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# New complementary problem formulation for the improved primal simplex

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**Abstract :** The primal simplex algorithm is still one of the most used algorithms by the operations research community. It moves from basis to adjacent one until optimality. The number of bases can be very huge, even exponential, due to degeneracy or when we have to go through all of the extreme points very close to each other. The improved primal simplex algorithm (IPS) is efficient against degeneracy but when there is no degeneracy, it behaves exactly as a primal simplex and consequently, it may suffer from the same limitations.

We present a new formulation of the complementary problem, i.e., the auxiliary subproblem used by the improved primal simplex to find descent directions, that guarantees a significant improvement of the objective value at each iteration until we reach an  $\epsilon$ -approximation of the optimal value. We prove that the number of needed directions is polynomial.

**Keywords:** Linear programming, simplex, degeneracy, decomposition, primal algorithms

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# 1 Introduction

Consider the linear program

$$(\mathcal{LP}) : \begin{array}{ll} \min_{x \in \mathbb{R}^n} & c^T x \\ & Ax = b \\ & x \geq 0 \end{array}$$

where  $A$  is an  $m \times n$  matrix,  $c$  an arbitrary cost vector of dimension  $n$  and  $b$  a vector of dimension  $m$ . We suppose that  $(\mathcal{LP})$  is feasible and  $A$  is of full rank. If the program is not or slightly degenerate, it can be easily solved by any standard linear programming solver using the primal simplex algorithm, which is still one of the most used algorithms in Operations Research even if the dual simplex and the interior point methods are faster in some cases. The primal simplex algorithm is well suited for re-optimization, especially when new columns are iteratively added in the column generation context. Actually, many problems in the transportation industry are for instance solved that way to profit from the problem's specific structure. In this case, the number of variables can be huge but only a limited number of promising variables are generated at each iteration. We discuss below two difficulties of the primal simplex algorithm and how to deal with them.

## 1.1 Primal degeneracy: a first difficulty

A first difficulty for the primal simplex is degeneracy. When there are many bases for the same extreme point, the primal simplex algorithm can do many pivots without improving the objective value. In past years, several techniques have been developed to deal with degeneracy. Large group of these methods focuses on the selection of the variable entering the basis. In one hand, [Greenberg \(1978\)](#) introduced a method that computes a vector such that each of its positive entries corresponds to a nonbasic variable that will necessarily lead to a degenerate pivot if it is chosen to enter the basis. On the other hand, [Bénichou et al. \(1977\)](#) uses another idea based on perturbation. Indeed, they apply a random modification of either the values of the degenerate variables or their bounds to put an end to a sequence of nonimproving pivots. The random modification avoids performing iterations without improvement, but it usually replaces them with small steps. For further information, an excellent study of degeneracy emphasizing its computational aspects is given by [Maros \(2012\)](#).

Another approach aims to take advantage of degeneracy rather than just to minimize its negative effects. Geometrically, a degenerate vertex of the  $n$ -dimensional polyhedron described by the LP's constraints is the crossing point of more than  $n - 1$  facets of the polyhedron. Degeneracy thus corresponds to a local excess of information, which suggests that a smaller problem may be considered locally to make a progress from a degenerate solution. In [Perold \(1980\)](#), they introduce a particular degeneracy structure in the LU decomposition of the basis, which involves fewer calculations when performing degenerate pivots. The paper [Pan \(2008\)](#) generalizes the definition of basis to include deficient bases containing  $p$  independent columns, with  $p$  is lower than the number of rows. When degeneracy occurs, a deficient basis, smaller than the usual square basis matrix, may be used to perform the pivot calculations.

The Improved Primal Simplex algorithm IPS ([Elhallaoui et al., 2011](#)) was introduced recently to deal with degeneracy in linear programming. The IPS algorithm permits to move from an extreme point to a strictly better adjacent extreme point. This algorithm identifies the compatible variables, i.e., those in the span of non-degenerate variables, to enter into the basis to obtain non-degenerate pivots. When no compatible variables with negative reduced costs exist these algorithms use a complementary problem to find an improving direction moving to a strictly better adjacent extreme point. The complementary problem, which is not dual degenerate, can be solved easily with the dual simplex. The process continues decreasing the objective value at each iteration until an optimal solution is reached. IPS is a generalization of previous works on dynamic constraint aggregation showing clear success for solving set partitioning problems (see [Elhallaoui et al. \(2008\)](#), [Saddoune et al. \(2011\)](#)). It has been already shown in [Metrane et al. \(2010\)](#) that IPS is a Dantzig-Wolf decomposition. The IPS algorithm reduces the computation time by a factor of more than 4 on driver and bus scheduling problems of

2000 constraints and 6000 – 10000 variables with in the order of 50% of degenerated constraints. It reduces the computation time by factors of 10 to 30 on real life fleet assignment and aircraft routing problems of 5000 constraints and 17000 – 25000 variables with 65% – 70% of degenerated constraints (Metrane et al., 2010).

## 1.2 Small improvement: a second difficulty

A second difficulty for the primal simplex is the small improvement at some pivots with pricing criteria based on reduced cost. The steep size and the improvement can be small. Using steepest edge as pricing criterion selects the best improvement but it is time consuming. At some extreme points, the best improvement can be small when all neighbors in the polyhedron are very close. For example, when a very degenerate problem is perturbed to escape degeneracy, the huge number of bases at an extreme point is transformed into a huge number of very close extreme points. In the worst case, the primal simplex algorithm could need an exponential number of pivots to solve the problem (Smale, 1998). Finding a polynomial adaptation of the primal simplex is classified as an important problem for the 21<sup>st</sup> century (Smale, 1998).

## 1.3 Contribution and organization of the paper

IPS is good for degeneracy but does not ensure a significant decrease at each iteration and when there is no degeneracy, IPS behaves exactly as the standard primal simplex algorithm. The contribution of this paper is to introduce a new variant of the complementary problem of IPS that ensures a significant decrease at each iteration even if the problem is degenerate or all neighbors are not significantly better. In the latter case, the resulting Improved Primal Simplex can jump to a non-neighbor extreme point which is significantly better. Thus, we reach an  $\epsilon$ -optimal solution in visiting only a polynomial number of extreme points.

The paper is organized as follows. In Section 2 we give the standard formulation of the Improved Primal Simplex, while the Section 3 introduces a general parametric formulation of the new complementary problem and in the same Section we give the choice of the parameters that ensure the desired results. The Section 4 provides the results of the paper and their proofs. A numerical example is studied in the Section 6 to illustrate the benefit of the new formulation introduced in the Section 3 on the standard formulation of IPS.

## 2 Improved Primal Simplex generic framework

For  $\mathcal{I} \subset \{1, \dots, m\}$  and  $\mathcal{J} \subset \{1, \dots, n\}$ , the submatrix of  $A$  with rows indexed by  $\mathcal{I}$  and columns indexed by  $\mathcal{J}$  is denoted  $A_{\mathcal{I}, \mathcal{J}}$ . Similarly, for any vector  $v \in \mathbb{R}^n$ ,  $v_{\mathcal{J}}$  is the subvector of all  $v_j, j \in \mathcal{J}$ . For the remainder of the paper,  $x \in \mathbb{R}^n$  is a feasible basic solution of  $(\mathcal{LP})$  where  $\mathcal{B}$  and  $\mathcal{N}$  are the index sets of basic and non-basic variables, respectively.

The main idea of the Improved Primal Simplex is to divide the set  $\mathcal{B}$  into two subsets  $\mathcal{P}$  and  $\mathcal{Z}$  and hence decompose the problem into two subproblems: the reduced problem containing the compatible variables with  $\mathcal{P}$  and a complementary problem containing the rest of variables. The decomposition can be explained as follows. We start by defining  $\mathcal{P} := \{j \in \mathcal{B}, x_j > 0\}$  as the index set of positive variables. The index set of compatible variables with  $\mathcal{P}$  is defined as the following: set  $\mathcal{C} := \{j \in \mathcal{N}, A_j \in \mathbf{Span}(A_l)_{l \in \mathcal{P}}\}$ . Finally, by defining  $\mathcal{L} = \mathcal{N} \setminus \mathcal{C}$ , we can formulate the two subproblems  $\mathcal{RLP}$  and  $\mathcal{CLP}_{\mathcal{B}}$  formulated in the next two subsections.

## 2.1 Reduced problem

The reduced problem can be simply seen as a restriction of the original problem to compatible variables as defined above.

$$(\mathcal{RLP}) : \quad \begin{aligned} \rho &:= \min_{x_C \in \mathbb{R}^C} c_C^T x_C \\ \begin{bmatrix} [A_B]^{-1} A \end{bmatrix}_{\mathcal{P}, C} x_C &= \begin{bmatrix} [A_B]^{-1} b \end{bmatrix}_{\mathcal{P}} \\ x_C &\geq 0 \end{aligned}$$

As we can observe, the rank of the constraint matrix is  $|\mathcal{P}|$ . Where there is degeneracy,  $|\mathcal{P}| < m$ . Consequently, there are  $m - |\mathcal{P}|$  redundant constraints that can be removed from the reduced problem. Obviously, the higher is degeneracy, the smaller is the reduced problem (so easier to solve). In [Elhallaoui et al. \(2011\)](#), we have shown that any pivot on a compatible variable having a negative reduced cost is non-degenerate. That means pivots in the reduced problem are fast and efficient. The reduced problem can be solved very easily by a standard primal simplex.

## 2.2 Complementary problem

The complementary problem seeks to combine incompatible variables to form a compatible combination having the least reduced cost according to the same definition of compatibility. The formulation is as follows:

$$(\mathcal{CLP}_B) : \quad \begin{aligned} \zeta &:= \min_{\substack{u \in \mathbb{R}^{\mathcal{P}} \\ v \in \mathbb{R}^{\mathcal{L}}}} -c_{\mathcal{P}}^T u + c_{\mathcal{L}}^T v \\ -A_{\mathcal{P}} u + A_{\mathcal{L}} v &= 0 \\ \mathbf{1}^T v &= 1 \\ v &\geq 0 \end{aligned}$$

Without the normalization constraint  $\mathbf{1}^T v = 1$ , the complementary problem is an unbounded cone (of directions); This constraint is used to bound it. We observed that the complementary problem is not dual degenerate. So, it is easily solvable by the dual simplex algorithm. Also, pivoting in the original problem on positive variables yields an improved basic solution. Remark here that columns of  $A_{\mathcal{P}}$  are linearly independent. So, the variables  $u$  can be substituted by variables  $v$ , reducing thus the dimension of the complementary problem (and reducing the number of constraints too by an equivalent number). In summary, the IPS algorithm solves successively the reduced problem  $\mathcal{RLP}$  (to optimality) then the complementary problem  $\mathcal{CLP}$  and reiterate until optimality is reached. In [Elhallaoui et al. \(2011\)](#), it has been proven that from iteration to iteration IPS goes from an extreme point to an adjacent extreme point and if  $\zeta \geq 0$  then the solution is optimal.

## 3 New formulation for the complementary problem

If the pivot selection criterion in the standard primal simplex selects a large valued exiting variable, the step size would be large enough to produce a significant improvement if the reduced cost of the entering variable is also large enough. If the exiting variables are not large enough, their reduced cost should compensate this. In the same way, when we move along a descent direction produced by the complementary problem in IPS, if the exiting variables have large values, the improvement will be significant (or at least the best we can do) because their reduced costs are minimum by construction. So, instead of considering the index set of positive valued variables  $\mathcal{P}$  as one set, we partitioned it into two subsets  $\mathcal{S}$  and  $\mathcal{R}$  where  $\mathcal{S}$  is the set of large valued variables. The objective of the paper is to provide a formal criterion to construct the set  $\mathcal{S}$  and to select a good improving direction permitting to obtain a significant improvement of the objective value by playing with the normalization constraint.

We redefine the set of compatible variables as

$$\mathcal{C} := \{j \in \mathcal{N}, A_j \in \mathbf{Span}(A_l)_{l \in \mathcal{S}}\}$$

(compatible with  $\mathcal{S}$  instead of  $\mathcal{P}$ ) and  $\mathcal{L} := \mathcal{N} \setminus \mathcal{C}$ . Let  $\lambda \in \mathbb{R}_{>0}^{\mathcal{L}}$  and  $\gamma \in \mathbb{R}_{>0}^{\mathcal{R}}$  be two vectors of weights we introduce in the normalization constraint of the complementary problem to obtain a descent direction with significant improvement. These weight vectors are computed using  $\mathcal{S}$  and the data of the problem. The formal definitions of  $S$ ,  $\lambda$  and  $\gamma$  will be given later in the proof of the Proposition 3. We formulate the parametric complementary problem as follows.

$$\begin{aligned}
 z &:= \min_{\substack{u \in \mathbb{R}^{\mathcal{S}} \\ v \in \mathbb{R}^{\mathcal{L}} \\ w \in \mathbb{R}^{\mathcal{R}}}} -c_{\mathcal{S}}^T u - c_{\mathcal{R}}^T w + c_{\mathcal{L}}^T v \\
 (\mathcal{NCLP}) : & \quad -A_{\mathcal{S}} u - A_{\mathcal{R}} w + A_{\mathcal{L}} v = 0 \\
 & \quad \lambda^T v + \gamma^T w \leq 1 \\
 & \quad v \geq 0 \\
 & \quad w \geq 0
 \end{aligned}$$

By duality, the dual problem of  $(\mathcal{CLP})$  is:

$$\begin{aligned}
 z &= \max_{y \in \mathbb{R}, \pi \in \mathbb{R}^m} -y \\
 (\mathcal{DCLP}) : & \quad c_j - \pi^T A_j = 0 \quad j \in \mathcal{S} \\
 & \quad c_j - \pi^T A_j \geq -\lambda_j y \quad j \in \mathcal{L} \\
 & \quad c_j - \pi^T A_j \leq \gamma_j y \quad j \in \mathcal{R} \\
 & \quad y \geq 0
 \end{aligned}$$

As will be shown in the next section, this new complementary problem ensures that basic variables indexed by  $\mathcal{S}$  can decrease significantly the objective value when they exit from the basis. In the same time, the normalization constraint will prevent the variables indexed by  $\mathcal{R}$  from getting out from the basis unless their reduced cost is important enough.

## 4 Theoretical results

In this section, we begin by formulating the main results of the paper and at the end, we propose a choice of  $\mathcal{S}$ ,  $\mathcal{R}$ ,  $\lambda$  and  $\gamma$  that ensures the desired behavior. We start by proving the following lemma.

**Lemma 1** *Let  $(y, \pi) \in \mathbb{R} \times \mathbb{R}^m$  a feasible solution of  $(\mathcal{DCLP})$  then*

$$c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}} \geq 0$$

**Proof.** The proof is straightforward as

$$\begin{aligned}
 c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}} &= c_{\mathcal{C}}^T - \pi^T A_{\mathcal{B}} [A_{\mathcal{B}}]^{-1} A_{\mathcal{C}} \\
 &= c_{\mathcal{C}}^T - \pi^T A_{\mathcal{S}} \left\{ [A_{\mathcal{B}}]^{-1} A \right\}_{\mathcal{S}, \mathcal{C}} \\
 &= c_{\mathcal{C}}^T - c_{\mathcal{S}}^T \left\{ [A_{\mathcal{B}}]^{-1} A \right\}_{\mathcal{S}, \mathcal{C}} + (c_{\mathcal{S}}^T - \pi^T A_{\mathcal{S}}) \left\{ [A_{\mathcal{B}}]^{-1} A \right\}_{\mathcal{S}, \mathcal{C}} \\
 &= c_{\mathcal{C}}^T - c_{\mathcal{S}}^T \left\{ [A_{\mathcal{B}}]^{-1} A \right\}_{\mathcal{S}, \mathcal{C}} \\
 &\geq 0.
 \end{aligned}$$

□

We then prove the following proposition that gives a bound on how far is the current basic solution from the optimal solution. Assume without loss of generality that the set of feasible solutions of the linear problem is bounded. We define  $\Delta$  as the upper bound on components of feasible solutions:

$$\Delta = \max_{i=1, \dots, n} \max_{Ax=b, x \geq 0} x_i$$

**Proposition 1** *Considering the (DCLP),  $x$  the current basic solution and  $x^*$  the optimal solution of (LP) then,*

$$c^T (x - x^*) \leq [m\Delta \|\lambda\|_\infty + \gamma^T x_{\mathcal{R}}] |z|$$

**Proof.** Let  $(y, \pi) \in \mathbb{R} \times \mathbb{R}^m$  a feasible solution of (DCLP),

$$\begin{aligned} c^T x^* &= c_{\mathcal{C}}^T x_{\mathcal{C}}^* + c_{\mathcal{L}}^T x_{\mathcal{L}}^* \\ &\geq c_{\mathcal{C}}^T x_{\mathcal{C}}^* + (-y\lambda^T + \pi^T A_{\mathcal{L}}) x_{\mathcal{L}}^* \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + c_{\mathcal{C}}^T x_{\mathcal{C}}^* + \pi^T A_{\mathcal{L}} x_{\mathcal{L}}^* \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + c_{\mathcal{C}}^T x_{\mathcal{C}}^* + \pi^T (b - A_{\mathcal{C}} x_{\mathcal{C}}^*) \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + \pi^T b + (c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}}) x_{\mathcal{C}}^* \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + \pi^T (A_{\mathcal{C}} x_{\mathcal{C}} + A_{\mathcal{L}} x_{\mathcal{L}}) + (c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}}) x_{\mathcal{C}}^* \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + \pi^T A_{\mathcal{S}} x_{\mathcal{S}} + \pi^T A_{\mathcal{R}} x_{\mathcal{R}} + (c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}}) x_{\mathcal{C}}^* \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + c_{\mathcal{S}}^T x_{\mathcal{S}} + c_{\mathcal{R}}^T x_{\mathcal{R}} - (c_{\mathcal{R}}^T - \pi^T A_{\mathcal{R}}) x_{\mathcal{R}} + (c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}}) x_{\mathcal{C}}^* \\ &= -y [\lambda^T x_{\mathcal{L}}^*] + c^T x - (c_{\mathcal{R}}^T - \pi^T A_{\mathcal{R}}) x_{\mathcal{R}} + (c_{\mathcal{C}}^T - \pi^T A_{\mathcal{C}}) x_{\mathcal{C}}^* \end{aligned}$$

By using the results of the Lemma 1 and the fact that  $c_{\mathcal{R}}^T - \pi^T A_{\mathcal{R}} \leq y\gamma^T$ , we have

$$c^T (x - x^*) \leq y [\lambda^T x_{\mathcal{L}}^* + \gamma^T x_{\mathcal{R}}]$$

Since  $|\lambda^T x_{\mathcal{L}}^*| \leq \|\lambda\|_\infty \sum_{j \in \mathcal{L}} x_j^* \leq m \|\lambda\|_\infty \Delta$  we have

$$c^T (x - x^*) \leq y (m\Delta \|\lambda\|_\infty + \gamma^T x_{\mathcal{R}}).$$

□

Now, we state a proposition quantifying the improvement obtained from the new formulation after one iteration. To ease notations, we define

$$\beta_j := \min \left\{ \frac{\lambda_l}{\left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l}} : l \in \mathcal{L}; \left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l} > 0 \right\}.$$

**Proposition 2** *Consider the (NCLP). If  $z < 0$ , then the improvement of the current iteration is equal to  $z \times \alpha$  where*

$$\alpha \geq \min \left\{ \min \left[ \beta_j \left[ [A_{\mathcal{B}}]^{-1} b \right]_j : j \in \mathcal{S} \right], \min \left[ \gamma_j \left[ [A_{\mathcal{B}}]^{-1} b \right]_j : j \in \mathcal{R} \right] \right\}$$

**Proof.** Let  $(u, w, v) \in \mathbb{R}^{\mathcal{S}} \times \mathbb{R}^{\mathcal{R}} \times \mathbb{R}^{\mathcal{L}}$  be the optimal solution of the complementary problem (NCLP). The (maximum) step through this direction is:

$$\alpha = \max \left\{ t > 0 : x_{\mathcal{S}} - tu \geq 0, x_{\mathcal{R}} - t(w - v_{\mathcal{R}}) \geq 0 \right\}$$

So,

$$\alpha = \max \left\{ t > 0 : t \leq \frac{x_j}{u_j}; j \in \mathcal{S}, u_j > 0, t \leq \frac{x_j}{w_j - v_j}; j \in \mathcal{R}, w_j - v_j > 0 \right\}$$

On one hand, we have  $\gamma^T w \leq 1$  then for  $j \in \mathcal{R}$ ,  $\gamma_j (w_j - v_j) \leq \gamma_j w_j \leq 1$ . If in addition  $w_j - v_j > 0$  then  $\gamma_j x_j \leq \frac{x_j}{w_j - v_j}$ . On other hand, since  $-A_{\mathcal{S}} u - A_{\mathcal{R}} w + A_{\mathcal{L}} v = 0$ , by multiplying by  $[A_{\mathcal{B}}]^{-1}$ ,



$$u = \left[ [A_{\mathcal{B}}]^{-1} \right]_{\mathcal{S},\cdot} A_{\mathcal{L}} v = \left[ [A_{\mathcal{B}}]^{-1} A \right]_{\mathcal{S},\mathcal{L}} v$$

Furthermore, for  $j \in \mathcal{P}$ ,

$$\begin{aligned} u_j &\leq \sum_{l \in \mathcal{L}; [A_{\mathcal{B}}]^{-1} A]_{j,l} > 0} \left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l} v_l \\ &= \sum_{l \in \mathcal{L}; [A_{\mathcal{B}}]^{-1} A]_{j,l} > 0} \lambda_l^{-1} \left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l} \lambda_l v_l \\ &\leq \beta_j^{-1} \sum_{l \in \mathcal{L}_{\mathcal{B}}} \lambda_l v_l \\ &\leq \beta_j^{-1} \end{aligned}$$

If in addition  $u_j > 0$  then  $\beta_j x_j \leq \frac{x_j}{u_j}$ . Using the fact that  $x = [A_{\mathcal{B}}]^{-1} b$ , we have directly the result.  $\square$

Henceforth, we assume that  $c \geq 0$ . This assumption is in fact a weak assumption since  $c_j < 0$  becomes  $c_j > 0$  by using  $y_j = \Delta - x_j$  instead of  $x_j$ . In the following proposition, we prove that there is a choice of all previous parameters such that either the solution is a good approximation of the optimal solution or the improvement after one iteration is significant. Let  $x^+$  be the feasible solution obtained by moving from the current basic solution  $x$  along the optimal direction returned by the  $(\mathcal{NCLP})$ .

**Proposition 3** *Let  $\epsilon \in ]0, \frac{1}{m}[$ ,  $x$  the current basic solution and  $x^*$  the optimal solution. There is a choice of  $\mathcal{S}$ ,  $\mathcal{R}$ ,  $\lambda \in \mathbb{R}_{>0}^{\mathcal{L}}$  and  $\gamma \in \mathbb{R}_{>0}^{\mathcal{R}}$  such that either  $c^T x \leq \frac{1}{1-\epsilon} c^T x^*$  or  $c^T x^+ \leq c^T x (1 - \epsilon^2)$ .*

**Proof.** Let  $a := \left( \left\| [A_{\mathcal{B}}]^{-1} A_j \right\|_{\infty} \right)_{j \in \mathcal{N}}$ ,  $\mu := \frac{m\Delta \|a\|_{\infty} \epsilon}{(1-m\epsilon)}$ ,  $\eta := \frac{\epsilon(1-m\epsilon)}{m\Delta \|a\|_{\infty}}$ ,  $\mathcal{S} := \{j \in \mathcal{P} : x_j > \mu\}$ ,  $\mathcal{R} := \{j \in \mathcal{P} : x_j \leq \mu\}$ ,  $\lambda := (a_j)_{j \in \mathcal{L}}$  and  $\gamma = \left( \frac{\mu}{\left[ [A_{\mathcal{B}}]^{-1} b \right]_j} \right)_{j \in \mathcal{R}}$ . We start by proving that  $\beta_j \geq 1$ ,  $\forall j \in \mathcal{S}$ . Indeed,

$$\beta_j := \min \left\{ \frac{\lambda_l}{\left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l}} : l \in \mathcal{L}; \left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l} > 0 \right\}$$

and for  $l \in \mathcal{L}$ , if  $\left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l} > 0$  then

$$\frac{\lambda_l}{\left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l}} = \frac{\left\| [A_{\mathcal{B}}]^{-1} A_l \right\|_{\infty}}{\left[ [A_{\mathcal{B}}]^{-1} A \right]_{j,l}} \geq 1.$$

Since  $x_j \geq \mu$  for  $j \in \mathcal{S}$ , we have  $\beta_j x_j \geq \mu$ . On the other hand, for  $j \in \mathcal{R}$ ,  $\left[ [A_{\mathcal{B}}]^{-1} b \right]_j = x_j$ , then  $\gamma_j x_j = \mu$ . By using the result of Proposition 2, we can prove that the step  $\alpha$  is at least equal to  $\mu$ . Let  $z$  be the optimal value of  $(\mathcal{NCLP})$ . If  $z \geq -\eta c^T x$ , by using the result from the Proposition 1 we have,

$$\begin{aligned} c^T x - c^T x^* &\leq [m\Delta \| \lambda \|_{\infty} + \gamma^T x_{\mathcal{R}}] |z| \\ &\leq [m\Delta \|a\|_{\infty} + |\mathcal{R}| \mu] \eta c^T x \\ &\leq m \left( \Delta \|a\|_{\infty} + \frac{m\Delta \|a\|_{\infty} \epsilon}{(1-m\epsilon)} \right) \frac{\epsilon(1-m\epsilon)}{m\Delta \|a\|_{\infty}} c^T x \\ &\leq \epsilon c^T x. \end{aligned}$$

This implies that  $c^T x^* \geq (1 - \epsilon)c^T x$ . On the other hand, if  $z \leq -\eta c^T x$ , using the Proposition 2, one can prove that the improvement  $c^T x^+ - c^T x = \alpha z \leq -\mu \times \eta c^T x = -\epsilon^2 c^T x$ . Which implies that  $c^T x^+ \leq (1 - \epsilon^2)c^T x$ .  $\square$

Since the construction of our formulation is based on the fact that  $x$  is a basic solution, it would be great if  $x^+$  is a basic solution as well. Unfortunately, this is not the case in general (see the example introduced in Section 6). However, from  $x^+$ , we can easily construct a better cost basic solution with a procedure of low polynomial complexity. We then can restart our algorithm. Before introducing how to get a basic solution from  $x^+$ , let us prove the following lemma that gives an upper bound on the number of positive variables in  $x^+$ .

**Lemma 2** *We have:*

$$|\{j \in \mathcal{N} : x_j^+ > 0\}| \leq m + |\mathcal{R}| \leq 2m.$$

**Proof.** The last inequality is direct from the fact that  $|\mathcal{R}| \leq m$ .  $\square$

**Proposition 4** *If  $y \in \mathbb{R}^n$  is a non basic feasible solution of  $(\mathcal{LP})$ . We can construct, in polynomial time, a feasible solution  $y^+$  of  $(\mathcal{LP})$  such that  $c^T y^+ \leq c^T y$  and*

$$|\{j \in \mathcal{N} : y_j^+ > 0\}| < |\{j \in \mathcal{N} : y_j > 0\}|.$$

**Proof.** The idea of the construction is simple. First, we start by finding a non-increasing feasible direction. Then we apply a simplex-pivot-like method to take out a variable from the set of positive variables.

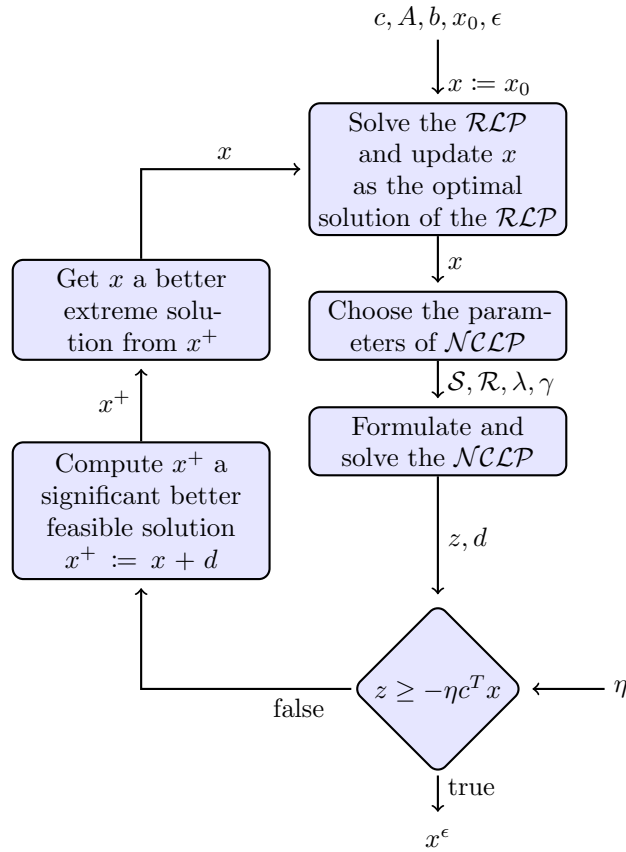
Formally, let  $\mathcal{P} := |\{j \in \mathcal{N} : y_j > 0\}|$ . The  $(A_j)_{j \in \mathcal{P}}$  is a set of linearly dependent vectors. There is  $l \in \mathcal{P}$  and  $(\delta_j)_{j \in \mathcal{P} \setminus \{l\}} \in \mathbb{R}^{\mathcal{P} \setminus \{l\}}$  such that  $A_l = \sum_{j \in \mathcal{P} \setminus \{l\}} \delta_j A_j$ . Consider the vector  $d \in \mathbb{R}^n$  such that  $d_l = -1$ ,  $d_j = \delta_j$  for  $j \in \mathcal{P} \setminus \{l\}$  and  $d_j = 0$  otherwise. It is obvious that  $Ad = 0$ . We can assume that  $c^T d \leq 0$ , by replacing  $d$  by  $-d$ , if it was not the case. For  $t > 0$ , let  $y^t = y + td$ . Using the fact that  $Ad = 0$  and  $c^T d \leq 0$ , one can easily prove that  $Ay^t = Ay = b$ ,  $c^T y^t = c^T y + tc^T d \leq c^T y$  and  $\{j \in \mathcal{N} : y_j^t > 0\} \subset \{j \in \mathcal{N} : y_j > 0\}$ . If  $\{j \in \mathcal{N} : d_j < 0\}$  is empty then the sequence  $(y^t)_t$  will be an unbounded sequence of feasible solutions of  $(\mathcal{LP})$ . Since the set of feasible solutions of  $(\mathcal{LP})$  is bounded then  $\{j \in \mathcal{N} : d_j < 0\}$  is nonempty. Let  $s = \max\{t : y_j^t \geq 0, \forall j \in \mathcal{N}\}$ , then there is  $l' \in \mathcal{P}$  such that  $y_{l'}^s = 0$  and hence  $|\{j \in \mathcal{N} : y_j^s > 0\}| < |\{j \in \mathcal{N} : y_j > 0\}|$ .  $\square$

**Proposition 5** *If  $y \in \mathbb{R}^n$  is a non basic feasible solution of  $(\mathcal{LP})$ . We can construct, in polynomial time, a feasible basic solution of  $(\mathcal{LP})$   $y^*$  such that  $c^T y^* \leq c^T y$ .*

**Proof.** We repeat the idea proved in the Proposition 4 until obtaining a basic solution. Since every step of the procedure is a polynomial time and since the number of positive variables is at most  $2m$ , then the procedure will take a polynomial time as well.  $\square$

## 5 Polynomial IPS

Now we can summarize our new algorithm in the following diagram:



subproblems	variables
$\mathcal{RLP}$ : Reduced problem	$x_0$ : initial basic solution
$\mathcal{NCLP}$ : New Complementary problem	$x$ : current basic solution
<b>parameters:</b> The parameters of $\mathcal{NCLP}$ are chosen from the current basic solution $x$ as explained in the proof of the Proposition 3.	$z$ : optimal solution of $\mathcal{NCLP}$
	$d$ : optimal direction from $\mathcal{NCLP}$
	$x^\epsilon$ : $\epsilon$ -optimal solution.

**Proposition 6** *The algorithm converges in polynomial time, i.e., the number of iterations (calls to the complementary problem) is polynomial.*

**Proof.** As the problem is assumed to be bounded, the result can be deduced directly from Propositions 3 and 5. □

## 6 Numerical example

In this section, we study a simple numerical example that illustrates the benefit of using our new formulation instead of the standard formulation of IPS. The example is as follows. For  $h, H > 0$  such that  $h < 1$  and  $h < \frac{H}{10}$ , consider the linear program:

$$\begin{aligned}
 (\mathcal{LP}) : \quad \min \quad & -x_1 - x_2 \\
 & x_1 + x_2 \geq h \\
 & x_1 - x_2 \leq h \\
 & -(H - h)x_1 + Hx_2 \leq hH \\
 & -(H - h)x_1 + (H - 2h)x_2 \geq -hH
 \end{aligned}$$

this example can be formulated using the slack variables as

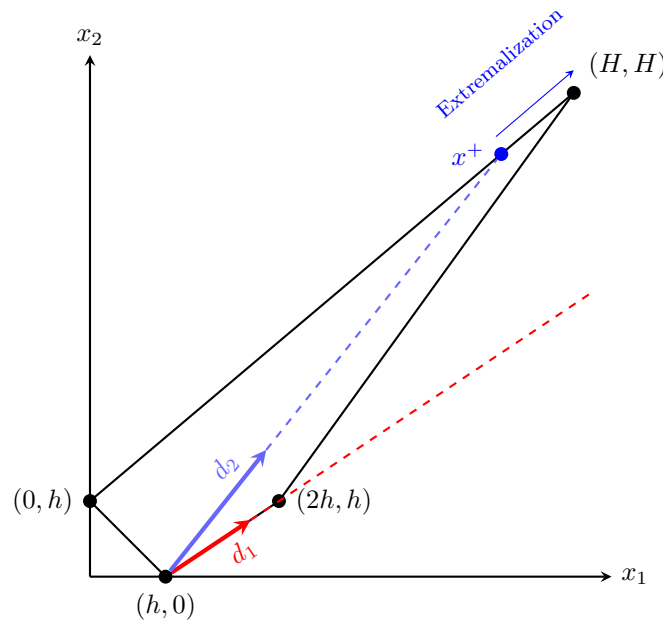
$$\begin{aligned}
 & \min && -x_1 - x_2 \\
 (\mathcal{LP}) : & \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ -(H-h) & H & 0 & 0 & 1 & 0 \\ -(H-h) & (H-2h) & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} h \\ h \\ hH \\ -hH \end{bmatrix}
 \end{aligned}$$

In Table 1, we summarize the results obtained by applying the standard formulation of IPS and our new formulation with  $h = 0.1$ ,  $\epsilon = 0.01$  and for several values of  $H$  starting from the basic solution  $x = (h, 0, 0, 0, h(2H - h), h^2)$ . The improvement factor in the Table 1 is defined as the ratio obtained by dividing the difference of the optimal value and the value after the first iteration by the difference of the optimal value and the value of the objective value before iteration :  $\left| \frac{c^T(x_0 - x^+)}{c^T(x_0 - x^*)} \right|$ .

**Table 1: Improvement factor for  $h = 0.1$ ,  $\epsilon = 0.01$**

$H$	IPS standard	New formulation
2	5.12%	94.79%
24	0.42%	99.58%
46	0.21%	99.78%
68	0.14%	99.78%

To explain why the new formulation gets a solution that is relatively close to the optimal, one can draw the set of feasible solutions and the two directions obtained by each method after one iteration. In red, we see the direction  $d_1$  given by the standard IPS heading from the basic solution  $(h, 0)$  to the basic solution  $(2h, h)$ . On the other hand, the new formulation gives us the direction  $d_2$  which does not ensure reaching a basic solution but ensures that the improvement factor is much larger than the factor obtained by standard IPS.



**Figure 1: Descent directions given by standard IPS and IPS with new complementary problem formulation**

## 7 Summary of results and future research

In this paper, we provided a theoretical proof that an IPS with the proposed formulation of the complementary problem can ensure a significant improvement of the objective function at each iteration, guaranteeing hence a polynomial behavior, i.e., the new IPS finds an  $\epsilon$ -approximation of the optimal solution in a polynomial number of iterations. At each iteration the complementary problem to solve is smaller than the original problem and can be efficiently solved with the dual simplex because it is not dual degenerated.

In the future, it would be interesting to discuss the practical relevance of the new IPS to numerically solve difficult problems and compare it to state of the art solvers including interior-point ones. We think that acceleration strategies like the warm starts and the multiphase strategies proposed in [Saddoune et al. \(2011\)](#) would reduce deeply the solution time of the complementary problem and make the new IPS very competitive.

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