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U. Sadana, P.V. Reddy, T. Başar,
G. Zaccour

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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Sampled-data Nash equilibria in differential games with impulse controls

Utsav Sadana [a]

Puduru Viswanadha Reddy [b]

Tamer Başar [c]

Georges Zaccour [a]

[a] *GERAD & Department of Decision Sciences, HEC Montréal, Montréal (Québec), Canada, H3T 2A7*

[b] *Department of Electrical Engineering, Indian Institute of Technology Madras, Chennai 600 036, India*

[c] *Coordinated Science Laboratory & Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana 61801, IL, USA*

utsav.sadana@hec.ca
vishwa@ee.iitm.ac.in
basar1@illinois.edu
georges.zaccour@gerad.ca

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Abstract: We study a class of deterministic two-player nonzero-sum differential games where one player uses piecewise-continuous controls to affect the continuously evolving state while the other player uses impulse controls at certain discrete instants of time to shift the state from one level to another. The state measurements are made at some given instants of time, and players determine their strategies using the last measured state value. We provide necessary conditions for the existence of sampled-data Nash equilibrium for a general class of differential games with impulse controls. We specialize our results for a scalar linear-quadratic differential game, and show that the equilibrium impulse timing can be obtained by determining a fixed point of a Riccati like system of differential equations with jumps coupled with a system of non-linear equality constraints. By reformulating our problem as a constrained non-linear optimization problem, we compute the equilibrium timing and level of impulses. We find that the equilibrium piecewise continuous control is a linear function of the last measured state value. For linear-state differential games, we obtain analytical characterizations of equilibrium number, timing and levels of impulses in terms of the problem data, and provide an extension of our results for the case with piecewise constant time-varying problem parameters. In particular, there can be at most one impulse in the game when the problem parameters are fixed while each sampling interval can contain at most one impulse when the problem parameters differ between the sampling intervals. Using a numerical example, we illustrate our results.

Keywords: Impulse control, sampled-data, linear-quadratic, differential games, Nash equilibrium

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1 Introduction

Recently, there has been renewed interest in the study of differential games with impulse controls where the state is controlled by two players, at least one of whom can affect the continuously evolving state variable at certain discrete instants of time only Aïd et al. (2020), Ferrari and Koch (2019), Sadana et al. (2020). The number and timing of interventions besides their level are also decision variables in the game. This allows for studying dynamic interactions in option pricing El Farouq et al. (2010), pollution regulation Ferrari and Koch (2019), exchange rate interventions Aïd et al. (2020), cybersecurity Sadana et al. (2019), and related problems. A solution concept for these games involves determining the Nash equilibrium which depends on the information that is available to the players when they make their decisions Başar and Olsder (1999). Nash equilibrium in differential games with impulse controls have been obtained under two information structures, namely, open-loop and feedback information structures. In the open-loop information structure, the equilibrium controls of the players are obtained assuming that players have access to only the initial state, whereas with the feedback information structure, players make their decisions using the state measurements at each instant of time in the game. One limitation of using open-loop strategies is that they are not strongly time-consistent (Başar and Olsder (1999), Başar et al. (2018)), whereas the feedback equilibrium strategies require state measurements to be made at each instant of time in the game. In many real-world problems, for instance, economic data from the surveys, position of players in pursuit-evasion games, quality of goods, the state measurement is costly. As a result, state information is available to the players at the (discrete) sampling instants only, and the players determine their sampled-data controls (Simaan and Cruz Jr. (1973), Başar (1991)), using the previous state measurements. To the best of our knowledge, Nash equilibrium in differential games with impulse controls and sampling has not been studied in the literature.

In Simaan and Cruz Jr. (1973), the authors introduced a deterministic two-player nonzero-sum differential game where state measurement is made at discrete instants of time, and both players use piecewise-continuous strategies. The sampled-data controls of the players are assumed to be functions of the last measured state value, and players implement open-loop controls between the sampling instants. The authors showed that the equilibrium of linear-quadratic differential games can be obtained by solving a system of Riccati equations coupled with a system of differential equations that determine the terminal conditions on the Riccati equations. In Başar (1980), the author studies a stochastic linear-quadratic differential game where players have access to the sampled-data state information as well as the sampling times. A zero-sum linear-quadratic differential game with linear time-varying parameters was studied in Başar (1991) where it is shown that the optimal minimax sampled-data controller can be obtained by solving a generalized Riccati-differential equation. In Başar (1995), the author provided a characterization of the minimax controller of a switching system with sampled state information. In contrast to the aforementioned research that deals with piecewise continuous controls, Drăgan et al. (2019) derived the Nash equilibrium of the stochastic linear-quadratic differential game assuming that the admissible strategies are constant between the state measurements.

In this paper, we consider a general class of deterministic two-player nonzero-sum differential games where the two players are endowed with different kinds of controls (discrete and piecewise-continuous). In particular, Player 1 uses piecewise continuous controls to affect the continuous evolution of state whereas Player 2 uses impulse controls to shift the state value instantaneously from one level to another at the impulse instants that are endogenously determined by Player 2 in addition to the number of impulse instants. The more general case with both players using continuous and impulse controls can be easily studied using our model. However, for the application of our work in problems involving regulation and cybersecurity, we can restrict our focus to our canonical game model with one player using piecewise-continuous controls and the other player using impulse controls.

The objectives of this research are three-fold: First, we aim to provide necessary conditions for the existence of Nash equilibrium. Our second objective is to specialize our results for scalar linear-quadratic differential games (LQDGs) which are widely used in economics, engineering and management domains (see Başar and Olsder (1999), Haurie et al. (2012), Başar et al. (2018)) as they allow the possibility to model real-world problems involving non-linear returns to scale. Also, linear dynamics can approximate sufficiently well the non-linear dynamics, at least in some applications. Third, we aim to determine analytical solutions for equilibrium number, timing and levels of impulses in scalar linear-state differential games (see Başar and

Olsder (1999), Dockner et al. (2000), Engwerda (2005), and Haurie et al. (2012)) where we restrict the payoff functions to be linear in state, and state dynamics to be linear in both state and controls of the players.

Our contributions can be summarized as follows:

(i) For the first time, our paper provides necessary conditions for the existence of Nash equilibrium in a differential game with impulse controls where the players' strategies are functions of the state values measured at certain discrete time instants; see Theorem 1.

(ii) For the case of LQDGs with exogenously given impulse instants, Theorem 2 provides a system of Riccati like equations with jumps which characterize the sampled-data Nash equilibrium.

(iii) For LQDGs with a given number of impulses in each sampling interval, Theorem 3 shows that the equilibrium timing of impulses can be obtained as a solution of a system of Riccati equations (with jumps) provided that the impulse instants satisfy a system of non-linear equality constraints. In particular, we show that an impulse occurs when the state trajectory hits a time-varying function of the gradient of the value function of Player 2.

(iv) In Theorem 4, we show that there can be at most one impulse in the sampled-data Nash equilibrium of a scalar linear-state differential game. When the problem parameters are piecewise constant functions of time, we show that in the scalar linear-state differential game, the number of impulses is at most equal to the number of sampling intervals; see Theorem 5.

The rest of this paper is organized as follows: In Section 2, we introduce our canonical two-player differential game model. Section 3 provides necessary conditions for the existence of sampled-data Nash equilibrium for our canonical model. In Section 4, we specialize the necessary conditions to a scalar linear-quadratic differential game. We further specialize our results to a scalar linear-state differential game in Section 5, and also provide an extension of our game to problems with time-varying parameters. Further, we illustrate the theoretical results using a numerical example in Section 6. Finally, Section 7 provides concluding remarks, and the paper ends with an appendix, which details the proof of Theorem 2.

2 Model

In this paper, we consider a deterministic two-player differential game of finite duration $T < \infty$ where both players can affect a continuously evolving state variable $x(t) \in \mathbb{R}^n$ to maximize their payoffs. However, the two players are endowed with different kinds of controls. Player 1 can continuously influence the dynamics of the state variable using her piecewise continuous controls $u(t) \in \Omega_u$ while Player 2 can intervene and cause jumps in the state variable at certain discrete instants of time $\tau_i\,(i = 1, 2, \cdots, k)$. We assume that Ω_u is a compact and convex subset of $\mathbb{R}^{m_1}$. When Player 2 does not intervene in the game, the state variable is continuous and its dynamics are controlled entirely by Player 1 so that the state variable evolves as follows:

$$\dot{x}(t) = f(x(t), u(t)),\ x(0^-) = x_0,\ \text{for}\ t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{1}$$

where $f : \mathbb{R}^n \times \Omega_u \to \mathbb{R}^n$, the initial value of state variable is given by $x_0 \in \mathbb{R}^n$ (a known parameter), $x(\tau_i^-) = \lim_{t\uparrow\tau_i} x(t)$, $x(\tau_i^+) = \lim_{t\downarrow\tau_i} x(t)$, and 0^- denotes the time instant just before 0. At the impulse instants τ_i, Player 2 intervenes in the game to shift the state from $x(\tau_i^-)$ to $x(\tau_i^+)$ by using an impulse of size $v_i \in \Omega_v$, that is,

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i),\ i \in \{1, 2, \cdots, k\}, \tag{2}$$

where $g : \mathbb{R}^n \times \Omega_v \to \mathbb{R}^n$. We assume that Ω_v is a compact and convex subset of $\mathbb{R}^n$. The number of impulses $k \in \mathbb{N}$ (the set of natural numbers), and timing of impulses τ_i are decision variables of Player 2 in addition to the levels of impulses. The impulse controls are denoted by $\tilde{v} = \{(\tau_i, v_i),\ i = \{1, 2, \cdots, k\}, k\}$.

In this differential game, Player 1 maximizes the following objective:

$$J_1(x_0, u(\cdot), \tilde{v}) = \int_0^T F_1(x(t), u(t))dt + \sum_{i=1}^{k} G_1(x(\tau_i^-), v_i) + S_1(x(T^+)), \tag{3}$$

and Player 2 uses the impulse controls (τ_i, v_i) to maximize the objective

$$J_2(x_0, u(\cdot), \tilde{v}) = \int_0^T F_2(x(t), u(t))dt + \sum_{i=1}^{k} G_2(x(\tau_i^-), v_i) + S_2(x(T^+)), \tag{4}$$

where F_1, $F_2 : \mathbb{R}^n \times \Omega_u \to \mathbb{R}$, G_1, $G_2 : \mathbb{R}^n \times \Omega_v \to \mathbb{R}$, and S_1, $S_2 : \mathbb{R}^n \to \mathbb{R}$. For Player 1, F_1 denotes the running payoff, G_1 denotes the intervention cost at the impulse instants, and S_1 is the terminal payoff. For Player 2, the running payoff is given by F_2, while G_2 represents the intervention costs at the impulse instants, and S_2 denotes the terminal payoff.

In a differential game, the Nash equilibrium depends on the state information that the players use to determine their strategies (see Başar and Olsder (1999), Haurie et al. (2012)). We assume that the state measurement is made at certain discrete instants of time $t_n, n \in \{1, 2, \cdots, N\}$, with the corresponding state values denoted by $x_1, x_2, \cdots, x_N$ such that $0 = t_1 < t_2 < \cdots < t_{N-1} < t_N = T$. The sampled-data controls of Player 1 are given by

$$u(t) = \gamma(t; x(t_n)) \in \Omega_u, \text{ for } t_n \leq t < t_{n+1},\ n \in \mathcal{N}' = \{1, 2, \cdots, N-1\},\ \gamma \in \Gamma, \tag{5}$$

where $\gamma : [t_n, t_{n+1}] \times \mathbb{R}^n \to \Omega_u$ is a sampled-data state feedback controller of Player 1 and the strategy set of Player 1 is denoted by Γ. Similarly, the impulse levels of Player 2 are given by

$$v_{i,n} = \delta(\tau_{i,n}; x(t_n)) \in \Omega_v, \text{ for } t_n \leq \tau_{i,n} < t_{n+1},\ n \in \mathcal{N}',\ \delta \in \Delta, \tag{6}$$

where $\delta : [t_n, t_{n+1}] \times \mathbb{R}^n \to \Omega_u$ is a sampled-data state feedback controller for Player 2 and Δ denotes the strategy set of Player 2.

The objective functions of the players over the sub-interval $[t_n, T]$, initialized at the sampling instant t_n with the corresponding state $x(t_n) = x_n$ are given by

$$\begin{aligned} &J_1(x_n, \gamma_{[t_n,T]}, \delta_{[t_n,T]}) \\ &= \sum_{j=n}^{N-1} \left(\int_{t_j}^{t_{j+1}} F_1(x(t), \gamma(t; x(t_j)))dt + \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_n} G_1(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x(t_j))) \right) + S_1(x(T^+)), \end{aligned} \tag{7a}$$

$$\begin{aligned} &J_2(x_n, \gamma_{[t_n,T]}, \delta_{[t_n,T]}) \\ &= \sum_{j=n}^{N-1} \left(\int_{t_j}^{t_{j+1}} F_2(x(t), \gamma(t; x(t_j)))dt + \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_n} G_2(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x(t_j))) \right) + S_2(x(T^+)), \end{aligned} \tag{7b}$$

where the strategies $\gamma_{[t_n,T]}$ and $\delta_{[t_n,T]}$ are restrictions of γ and δ to the interval $[t_n, T]$, and $\Gamma_{[t_n,T]}$ and $\Delta_{[t_n,T]}$ denote the corresponding admissible strategy sets of Player 1 and Player 2, respectively. The state dynamics are given by

$$\dot{x}(t) = f(x(t), \gamma(t; x_n)),\ x(t_n^-) = x_n, \text{ for } t_n \leq t < t_{n+1},\ n \in \mathcal{N}', \tag{7c}$$

$$x(\tau_{i,n}^+) - x(\tau_{i,n}^-) = g(x(\tau_{i,n}^-), \delta(\tau_{i,n}; x_n)), \text{ for } i \in \mathcal{I}^n = \{1, 2, \cdots, k_n\}, \tag{7d}$$

where k_n denotes the equilibrium number of impulses in the sampling interval $[t_n, t_{n+1}]$. From (7a)-(7b), it is clear that each player can influence the payoff of their opponent directly through their controls, and indirectly by changing the state variable.

Remark 1 *The above canonical differential game model (7a–7d) can be used to study problems in cybersecurity and pollution regulation where the running payoff of one player, say Player 1, decreases with state and Player 2's running payoff is increasing with state. Player 1 continuously invests in reducing the state except at the impulse instants wherein Player 2 intervenes in the game to instantaneously shift the state to a higher value. Consequently, Player 1 incurs a state-dependent cost at the impulse instant.*

Clearly, the admissible controls in the aforementioned real-world applications satisfy the following definition:

Definition 1 $(\tau_{i,n}, v_{i,n})$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$, *is an admissible impulse control of Player 2 if the impulse instants satisfy the following increasing monotone sequence property:*

$$t_n < \tau_{1,n} < \tau_{2,n} < \cdots < \tau_{k_n,n} < t_{n+1}, \tag{8}$$

where $k_n < \infty$, $v_{i,n} \neq 0$, *and it is assumed that the impulse instants are interior, that is,* $\tau_{i,n} \in (t_n, t_{n+1})$.

In this paper, we seek to determine the sampled-data Nash equilibrium of the differential game (7a–7d), which is defined as follows:

Definition 2 *The strategy profile* (γ^*, δ^*) *is a sampled-data Nash equilibrium of the differential game (7a–7d), if the restrictions of* γ^* *and* δ^*, *denoted by* $\gamma^*_{[t_n,T]}$ *and* $\delta^*_{[t_n,T]}$, *to any subgame that starts at the sampling time* t_n *with state measurement* x_n *satisfy the following inequalities:*

$$J_1(x_n, \gamma^*_{[t_n,T]}, \delta^*_{[t_n,T]}) \geq J_1(x_n, \gamma_{[t_n,T]}, \delta^*_{[t_n,T]}),\ \forall \gamma_{[t_n,T]} \in \Gamma_{[t_n,T]}, \tag{9a}$$

$$J_2(x_n, \gamma^*_{[t_n,T]}, \delta^*_{[t_n,T]}) \geq J_2(x_n, \gamma^*_{[t_n,T]}, \delta_{[t_n,T]}),\ \forall \delta_{[t_n,T]} \in \Delta_{[t_n,T]}. \tag{9b}$$

Remark 2 *The sampled-data Nash equilibrium strategies of the differential game (7a–7d) for* $t \in [0, T]$ *when restricted to* $[t_n, T]$ *are also the Nash equilibrium strategies of the subgame that starts at* t_n. *As a result, the sampled-data Nash equilibrium strategies are strongly time-consistent Başar (2018) if the perturbation of state can occur only at the sampling instants* t_n, $n \in \mathcal{N} = \{1, 2, \cdots, N\}$. *At all other time instants, that is,* $t \neq t_n, n \in N$, *the sampled-data Nash equilibrium strategies are weakly time-consistent Başar (2018).*

Remark 3 *When sampling is done at the initial and final time only, then the sampled-data Nash equilibrium coincides with the open-loop Nash equilibrium of a differential game. It is shown in Simaan and Cruz Jr. (1973) that the sampled-data equilibrium controls approach the closed-loop controls as the number of sampling intervals increases.*

3 Necessary conditions

In this section, we derive the necessary conditions for the existence of sampled-data Nash equilibrium in differential games with impulse controls.

The approach to determine the sampled-data Nash equilibrium can be summarized as follows. Suppose the sampling instants are given by $t_1, t_2, \cdots, t_N$. For $t \in [t_n, t_{n+1}]$, players use open-loop strategies $\gamma^*(t; x_n)$ and $\delta^*(t; x_n)$, which are functions of last measured state value x_n, that is, for any given initial state x_n, Player 1 determines the open-loop controls in the sampling interval and Player 2 determines the equilibrium number, timing and levels of impulses. The payoff of each player at $(t_n, x(t_n))$ is a salvage value for the open-loop game between t_{n-1} and t_n. Therefore, starting from the last sampling interval $[t_{N-1}, T]$ with salvage values S_1 and S_2, we can recursively obtain the equilibrium strategies for all the sampling intervals $[t_n, t_{n+1}]$, $n \in \mathcal{N}'$.

First, we define the Hamiltonians of the two players that will be used in the necessary conditions for the existence of sampled-data Nash equilibrium. The continuous Hamiltonians of Player 1 and Player 2 are given by

$$H_1(x(t), u(t), \lambda_1(t)) = F_1(x(t), u(t), \lambda_1(t)) + \lambda_1(t)^T f(x(t), u(t)), \tag{10}$$

$$H_2(x(t), u(t), \lambda_2(t)) = F_2(x(t), u(t), \lambda_2(t)) + \lambda_2(t)^T f(x(t), u(t)), \tag{11}$$

where $\lambda_1(.)$ and $\lambda_2(.)$ denote the co-states of Player 1 and Player 2, respectively. The impulse Hamiltonian of Player 2 is given by

$$H_2^I(x(t), v, \lambda_2(t)) = G_2(x(t), v) + \lambda_2(t)^T g(x(t), v). \tag{12}$$

Given the strategies, γ and δ, the value-to-go functions of Player 1 and Player 2 at the sampling instants t_{n+1}, $n \in \mathcal{N}'$ are given by

$$V_1(t_{n+1}, x_{n+1}) = \sum_{j=n+1}^{N-1} \left(\int_{t_j}^{t_{j+1}} F_1(x(t), \gamma(t; x_j))dt + \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_{n+1}} G_1(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x_j)) \right) + S_1(x(T)), \tag{13a}$$

$$V_2(t_{n+1}, x_{n+1}) = \sum_{j=n+1}^{N-1} \left(\int_{t_j}^{t_{j+1}} F_2(x(t), \gamma(t; x_j))dt + \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_{n+1}} G_2(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x_j)) \right) + S_2(x(T)), \tag{13b}$$

with $V_1(T, x(T)) = S_1(x(T))$, and $V_2(T, x(T)) = S_2(x(T))$. We denote the equilibrium payoffs of Player 1 and Player 2 at t_{n+1} by $V_1^*(t_{n+1}, x_{n+1})$ and $V_2^*(t_{n+1}, x_{n+1})$, respectively.

To derive a set of necessary conditions for the existence of Nash equilibrium, we make the following assumptions:

Assumption 1 *(a) The function $f : \mathbb{R}^n \times \Omega_u \to \mathbb{R}^n$ is Lipschitz continuous in x for all u.*

(b) The functions of time are piecewise continuous, and left continuous at points of discontinuity, e.g., $x(\tau_{i,n}^-) = x(\tau_{i,n})$, $i = \mathcal{I}^n$, $n = \mathcal{N}'$.

(c) Between the sampling instants, the functions F_1, F_2, G_1, G_2 are continuous, and have continuous partial derivatives with respect to their arguments. The value-to-go functions V_1 and V_2 are continuous, and have continuous partial derivatives with respect to the state at the sampling instants.

The following theorem gives the necessary conditions for the existence of sampled-data Nash equilibrium of the differential game (7a–7d).

Theorem 1 *Suppose the sampling instants are given by $t_1, t_2, \cdots, t_N$ with $0 = t_1 < t_2 < \cdots < t_N = T$, and Assumption 1 holds. Let (γ^*, δ^*) be the sampled-data Nash equilibrium of the differential game described by (7a–7d). Then, there exist piecewise continuous and piecewise differentiable functions $\lambda_1(.)$ and $\lambda_2(.)$ with $\lambda_1(t) \in \mathbb{R}^n$ and $\lambda_2(t) \in \mathbb{R}^n$ such that the following conditions hold for $t \in [t_n, t_{n+1}), n \in \mathcal{N}'$:*
The equilibrium control of Player 1 satisfies

$$u^*(t) = \arg\max_{u \in \Omega_u} H_1(x^*(t), u(t), \lambda_1(t)), \forall t \notin \mathcal{T}^n = \{\tau_{1,n}^*, \tau_{1,n}^*, \cdots, \tau_{k_n^*,n}^*\}. \tag{14a}$$

At the impulse instant $\tau_{i,n}^*, i \in \mathcal{I}^n$, the equilibrium control of Player 2 satisfies

$$v_{i,n}^* = \arg\max_{v_{i,n} \in \Omega_v} H_2^I(x^*(\tau_{i,n}^{*-}), v_{i,n}, \lambda_2(\tau_{i,n}^{*+})). \tag{14b}$$

The equilibrium strategies of Player 1 and Player 2 are given by $\gamma^*(t; x_n) = u^*(t), \forall t \in [t_n, t_{n+1}]$, $t \notin \mathcal{T}^n$ and $\delta^*(\tau_{i,n}^*; x_n) = v_{i,n}^*$, $\forall i \in \mathcal{I}^n$.

The maximized Hamiltonian and impulse Hamiltonian functions are given by

$$H_1^*(x^*(t), \lambda_1(t)) = H_1(x^*(t), u^*(t), \lambda_1(t)), \ \forall t \notin \mathcal{T}^n, \tag{14c}$$

$$H_2^{I^*}(x^*(\tau_{i,n}^{*-}), \lambda_2(\tau_{i,n}^{*+})) = H_2^I(x^*(\tau_{i,n}^{*-}), v_{i,n}^*, \lambda_2(\tau_{i,n}^{*+})), \ i \in \mathcal{I}^n, \tag{14d}$$

the equilibrium state and co-state equations satisfy for $t \notin \mathcal{T}^n$,

$$\dot{x}^*(t) = f(x^*(t), u^*(t)), \ x^*(t_n) = x_n, \tag{14e}$$

$$\dot{\lambda}_1(t) = -H_{1x}^*(x^*(t), \lambda_1(t)), \ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x(t_{n+1}))}{\partial x}, \ V_1^*(T, x(T)) = S_1(x(T)), \tag{14f}$$

$$\dot{\lambda}_2(t) = -H_{2x}^*(x^*(t), u^*(t), \lambda_2(t)), \ \lambda_2(t_{n+1}) = \frac{\partial V_2^*(t_{n+1}, x(t_{n+1}))}{\partial x}, \ V_2^*(T, x(T)) = S_2(x(T)), \tag{14g}$$

the jumps in the state and co-state variables satisfy for $i \in \mathcal{I}^n$

$$x^*(\tau_{i,n}^{*+}) = x^*(\tau_{i,n}^{*-}) + g(x^*(\tau_{i,n}^{*-}), v_{i,n}^*), \tag{14h}$$

$$\lambda_1(\tau_{i,n}^{*-}) = (I + (g_x(x^*(\tau_{i,n}^{*-}), v_{i,n}^*))^T)\lambda_1(\tau_{i,n}^{*+}) + G_{1x}(x^*(\tau_{i,n}^{*-}), v_{i,n}^*), \tag{14i}$$

$$\lambda_2(\tau_{i,n}^{*-}) = \lambda_2(\tau_{i,n}^{*+}) + H_{2x}^{I*}(x^*(\tau_{i,n}^{*-}), \lambda_2(\tau_{i,n}^{*+})), \tag{14j}$$

and the following Hamiltonian continuity condition holds:

$$H_2(x^*(\tau_{i,n}^{*+}), u^*(\tau_{i,,n}^{*+}), \lambda_2(\tau_{i,n}^{*+})) = H_2(x^*(\tau_{i,n}^{*-}), u^*(\tau_{i,n}^{*-}), \lambda_2(\tau_{i,n}^{*-})). \tag{14k}$$

Proof. For $t \in [t_n, t_{n+1}]$, Player 1 and Player 2 play their open-loop Nash equilibrium strategies, $\gamma^*(t; x_n)$ and $\delta^*(\tau_{i,n}; x_n)$, that depend on the last measured state value x_n. The salvage values of the two players at t_{n+1} are given by (13a) and (13b).

Given the equilibrium strategy $\delta^*(\tau_{i,n}^*, x_n)$ of Player 2 in the sampling interval $[t_n, t_{n+1}]$, Player 1 solves a non-standard optimal control problem given in (9a) due to jumps in the state and the additional cost at the impulse instant. Suppose Assumption 1 holds. Then, the optimality conditions for Player 1 are given in (14a), (14e), (14f), (14h), (14i) (see Geering (1976), Sadana et al. (2019)), with co-state at t_{n+1} given by the gradient of the equilibrium payoff of Player 1 at t_{n+1}. Next, for Player 1's open-loop equilibrium strategy, $\gamma^*(t; x_n)$ in $[t_n, t_{n+1}]$, Player 2 solves the impulse optimal control problem (9b). The necessary conditions for the existence of the impulse controls follow from Blaquière (1977a), Blaquière (1977b), Chahim et al. (2012), and are given by (14b), (14e), (14h), (14g), (14j), (14k), where the co-state at t_{n+1} is given by the gradient of the equilibrium payoff of Player 2 at t_{n+1}. □

The necessary conditions yield the candidates for the sampled-data Nash equilibrium. In each sampling interval, the players use open-loop Nash equilibrium strategies, and the game is solved using backward translation starting from the last sampling interval. Consequently, if the sufficient conditions for the open-loop Nash equilibrium are satisfied in each sampling interval, then the candidate solutions identified by using the necessary conditions are indeed the sampled-data Nash equilibrium strategies.

Sufficient conditions for the existence of sampled-data Nash equilibrium for the differential game described by (7a–7d) are given as follows:

Proposition 1 (Theorem 3, Sadana et al. (2019)) *Let Assumption 1 hold. Suppose in each sampling interval $[t_n, t_{n+1}], n \in \mathcal{N}'$, the initial state is given by x_n, and there exist feasible solutions $(\gamma^*(t; x_n), \delta^*(\tau_{i,n}^*; x_n))$ with corresponding state trajectory $x^*(.)$, and co-state trajectories $\lambda_1(.)$ and $\lambda_2(.)$, such that the conditions given in Theorem 1 are satisfied. Also, if in each sampling interval, the maximized Hamiltonian $H_1^*(x(t), \lambda_1(t))$ of Player 1 is concave in $x(t)$ for all $\lambda_1(t)$, the Hamiltonian $H_2(x(t), u^*(t), \lambda_2(t))$ of Player 2 is concave in $x(t)$, the value-to-go functions for Player 1 and Player 2 given by (13a) and (13b) are concave in $x(t_{n+1})$, $G_1(x(t), v) + \lambda_1^T g(x(t), v)$ is concave in $x(t)$, and the impulse Hamiltonian $H_2^I(x(t), v, \lambda_2(t))$ of Player 2 is concave in $(x(t), v)$, then (γ^*, δ^*), obtained by concatenating the (open-loop) strategies $(\gamma^*(t; x_n)$, $\delta^*(\tau_{i,n}^*; x_n))$ for $t \in [t_n, t_{n+1}]$, are indeed the sampled-data Nash equilibrium strategies of the differential game described by (7a–7d).*

4 Scalar linear-quadratic differential game

In this section, we specialize our results in Theorem 1 to a one-dimensional linear-quadratic differential game with impulse controls, where state measurements are made at the sampling instants t_n, $n \in \mathcal{N} = \{1, 2, \cdots, N\}$ such that $0 = t_1 < t_2 < \cdots < t_N = T$.

We study the following scalar linear-quadratic differential game with impulse controls (referred to as iLQDG from here on):

$$
\text{(iLQDG)}\quad
\begin{aligned}
J_1(x_0,u(\cdot),\tilde{v}) =& \frac{1}{2}\Big[\sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}}\big(h_1x(t)^2+2w_1x(t)+c_uu(t)^2\big)\,dt \\
&+\sum_{i=1}^{k_n}\big(z_1x(\tau_{i,n}^-)^2+2d_1x(\tau_{i,n}^-)\big)\Big)+f_1x(T)^2+2s_1x(T)\Big],\\
J_2(x_0,u(\cdot),\tilde{v}) =& \sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}}w_2x(t)dt+\sum_{i=1}^{k_n}\Big(\frac{1}{2}c_vv_{i,n}^2\Big)\Big)+s_2x(T),\\
\dot{x}(t) =& ax(t)+bu(t),\ \forall t\notin\mathcal{T}^n,\ n\in\mathcal{N}',\ x(0)=x_0,\\
x(\tau_{i,n}^+) =& x(\tau_{i,n}^-)+gv_{i,n},\ \forall i\in\mathcal{I}^n=\{1,2,\cdots,k_n\},\ n\in\mathcal{N}',
\end{aligned}
\tag{15}
$$

where $b\neq 0$, $g\neq 0$, $c_u<0$, $c_v<0$, and the state at the sampling instants $t_1,t_2,\cdots,t_N$ is denoted by $x_1,x_2,\cdots,x_N$.

We make the following assumptions on the equilibrium controls of the players:

Assumption 2 *In each sampling interval, Player 1's strategy space $\Gamma_{[t_n,t_{n+1}]}$ is the set of locally square-integrable functions, that is,*

$$
\Gamma_{[t_n,t_{n+1}]} := \Big\{u(t)\in\mathbb{R},\ t\in[t_n,t_{n+1}]\ \Big|\ \int_{t_n}^{t_{n+1}}u^T(t)u(t)dt<\infty\Big\},
\tag{16}
$$

and Player 2's controls satisfy Definition 1.

Assumption 3 *The equilibrium controls $u^*(t)$ of Player 1 and equilibrium impulse levels v_i^* of Player 2 lie in the interior of the control sets Ω_u and Ω_v.*

4.1 Necessary conditions

Before considering the case where the number, timing and levels of impulses are determined by Player 2, we consider the differential game (15) with exogenously given impulse instants.

Theorem 2 *Let $t_1,t_2,\cdots,t_N$ denote the sampling instants, and suppose that Assumptions 2 and 3 hold. Let the equilibrium impulse instants be given by $\tau_{i,n}^*,\forall i\in\mathcal{I}^n=\{\tau_1^*,\tau_2^*,\cdots,\tau_{k_n^*,n}^*\},n\in\mathcal{N}'=\{1,2,\cdots,N-1\}$. Then γ^* and δ^* are the equilibrium strategies of Player 1 and Player 2, respectively if the following Riccati system for $n\in\mathcal{N}'$ has a solution with no finite escape time in all the sampling intervals $[t_n,t_n+1]$:*

$$\dot{\alpha}_{1,n}(t)=-2\alpha_{1,n}(t)a+\frac{b^2}{c_u}\alpha_{1,n}(t)^2-h_1,\ \forall t\notin\mathcal{T}^n,\ \alpha_1(t_{n+1})=p_{1,n+1}(t_{n+1}),\ \alpha_{1,N}(T)=f_1, \tag{17a}$$

$$\dot{\beta}_{1,n}(t)=\beta_{1,n}(t)\left(\frac{b^2}{c_u}\alpha_{1,n}(t)-a\right)-w_1,\ \forall t\notin\mathcal{T}^n,\ \beta_1(t_{n+1})=q_{1,n+1}(t_{n+1}),\beta_{1,N}(T)=s_1, \tag{17b}$$

$$\alpha_{1,n}(\tau_{i,n}^{*-})=\alpha_{1,n}(\tau_{i,n}^{*+})+z_1,\ \forall i\in\mathcal{I}^n, \tag{17c}$$

$$\beta_{1,n}(\tau_{i,n}^{*-})=\beta_{1,n}(\tau_{i,n}^{*+})-\alpha_{1,n}(\tau_{i,n}^{*+})\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+})+d_1,\ \forall i\in\mathcal{I}^n, \tag{17d}$$

$$\dot{p}_{1,n}(t)=-h_1-2(a-\frac{b^2}{c_u}\alpha_{1,n}(t))p_{1,n}(t)-\frac{b^2}{c_u}\alpha_{1,n}(t)^2,\ \forall t\notin\mathcal{T}^n, \tag{17e}$$

$$\dot{q}_{1,n}(t)=-w_1+\frac{b^2}{c_u}p_{1,n}(t)\beta_{1,n}(t)-q_{1,n}(t)(a-\frac{b^2}{c_u}\alpha_{1,n}(t))-\frac{b^2}{c_u}\alpha_{1,n}(t)\beta_{1,n}(t),\ \forall t\notin\mathcal{T}^n, \tag{17f}$$

$$p_{1,n}(\tau_{i+1,n}^{*-})=p_{1,n}(\tau_{i+1,n}^{*+})+z_1,\ \forall i\in\mathcal{I}^n, \tag{17g}$$

$$q_{1,n}(\tau_{i+1,n}^{*-})=-p_{1,n}(\tau_{i+1,n}^{*+})\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+})+q_{1,n}(\tau_{i+1,n}^{*+})+d_1,\ \forall i\in\mathcal{I}^n, \tag{17h}$$

$$p_{1,n}(t_{n+1}) = p_{1,n+1}(t_{n+1}),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}),\ p_{1,N}(T) = f_1,\ q_{1,N}(T) = s_1, \tag{17i}$$

$$\dot{q}_{2,n}(t) = -w_2 - (a - \frac{b^2}{c_u}\alpha_{1,n}(t))q_{2,n}(t), \forall t \notin \mathcal{T}^n,\ q_{2,N}(T) = s_2, \tag{17j}$$

$$\lambda_2(t_{n+1}) = q_{2,n}(t_{n+1}), \tag{17k}$$

$$\lambda_2(t) = -\frac{w_2}{a} + (\lambda_2(t_{n+1}) + \frac{w_2}{a})e^{a(t_{n+1}-t)},\ \forall t \in [t_n, t_{n+1}), \tag{17l}$$

$$q_{2,n}(\tau^{*-}_{i+1,n}) = q_{2,n}(\tau^{*+}_{i+1,n}),\ \forall i \in \mathcal{I}^n, \tag{17m}$$

$$q_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ q_{2,N}(T) = s_2. \tag{17n}$$

The equilibrium strategies of Player 1 and Player 2 are given by

$$\begin{aligned}\gamma^*(t; x_n) = &-\frac{b}{c_u}\Big(\alpha_{1,n}(t)\big(\phi(t,\tau^{*+}_{i,n})(\phi(\tau^{*-}_{i,n},t_n)x_n + gv^*_{i,n}\mathbb{1}_{t>\tau^*_{i,n}} + \varphi(\tau^{*-}_{i,n},t_n))\\ &+ \varphi(t,\tau^{*+}_{i,n})\big) + \beta_{1,n}(t)\Big), \forall t \in [\tau^*_{i,n}, \tau^*_{i+1,n}), i \in \mathcal{I}^n \cup \{0\},\end{aligned} \tag{18a}$$

$$\delta^*(\tau^*_{i,n}; x_n) = \frac{g}{c_v}\left(\frac{w_2}{a} - (\lambda_2(t_{n+1}) + \frac{w_2}{a})e^{a(t_{n+1}-\tau^*_{i,n})}\right), \tag{18b}$$

where $\tau^*_{0,n} := t_n$, $\tau^*_{k^*_n+1,n} := t_{n+1}$, *and* $\forall i \in \{0\} \cup \mathcal{I}^n$,

$$\dot{\phi}(t,\tau^*_{i,n}) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t,\tau^*_{i,n}), \forall t \in (\tau^*_{i,n}, \tau^*_{i+1,n}), \phi(\tau^*_{i,n},\tau^*_{i,n}) = 1, \tag{19a}$$

$$\varphi(t,\tau^{*-}_{i,n}) = -\int_{\tau^{*-}_{i,n}}^{t} \phi(h,\tau^{*-}_{i,n})\frac{b^2}{c_u}\beta_{1,n}(h)dh, \forall t \in (\tau^*_{i,n}, \tau^*_{i+1,n}), \tag{19b}$$

$$\phi(\tau^{*-}_{i+1,n}, t_n) = \phi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n})\phi(\tau^{*-}_{i,n}, t_n), \tag{19c}$$

$$\varphi(\tau^{*-}_{i+1,n}, t_n) = \phi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n})\varphi(\tau^{*-}_{i,n}, t_n) - \phi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n})\frac{g^2}{c_v}\lambda_2(\tau^{*+}_{i,n}) + \varphi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n}). \tag{19d}$$

Proof. See Appendix. □

Remark 4 *Even when the timing of impulses is given, the Riccati like system of equations (17a)–(17n) differ from those obtained for classical differential games without impulse controls because of jumps in state and additional costs incurred by the players at the impulse instants.*

The above theorem characterizes the equilibrium with exogenously given impulse instants. If the number and timing of impulses are determined by Player 2, the impulse instants must satisfy the Hamiltonian continuity condition (14k) in addition to (17a)–(17n).

Theorem 3 *Suppose* $t_1, t_2, \cdots, t_N$ *are the sampling instants, and Assumptions 2 and 3 hold. Then* $\tau^*_{i,n}$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$ *are the equilibrium impulse instants if*

$$x(\tau^*_{i,n}) = \phi(\tau^{*-}_{i,n}, t_n)x_n + \varphi(\tau^{*-}_{i,n}, t_n) = \frac{\left(\frac{c_u g^2}{c_v b^2}(aq_{2,n+1}(t_{n+1}) + w_2)e^{a(t_{n+1}-\tau^*_{i,n})}\right) - d_1}{z_1}, \tag{20a}$$

where $q_{2,n+1}(t_{n+1})$ *is the gradient of the value function of Player 2 at the sampling instant* t_{n+1}, ϕ *and* φ *satisfy (19a)-(19d), and the Riccati system (17a)–(17n) has no finite escape time.*

Proof. From the continuity condition (14k) on the Hamiltonian, we have for $i \in \mathcal{I}^n, n \in \mathcal{N}'$,

$$w_2 x(\tau^{*+}_{i,n}) + \lambda_2(\tau^{*+}_{i,n})(ax(\tau^{*+}_{i,n}) + bu(\tau^{*+}_{i,n})) = w_2 x(\tau^{*-}_{i,n}) + \lambda_2(\tau^{*-}_{i,n})(ax(\tau^{*-}_{i,n}) + bu(\tau^{*-}_{i,n})).$$

Using the continuity of the co-state of Player 2 (53), we can write the above equation as

$$w_2(x(\tau^{*+}_{i,n}) - x(\tau^{*-}_{i,n})) + a\lambda_2(\tau^*_{i,n})(x(\tau^{*+}_{i,n}) - x(\tau^{*-}_{i,n})) + \lambda_2(\tau^*_{i,n})b(u(\tau^{*+}_{i,n}) - u(\tau^{*-}_{i,n})) = 0.$$

On substituting $x(\tau_{i,n}^{*+}) - x(\tau_{i,n}^{*-}) = gv_{i,n}^*$, (43), and (47) in the above equation, we have

$$w_2 g v_{i,n}^* + a\lambda_2(\tau_{i,n}^*) g v_{i,n}^* - \frac{b^2}{c_u}\lambda_2(\tau_{i,n}^*)(\lambda_1(\tau_{i,n}^{*+}) - \lambda_1(\tau_{i,n}^{*-})) = 0, \tag{21}$$

which on substituting (46) and (52) simplifies to

$$-w_2\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^*) - a\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^*)^2 + \frac{b^2}{c_u}\lambda_2(\tau_{i,n}^*)\left(z_1 x(\tau_{i,n}^*) + d_1\right) = 0,$$
$$\Rightarrow \lambda_2(\tau_{i,n}^*)\left(\frac{c_u g^2}{b^2 c_v}(-w_2 - a\lambda_2(\tau_{i,n}^*)) + z_1 x(\tau_{i,n}^*) + d_1\right) = 0.$$

$\lambda_2(\tau_{i,n}^*) = 0$ implies that the equilibrium impulse level is zero. From Definition 1, $v_{i,n}^*$ cannot be equal to zero if $\tau_{i,n}^*$ is an admissible impulse instant. So, an impulse occurs if

$$x(\tau_{i,n}^*) = \frac{\frac{c_u g^2}{c_v b^2}(w_2 + a\lambda_2(\tau_{i,n}^*)) - d_1}{z_1}.$$

We can rewrite (54) as

$$a\lambda_2(\tau_{i,n}^*) + w_2 = (a\lambda_2(t_{n+1}) + w_2)e^{a(t_{n+1}-\tau_{i,n}^*)},$$

and substitute in the above equation to obtain

$$x(\tau_{i,n}^*) = \frac{\left(\frac{c_u g^2}{c_v b^2}(a\lambda_2(t_{n+1}) + w_2)e^{a(t_{n+1}-\tau_{i,n}^*)}\right) - d_1}{z_1},\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \tag{22}$$

On substituting (57) and (60a) in the above equation, we arrive at (20a). ☐

Remark 5 *An impulse occurs at equilibrium whenever the state trajectory intersects the time varying function of gradient of the value function of Player 2, $\xi(t)$, given by*

$$\xi(t) = \frac{\left(\frac{c_u g^2}{c_v b^2}(aq_{2,n+1}(t_{n+1}) + w_2)e^{a(t_{n+1}-t)}\right) - d_1}{z_1}.$$

4.2 Non-linear optimization

Let $\tau_{1,n}, \tau_{2,n}, \cdots, \tau_{k_n,n}$ denote the admissible impulse instants for a given number of impulses, k_n, in each sampling interval $[t_n, t_{n+1}]$, $n \in \mathcal{N}'$. From Definition 1, we have

$$\tau_{1,n} < \tau_{2,n} < \cdots < \tau_{k_n,n}.$$

The above constraint can be represented as

$$D_n \boldsymbol{\tau}_n < \mathbf{0}, \tag{23}$$

where

$$D_n := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(k_n-1)\times k_n}, \quad \boldsymbol{\tau}_n := \begin{bmatrix} \tau_{1,n} \\ \vdots \\ \tau_{k_n,n} \end{bmatrix}, \forall n \in \mathcal{N}'.$$

At the equilibrium impulse instants, the Hamiltonian continuity condition (20a) holds for the iLQDG formulated by (15). The equilibrium impulse instants are obtained by finding the fixed-point solution of the Riccati like system of equations (17a)–(17n) and the system of non-linear equality constraints (20a). Alternatively, this problem can be viewed as the following constrained non-linear optimization problem:

$$\underset{\{\boldsymbol{\tau}_n\}_{n\in\mathcal{N}'}}{\operatorname{argmin}} \quad \sum_{n=1}^{N-1}\sum_{i=1}^{k_n}\left(x(\tau_{i,n})-\xi(\tau_{i,n})\right)^2 \tag{24a}$$

$$\text{subject to} \quad \mathbf{1}.(t_n+s)\le \boldsymbol{\tau}_n \le \mathbf{1}.(t_{n+1}-s) \qquad \forall n\in\mathcal{N}' \tag{24b}$$

$$D_n\boldsymbol{\tau}_n \le -\mathbf{1}.s \qquad \forall n\in\mathcal{N}', \tag{24c}$$

where $s>0$ is a slack variable, and

$$\xi(\tau_{i,n}) = \frac{\left(\frac{c_u g^2}{c_v b^2}(aq_{2,n+1}(t_{n+1})+w_2)e^{a(t_{n+1}-\tau_{i,n})}\right)-d_1}{z_1}. \tag{24d}$$

The above problem can be solved using interior point algorithms Byrd et al. (1999) or sequential quadratic programming methods Büskens and Maurer (2000).

5 Impulse linear-state differential game

In this section, we derive analytical expressions for the equilibrium number, timing and levels of impulses for the scalar linear-state differential game that can be obtained by setting $h_1=z_1=f_1=0$ in (15).

Theorem 4 *Let Assumptions 2 and 3 hold. Suppose that $t_1,t_2,\dots,t_N$ are the sampling instants. Then, there can be at most one impulse in the sampled-data Nash equilibrium with the timing and level of impulse given by*

$$\tau^* = T-\frac{1}{a}\ln\left(\frac{b^2c_v}{g^2c_u}\frac{d_1}{as_2+w_2}\right), \tag{25a}$$

$$v = \frac{gw_2}{c_v a}-\frac{b^2d_1}{agc_u}. \tag{25b}$$

Further, τ^* is the equilibrium impulse instant if the following condition holds for some $n\in\mathcal{N}'$:

$$t_n < T-\frac{1}{a}\ln\left(\frac{b^2c_v}{g^2c_u}\frac{d_1}{as_2+w_2}\right) < t_{n+1}. \tag{25c}$$

The equilibrium control of Player 1 is given by

$$u^*(t) = -\frac{b}{c_u}\lambda_1(t), \tag{25d}$$

where

$$\begin{aligned}
&\dot\lambda_1(t) = -a\lambda_1(t)-w_1,\ \lambda_1(t_{n+1}) = q_{1,n+1}(t_{n+1}),\ n\in\mathcal{N}',\\
&\dot q_{1,n}(t) = -w_1-aq_{1,n}(t),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}), n\in\mathcal{N}',\ q_{1,N}(T)=s_1,\\
&q_{1,n}(\tau^{*-}) = q_{1,n}(\tau^{*+})+d_1.
\end{aligned}$$

Proof. The Hamiltonian of Player 1 is given by

$$H_1(x(t),u(t),\lambda_1(t)) := w_1x(t)+\frac{1}{2}c_uu(t)^2+\lambda_1(t)(ax(t)+bu(t)).$$

Using (14a) and Assumption 3 on interior solutions, the first-order condition yields

$$H_{1u}(x^*(t),u^*(t),\lambda_1(t)) = 0 \Rightarrow u^*(t) = -\frac{b}{c_u}\lambda_1(t). \tag{26}$$

From (14e), (14f), and (14i), the equilibrium state and co-state trajectories during the non-impulse instants evolves as follows:

$$\dot{x}^*(t) = ax^*(t) - \frac{b^2}{c_u}\lambda_1(t),\ x^*(t_n) = x_n,\ n \in \mathcal{N}', \tag{27}$$

$$\dot{\lambda}_1(t) = -a\lambda_1(t) - w_1,\ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x(t_{n+1}))}{\partial x},\ n \in \mathcal{N}', \tag{28}$$

and at the impulse instants, the co-state jumps according to

$$\lambda_1(\tau_{i,n}^{*-}) = \lambda_1(\tau_{i,n}^{*+}) + d_1,\ i \in \mathcal{I}^n, n \in \mathcal{N}'. \tag{29}$$

Using the approach in Theorem 2, it can be shown that the equilibrium value-to-go of Player 1 is given by $V_1^*(t_n, x_n) = q_{1,n}(t_n)x_n + r_{1,n}(t_n)$, $\forall n \in \mathcal{N}'$, such that

$$\dot{q}_{1,n}(t) = -w_1 - aq_{1,n}(t),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}), n \in \mathcal{N}',\ q_{1,N}(T) = s_1, \tag{30a}$$

$$\dot{r}_{1,n}(t) = -\left(\frac{\lambda_1(t)}{2} - q_{1,n}(t)\right)\frac{b^2}{c_u}\lambda_1(t),\ r_{1,n}(t_{n+1}) = r_{1,n+1}(t_{n+1}),\ r_{1,N}(T) = 0, \tag{30b}$$

$$q_{1,n}(\tau_{i+1,n}^{*-}) = q_{1,n}(\tau_{i+1,n}^{*+}) + d_1,\ i \in \mathcal{I}^n, \tag{30c}$$

$$r_{1,n}(\tau_{i+1,n}^{*-}) = r_{1,n}(\tau_{i+1,n}^{*+}) + r_{1,n}(\tau_{i+1,n}^{*+})gv_{i,n}^*,\ i \in \mathcal{I}^n. \tag{30d}$$

From (28), we obtain

$$\lambda_1(t_{n+1}) = q_{1,n+1}(t_{n+1}), \forall n \in \mathcal{N}'. \tag{31}$$

Given the equilibrium control $u^*(.)$ of Player 1, the necessary optimality conditions for Player 2 are given by (14b), (14g), (14h), (14j), (14k). The co-state of Player 2 evolves according to the following equation

$$\lambda_2(t) = -a\lambda_2(t) - w_2,\ \forall t \notin \mathcal{T}^n,\ n \in \mathcal{N}', \lambda_2(T) = s_2, \tag{32}$$

$$\lambda_2(\tau_{i,n}^{*+}) = \lambda_2(\tau_{i,n}^{*-}),\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \tag{33}$$

The jump in the state at the impulse instant is given

$$x(\tau_{i,n}^{*+}) = x(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+}),\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \tag{34}$$

From the proof of Theorem 2, we can show that the value-to-go for Player 2 at any time t is given by $V_2(t_n, x_n) = q_{2,n}(t_n)x_n + r_{2,n}(t_n)$ such that

$$\dot{q}_{2,n}(t) = -w_2 - aq_{2,n}(t),\ q_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ q_{2,N}(T) = s_2, \tag{35a}$$

$$\dot{r}_{2,n}(t) = q_{2,n}(t)\frac{b^2}{c_u}\lambda_1(t),\ r_{2,n}(t_{n+1}) = r_{2,n+1}(t_{n+1}),\ \forall n \in \mathcal{N}',\ r_{2,N}(T) = 0, \tag{35b}$$

$$q_{2,n}(\tau_{i,n}^{*-}) = q_{2,n}(\tau_{i,n}^{*+}),\ i \in \mathcal{I}^n, \tag{35c}$$

$$r_{2,n}(\tau_{i+1,n}^{*-}) = r_{2,n}(\tau_{i+1,n}^{*+}) + \frac{g^2}{2c_v}\lambda_2(\tau_{i+1,n}^{*+})^2 - q_{2,n}(\tau_{i+1,n}^{*+})\left(\frac{g^2}{c_v}\lambda_2(\tau_{i+1,n}^{*+})\right),\ i \in \mathcal{I}^n. \tag{35d}$$

From the above equations for the evolution of q_2, and (32), (33), we can see that $q_2(.)$ and $\lambda_2(.)$ have the same dynamics and terminal conditions in each sampling interval, and are continuous functions of time, and thus we obtain

$$\lambda_2(t) = q_2(t) = -\frac{w_2}{a} + (s_2 + \frac{w_2}{a})e^{a(T-t)},\ \forall t \in [0, T]. \tag{36}$$

At the impulse instants $\tau_{i,n}^*$, the Hamiltonian continuity condition (14k) holds, which implies

$$w_2x(\tau_{i,n}^{*+}) + \lambda_2(\tau_{i,n}^{*+})(ax(\tau_{i,n}^{*+}) + bu(\tau_{i,n}^{*+})) = w_2x(\tau_{i,n}^{*-}) + \lambda_2(\tau_{i,n}^{*-})(ax(\tau_{i,n}^{*-}) + bu(\tau_{i,n}^{*-})).$$

Using the conditions (29), (33), (34), we can rewrite the Hamiltonian continuity condition as

$$-\left((w_2 + a\lambda_2(\tau^*_{i,n}))\frac{g^2}{c_v} - \frac{b^2}{c_u}d_1\right)\lambda_2(\tau^*_{i,n}) = 0,$$

This implies that if an impulse occurs at $\tau^*_{i,n}$, then $\lambda(\tau^*_{i,n})$ can take the following two values:

$$\lambda_2(\tau^*_{i,n}) = 0, \frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a} - \frac{w_2}{a}.$$

$\lambda_2(\tau^*_{i,n}) = 0$ implies that the impulse level is zero. The admissible impulse instants in Definition 1 are such that if $\tau^*_{i,n}$ is an equilibrium impulse instant, then the impulse level is not equal to 0. Since $\lambda_2(t)$ is strictly monotone for $t \in [0, T]$, we obtain a unique solution:

$$\lambda_2(\tau^*_{i,n}) = \frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a} - \frac{w_2}{a}. \tag{37}$$

From (36) and (37), we obtain a unique equilibrium impulse instant

$$\tau^* = T - \frac{1}{a}\ln\left(\frac{b^2 c_v}{g^2 c_u}\frac{d_1}{as_2 + w_2}\right).$$

For τ^* to be an interior impulse, we must have for some $n \in \mathcal{N}'$,

$$t_n < \tau^* < t_{n+1} \Rightarrow t_n < T - \frac{1}{a}\ln\left(\frac{b^2 c_v}{g^2 c_u}\frac{d_1}{as_2 + w_2}\right) < t_{n+1}.$$

The equilibrium impulse level is given by

$$v^* = -\frac{g}{c_v}\lambda_2(\tau^*) = -\frac{g}{c_v}\left(\frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a} - \frac{w_2}{a}\right). \tag{38}$$

□

Clearly, there can be at most one impulse in the sampled-data Nash equilibrium of our specialized scalar linear-state differential game with impulse controls. Next, we consider a variation of the game where the problem parameters of Player 1 and Player 2 vary with time, and are constant between the sampling instants:

$$\begin{aligned}
J_1(x_0, u(\cdot), \tilde{v}) &= \sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}}\left(w_{1,n}x(t) + \frac{1}{2}c_{u,n}u(t)^2\right)dt + \sum_{i=1}^{k_n} d_{1,n}x(\tau^-_{i,n})\Big) + s_1 x(T),\\
J_2(x_0, u(\cdot), \tilde{v}) &= \sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}} w_{2,n}x(t)dt + \sum_{i=1}^{k_n}\left(\frac{1}{2}c_{v.n}v^2_{i,n}\right)\Big) + s_2 x(T),\\
\dot{x}(t) &= a_n x(t) + b_n u(t),\ t \neq \mathcal{T}^n,\ x(0) = x_0,\\
x(\tau^+_{i,n}) &= x(\tau^-_{i,n}) + g_n v_{i,n},\ i \in \mathcal{I}^n,
\end{aligned} \tag{39}$$

where the state at the sampling instants $t_1, t_2, \cdots, t_N$ is denoted by $x_1, x_2, \cdots, x_N$.

Theorem 5 *Let Assumptions 2 and 3 hold. Suppose that $t_1, t_2, \ldots, t_N$ are the sampling instants. Then, there can be at most one impulse in each sampling interval, and at most N impulses in the sampled-data Nash equilibria with the timing and level of impulses given by*

$$\tau^*_n = t_{n+1} - \frac{1}{a_n}\ln\left(\frac{b^2_n c_{v,n}}{g^2_n c_{u,n}}\frac{d_{1,n}}{a_n\lambda_2(t_{n+1}) + w_{2,n}}\right), \tag{40a}$$

$$v^*_n = \frac{g_n w_{2,n}}{c_{v,n}a_n} - \frac{b^2_n d_{1,n}}{a_n g_n c_{u,n}}, \tag{40b}$$

where

$$\lambda_2(t) = -\frac{w_{2,n}}{a_n} + (\lambda_2(t_{n+1}) + \frac{w_{2,n}}{a_n})e^{a_n(t_{n+1}-t)}, \forall t \in [t_n, t_n+1),\ n \in \mathcal{N}',\ \lambda_2(T) = s_2. \tag{40c}$$

Further, τ_n^* is an equilibrium impulse instant if the following conditions hold:

$$t_n < t_{n+1} - \frac{1}{a_n}\ln\left(\frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}} \frac{d_{1,n}}{a_n\lambda_2(t_{n+1}) + w_{2,n}}\right) < t_{n+1},\ n \in \mathcal{N}'.$$

The equilibrium strategy of Player 1 is given by

$$u^*(t) = -\frac{b_n}{c_{u,n}}\lambda_1(t), \tag{40d}$$

where, for $n \in \mathcal{N}'$,

$$\begin{aligned}
&\dot{\lambda}_1(t) = -a_n\lambda_1(t) - w_{1,n},\ \lambda_1(t_{n+1}) = q_{1,n+1}(t_{n+1}),\\
&\dot{q}_{1,n}(t) = -w_{1,n} - a_n q_{1,n}(t),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}),\ q_{1,N}(T) = s_1,\\
&q_{1,n}(\tau_n^{*-}) = q_{1,n}(\tau_n^{*+}) + d_{1,n}.
\end{aligned}$$

Proof. Using the proof of Theorem 4, we obtain

$$\begin{aligned}
\lambda_2(t) = -\frac{w_{2,n}}{a_n} + (\lambda_2(t_{n+1}) + \frac{w_{2,n}}{a_n})e^{a_n(t_{n+1}-t)}, \forall t \in [t_n, t_n+1),\\
n \in \mathcal{N}',\ \lambda_2(T) = s_2.
\end{aligned} \tag{41}$$

Between the sampling instants t_n and t_{n+1}, the Hamiltonian continuity condition holds at the impulse instants, which implies

$$-\left((w_{2,n} + a_n\lambda_2(\tau_{i,n}^*))\frac{g_n^2}{c_{v,n}} - \frac{b_n^2}{c_{u,n}}d_{1,n}\right)\lambda_2(\tau_{i,n}^*) = 0,$$

From the continuity and strict monotonicity of co-state in each sampling interval, we obtain a unique value of co-state in each sampling interval

$$\lambda_2(\tau_n^*) = \frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}}\frac{d_{1,n}}{a_n} - \frac{w_{2,n}}{a_n}. \tag{42}$$

Substituting (42) in (41), we obtain

$$\tau_n^* = t_{n+1} - \frac{1}{a_n}\ln\left(\frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}}\frac{d_{1,n}}{a_n\lambda_2(t_{n+1}) + w_{2,n}}\right).$$

The equilibrium impulse level is then given by

$$v_n^* = -\frac{g_n}{c_{v,n}}\lambda_2(\tau_n^*) = \frac{g_n w_{2,n}}{c_{v,n}a_n} - \frac{b_n^2 d_{1,n}}{a_n g_n c_{u,n}}.$$

From the proof of Theorem 2, we also obtain the equilibrium controls of Player 1 by replacing the problem parameters in each sampling interval by the time-varying parameters. □

6 A numerical example

In this section, we illustrate the theory developed in the previous sections using a numerical example.

Consider a dynamic game where Player 1's profit is decreasing quadratically with the state while Player 2's profit increases linearly with the state. The time horizon of the game is $T = 20$. Player 1 uses piecewise continuous sampled-data state feedback controls while Player 2 uses impulse controls. The state measurements are made at given instants of time $t_1 = 0$, $t_2 = 10$, $t_3 = 20$. Player 1 and Player 2 maximize their respective objective functions

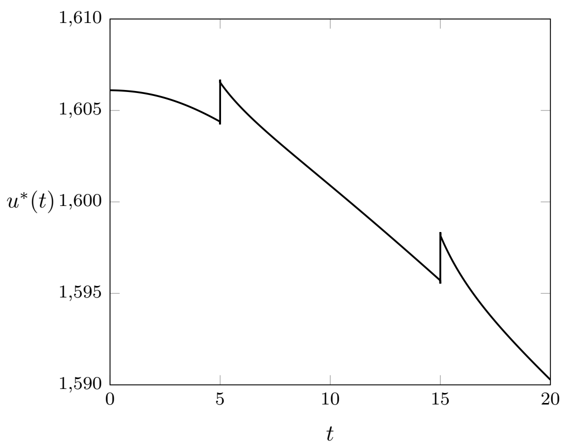
$$J_1(x_0, u(\cdot), \tilde{v}) = \sum_{n=1}^{2} \Big(\int_{t_n}^{t_{n+1}} \big(-x(t)^2 - 4x(t) - 3u(t)^2\big)\, dt - 4x(\tau_n^-)^2 \Big) - 2x(20)(x(20)+1)$$

$$J_2(x_0, u(\cdot), \tilde{v}) = \sum_{n=1}^{2} \Big(\int_{t_n}^{t_{n+1}} 10x(t)dt - 0.25v_n^2 \Big) + 6x(20),$$

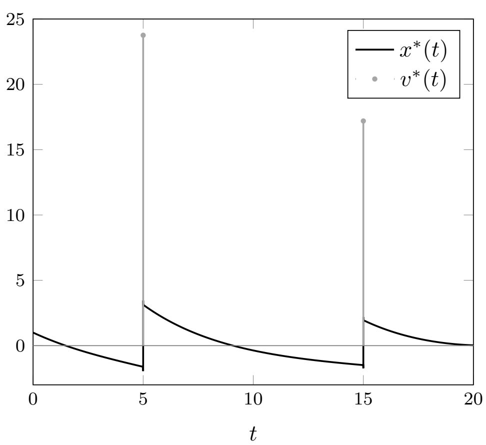
and the state dynamics are given by

$$\dot{x}(t) = -0.1x(t) + 0.4u(t),\ t \notin \{\tau_1, \tau_2\},\ x(0) = 1,$$
$$x(\tau_i^+) = x(\tau_i^-) + 0.2v_i,\ i \in \{1, 2\}.$$

First, we analyze the case where the impulses are periodic, that is, $\tau_1 = 5$ and $\tau_2 = 15$. The equilibrium control of Player 1, given in Figure 1a, jumps at the impulse instants because of the jump in her co-state caused by the impulse control of Player 2. The state trajectory, and the equilibrium impulse levels of Player 2 are shown in Figure 1b. At equilibrium, Player 1 incurs a loss of 238.37, while Player 2 incurs a loss of 203.09.



(a) $u^*(t)$



(b) $x^*(t)$ **and** v^*

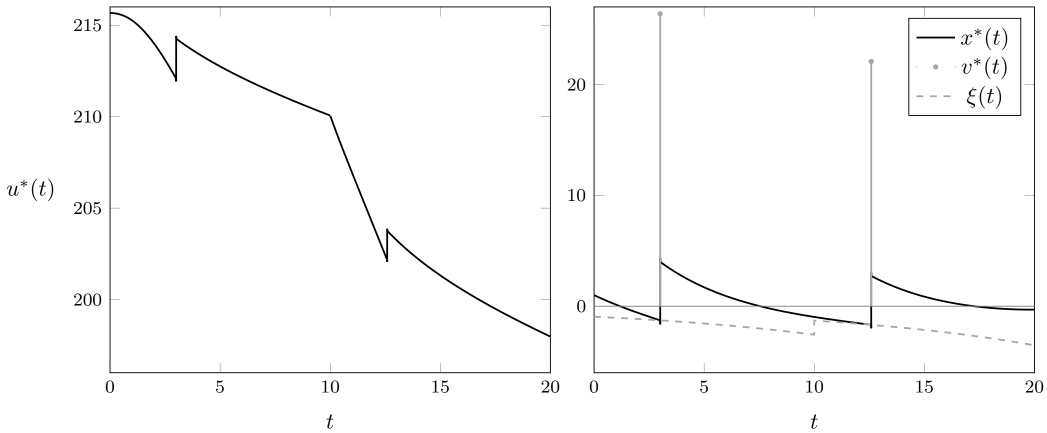
Figure 1: Equilibrium controls, and state trajectory with periodic impulses.

Next, we determine the equilibrium when the impulse instants in each sampling interval are determined by Player 2, and there is one impulse in each sampling interval. The impulse timing is characterized by the Hamiltonian continuity condition (21) which reduces to determining the time at which state trajectory intersects $\xi(t)$ as shown in Figure 2b

$$\xi(t) = \begin{cases} -3.52e^{-0.1(10-t)} & t \in [0, 10) \\ -2.57e^{-0.1(20-t)} & t \in (10, 20] \end{cases}.$$

The equilibrium impulses occur at $\tau_1^* = 3$ and $\tau_2^* = 12.59$, and at equilibrium, the losses of Player 1 and Player 2 are given by 311.64 and 232.83, respectively. The piecewise continuous equilibrium control of Player 1 is shown in Figure 2a and equilibrium impulse levels of Player 2 are shown in Figure 2b.

Clearly, both players incur higher loss if Player 2 determines the timing of impulses when compared with the case where impulse timings are periodic. This illustrates a well-known result that enlarging the strategy space of a player does not necessarily benefit the player in a game problem.



(a) $u^*(t)$ **(b)** $x^*(t)$, v^* **and** ξ

Figure 2: Equilibrium controls, and state and co-state trajectories.

7 Conclusions

In this paper, we have derived necessary conditions for the existence of sampled-data Nash equilibrium in a general class of two-player nonzero-sum differential games with impulse controls, where only one of the players controls the impulses (their number, timing and magnitudes). For a scalar linear-quadratic differential game, we have shown that the sampled-data Nash equilibrium can be obtained by determining the fixed point of a system of Riccati like equations with jumps coupled with non-linear equality constraints. We have also shown that the equilibrium piecewise continuous control of Player 1 is linear in the most recently measured state value, and provide a numerical procedure to determine the equilibrium. Further, we have shown that there can be at most one impulse in the sampled-data Nash equilibrium of a scalar linear-state differential game with impulse controls, and for the case with time-varying parameters, there can be at most one impulse in each sampling interval, and we have obtained analytical expressions for equilibrium timing and level of impulses.

For the future, it would be interesting to apply our results to case studies in pollution regulation, exchange rate interventions, and cybersecurity. One extension of our work would be to differential games where both players use continuous as well as impulse controls. Another extension would be to differential games with more than two players.

A Appendix

A.1 Proof of Theorem 2

Given the equilibrium control of Player 2, we obtain necessary conditions for iLQDG using (14a), (14e), (14f), (14i). The Hamiltonian of Player 1 is given by

$$H_1(x(t), u(t), \lambda_1(t)) := \frac{1}{2}h_1 x(t)^2 + w_1 x(t) + \frac{1}{2}c_u u(t)^2 + \lambda_1(t)(ax(t) + bu(t)),$$

where $\lambda_1(t)$ is the co-state of Player 1. Using (14a) and Assumption 3 on interior solutions, the first-order condition yields

$$H_{1u}(x^*(t), u^*(t), \lambda_1(t)) = 0 \Rightarrow u^*(t) = -\frac{b}{c_u}\lambda_1(t). \tag{43}$$

From (14e) and (14f), the equilibrium state and co-state trajectory at the non-impulse instants evolve as follows:

$$\dot{x}^*(t) = ax^*(t) - \frac{b^2}{c_u}\lambda_1(t),\ x^*(t_{n+1}) = x_{n+1}, \tag{44}$$

$$\dot{\lambda}_1(t) = -a\lambda_1(t) - h_1 x^*(t) - w_1,\ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x(t_{n+1}))}{\partial x}. \tag{45}$$

From (14i), the jump in the co-state at the impulse instants is given by

$$\lambda_1(\tau_{i,n}^{*-}) = \lambda_1(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1. \tag{46}$$

Given that the objective of Player 1 is quadratic in state, we can guess the form of co-state to be linear in state so that

$$\lambda_1(t) = \alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t),\ \forall t \in [t_n, t_{n+1}),\ n \in \mathcal{N}'. \tag{47}$$

We substitute (47) in (46) to obtain the following relation at the impulse instants:

$$\begin{aligned}\alpha_{1,n}(\tau_{i,n}^{*-})x^*(\tau_{i,n}^{*-}) + \beta_{1,n}(\tau_{i,n}^{*-}) &= \alpha_{1,n}(\tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \beta_{1,n}(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1\\ &= \alpha_{1,n}(\tau_{i,n}^{*+})(x^*(\tau_{i,n}^{*-}) + g v_{i,n}^*) + \beta_{1,n}(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1,\end{aligned}$$

where $v_{i,n}^*$ denotes the equilibrium impulse level of Player 2 at the impulse instant $\tau_{i,n}^*$. On comparing the coefficients, we obtain

$$\begin{aligned}&\alpha_{1,n}(\tau_{i,n}^{*-}) = \alpha_{1,n}(\tau_{i,n}^{*+}) + z_1,\ \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}',\\ &\beta_{1,n}(\tau_{i,n}^{*-}) = \beta_{1,n}(\tau_{i,n}^{*+}) + \alpha_{1,n}(\tau_{i,n}^{*+}) g v_{i,n}^* + d_1,\ \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}'.\end{aligned}$$

Taking the derivative of (47) with respect to time, we obtain

$$\dot{\lambda}_1(t) = \dot{\alpha}_{1,n}(t)x^*(t) + \alpha_{1,n}(t)\dot{x}^*(t) + \dot{\beta}_{1,n}(t).$$

Using the derivatives of state and co-state from (44) and (45) in the above equation, we get

$$-a\lambda_1(t) - h_1 x^*(t) - w_1 = \dot{\alpha}_{1,n}(t)x^*(t) + \alpha_{1,n}(t)\left(a x^*(t) - \frac{b^2}{c_u}\lambda_1(t)\right) + \dot{\beta}_{1,n}(t).$$

Substitute (47) in the above equation to obtain

$$\begin{aligned}&- a(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t)) - h_1 x^*(t) - w_1\\ &\quad = \dot{\alpha}_{1,n}(t)x^*(t) + \alpha_{1,n}(t)\left(a x^*(t) - \frac{b^2}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))\right) + \dot{\beta}_{1,n}(t).\end{aligned}$$

On comparing the coefficients, we obtain

$$\begin{aligned}\dot{\alpha}_{1,n}(t) &= -2\alpha_{1,n}(t)a + \frac{b^2}{c_u}\alpha_{1,n}(t)^2 - h_1,\ \forall t \notin \mathcal{T}^n, n \in \mathcal{N},\ \alpha_{1,n}(T) = f_1,\\ \dot{\beta}_{1,n}(t) &= \beta_{1,n}(t)\left(\frac{b^2}{c_u}\alpha_{1,n}(t) - a\right) - w_1,\ \forall t \notin \mathcal{T}^n,\ n \in \mathcal{N},\ \beta_{1,n}(T) = s_1,\end{aligned}$$

where $\alpha_{1,n}(t_{n+1})x(t_{n+1}) + \beta_{1,n}(t_{n+1}) = \frac{\partial V_1^*(t_{n+1},x_{n+1})}{\partial x}$. The value-to-go for Player 1 is given by

$$\begin{aligned}V_1(t_n, x_n) = &\sum_{i=1}^{k_n^*}\Bigg(\frac{1}{2}\left(\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}}\left(h_1 x(t)^2 + 2w_1 x(t) + c_u u(t)^2\right)dt\right)\\ &+ \frac{1}{2}z_1 x(\tau_{i,n}^{*-})^2 + d_1 x(\tau_{i,n}^{*-})\Bigg) + V_1(t_{n+1}, x_{n+1}),\end{aligned} \tag{48}$$

where $\tau_{k_n^*+1}^* := t_{n+1}$. Next, we know that for all x,

$$\begin{aligned}&\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}}\left(\frac{1}{2}\dot{p}_{1,n}(t)x(t)^2 + p_{1,n}(t)x(t)\dot{x}(t) + \dot{q}_{1,n}(t)x(t) + q_{1,n}(t)\dot{x}(t) + \dot{r}_{1,n}(t)\right)dt\\ &\quad - \frac{1}{2}p_{1,n}(t)x(t)^2\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - q_{1,n}(t)x(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - r_{1,n}(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} = 0, i \in \mathcal{I}^n,\ n \in \mathcal{N}'.\end{aligned}$$

Substituting $\dot{x}(t) = ax(t) + bu(t)$ in the above equation and adding it to (48) gives

$$\begin{aligned} V_1(t_n, x_n) = &\sum_{i=1}^{k_n^*} \Big(\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \left(\frac{1}{2}c_u u(t)^2 + p_{1,n}(t)x(t)bu(t) + q_{1,n}(t)bu(t)\right) dt \\ &+ \int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \Big(\frac{1}{2}h_1 x(t)^2 + w_1 x(t) + \frac{1}{2}\dot{p}_{1,n}(t)x(t)^2 + p_{1,n}(t)ax(t)^2 + \dot{q}_{1,n}(t)x(t) \\ &+ q_{1,n}(t)ax(t) + \dot{r}_{1,n}(t)\Big)dt - \frac{1}{2}p_{1,n}(t)x(t)^2\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - q_{1,n}(t)x(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - r_{1,n}(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \\ &+ \frac{1}{2}z_1 x(\tau_{i,n}^{*-})^2 + d_1 x(\tau_{i,n}^{*-})\Big) + V_1(t_{n+1}, x_{n+1}). \end{aligned}$$

Substituting the equilibrium control $u^*(t) = -\frac{b}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))$, we obtain

$$\begin{aligned} V_1^*(t_n, x_n) = &\sum_{i=1}^{k_n} \Big(\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \Big(\frac{1}{2}c_u(-\frac{b}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t)))^2 \\ &- (p_{1,n}(t)x^*(t) + q_{1,n}(t))\frac{b^2}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))\Big)dt + \int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \Big(\frac{h_1 x^*(t)^2}{2} + w_1 x^*(t) \\ &+ \frac{1}{2}\dot{p}_{1,n}(t)x^*(t)^2 + p_{1,n}(t)ax^*(t)^2 + \dot{q}_{1,n}(t)x^*(t) + q_{1,n}(t)ax^*(t) + \dot{r}_{1,n}(t)\Big)dt \\ &- \frac{1}{2}p_{1,n}(t)x^*(t)^2\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - q_{1,n}(t)x^*(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - r_{1,n}(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} + \frac{1}{2}z_1 x^*(\tau_{i,n}^{*-})^2 \\ &+ d_1 x^*(\tau_{i,n}^{*-})\Big) + V_1^*(t_{n+1}, x_{n+1}). \end{aligned}$$

Since the equilibrium control maximizes the value-to-go for Player 1, the following relations hold for all $n \in \mathcal{N}'$:

$$\begin{aligned} &\dot{p}_{1,n}(t) = -h_1 - 2(a - \frac{b^2}{c_u}\alpha_{1,n}(t))p_{1,n}(t) - \frac{b^2}{c_u}\alpha_{1,n}(t)^2, \ \forall t \notin \mathcal{T}^n, \\ &\dot{q}_{1,n}(t) = -w_1 + \frac{b^2}{c_u}p_{1,n}(t)\beta_{1,n}(t) - q_{1,n}(t)(a - \frac{b^2}{c_u}\alpha_{1,n}(t)) - \frac{b^2}{c_u}\alpha_{1,n}(t)\beta_{1,n}(t), \ \forall t \notin \mathcal{T}^n, \\ &\dot{r}_{1,n}(t) = \frac{b^2}{c_u}\left(q_{1,n}(t)\beta_{1,n}(t) - \beta_{1,n}(t)^2\right), \ \forall t \notin \mathcal{T}^n, \\ &p_{1,n}(\tau_{i+1,n}^{*-}) = p_{1,n}(\tau_{i+1,n}^{*+}) + z_1, \ \forall i \in \mathcal{I}^n, \\ &q_{1,n}(\tau_{i+1,n}^{*-}) = p_{1,n}(\tau_{i+1,n}^{*+})gv_{i,n}^* + q_{1,n}(\tau_{i+1,n}^{*+}) + d_1, \ \forall i \in \mathcal{I}^n, \\ &r_{1,n}(\tau_{i+1,n}^{*-}) = r_{1,n}(\tau_{i+1,n}^{*+}) + \frac{1}{2}p_{1,n}(\tau_{i+1,n}^{*+})g^2 v_{i,n}^{*2} + q_{1,n}(\tau_{i+1,n}^{*+})gv_{i,n}^*, \ \forall i \in \mathcal{I}^n, \\ &p_{1,n}(t_{n+1}) = p_{1,n+1}(t_{n+1}), \ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}), \ r_{1,n}(t_{n+1}) = r_{1,n+1}(t_{n+1}), \end{aligned}$$

where the last set of equations hold for $n \in \mathcal{N}'$ because there are no impulses at the sampling instants (see Definition 1). Therefore, the equilibrium value-to-go is given by

$$V_1^*(t_n, x_n) = \frac{1}{2}p_{1,n}(t_n)x_n^2 + q_{1,n}(t_n)x_n + r_{1,n}(t_n), \ \forall n \in \mathcal{N}'. \tag{49}$$

The Hamiltonian, and the impulse Hamiltonian of Player 2 are given by

$$\begin{aligned} &H_2(x(t), u(t), \lambda_2(t)) := w_2 x(t) + \lambda_2(t)(ax(t) + bu(t)), \\ &H_2^I(v_i, \lambda_2(\tau_i^+)) := \frac{1}{2}c_v v_i^2 + \lambda_2(\tau_{i,n}^+)gv_i, \end{aligned}$$

where $\lambda_2(t)$ is the co-state of Player 2. From (14g), we obtain the dynamics of the co-state of Player 2 at the non-impulse instants as follows:

$$\dot{\lambda}_2(t) = -a\lambda_2(t) - w_2, \ \forall t \in (t_n, t_{n+1}), \ n \in \mathcal{N}, \ \lambda_2(t_{n+1}) = \frac{\partial V_2^*(t_{n+1}, x_{n+1})}{\partial x}. \tag{50}$$

The co-state is equal to the gradient of the value function of Player 2 at the sampling instants because of our assumption that there are no impulses at the sampling instants. Using the necessary condition (14b) and Assumption 3 on interior impulse levels, the first-order condition yields

$$H_{1v_i}(v^*_{i,n}, \lambda_2(\tau^*_{i,n})) = 0 \Rightarrow v^*_{i,n} = -\frac{g}{c_v}\lambda_2(\tau^{*+}_{i,n}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}'. \tag{51}$$

Since $v^*_{i,n}$ are the equilibrium impulse levels, it follows from (14h) that the jump in the state is given by

$$x(\tau^{*+}_{i,n}) = x(\tau^{*-}_{i,n}) - \frac{g^2}{c_v}\lambda_2(\tau^{*+}_{i,n}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}', \tag{52}$$

and from (14j), we have that the co-state of Player 2 is continuous, that is

$$\lambda_2(\tau^{*-}_{i,n}) = \lambda_2(\tau^{*+}_{i,n}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}. \tag{53}$$

Also, from the continuity of co-state at the impulse instants and (50), we obtain

$$\begin{aligned} \lambda_2(t) &= -\frac{w_2}{a} + (\lambda_2(t_{n+1}) + \frac{w_2}{a})e^{a(t_{n+1}-t)},\ \forall t \in [t_n, t_{n+1}), \\ \lambda_2(t_{n+1}) &= \frac{\partial V^*_2(t_{n+1}, x_{n+1})}{\partial x},\ n \in \mathcal{N}', \lambda_2(T) = s_2. \end{aligned} \tag{54}$$

The value-to-go for Player 2 is given by

$$V_2(t_n, x_n) = \sum_{i=1}^{k_n}\left(\int_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} w_2 x(t)dt + \frac{1}{2}c_v v^2_{i,n}\right) + V_2(t_{n+1}, x_{n+1}). \tag{55}$$

For all x, we have

$$\begin{aligned} &\int_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} (\dot q_{2,n}(t)x(t) + q_{2,n}(t)\dot x(t) + \dot r_{2,n}(t))\, dt - q_{2,n}(t)x(t)\Big|_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} \\ &\qquad - r_{2,n}(t)\Big|_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} = 0,\ \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \end{aligned}$$

Substituting $\dot x(t) = ax(t) + bu^*(t)$ in the above equation and adding it to (55) yields

$$\begin{aligned} V_2(t_n, x_n) = &\sum_{i=1}^{k_n}\Big(\int_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} (w_2 x(t) + \dot q_{2,n}(t)x(t) + q_{2,n}(t)ax(t) + q_{2,n}(t)bu^*(t) + \dot r_{2,n}(t))\, dt \\ &- q_{2,n}(t)x(t)\Big|_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} - r_{2,n}(t)\Big|_{\tau^-_{i,n}}^{\tau^+_{i+1,n}} + \frac{1}{2}c_v v^2_{i,n}\Big) + V_2(t_{n+1}, x_{n+1}). \end{aligned}$$

On substituting the equilibrium controls $(\tau^*_{i,n}, v^*_{i,n})$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$, we obtain the equilibrium value-to-go $V^*_2(t_n, x_n)$, so that

$$\begin{aligned} V^*_2(t_n, x_n) = &\sum_{i=1}^{k^*_n}\Big(\int_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \Big((w_2 + \dot q_{2,n}(t) + q_{2,n}(t)(a - \frac{b^2}{c_u}\alpha_{1,n}(t)))x^*(t) + \dot r_{2,n}(t) - q_{2,n}(t)\frac{b^2}{c_u}\beta_{1,n}(t)\Big)\, dt \\ &- q_{2,n}(t)x^*(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} - r_{2,n}(t)\Big|_{\tau^{*-}_{i,n}}^{\tau^{*+}_{i+1,n}} + \frac{1}{2}c_v v^{*2}_{i,n}\Big) + V^*_2(t_{n+1}, x_{n+1}). \end{aligned}$$

Taking $u^*(t)$ as given, the equilibrium control of Player 2 maximizes the value-to-go for Player 2 for all x, so that the following relations hold:

$$\begin{aligned} \dot q_{2,n}(t) &= -w_2 - (a - \frac{b^2}{c_u}\alpha_{1,n}(t))q_{2,n}(t), \forall t \notin \mathcal{T}^n, \\ \dot r_{2,n}(t) &= q_{2,n}(t)\frac{b^2}{c_u}\beta_{1,n}(t),\ \forall t \notin \mathcal{T}^n, \end{aligned}$$

$$
\begin{aligned}
&q_{2,n}(\tau_{i+1,n}^{*-}) = q_{2,n}(\tau_{i+1,n}^{*+}),\ \forall i \in \mathcal{I}^n,\\
&r_{2,n}(\tau_{i+1,n}^{*-}) = r_{2,n}(\tau_{i+1,n}^{*+}) + \frac{1}{2}\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+})^2 - q_{2,n}(\tau_{i+1,n}^{*+})\lambda_2(\tau_{i,n}^{*+})\frac{g^2}{c_v},\ \forall i \in \mathcal{I}^n,\\
&q_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ r_{2,n}(t_{n+1}) = r_{2,n+1}(t_{n+1}),\ q_{2,N}(T) = s_2,\ r_{2,N}(T) = 0,
\end{aligned}
$$

and the profit-to-go is given by

$$V_2^*(t_n, x_n) = q_{2,n}(t_n)x_n + r_{2,n}(t_n),\ \forall n \in \mathcal{N}'. \tag{56}$$

Since co-state is equal to the gradient of value function at the sampling instants, we have

$$\lambda_2(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ \forall n \in \mathcal{N}'. \tag{57}$$

Using (47) in (44), we obtain

$$\dot{x}^*(t) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)x^*(t) - \frac{b^2}{c_u}\beta_{1,n}(t) \tag{58}$$

$$\Rightarrow x^*(\tau_1^{*-}) = \phi(\tau_{1.n}^{*-}, t_n)x_n + \varphi(\tau_{1,n}^{*-}, t_n), \tag{59}$$

where

$$
\begin{aligned}
&\dot{\phi}(t, t_n) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t, t_n), \forall t \in (t_n, \tau_{1,n}^*),\ \phi(t_n, t_n) = 1,\ \forall n \in \mathcal{N}'\\
&\varphi(\tau_{1,n}^{*-}, t_n) = -\int_{t_n}^{\tau_{1,n}^{*-}} \phi(h, t_n)\frac{b^2}{c_u}\beta_{1,n}(h)dh, \forall n \in \mathcal{N}',\\
&\dot{\phi}(t, \tau_{i,n}^*) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t, \tau_{i,n}^*), \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*),\ \phi(\tau_{i,n}^*, \tau_{i,n}^*) = 1, \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}',\\
&\varphi(t, \tau_{i,n}^*) = -\int_{\tau_{i,n}^*}^{t} \phi(h, \tau_{i,n}^*)\frac{b^2}{c_u}\beta_{1,n}(h)dh, \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}',
\end{aligned}
$$

and $\tau_{k_n^*+1,n} := t_{n+1}$. Define

$$x^*(\tau_{i,n}^{*-}) = \phi(\tau_{i,n}^{*-}, t_n)x_n + \varphi(\tau_{i,n}^{*-}, t_n),\ \forall i = \mathcal{I}^n, \tag{60a}$$

$$x^*(\tau_{i+1,n}^{*-}) = \phi(\tau_{i+1,n}^{*-}, t_n)x_n + \varphi(\tau_{i+1,n}^{*-}, t_n),\ \forall i = \mathcal{I}^n \backslash \{k_n\}. \tag{60b}$$

From (58), we obtain

$$
\begin{aligned}
x^*(\tau_{i+1,n}^{*-}) &= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\\
&= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})(x^*(\tau_{i,n}^{*-}) + gv_{i,n}^*)) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\\
&= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-}, t_n)x_n + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-}, t_n)\\
&\quad + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})gv_{i,n}^* + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}).
\end{aligned}
$$

On comparing with (60b), we obtain

$$
\begin{aligned}
&\phi(\tau_{i+1,n}^{*-}, t_n) = \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-}, t_n),\\
&\varphi(\tau_{i+1,n}^{*-}, t_n) = \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-}, t_n) + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})gv_{i,n}^* + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}).
\end{aligned}
$$

The equilibrium state evolves according to the following equation:

$$
\begin{aligned}
x(t) =& \phi(t, \tau_{i,n}^*)(\phi(\tau_{i,n}^{*-}, t_n)x_n + gv_{i,n}^*\mathbb{1}_{t>\tau_{i,n}} + \varphi(\tau_{i,n}^{*-}, t_n))\\
&+ \varphi(t, \tau_{i,n}^*),\ \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), i \in \mathcal{I}^n \cup \{0\}, n \in \mathcal{N}'.
\end{aligned} \tag{61}
$$

where $\tau_{0,n}^* := 0$. Then, from (43), the equilibrium control of Player 1 is given by

$$
\begin{aligned}
u^*(t) =& -\frac{b}{c_u}\alpha_{1,n}(t)x(t) + \beta_{1,n}(t)\\
=& -\frac{b}{c_u}\Big(\alpha_{1,n}(t)\big(\phi(t, \tau_{i,n}^{*+})(\phi(\tau_{i,n}^{*-}, t_n)x_n + gv_{i,n}^*\mathbb{1}_{t>\tau_{i,n}} + \varphi(\tau_{i,n}^{*-}, t_n))\\
&+ \varphi(t, \tau_{i,n}^{*+})\big) + \beta_{1,n}(t)\Big),\ \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), i \in \mathcal{I}^n \cup \{0\}, n \in \mathcal{N}'.
\end{aligned} \tag{62}
$$

□

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