{\beta}{2} I_j(t)^2)\right) + \lambda(t)\left(I_j(t) - \delta K_j(t)\right),$$

where λ denotes the costate variable associated with the state variable K_j. The first-order conditions are given by:

$$\lambda(t) = (\alpha + \beta I_j(t))e^{-rt}, \tag{5}$$

$$\dot{\lambda}(t) = \delta\lambda(t),$$
$$\dot{K}_j(t) = I_j(t) - \delta K_j(t), \quad K_j(0) = K_{j0}.$$

Because firm j's objective is strictly concave, the solution $I_j(t)$ is unique and differentiable.[9] Thus, upon differentiating Equation (5) and using the two other first-order conditions we obtain

$$\begin{aligned}\dot{\lambda}(t) &= -re^{-rt}\left(\alpha + \beta\big(\dot{K}_j(t) + \delta K_j(t)\big)\right) + \beta e^{-rt}\left(\ddot{K}_j(t) + \delta\dot{K}_j(t)\right)\\ &= \delta\lambda(t) = \delta\left(\alpha + \beta\big(\dot{K}_j(t) + \delta K_j(t)\big)\right)e^{-rt}.\end{aligned}$$

Rearranging, we arrive at the following differential equation

$$\ddot{K}_j(t) - r\dot{K}_j(t) - K_j(t)\delta(r+\delta) = \alpha\frac{(r+\delta)}{\beta}.$$

The general solution of the equation above is given by

$$K_j(t) = \left(K_{j0} + \frac{\alpha}{\beta\delta}\right)e^{-\delta t} - \frac{\alpha}{\beta\delta}, \tag{6}$$

where $-\delta$ is the negative root of the following characteristic equation

$$s^2 - rs - (r+\delta)\delta = 0.$$

That is

$$\frac{r - \sqrt{r^2 + 4(r+\delta)\delta}}{2} = -\delta.$$

It is clear, however, that $K_j(t)$ goes to a negative value, which contradicts the assumption that $K_j^*(t) > K_i^*(t)$ for all t and the Proposition follows. □

The gist of the Proposition is that because firm j's objective does not depend upon its capacity, the best decision is to downsize this capacity as much as possible. Selling its capital is indeed the only way for firm j to increase its instant profit. But by so doing, firm j's capital stock is soon lower than firm i's, and this is inconsistent with the assumption that firm j's capacity is always the largest.

Proposition 5. *There is no open-loop Nash equilibrium such that $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, and $K_j^*(t) < K_i^*(t)$ on $]\underline{t}, \bar{t}[$.*

Proof. From Proposition 1, over the interval $[\underline{t}, \bar{t}]$ firm i's profit is written

$$\int_{\underline{t}}^{\bar{t}} e^{-rt}\left(b(K_j^*(t))^2 - C(I_i^*(t))\right)dt.$$

Since for all $t \in]\underline{t}, \bar{t}[$, $K_j^*(t) < K_i^*(t)$ this implies that

$$\int_{\underline{t}}^{\bar{t}} I_j^*(s)e^{\delta(s-t)}ds + e^{-\delta t}K_j^*(\underline{t}) < \int_{\underline{t}}^{\bar{t}} I_i^*(s)e^{\delta(s-t)}ds + e^{-\delta t}K_i^*(\underline{t}).$$

As firms have the same capacity at date $\underline{t}$, the inequality above reduces to

$$\int_{\underline{t}}^{\bar{t}} I_j^*(s)e^{\delta(s-t)}ds < \int_{\underline{t}}^{\bar{t}} I_i^*(s)e^{\delta(s-t)}ds.$$

[9] See, e.g., Corollary 3.3, page 8 in Fleming and Rishel (1975). The corollary, which pertains to a calculus of variations approach of the problem, applies since we can express the integrand as a strictly concave function of $\dot{K}_j(t)$.

This inequality implies that there is a sub-interval A of $]\underline{t},\bar{t}[$ on which $I_j^*(t) < I_i^*(t)$. Suppose now that firm i chooses the alternative policy $I_i'(\cdot)$ defined as follows: $\forall t \in A, I_i'(t) = I_i^*(t) - \epsilon$, $\forall t \notin A, I_i'(t) = I_i^*(t)$. Notice that we can always choose ϵ such that $I_i^*(t) - \epsilon > I_j^*(t)$ on A. Therefore, for ϵ small enough we still have $K_j^*(t) < K_i'(t)$ on $]\underline{t},\bar{t}[$ where $K_i'(t)$ is obtained from the decision $I_i'(t)$. Thus, firm i's receipts still depend on firm j's capacity only (see Proposition 1). But as $C(\cdot)$ is increasing this proves that the alternative policy increases firm i's objective by decreasing its investment expenditures (or by disinvesting more). This is a contradiction. □

The intuition behind Proposition 5 is that if firms' capacities are equal at two different dates, and ordered in the same manner between these dates, the firm having the largest capacity can always decrease it and therefore saves on investment expenditures. To put it another way, it does not pay for a firm to have a larger capacity than the other firm because the extra capacity is costly to build and maintain and because it does not yield higher revenues.

We are now in position to prove Theorem 1.

Proof of Theorem 1. From Propositions 2, 3 and 4, firms' capacities cannot always be the same, nor always be ordered in the same manner. Therefore, if there is an equilibrium, there must be at least two crossings similar to those considered in Proposition 5. But this very Proposition rules out such case and we are done. □

4 Open-loop Nash Equilibrium when investment is irreversible

In this section, we no longer assume that firms can disinvest. Disinvesting is not always possible, especially if we construe firms' capacity as including non-material assets, that is, intangible capital accumulating from R&D or advertising. As a result, our previous results do not apply to study open-loop equilibria. Disinvesting was used to get rid of excess capacity which may arise because firms produce complements. A major difference with the previous section is that existence of open-loop equilibrium is possible, albeit only when special and somewhat restrictive conditions are met.

Theorem 2. *Assume that $K_{i0} = K_{j0} > 0$ and $K_{i0} \le \frac{a}{3b}$. Then there is an open-loop Nash equilibrium where $I_i^*(t) = I_j^*(t) = 0$, for all t. Firms' capital stocks are equal at each date, and their common value goes to 0. So do their profits. Moreover, the sum of firms' prices goes to a.*

Proof. Assume that firm j's investment is always nil, that is, $I_j^*(t) = 0$ for all t. Consider any investment policy $I_i(t)$ for firm i that is piece-wise continuous, with $I_i(t) \neq 0$. As a result, firm i's capital stock reads

$$K_i(t) = K_{i0}e^{-\delta t} + \int_0^t I_i(s)e^{\delta(s-t)}ds.$$

Since firm i's investment is non-negative, we have

$$K_i(t) \ge K_j^*(t) = e^{-\delta t}K_{j0} = e^{-\delta t}K_{i0}.$$

By Proposition 1, in equilibrium firm i's profit satisfies

$$\begin{aligned}\int_0^\infty e^{-rt}\left[R^i(K_i(t), K_j^*(t)) - C(I_i(t))\right]dt &\le \int_0^\infty e^{-rt}\left[\frac{(a-K_j^*(t))}{2}K_j^*(t) - C(I_i(t))\right]dt \\ &\le \int_0^\infty e^{-rt}\left[\frac{(a-K_j^*(t))}{2}K_j^*(t) - C(0)\right]dt \\ &= \int_0^\infty e^{-rt}\left[\frac{(a-K_i^*(t))}{2}K_i^*(t) - C(I_i^*(t))\right]dt.\end{aligned}$$

The first inequality stems from the fact that 1) $K_j^*(t) \leq \frac{a}{3b}$ for all t, since firm j's investment is nil and capital depreciates, and 2) $K_j^*(t) \leq K_i(t)$. Then from Proposition 1, it can readily be seen that $R^i(K_i(t), K_j^*(t)) \leq \frac{(a-K_j^*(t))}{2}K_j^*(t)$. The second inequality results from the assumption that $I_i(t)$ is non-negative. The last inequality is just a consequence of the assumption that equilibrium strategies are symmetric.

The last inequality also shows that $I_i^*(t) = 0$ for all t is the best response to $I_j^*(t) = 0$ for all t (since firm i's profit is strictly concave). Thus, along the equilibrium path the capital $K(t)$ of both firms goes to zero. As the sum of the firms' prices satisfies the equation $a - (p_i(t) + p_j(t)) = bK(t)$, the sum of the firms' price goes to a. □

The crux of the result is that when firms have initially the same capacities and produce complements, increasing capacity is not profitable. Accumulating more capital will not result in an increase in sales, only in a rise in costs. Given the discontinuity of the instant profit function, firms would actually be better-off by slightly decreasing their capacities in order to raise their revenues. But we have ruled out disinvestment. Thus, the best decision for each firm is to let its capital depreciate.

Corollary 1. *Assume that there exists $\tilde{t}$ such that $K_i(\tilde{t}) = K_j(\tilde{t}) > 0$. Then there is an open-loop Nash equilibrium where $I_i^*(t) = I_j^*(t) = 0$, $\forall t \geq \tilde{t}$ and $K_i(t) = K_j(t)$, $\forall t \geq \tilde{t}$.*

Proof. The proof goes like the proof of Theorem 2. □

The next result addresses the double-crossing property.

Proposition 6. *There is no open-loop Nash equilibrium such that $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, and $K_j^*(t) < K_i^*(t)$ on $]\underline{t}, \bar{t}[$, and $K_h^*(t) \leq \frac{a}{3b}$, for all t, $h = i, j$.*

Proof. The proof relies on the same argument as that used in the proof of Proposition 5. □

The intuition of the result is exactly the same as that given for Proposition 5. Now, we have seen in Theorem 2 above that there is a symmetric open-loop equilibrium when firms' initial capacities are equal. One may wonder if there exist other kinds of equilibria under this assumption. The following result gives a negative answer to this question.

Theorem 3. *Assume that $K_{i0} = K_{j0}$. Then, the only open-loop Nash equilibrium is the symmetric one, where $I_i^*(t) = I_j^*(t) = 0$ for all t.*

Proof. Suppose that there is an asymmetric equilibrium. Then, by continuity of $K_i(\cdot)$ there is a date $\hat{t}$ such that, say $K_i^*(\hat{t}) < K_j^*(\hat{t})$ on an open neighborhood of $\hat{t}$. Now, it must be the case that $K_i^*(t) < K_j^*(t)$ for all t larger than $\hat{t}$. Assume, for the sake of contradiction, that this assertion is false. Therefore, there exists a date $\bar{t}$ such that $\hat{t} < \bar{t}$ and $K_j^*(\bar{t}) \leq K_i^*(\bar{t})$. Without loss of generality, let $\bar{t}$ be the smallest value of t such that $\hat{t} < t$ and $K_i^*(t) = K_j^*(t)$ (by continuity of $K_i^*(\cdot)$ and $K_j^*(\cdot)$ this date is well-defined). Similarly, let $\underline{t}$ be the largest value of t such that $t < \hat{t}$ and $K_i^*(t) = K_j^*(t)$ (again, by continuity of $K_i^*(\cdot)$ and $K_j^*(\cdot)$ this date is well-defined). By construction we have $K_i^*(t) < K_j^*(t)$ on $]\underline{t}, \bar{t}\,[$ and $K_i^*(\underline{t}) = K_j^*(\underline{t})$ as well as $K_i^*(\bar{t}) = K_j^*(\bar{t})$. But from Proposition 6, this is impossible. Therefore, $K_i^*(t) < K_j^*(t)$ for all $t > \hat{t}$. Now, let t_1 be the largest date such that $K_i^*(t_1) = K_j^*(t_1)$. From what we have just seen above, $K_i^*(t) < K_j^*(t)$ for all t strictly larger that t_1. Notice that for all $t > t_1$,

$$K_i^*(t) = e^{\delta(t_1-t)}K_i^*(t_1) + \int_{t_1}^{t} e^{-\delta(t-t_1)}I_i^*(t)dt < K_j^*(t) = e^{\delta(t_1-t)}K_j^*(t_1) + \int_{t_1}^{t} e^{-\delta(t-t_1)}I_j^*(t)dt.$$

Now, since firm j's receipts do not depend on K_j the best decision for this firm is to choose $I_j^*(t) = 0$ for all $t > t_1$. But this contradicts the previous inequality since $I_i^*(t) \geq 0$. □

The uniqueness property depends on the fact that the goods produced by the two firms are complements. If firms' capacities are initially identical, no firm has ever an incentive to increase its capital. A capacity increase will hardly give any additional income, and always entails more costs.

Corollary 2. *Suppose that* $K_{i0} < K_{j0}$*. Then in any open-loop Nash equilibrium we have* $K_i^*(t) \leq K_j^*(t)$ *for all* t*.*

Proof. The proof is again by way of a contradiction. If there were a date $\underline{t}$, such that $K_j^*(\underline{t}) < K_i^*(\underline{t})$, since $K_{i0} < K_{j0}$, there would exist an earlier date t_1 ($t_1 < \underline{t}$) such that $K_i^*(t_1) = K_j^*(t_1)$. But from Proposition 3 it holds that $K_i^*(t) = K_i^*(t)$ for all $t \geq t_1$. This is a contradiction. □

As the next Theorem states, it turns out that the cases considered above are the only ones where an open-loop Nash equilibrium exists.

Theorem 4. *Suppose that* $K_{i0} < K_{j0} \leq \frac{a}{3b}$*. Then there is no open-loop Nash equilibrium.*

We shall prove this theorem in several steps, each of them being of independent interest. The next Proposition addresses the existence of an open-loop Nash equilibrium when firms have different initial capacities. In that case, the firm with the largest capacity is impelled to reduce it. The question arises as to whether in equilibrium this firm's capacity will catch-up the capacity of the other one. As the next result shows, that never occurs.

Proposition 7. *Assume that* $K_{i0} < K_{j0} \leq \frac{a}{3b}$*. Then there is no open-loop Nash equilibrium such that:* $I_j^*(t) = 0$*, for all* t*,* $I_i^*(t) > 0$*,* $t \in [0, t_1]$*,* $I_i^*(t) = I_j^*(t) = 0$*,* $t \in [t_1, \infty[$*, and* $K_i^*(t) = K_j^*(t)$*,* $t \geq t_1$*.*

Proof. See the appendix. □

The Proposition above strongly depends on the fact that the revenue function is discontinuous when capacities are equal. The proof relies on the fact that it always pays for the firm having the lowest capacity to postpone a catching up with the capacity of the other firm. Not only does this postponement prevent a drop in firm's revenues, but it also allows it to reduce investment expenditures. Altogether, postponing a catch-up always brings about an increase in profit.

The next result tackles the cases where firm i only invests, or on the contrary, does not invest, for a short time only.

Proposition 8. *Assume that* $K_{i0} < K_{j0}$ *and* $K_{i0} \leq \frac{a}{3b}$*. Then there is no open-loop Nash equilibrium such that simultaneously* $K_i^*(t) < K_j^*(t)$ *for all* t *and: either* $I_i^*(t) > 0$ *for all* t*, or* $I_i^*(t) = 0$ *for* $[0, \tilde{t}]$ *and* $I_i^*(t) > 0$ *for* $t > \tilde{t}$*, or* $I_i^*(t) > 0$ *for* $t < \tilde{t}$ *and* $I_i^*(t) = 0$ *for* $t \geq \tilde{t} \geq 0$*.*

Proof. See the appendix. □

The Proposition above tells us that there is no equilibrium where the firm with the lowest initial capacity always invests (except perhaps for a short time) and always has the lowest capacity. That is because, as its competitor's capacity always decreases there is necessarily a date at which firms' capacities are equal. What is probably more surprising is that there is no equilibrium where the firm with the lowest capacity only invests for a short time. By doing this, it endeavors to increase its instant profits while still avoiding being caught up by its competitor. Yet, such policy move is never optimal. For if it is profitable—and this is certainly never the case whenever $K_{i0} > \frac{a}{4b}$, because then sales proceeds *increase* when firm i's capacity decreases—it would always be worth slightly increasing the time during which firm i invests.

In the results above, we have overlooked the fact that firm i's investment policy could take positive values intermittently. We address this case next.

Proposition 9. *Assume that* $K_{i0} < K_{j0}$*,* $K_{j0} \leq \frac{a}{3b}$*. Then there is no open-loop Nash equilibrium such that* $K_i^*(t) < K_j^*(t)$ *for all* t *and for all* T*, there exists* $t, t' > T$*, such that* $I_i^*(t) = 0$ *and* $I_i^*(t') > 0$*.*

Proof. Suppose that there exists such an equilibrium. Then we would have $I_j^*(t) = 0$ for all t and thus $K_j^*(t) = e^{-\delta t} K_{j0}$. Moreover, since the objective is strictly concave, there would also exist a unique solution to the problem:

$$\max_{I_i(t)\geq 0} \int_0^\infty e^{-rt}\Big(\big(a - 2bK_i(t)\big)K_i(t) - C(I_i(t)\big)\,dt,$$
$$\text{s.t.:} \qquad \dot{K}_i(t) = I_i(t) - \delta K_i(t), \quad K_i(0) = K_{i0}, \quad \lim_{t\to\infty} K_i(t) = 0.$$

Now, since the solution of the problem above is unique, from Hartl (1987)'s theorem we know that the state variable $K_i^*(t)$ is monotonic. It cannot be monotonically increasing since $K_i^*(t) < K_j^*(t) = e^{-\delta t} K_{j0}$. So $K_i^*(t)$ is monotonically decreasing. Moreover, the limit of $K_i^*(\cdot)$ is nil.

Then consider the Hamiltonian associated with firm i's problem

$$H = e^{-rt}\left((a - 2bK_i(t))K_i(t) - (\alpha I_i(t) + \beta\frac{(I_i(t))^2}{2})\right) + \lambda(t)(I_i(t) - \delta K_i(t)) + \mu(t) I_i(t).$$

The first-order conditions are as follows.

$$\begin{aligned}
&\mu(t) \geq 0,\\
&\dot{\lambda}(t) = -e^{-rt}(a - 4bK_i(t)) + \delta\lambda(t),\\
&e^{-rt}(-\alpha - \beta I_i(t)) + \lambda(t) + \mu(t) = 0,\ \mu(t) I_i(t) = 0.
\end{aligned} \tag{7}$$

As was seen in the first part of Proposition 8 the investment rate $I_i^*(t)$ cannot be strictly positive for all t.

Because I_i^* is continuous[10], it reaches a maximum value at some point of every closed interval on which it is non-constant and vanishes at the end-points thereof (there are many such intervals, because, as was already mentioned, from the first part of the proof of Proposition 8 we know that for all T, there always exists $t > T$, such that $I_i^*(t) = 0$). Consider one such interval. At the point t_0 where $I_i^*(t)$ realizes its (local) maximum, we get from the first-order conditions that $I_i^*(t_0) = \frac{e^{rt_0}\lambda^*(t_0) - \alpha}{\beta}$, and as I^* is clearly differentiable at t_0, we have $\frac{dI_i^*(t)}{dt}\mid_{t=t_0} = 0$ and thus

$$\frac{d\big(e^{rt}\lambda^*(t)\big)}{dt}\mid_{t=t_0} = 0,$$

or

$$re^{rt_0}\lambda^*(t_0) + e^{rt_0}\dot{\lambda}^*(t_0) = 0.$$

Using Equation (7) in the equation above we arrive at

$$\begin{aligned}
r\lambda^*(t_0) - e^{-rt_0}\big(a - 4bK_i^*(t_0)\big) + \lambda^*(t_0)\delta = 0\\
\Rightarrow \lambda^*(t_0) = \frac{e^{-rt_0}(a - 4bK_i^*(t_0))}{r + \delta}.
\end{aligned}$$

Thus when I_i^* reaches a maximum at a date t on an interval its value is given by

$$I_i^*(t) = \frac{e^{rt}\lambda^*(t) - \alpha}{\beta} = \frac{\frac{a - 4bK_i^*(t)}{r+\delta} - \alpha}{\beta}.$$

[10]Here, we can apply Theorem 6.1, page 75, in Fleming and Rishel (1975). To apply this Theorem, we need to make sure that $I_i(t)$ lies in a compact set. One such compact set can be constructed as follows. We know that the optimal solution is such that $K_i^*(\cdot)$ is non-increasing. Thus $I_i^*(t) \leq \delta K_i^*(t) \leq \delta K_{i0}$. Therefore there is no loss of generality in assuming that $I_i \in [0, \delta K_{i0}]$.

We can then build a sequence of dates t_0, t_1, ..., t_k,..., such that $I_i^*(\cdot)$ reaches a local maximum and takes the value above. Notice that the sequence of values $\left(I_i^*(t_k)\right)_k$ is non-decreasing since we have seen that the optimal path $K_i^*(t)$ is monotonically decreasing. The limit of this sequence is $I_i^* = \frac{\frac{a}{r+\delta}-\alpha}{\beta}$ (since $\lim_{t\to\infty} K_i^*(t) = 0$). But as $\dot{K}_i^*(t) \leq 0$, we have $I_i^*(t) \leq \delta K_i^*(t) < \delta e^{-\delta t} K_{j0}$, and we see that $\lim_{t\to+\infty} I_i^*(t) = 0$, which is a contradiction. □

The reason why there is no equilibrium with intermittent investment is the same that explains why there is no equilibrium with permanent investment. Investing intermittently prevents the capacity from decreasing "naturally", that is, at a rate equal to δ. This is pointless if this policy entails that firms' capacities become equal in finite time (which makes firm i always worse off). It is also pointless if firms' capacities never meet because firm i can always slightly increase its capacity in a profitable way.

We can now prove the major result of this section.

Proof of Theorem 4. The case where I_i^* only takes positive values on a finite number of interval can be handled with the same arguments as those of the proof of Proposition 8. Then the theorem results in turn from Propositions 7, 8, and 9. □

To wrap up, only if firms' capacities are equal at the initial date will an open-loop Nash equilibrium exist.

5 Different disinvestment possibilities

We now study the robustness of the inexistence of open-loop equilibrium in the Cournot's duopoly model of complements when firms face different disinvestment possibilities. More formally, we assume that there always is a firm h such that $I_h(t) \geq 0$ for all t, and another one for which there is no sign constraint on its investment decisions.

5.1 Initial capacities are equal

Let us first consider the case where initial capacities are equal. We have seen that when no firm can disinvest, there is an open-loop Nash equilibrium, whereas there is no such an open-loop Nash equilibrium when both can disinvest. Here we have the following result.

Proposition 10. *When initial capacities are the same and no greater than $a/3b$, there is no symmetric open-loop Nash equilibrium where firms' capacities remain equal and strictly positive over a time interval $[\underline{t}, \bar{t}]$, with $\underline{t} < \bar{t}$.*

Proof. The proof is exactly the same as that of Propositions 2 and 3 (at least one firm can disinvest and find a profitable deviating strategy). □

5.2 An intermediate result

Proposition 11. *Assume that firm i can disinvest. Then there is no open-loop Nash equilibrium such that $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, and $K_j^*(t) < K_i^*(t)$ on $]\underline{t}, \bar{t}[$.*

Proof. The proof is the same as that of Proposition 5. □

5.3 The firm with the lowest initial capacity can disinvest

Proposition 12. *Assume that* $K_{i0} < K_{j0} \leq \frac{a}{3b}$. *Then there is no open-loop Nash equilibrium such that:* $I_j^*(t) = 0$, *for all* t, $I_i^*(t) > 0$, $t \in [0, t_1]$, $I_i^*(t) = I_j^*(t) = 0$, $t \in [t_1, \infty[$, *and* $K_i^*(t) = K_j^*(t)$, $t \geq t_1$.

Proof. It is similar to the proof of Proposition 7 (firm i has a profitable deviating strategy). □

We also have:

Proposition 13. *Assume that* $K_{i0} < K_{j0}$ *and* $K_{i0} \leq \frac{a}{3b}$. *Then there is no open-loop Nash equilibrium such that simultaneously* $K_i^*(t) < K_j^*(t)$ *for all* t *and: either* $I_i^*(t) > 0$ *for all* t, *or* $I_i^*(t) = 0$ *for* $[0, \tilde{t}]$ *and* $I_i^*(t) > 0$ *for* $t > \tilde{t}$, *or* $I_i^*(t) > 0$ *for* $t < \tilde{t}$ *and* $I_i^*(t) = 0$ *for* $t \geq \tilde{t} \geq 0$.

Proof. It is similar to that of Proposition 8. □

Finally, we get

Proposition 14. *Let assume that* $K_{i0} < K_{j0}$, $K_{j0} \leq \frac{a}{3b}$. *Then there is no open-loop Nash equilibrium such that* $K_i^*(t) < K_j^*(t)$ *for all* t *and for all* T, *there exists* $t, t' > T$, *such that* $I_i^*(t) = 0$ *and* $I_i^*(t') > 0$.

Proof. It is similar to that of Proposition 9 (firm i has a profitable deviating strategy). □

Notice that the proofs of the last three Propositions rely on the fact that a firm can diminish its investment.

From the above Propositions we have the following result

Theorem 5. *Assume that* $K_{i0} < K_{j0}$ *and that firm* i *can disinvest. Then there is no open-loop Nash equilibrium.*

Proof. According to the Propositions above, the equilibrium values of the capital stocks cannot always be ranked in the same manner. Moreover, the equilibrium time paths cannot cross twice. They cannot be equal over a non-trivial interval either. It is also clear that firm i capital cannot be larger than firm j's over an infinite horizon. The conclusion follows. □

5.4 The firm with the lowest initial capacity cannot disinvest

Let us now turn to the case where the firm with the lowest initial capacity cannot disinvest ($I_i(t) \geq 0$ for all t).

Proposition 15. *Assume that* $K_{i0} < K_{j0} \leq \frac{a}{3b}$ *and that firm* i *cannot disinvest. Then, there is no open-loop Nash equilibrium with* $K_i^*(t) < K_j^*(t)$ *for all* $t > 0$.

Proof. Since firm j can disinvest, one can apply the proof of Proposition 4 and the result obtains. □

Proposition 16. *Assume that* $K_{i0} < K_{j0} \leq \frac{a}{3b}$ *and that firm* i *cannot disinvest. Then, there is no open-loop Nash equilibrium such that* $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, *and* $K_j^*(t) < K_i^*(t)$ *on* $]\underline{t}, \bar{t}[$.

Proof. Let t be in $]\underline{t}, \bar{t}[$. By assumption $K_j^*(t) < K_i^*(t)$ on $]\underline{t}, \bar{t}[$. But at this date t, we are exactly in the same case considered in the previous subsection. That is, there is no open-loop Nash equilibrium when the game starts with the initial condition $(K_i^*(t), K_j^*(t))$. □

Theorem 6. *Assume that* $K_{i0} < K_{j0}$ *and that firm* i *cannot disinvest. Then there is no open-loop Nash equilibrium.*

Proof. From the two Propositions above, we deduce that there is no open-loop Nash equilibrium where the optimal paths satisfy $K_i^*(t) < K_j^*(t)$ for all t, and there is no double-crossing. Single crossing, however, cannot happen in equilibrium. Indeed, we would be in the case where at some date t, $K_j^*(t) < K_i^*(t)$ and we know that an equilibrium does not exist in this case. Moreover, firms' capacities cannot be equal on a non-trivial interval. Then the result follows. □

The results of this section can be summarized as follows. The general inexistence of an open-loop Nash equilibrium in our dynamic version of Cournot model of complements does not depend on the assumption made in the previous sections that firms have the same disinvestment possibilities.

6 Conclusion

The main insight of this paper is that capital accumulation games have sharp different outcomes depending on whether or not the duopoly is comprised of complements firms. There generally does not exist an open-loop Nash equilibrium when firms are complements. Existence, however, is obtained, in the very special case where firms' capital stocks are equal and investment irreversible. In this peculiar case, the equilibrium outcome is dismal, and illustrates the tragedy of the prisoner's dilemma. That is because, firms let their capital depreciate and they end up with negligible profits in the long run.

These results ensue because firms' revenues are discontinuous functions of their capital stocks, and this discontinuity in turn stems from the fact that firms produce complement goods. This is an additional instance where there is no duality between the Cournot model with complements and the traditional quantity Cournot model.

Our results can also perhaps be considered as an additional instance where some firms need to cooperate in order to prevent chaotic outcomes that competition is likely to produce is some specific markets. Such view was notably held by Telser who argues that there is no equilibrium in markets where plants are large relative to demands, fixed costs are avoidable, technology displays increasing returns to scale and so on (see, e.g., Telser (1994), (1987), (2017) and McWilliams (1990)). Studying cooperation in our version of the dynamic Cournot duopoly model of complements is a natural topic for further research.

There are at least two additional avenues for future research. Firstly, it would be interesting to address the existence of open-loop Stackelberg Equilibrium (feedback solutions appear difficult to analyze since firms' objectives are not continuous). Secondly, it would be worthwhile to investigate other dynamic settings where the duality with the quantity Cournot model breaks down.

Appendix

Proof of Proposition 1.

Consider the problem given in Equations (1)–(2), where firm i takes as given the price of firm j. Denoting by λ_i the Lagrange multiplier associated with the capacity constraint, the first-order optimality conditions read:

$$a - 2p_i - p_j = \lambda_i, \quad \frac{a-(p_i+p_j)}{b} \le K_i, \quad \lambda_i\left(\frac{a-(p_i+p_j)}{b} - K_i\right) = 0, \quad \lambda_i \le 0. \tag{8}$$

Two possibilities arise:

1. $\lambda_i = 0$. In this case from (8), we get:

$$p_i = \frac{a-p_j}{2},$$

$$\frac{a-(p_i+p_j)}{b} - K_i = \frac{a-p_j}{2b} - K_i \le 0 \;\Leftrightarrow\; p_j \ge a - 2bK_i.$$

2. $\lambda_i < 0$. In this case from (8), we get:

$$\frac{a-(p_i+p_j)}{b} = K_i \;\Leftrightarrow\; p_i = a - p_j - bK_i,$$
$$\lambda_i = a - 2p_i - p_j = 2bK_i + p_j - a < 0 \;\Leftrightarrow\; p_j < a - 2bK_i.$$

Let us now consider firm j. By symmetry, denoting by λ_j the Lagrange multiplier associated with its capacity constraint, the first-order optimality conditions read:

$$a - 2p_j - p_i = \lambda_j, \quad \frac{a-(p_i+p_j)}{b} \le K_j, \quad \lambda_j\left(\frac{a-(p_i+p_j)}{b} - K_j\right) = 0, \quad \lambda_j \le 0. \tag{9}$$

Again two possibilities arise:

1. $\lambda_j = 0$. In this case from (9), we get:

$$p_j = \frac{a-p_i}{2},$$
$$\frac{a-(p_i+p_j)}{b} - K_j = \frac{a-p_i}{2b} - K_j \le 0 \;\Leftrightarrow\; p_i \ge a - 2bK_j.$$

2. $\lambda_j < 0$. In this case from (9), we get:

$$\frac{a-(p_i+p_j)}{b} = K_j \;\Leftrightarrow\; p_j = a - p_i - bK_j,$$
$$\lambda_j = a - 2p_j - p_i = 2bK_j + p_i - a < 0 \;\Leftrightarrow\; p_i < a - 2bK_j.$$

In order to characterize short-run Nash equilibria of the duopoly with perfect complements and capacity constraints we take into account conditions (8)–(9) and the following four cases arise:

1. $\lambda_i = \lambda_j = 0$. From $2p_i = a - p_j$ and $2p_j = a - p_i$, we get $p_i = p_j = \frac{a}{3}$ and conditions $\frac{a-(p_i+p_j)}{b} \le K_i$, $\frac{a-(p_i+p_j)}{b} \le K_j$ impose $K_i \ge \frac{a}{3b}, K_j \ge \frac{a}{3b}$.
2. $\lambda_i < 0$, $\lambda_j < 0$. In this case $\frac{a-(p_i+p_j)}{b} = K_i$ and $\frac{a-(p_i+p_j)}{b} = K_j$, therefore, $K_i = K_j = K$ and $p_i + p_j = a - bK < a$. Consequently,

$$\lambda_i = a - 2p_i - p_j = 2bK + p_j - a = bK + p_j - (p_i + p_j) = bK - p_i,$$
$$\lambda_j = a - p_i - 2p_j = 2bK + p_i - a = bK + p_i - (p_i + p_j) = bK - p_j.$$

In a symmetric equilibrium $p_i = p_j = p = \frac{a-bK}{2}$, and hence, $\lambda_i = \lambda_j = \lambda = \frac{3bK-a}{2} < 0$ and $K < \frac{a}{3b}$.
Defining as in Reynolds (1987) the set κ of capacity pairs by

$$\kappa = \{K \in \mathbb{R} \mid K \ge 0, \quad K < \frac{a}{3b}\},$$

we have that for $(K_1, K_2) = (K, K)$ and $K \in \kappa$ each firm sets price such that both firms are producing at full capacity $\frac{a-(p_i+p_j)}{b} = K$.
Therefore, under the symmetry assumption $p_i = p_j = p$, $p = \frac{a-bK}{2}$ and $\frac{a-2p}{b} = K$, and for $(K_1, K_2) = (K, K)$ and $K \in \kappa$, the net revenue function in terms of the firms' capacity K is

$$R(K) = \frac{a-bK}{2}K.$$

Notice that for $K \notin \kappa$, along any short-run equilibria at least one of the firm would have excess capacity.

3. Asymmetric equilibria: $\lambda_i = 0, \lambda_j < 0$ or $\lambda_i < 0, \lambda_j = 0$.

   (a) Case $\lambda_i = 0, \lambda_j < 0$.
   Because $\lambda_i = 0$, then $2p_i = a - p_j$ and $\frac{a-(p_i+p_j)}{b} \leq K_i$.
   Because $\lambda_j < 0$, $\frac{a-(p_i+p_j)}{b} = K_j$ and $\lambda_j = a - p_i - 2p_j$.
   Therefore, in this case $K_j \leq K_i$. From the expressions above we get:

$$p_j = a - 2bK_j, \quad p_i = bK_j, \quad \lambda_j = -a + 3bK_j.$$

   Condition $\lambda_j < 0$ establishes that $K_j < \frac{a}{3b}$.

   (b) Case $\lambda_i < 0, \lambda_j = 0$.
   This case implies $K_i \leq K_j$ and

$$p_i = a - 2bK_i, \quad p_j = bK_i, \quad \lambda_i = -a + 3bK_i.$$

   Condition $\lambda_i \leq 0$ establishes that $K_i < \frac{a}{3b}$.

□

Proof of Proposition 7.

The proof is by way of a contradiction. Suppose that $I_j^*(t) = 0$ for all t (this is the best decision for firm j since its capital stock is always no-lower than that of firm i, and thus it never pays to invest). Under the statement of the proposition above and Proposition 1, firm i's profit reads

$$\int_0^{t_1} e^{-rt}\Big(\big(a-2bK_i^*(t)\big)K_i^*(t) - C(I_i^*(t))\Big)dt + \frac{e^{-rt_1}}{2}\left(a\frac{K_i^*(t_1)}{\delta+r} - b\frac{(K_i^*(t_1))^2}{2\delta+r}\right).$$

Let ϵ be a positive real number such that $t_1-\epsilon>0$ and consider the alternative strategy $I_i'(t)$ for firm i:

$$I_i'(t) = \begin{cases} I_i^*(t),\ t\in[0,t_1-\epsilon[\\ 0,\ t\in[t_1-\epsilon,\infty[. \end{cases} \tag{10}$$

With this new policy, firm i's capital remains below firm j 's and from Proposition 1, firm i's instant receipts are always equal to $(a-2bK_i')K_i'$ instead of $\frac{(a-bK_i^*)}{2}K_i^*$ from date t_1 on when firms have forever the same capacities. Since $K_i'(t)$ is lower than $\frac{a}{3b}$, firm i gets more receipts when its capacity became less than its competitor.

The profit associated with the deviation strategy in 10 reads

$$\int_0^{t_1-\epsilon} e^{-rt}\Big(\big(a-2bK_i^*(t)\big)K_i^*(t) - C(I_i^*(t))\Big)dt + \int_{t_1-\epsilon}^{\infty} e^{-rt}\left(a-2bK_i'(t)\right)K_i'(t)dt$$

where $K_i'(t) = e^{\delta(t_1-\epsilon-t)}K_i^*(t_1-\epsilon)$. Therefore, the difference in profit resulting from the deviation strategy can be written as

$$\int_{t_1-\epsilon}^{t_1} e^{-rt}\Big(\big(a-2bK_i'(t)\big)K_i'(t) - \big(a-2bK_i^*(t)\big)K_i^*(t) + C(I_i^*(t))\Big)dt$$
$$+\int_{t_1}^{\infty} e^{-rt}\left(\big(a-2bK_i'(t)\big)K_i'(t) - \frac{\big(a-bK_i^*(t)\big)}{2}K_i^*(t)\right)dt. \tag{11}$$

One can see that

$$\lim_{\epsilon\to 0}\int_{t_1-\epsilon}^{t_1} e^{-rt}\Big(\big(a-2bK_i'(t)\big)K_i'(t) - \big(a-2bK_i^*(t)\big)K_i^*(t) + C(I_i^*(t))\Big)dt = 0.$$

Now consider the second term in the difference in profits. We have

$$\begin{aligned}\int_{t_1}^{\infty} e^{-rt}\big(a-2bK_i'(t))K_i'(t)\big)dt &= \int_{t_1}^{\infty} e^{-rt}\big(a-2bK_i^*(t_1-\epsilon)e^{\delta(t_1-\epsilon-t)}\big)e^{\delta(t_1-\epsilon-t)}K_i^*(t_1-\epsilon)dt \\ &= ae^{-\delta\epsilon}e^{-rt_1}\frac{K_i^*(t_1-\epsilon)}{r+\delta} - \left[\frac{2b}{r+2\delta}e^{-2\delta\epsilon}e^{-rt_1}\right](K_i^*(t_1-\epsilon))^2.\end{aligned}$$

On the other hand, since $K_i^*(t) = e^{-(t-t_1)\delta}K_i^*(t_1)$ from date t_1 on, we obtain

$$\int_{t_1}^{\infty} e^{-rt}\frac{\big(a-bK_i^*(t)\big)}{2}K_i^*(t)dt = \frac{a}{2(r+\delta)}e^{-rt_1}K_i^*(t_1) - \frac{b}{2(r+2\delta)}e^{-rt_1}(K_i^*(t_1))^2.$$

Building on these last two expressions, since $K_i^*(\cdot)$ is continuous we get that

$$\lim_{\epsilon\to 0}\left(\int_{t_1}^{\infty} e^{-rt}\left(a-2bK_i'(t)\right)K_i'(t)\right)dt - \int_{t_1}^{\infty} e^{-rt}\frac{\big(a-bK_i^*(t)\big)}{2}K_i^*(t)dt\right)$$
$$= \frac{e^{-rt_1}}{2}\left(a\frac{K_i^*(t_1)}{r+\delta} - \frac{3bK_i^*(t_1)^2}{r+2\delta}\right) > 0$$

where the inequality above stems from the fact that $K_i^*(t_1) < \frac{a}{3b}$. It follows that there is a profitable deviation. □

Proof of Proposition 8. If $K_i^*(t) < K_j^*(t)$ for all t, then $I_j^*(t) = 0$ for all t, $K_j^*(t) = e^{-\delta t} K_{j0}$ and firm i's instant profits are given by $(a - 2bK_i^*(t))K_i^*(t) - C(I_i^*(t))$. Then firm i faces the following standard dynamic optimization problem:

$$\max_{I_i(t)\geq 0} \int_0^\infty e^{-rt}\Big(\big(a - 2bK_i(t)\big)K_i(t) - C(I_i(t)\big)\,dt,$$

$$\text{s.t.:} \qquad \dot{K}_i(t) = I_i(t) - \delta K_i(t), \quad K_i(0) = K_{i0}, \quad K_{i0} \leq \frac{a}{3b}.$$

The Hamiltonian associated with this problem is given by

$$H = e^{-rt}\left((a - 2bK_i(t))K_i(t) - (\alpha I_i(t) + \beta\frac{(I_i(t))^2}{2})\right) + \lambda(t)(I_i(t) - \delta K_i(t)) + \mu(t) I_i(t),$$

where λ is the costate variable and μ the Lagrange multiplier function.

The firs-order conditions are as follows

$$\mu(t) \geq 0, \tag{12}$$
$$\dot{\lambda}(t) = -e^{-rt}(a - 4bK_i(t)) + \delta\lambda(t), \tag{13}$$
$$e^{-rt}(-\alpha - \beta I_i(t)) + \lambda(t) + \mu(t) = 0. \tag{14}$$

Let assume that there exists an equilibrium such that $I_i^*(t) > 0$ for all t. Then we have $\mu(t) = 0$ for all t and we have

$$I_i^*(t) = \frac{e^{rt}\lambda^*(t) - \alpha}{\beta},$$
$$\dot{K}_i^*(t) = \frac{e^{rt}\lambda^*(t) - \alpha}{\beta} - \delta K_i^*(t),$$
$$\dot{\lambda}^*(t) = -e^{-rt}(a - 4bK_i^*(t)) + \delta\lambda^*(t). \tag{15}$$

Since

$$\lambda^*(t) = e^{-rt}\big(\beta(\dot{K}_i^*(t) + \delta K_i^*(t)) + \alpha\big),$$

and the optimal policy I_i^* is differentiable (see footnote 9) we get

$$\dot{\lambda}^*(t) = -re^{-rt}(\beta(\dot{K}_i^*(t) + \delta K_i^*(t)) + \alpha) + e^{-rt}\beta(\ddot{K}_i^*(t) + \delta\dot{K}_i^*(t)).$$

Using Equation (15) in the expression above we obtain the following differential equation:

$$\ddot{K}_i^*(t) - r\dot{K}_i^*(t) - (\delta(r+1) + 4b/\beta)K_i^*(t) = (r+\delta)\alpha/\beta - a/\beta.$$

The solution to this equation is

$$K_i^*(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} - \frac{(r+\delta)\alpha - a}{\beta\delta(r+1) + 4b},$$

where $s_1 = \frac{r-\sqrt{\Delta}}{2} < 0$, $s_2 = \frac{r+\sqrt{\Delta}}{2} > 0$, $\Delta = r^2 + 4\big(\delta(r+1) + 4b/\beta\big) > 0$ and $c_1 + c_2$ satisfies the equation

$$K_{i0} + \frac{(r+\delta)\alpha - a}{\beta\delta(r+1) + 4b} = c_1 + c_2.$$

The costate variable reads:

$$\lambda^*(t) = e^{-rt}\left[\beta\left((s_1+\delta)c_1 e^{s_1 t} + (s_2+\delta)c_2 e^{s_2 t} - \delta\frac{(r+\delta)\alpha - a}{\beta\delta(r+1) + 4b}\right) + \alpha\right],$$

and the investment policy

$$I_i^*(t) = (s_1+\delta)c_1e^{s_1t} + (s_2+\delta)c_2e^{s_2t} - \delta\frac{(r+\delta)\alpha - a}{\beta\delta(r+1)+4b}.$$

Imposing the transversality condition $\lim_{t\to\infty}\lambda^*(t)K_i^*(t) = 0$ or $\lim_{t\to\infty}H^* = 0$ implies that $c_2 = 0$, and firm i's optimal capacity reads:

$$K_i^*(t) = \left(K_{i0} - \frac{a-(r+\delta)\alpha}{\beta\delta(r+1)+4b}\right)e^{s_1t} + \frac{a-(r+\delta)\alpha}{\beta\delta(r+1)+4b}.$$

The optimal time path of firm i's capacity converges towards the steady-state value $\frac{a-(r+\delta)\alpha}{\beta\delta(r+1)+4b}$. Since

$$(r+\delta)\alpha < a,$$

this steady-state value is non-negative, and thus K_i^* and K_j^* will intersect in some point in time, leading to a contradiction.

The assumption that there is an equilibrium such that there exists $\tilde{t} > 0$ with $I_i^*(t) = 0$ for all $t \in [0,\tilde{t}]$ and $I_i^*(t) > 0$ for all $t > \tilde{t}$ leads to a contradiction following the same reasoning as in the previous case, because now firm i's optimal capacity would read:

$$K_i^*(t) = \left(K_{i0}e^{-\delta\tilde{t}} - \frac{a-(r+\delta)\alpha}{\beta\delta(r+1)+4b}\right)e^{-s_1(\tilde{t}-t)} + \frac{a-(r+\delta)\alpha}{\beta\delta(r+1)+4b}.$$

Consider then the case where the equilibrium is such that there exists $\tilde{t} > 0$ with $I_i^*(t) > 0$ for all $t \in [0,\tilde{t}[$ and $I_i^*(t) = 0$ for all $t \geq \tilde{t}$. From $\tilde{t}$ upwards firm i's capacity is given by $K_i^*(t) = K_i^*(\tilde{t})e^{\delta(\tilde{t}-t)}$. Taking this expression into account into the differential equation in (13) and solving we get:

$$\lambda^*(t) = D_1e^{\delta t} + \frac{a}{r+\delta}e^{-rt} - \frac{4bK_i(\tilde{t})}{r+2\delta}e^{\delta\tilde{t}}e^{-(r+\delta)t}.$$

Imposing the transversality condition either $\lim_{t\to\infty}\lambda^*(t)K_i^*(t) = 0$ or $\lim_{t\to\infty}H^* = 0$ implies that $D_1 = 0$, and the costate variable thus reads:

$$\lambda^*(t) = \frac{a}{r+\delta}e^{-rt} - \frac{4bK_i(\tilde{t})}{r+2\delta}e^{\delta\tilde{t}}e^{-(r+\delta)t}.$$

From (14)

$$\mu^*(t) = \left(\alpha - \frac{a}{r+\delta}\right)e^{-rt} + \frac{4bK_i(\tilde{t})}{r+2\delta}e^{\delta\tilde{t}}e^{-(r+\delta)t}.$$

As $\alpha < \frac{a}{r+\delta}$, then $\mu^*(t) < 0$ for t big enough, which contradicts (12).

An identical argument applies if $\tilde{t} = 0$ and thus the equilibrium such that $I_i^*(t) = 0$ for all t can also be discarded. □

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