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Tight bounds on the maximal perimeter and the maximal width of convex small polygons

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Abstract: A small polygon is a polygon of unit diameter. The maximal perimeter and the maximal width of a convex small polygon with $n = 2^s$ vertices are not known when $s \geq 4$. In this paper, we construct a family of convex small n-gons, $n = 2^s$ and $s \geq 3$, and show that the perimeters and the widths obtained cannot be improved for large n by more than a/n^6 and b/n^4 respectively, for certain positive constants a and b. In addition, we formulate the maximal perimeter problem as a nonconvex quadratically constrained quadratic optimization problem and, for $n = 2^s$ with $3 \leq s \leq 7$, we provide near-global optimal solutions obtained with a sequential convex optimization approach.

Keywords: Planar geometry, convex small polygons, maximal perimeter, maximal width, quadratically constrained quadratic optimization, sequential convex optimization, concave-convex procedure

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1 Introduction

The *diameter* of a polygon is the largest Euclidean distance between pairs of its vertices. A polygon is said to be *small* if its diameter equals one. For a given integer $n \geq 3$, the maximal perimeter problem consists in finding the convex small n-gon with the longest perimeter. The problem was first investigated by Reinhardt [1] in 1922, and later by Datta [2] in 1997. They proved that

- for all $n \geq 3$, the value $2n \sin \frac{\pi}{2n}$ is an upper bound on the perimeter of a convex small n-gon;
- when n is odd, the regular small n-gon is an optimal solution, but it is unique only if n is prime;
- when n is even, the regular small n-gon is not optimal;
- when n has an odd factor, there are finitely many optimal solutions and there are all equilateral.

When n is a power of 2, the maximal perimeter problem is solved for $n \leq 8$. In 1987, Tamvakis [3] found the unique convex small 4-gon with the longest perimeter, shown in Figure 1b. In 2007, Audet, Hansen, and Messine [4] used both geometrical arguments and methods of global optimization to determine the unique convex small 8-gon with the longest perimeter, illustrated in Figure 3c.

The diameter graph of a small polygon is defined as the graph with the vertices of the polygon, and an edge between two vertices if the distance between these vertices equals one. Figure 1, Figure 2, and Figure 3 show diameter graphs of some convex small polygons. The solid lines illustrate pairs of vertices which are unit distance apart. Mossinghoff [5] conjectured that, for $n \geq 4$ power of 2, the diameter graph of a convex small n-gon with maximal perimeter has a cycle of length $n/2 + 1$, plus $n/2 - 1$ additional pendant edges, arranged so that all but two particular vertices of the cycle have a pendant edge. For example, Figure 1b and Figure 3c exhibit the diameter graphs of optimal n-gons when $n = 4$ and when $n = 8$ respectively.

The *width* of a polygon in some direction is the distance between two parallel lines perpendicular to this direction and supporting the polygon from below and above. The width of a polygon is the minimum width for all directions. For a given integer $n \geq 3$, the maximal width problem consists in finding the convex small n-gon with the largest width. This problem was partially solved by Bezdek and Fodor [6] in 2000. They proved that

- for all $n \geq 3$, the value $\cos \frac{\pi}{2n}$ is an upper bound on the width of a convex small n-gon;
- when n has an odd factor, a convex small n-gon is optimal for the maximal width problem if and only if it is optimal for the maximal perimeter problem;
- when $n = 4$, there are infinitely many optimal convex small 4-gons, including the 4-gon illustrated in Figure 1b.

When $n \geq 8$ is a power of 2, the maximal width is only known for the first open case $n = 8$. In 2013, Audet, Hansen, Messine, and Ninin [7] combined geometrical and analytical reasoning as well as methods of global optimization to prove that there are infinitely many optimal convex small 8-gons with largest width, including the 8-gon illustrated in Figure 3d.

For $n = 2^s$ with integer $s \geq 4$, exact solutions in both problems appear to be presently out of reach. However, tight lower bounds on the maximal perimeter and the maximal width can be obtained analytically. For instance, Mossinghoff [5] constructed convex small n-gons, for $n = 2^s$ with $s \geq 3$, and proved that the perimeters obtained cannot be improved for large n by more than c/n^5, for a certain positive constant c. We can also show that, when $n = 2^s$ with $s \geq 2$, the value $\cos \frac{\pi}{2n-2}$ is a lower bound on the maximal width and this bound cannot be improved for large n by more than d/n^3, for a particular positive constant d. In this paper, we propose tighter lower bounds on both the maximal perimeter and the maximal width of convex small n-gons when $n = 2^s$ and integer $s \geq 2$. Thus, the main result of this paper is the following:

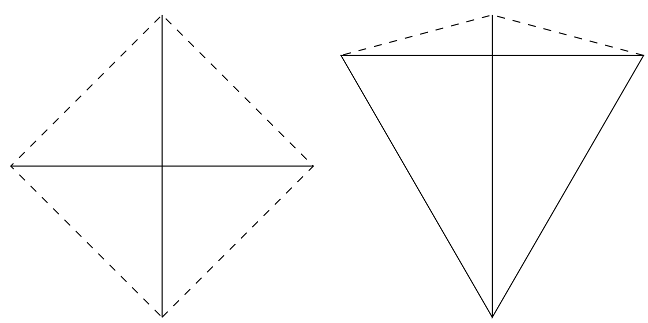
Theorem 1 *For a given integer $n \geq 3$, let $\overline{L}_n := 2n \sin \frac{\pi}{2n}$ denote an upper bound on the perimeter $L(\mathtt{P}_n)$ of a convex small n-gon $\mathtt{P}_n$, and $\overline{W}_n := \cos \frac{\pi}{2n}$ denote an upper bound on its width $W(\mathtt{P}_n)$. If $n = 2^s$ with $s \geq 3$, then there exists a convex small n-gon $\mathtt{B}_n$ such that*

$$L(\mathtt{B}_n) = 2n \sin \frac{\pi}{2n} \cos \left(\frac{\pi}{2n} - \frac{1}{2} \arcsin \left(\frac{1}{2} \sin \frac{2\pi}{n} \right) \right),$$
$$W(\mathtt{B}_n) = \cos \left(\frac{\pi}{n} - \frac{1}{2} \arcsin \left(\frac{1}{2} \sin \frac{2\pi}{n} \right) \right),$$

and

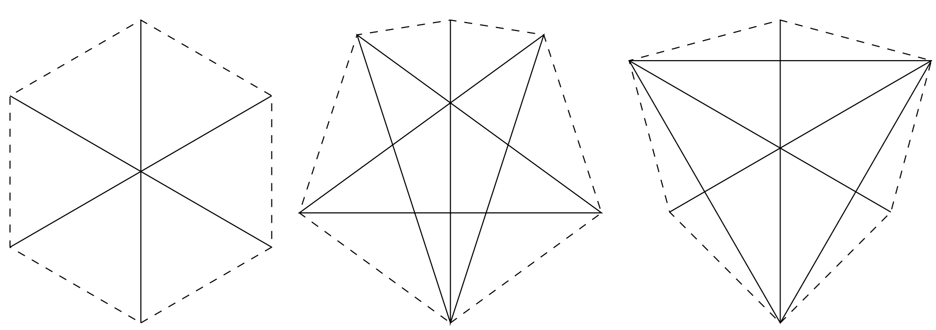
$$\overline{L}_n - L(\mathtt{B}_n) = \frac{\pi^7}{32n^6} + O\left(\frac{1}{n^8} \right),$$
$$\overline{W}_n - W(\mathtt{B}_n) = \frac{\pi^4}{8n^4} + O\left(\frac{1}{n^6} \right).$$

The remainder of this paper is organized as follows. Section 2 recalls principal results on the maximal perimeter and the maximal width of convex small polygons. We prove Theorem 1 In Section 3. Tight bounds on the maximal width of unit-perimeter n-gons, $n = 2^s$ and $s \geq 3$, are deduced from Theorem 1 in Section 4. A nonconvex quadratically constrained quadratic optimization problem is proposed for finding convex small polygons with longest perimeter in Section 5. Near-global optimal solutions obtained with a sequential convex optimization approach are given for $n = 2^s$ with $3 \leq s \leq 7$. This approach is an ascent algorithm guaranteeing convergence to a locally optimal solution and was used in [8] for finding the small n-gon with the largest area when $n \geq 6$ is even. Section 6 concludes the paper.



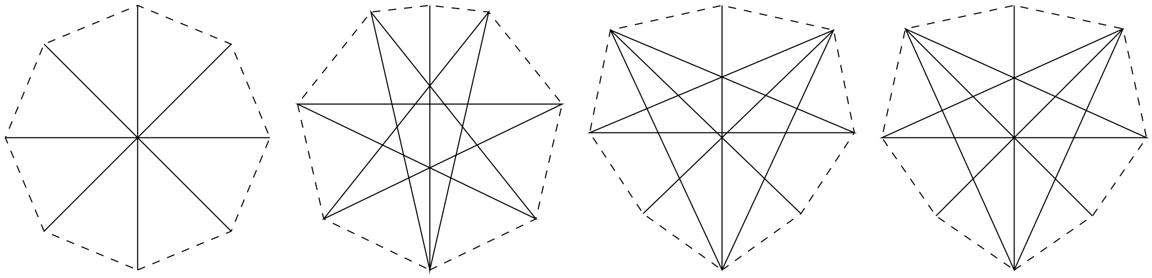
(a) $(\mathtt{R}_4, 2.8284, 0.7071)$ **(b)** $(\mathtt{R}_3^+, 3.0353, 0.8660)$

Figure 1: Two small 4**-gons** $(\mathtt{P}_4, L(\mathtt{P}_4)), W(\mathtt{P}_4))$



(a) $(\mathtt{R}_6, 3, 0.8660)$ **(b)** $(\mathtt{R}_5^+, 3.0979, 0.9511)$ **(c)** $(\mathtt{R}_{3,6}, 3.1058, 0.9659)$

Figure 2: Three small 6**-gons** $(\mathtt{P}_6, L(\mathtt{P}_6)), W(\mathtt{P}_6))$



(a) $(\mathtt{R}_8, 3.0615, 0.9239)$ **(b)** $(\mathtt{R}_7^+, 3.1181, 0.9749)$ **(c)** $(\mathtt{V}_8, 3.1211, 0.9764)$ **(d)** $(\mathtt{B}_8, 3.1211, 0.9776)$

Figure 3: Four small 8**-gons** $(\mathtt{P}_8, L(\mathtt{P}_8)), W(\mathtt{P}_8))$

2 Perimeters and widths of convex small polygons

2.1 Maximal perimeter and maximal width

Let $L(\mathtt{P})$ denote the perimeter of a polygon $\mathtt{P}$ and $W(\mathtt{P})$ its width. For a given integer $n \geq 3$, let $\mathtt{R}_n$ denote the regular small n-gon. We have

$$L(\mathtt{R}_n) = \begin{cases} 2n \sin \frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ n \sin \frac{\pi}{n} & \text{if } n \text{ is even,} \end{cases}$$

and

$$W(\mathtt{R}_n) = \begin{cases} \cos \frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ \cos \frac{\pi}{n} & \text{if } n \text{ is even.} \end{cases}$$

We remark that $L(\mathtt{R}_n) < L(\mathtt{R}_{n-1})$ and $W(\mathtt{R}_n) < W(\mathtt{R}_{n-1})$ for all even $n \geq 4$. This suggests that $\mathtt{R}_n$ does not have maximum perimeter nor maximum width for any even $n \geq 4$. Indeed, when n is even, we can construct a convex small n-gon with a longer perimeter and larger width than $\mathtt{R}_n$ by adding a vertex at distance 1 along the mediatrix of an angle in $\mathtt{R}_{n-1}$. We denote this n-gon by $\mathtt{R}_{n-1}^+$ and we have

$$\begin{aligned} L(\mathtt{R}_{n-1}^+) &= (2n-2) \sin \frac{\pi}{2n-2} + 4 \sin \frac{\pi}{4n-4} - 2 \sin \frac{\pi}{2n-2}, \\ W(\mathtt{R}_{n-1}^+) &= \cos \frac{\pi}{2n-2}. \end{aligned}$$

When n has an odd factor m, let construct another family of convex equilateral small n-gons as follows:

1. Consider a regular small m-gon $\mathtt{R}_m$;
2. Transform $\mathtt{R}_m$ into a Reuleaux m-gon by replacing each edge by a circle's arc passing through its end vertices and centered at the opposite vertex;
3. Add at regular intervals $n/m - 1$ vertices within each arc;
4. Take the convex hull of all vertices.

We denote these n-gons by $\mathtt{R}_{m,n}$ and we have

$$\begin{aligned} L(\mathtt{R}_{m,n}) &= 2n \sin \frac{\pi}{2n}, \\ W(\mathtt{R}_{m,n}) &= \cos \frac{\pi}{2n}. \end{aligned}$$

The 6-gon $\mathtt{R}_{3,6}$ is illustrated in Figure 2c.

Theorem 2 (Reinhardt [1], Datta [2]) *For all $n \geq 3$, let L_n^* denote the maximal perimeter among all convex small n-gons and let $\overline{L}_n := 2n \sin \frac{\pi}{2n}$.*

- *When n has an odd factor m, $L_n^* = \overline{L}_n$ is achieved by finitely many equilateral n-gons, including $\mathtt{R}_{m,n}$. The optimal n-gon $\mathtt{R}_{m,n}$ is unique if m is prime and $n/m \leq 2$.*

- *When $n = 2^s$ with integer $s \geq 2$, $L(\mathtt{R}_n) < L_n^* < \overline{L}_n$.*

When $n = 2^s$, the maximal perimeter L_n^* is only known for $s \leq 3$. Tamvakis [3] found that $L_4^* = 2 + \sqrt{6} - \sqrt{2}$, and this value is achieved only by $\mathtt{R}_3^+$, shown in Figure 1b. Audet, Hansen, and Messine [4] found that $L_8^* \approx 3.121147$, and this value is only achieved by $\mathtt{V}_8$, shown in Figure 3c.

Theorem 3 (Bezdek and Fodor [6]) *For all $n \geq 3$, let W_n^* denote the maximal width among all convex small n-gons and let $\overline{W}_n := \cos\frac{\pi}{2n}$.*

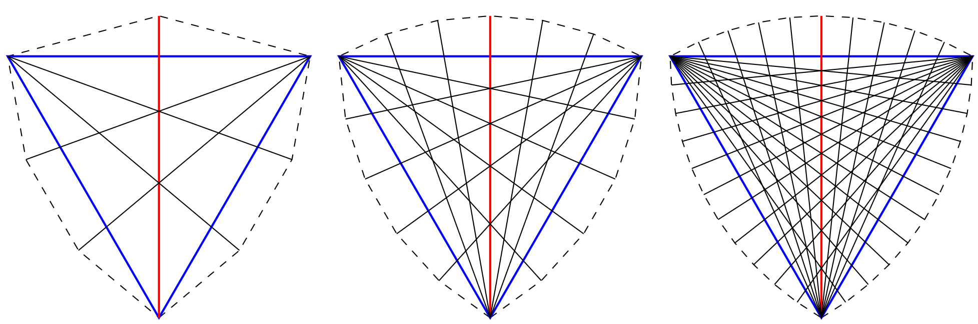
- *When n has an odd factor, $W_n^* = \overline{W}_n$ is achieved by a convex small n-gon with maximal perimeter $L_n^* = \overline{L}_n$.*
- *When $n = 2^s$ with integer $s \geq 2$, $W(\mathtt{R}_n) < W_n^* < \overline{W}_n$.*

When $n = 2^s$, the maximal width W_n^* is only known for $s \leq 3$. Bezdek and Fodor [6] showed that $W_4^* = \frac{1}{2}\sqrt{3}$, and this value is achieved by infinitely many convex small 4-gons, including $\mathtt{R}_3^+$ shown in Figure 1b. Audet, Hansen, Messine, and Ninin found that $W_8^* = \frac{1}{4}\sqrt{10 + 2\sqrt{7}}$, and this value is also achieved by infinitely many convex small 8-gons, including $\mathtt{B}_8$ shown in Figure 3d. It is interesting to note that while the optimal 4-gon for the maximal perimeter problem is also optimal for the maximal width problem, the optimal 8-gon for the maximal perimeter problem is not optimal for the maximal width problem.

2.2 Lower bounds on the maximal perimeter and the maximal width

For $n = 2^s$ with integer $s \geq 2$, let $\mathtt{T}_n$ denote the convex n-gon obtained by subviding each bounding arc of a such Reuleaux triangle into either $\lceil n/3 \rceil$ or $\lfloor n/3 \rfloor$ subarcs of equal length, then taking the convex hull of the endpoints of these arcs. We illustrate $\mathtt{T}_n$ for some n in Figure 4. For each n, the perimeter of $\mathtt{T}_n$ is given by

$$L(\mathtt{T}_n) = \begin{cases} \frac{4n-4}{3}\sin\frac{\pi}{2n-2} + \frac{2n+4}{3}\sin\frac{\pi}{2n+4} & \text{if } n = 3k+1, \\ \frac{4n+4}{3}\sin\frac{\pi}{2n+2} + \frac{2n-4}{3}\sin\frac{\pi}{2n-4} & \text{if } n = 3k+2. \end{cases}$$



(a) $(\mathtt{T}_8, 3.1191, 0.9659)$ **(b)** $(\mathtt{T}_{16}, 3.1364, 0.9945)$ **(c)** $(\mathtt{T}_{32}, 3.1403, 0.9986)$

Figure 4: Tamvakis polygons $(\mathtt{T}_n, L(\mathtt{T}_n), W(\mathtt{T}_n))$

We note that $\mathtt{T}_4$ is optimal for the maximal perimeter problem and we can show that

$$\overline{L}_n - L(\mathtt{T}_n) = \frac{\pi^3}{4n^4} + O\left(\frac{1}{n^5}\right)$$

for all $n = 2^s$ and $s \geq 2$. By contrast,

$$\overline{L}_n - L(\mathtt{R}_n) = \frac{\pi^3}{8n^2} + O\left(\frac{1}{n^4}\right),$$
$$\overline{L}_n - L(\mathtt{R}_{n-1}^+) = \frac{5\pi^3}{96n^3} + O\left(\frac{1}{n^4}\right)$$

for all even $n \geq 4$. Tamvakis asked if $\mathtt{T}_n$ is also optimal when $s \geq 3$. Obviously, $\mathtt{T}_8$ is not optimal.

For all $n = 2^s$ with integer $s \geq 2$, let $\mathtt{V}_n$ denote the convex small n-gon with the longest perimeter.

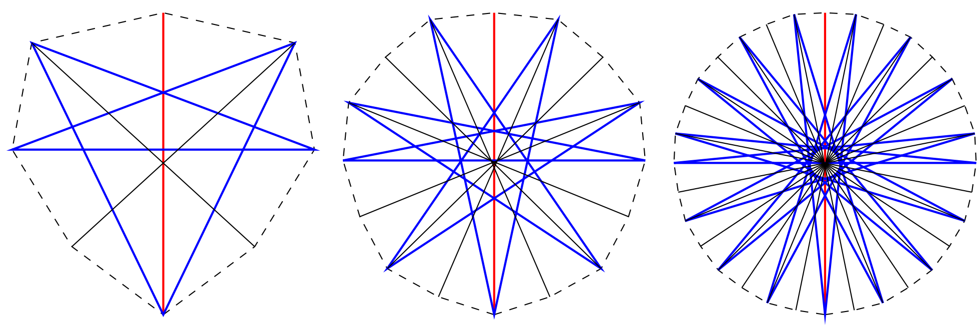
Conjecture 1 (Mossinghoff [5]) *For all $n = 2^s$ with integer $s \geq 2$, the diameter graph of $\mathtt{V}_n$ has a cycle of length $n/2+1$, plus $n/2-1$ additional pendant edges, arranged so that all but two particular vertices of the cycle have a pendant edge.*

Conjecture 2 (Mossinghoff [5]) *For all $n = 2^s$ with integer $s \geq 2$, $\mathtt{V}_n$ has an axis of symmetry corresponding to one particular pendant edge in its diameter graph.*

Conjecture 1 is proven for $n = 4$ [3] and $n = 8$ [4], and Conjecture 2 is only proven for $n = 4$ [3]. Assuming both conjectures, Mossinghoff [5] constructed a family of convex small n-gons $\mathtt{M}_n$ such that

$$\overline{L}_n - L(\mathtt{M}_n) = \frac{\pi^5}{16n^5} + O\left(\frac{1}{n^6}\right)$$

when $n = 2^s$ and $s \geq 3$. We show $\mathtt{M}_n$ for some n in Figure 5.



(a) $(\mathtt{M}_8, 3.1210, 0.9747)$ **(b)** $(\mathtt{M}_{16}, 3.1365, 0.9943)$ **(c)** $(\mathtt{M}_{32}, 3.1403, 0.9987)$

Figure 5: Mossinghoff polygons $(\mathtt{M}_n, L(\mathtt{M}_n), W(\mathtt{M}_n))$

On the other hand, for all $n = 2^s$ and integer $s \geq 3$,

$$W(\mathtt{T}_n) = \begin{cases} \cos\frac{\pi}{2n-2} & \text{if } n = 3k+1, \\ \cos\frac{\pi}{2n-4} & \text{if } n = 3k+2, \end{cases}$$

$$W(\mathtt{M}_n) = \cos\left(\frac{\pi}{2n} + \frac{\pi^2}{4n^2} - \frac{\pi^2}{2n^3}\right),$$

and we can show that $W(\mathtt{R}^+_{n-1}) \geq \max\{W(\mathtt{T}_n), W(\mathtt{M}_n)\}$. Note that

$$\overline{W}_n - W(\mathtt{R}_n) = \frac{3\pi^2}{8n^2} + O\left(\frac{1}{n^4}\right),$$

$$\overline{W}_n - W(\mathtt{R}^+_{n-1}) = \frac{\pi^2}{4n^3} + O\left(\frac{1}{n^4}\right)$$

for all even $n \geq 4$.

3 Proof of Theorem 1

We use cartesian coordinates to describe an n-gon $\mathtt{P}_n$, assuming that a vertex $\mathtt{v}_i$, $i = 0, 1, \ldots, n-1$, is positioned at abscissa x_i and ordinate y_i. Sum or differences of the indices of the coordinates are taken modulo n. Placing the vertex $\mathtt{v}_0$ at the origin, we set $x_0 = y_0 = 0$. We also assume that the n-gon $\mathtt{P}_n$ is in the half-plane $y \geq 0$ and the vertices $\mathtt{v}_i$, $i = 1, 2, \ldots, n-1$, are arranged in a counterclockwise order as illustrated in Figure 6, i.e., $x_i y_{i+1} \geq y_i x_{i+1}$ for all $i = 1, 2, \ldots, n-2$.

For all $n = 2^s$ with integer $s \geq 3$, consider the n-gon $\mathtt{B}_n$ having an $n/2+1$-length cycle: $\mathtt{v}_0 - \mathtt{v}_{\frac{n}{2}-1} - \ldots - \mathtt{v}_{k(\frac{n}{2}-1)} - \ldots - \mathtt{v}_{\frac{n}{4}(\frac{n}{2}-1)} - \mathtt{v}_{n-\frac{n}{4}(\frac{n}{2}-1)} - \ldots - \mathtt{v}_{n-k(\frac{n}{2}-1)} - \ldots - \mathtt{v}_{\frac{n}{2}+1} - \mathtt{v}_0$ plus $n/2-1$ pendant edges: $\mathtt{v}_0 - \mathtt{v}_{\frac{n}{2}}$, $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}}$, $\mathtt{v}_{n-k(\frac{n}{2}-1)} - \mathtt{v}_{\frac{n}{2}-k(\frac{n}{2}-1)}$, $k = 0, 1, \ldots, n/4-1$. We assume

- $\mathtt{B}_n$ has the edge $\mathtt{v}_0 - \mathtt{v}_{\frac{n}{2}}$ as axis of symmetry;
- for all $k = 1, \dots, n/4 - 1$, the pendant edge $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}}$ bissects the angle formed by the edge $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}-1}$ and the edge $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}+1}$. By symmetry, for all $k = 1, \dots, n/4 - 1$, the pendant edge $\mathtt{v}_{n-k(\frac{n}{2}-1)} - \mathtt{v}_{\frac{n}{2}-k(\frac{n}{2}-1)}$ bissects the angle formed by the edge $\mathtt{v}_{n-k(\frac{n}{2}-1)} - \mathtt{v}_{\frac{n}{2}-k(\frac{n}{2}-1)+1}$ and the edge $\mathtt{v}_{n-k(\frac{n}{2}-1)} - \mathtt{v}_{\frac{n}{2}-k(\frac{n}{2}-1)-1}$.

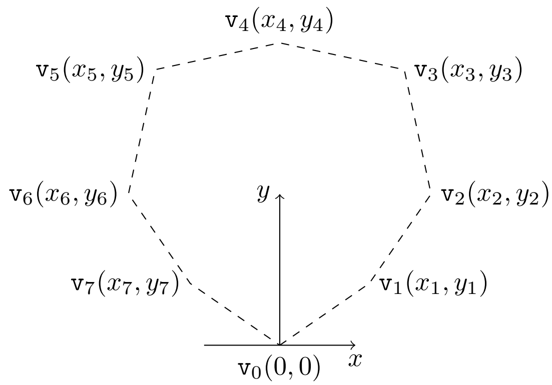
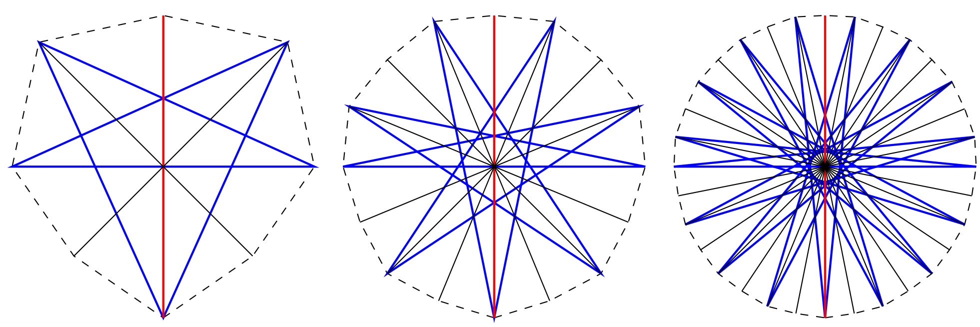


Figure 6: Definition of variables: Case of $n = 8$ vertices

We illustre $\mathtt{B}_n$ for some n in Figure 7.



(a) $(\mathtt{B}_8, 3.1211, 0.9776)$ **(b)** $(\mathtt{B}_{16}, 3.1365, 0.9950)$ **(c)** $(\mathtt{B}_{32}, 3.1403, 0.9988)$

Figure 7: $(\mathtt{B}_n, L(\mathtt{B}_n), W(\mathtt{B}_n))$

Let $2\alpha_k$ denote the angle formed the edge $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}-1}$ and the edge $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}+1}$ for all $k = 0, 1, \dots, n/4 - 1$, and let $\alpha_{\frac{n}{4}}$ denote the angle formed by the edge $\mathtt{v}_{\frac{3n}{4}} - \mathtt{v}_{\frac{n}{4}}$ and the edge $\mathtt{v}_{\frac{3n}{4}} - \mathtt{v}_{\frac{n}{4}+1}$. Since $\mathtt{B}_n$ is symmetric, we have

$$\alpha_0 + 2\sum_{k=1}^{n/4-1} \alpha_k + \alpha_{n/4} = \frac{\pi}{2}, \tag{1}$$

and

$$L(\mathtt{B}_n) = 4\sin\frac{\alpha_0}{2} + 8\sum_{k=1}^{n/4-1} \sin\frac{\alpha_k}{2} + 4\sin\frac{\alpha_{n/4}}{2}, \tag{2a}$$

$$W(\mathtt{B}_n) = \min_{k=0,1,\dots,n/4} \cos\frac{\alpha_k}{2}. \tag{2b}$$

Let place the vertex $\mathtt{v}_0$ at $(0,0)$ in the plane, and the vertex $\mathtt{v}_{\frac{n}{2}}$ at $(0,1)$. We have

$$x_{\frac{n}{2}-1} = \sin\alpha_0 = -x_{\frac{n}{2}+1}, \tag{3a}$$
$$y_{\frac{n}{2}-1} = \cos\alpha_0 = y_{\frac{n}{2}+1}, \tag{3b}$$

$$x_{k\left(\frac{n}{2}-1\right)} = x_{(k-1)\left(\frac{n}{2}-1\right)} - (-1)^k \sin\left(\alpha_0 + 2\sum_{j=1}^{k-1}\alpha_j\right) = -x_{n-k\left(\frac{n}{2}-1\right)} \tag{3c}$$
$$y_{k\left(\frac{n}{2}-1\right)} = y_{(k-1)\left(\frac{n}{2}-1\right)} - (-1)^k \cos\left(\alpha_0 + 2\sum_{j=1}^{k-1}\alpha_j\right) = y_{n-k\left(\frac{n}{2}-1\right)} \tag{3d}$$

for all $k = 2, 3, \ldots, n/4$, and

$$x_{k\left(\frac{n}{2}-1\right)+\frac{n}{2}} = x_{k\left(\frac{n}{2}-1\right)} + (-1)^k \sin\left(\alpha_0 + 2\sum_{j=1}^{k-1}\alpha_j + \alpha_k\right) = -x_{\frac{n}{2}-k\left(\frac{n}{2}-1\right)}, \tag{3e}$$
$$y_{k\left(\frac{n}{2}-1\right)+\frac{n}{2}} = y_{k\left(\frac{n}{2}-1\right)} + (-1)^k \cos\left(\alpha_0 + 2\sum_{j=1}^{k-1}\alpha_j + \alpha_k\right) = y_{\frac{n}{2}-k\left(\frac{n}{2}-1\right)} \tag{3f}$$

for all $k = 1, 2, \ldots, n/4 - 1$.

For all $k = 0, 1, \ldots, n/4$, suppose $\alpha_k = \frac{\pi}{n} + (-1)^k\delta$ with $|\delta| < \frac{\pi}{n}$. Then (1) is verified and (2) becomes

$$L(\mathsf{B}_n) = n\sin\left(\frac{\pi}{2n} + \frac{\delta}{2}\right) + n\sin\left(\frac{\pi}{2n} - \frac{\delta}{2}\right) = 2n\sin\frac{\pi}{2n}\cos\frac{\delta}{2}, \tag{4a}$$
$$W(\mathsf{B}_n) = \cos\left(\frac{\pi}{2n} + \frac{|\delta|}{2}\right). \tag{4b}$$

Coordinates (x_i, y_i) in (3) are given by

$$x_{k\left(\frac{n}{2}-1\right)} = \sum_{j=1}^{k}(-1)^{j-1}\sin\left((2j-1)\frac{\pi}{n} + (-1)^{j-1}\delta\right) = \frac{\sin\frac{2k\pi}{n}\sin\left(\delta - (-1)^k\frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}} = -x_{n-k\left(\frac{n}{2}-1\right)}, \tag{5a}$$
$$y_{k\left(\frac{n}{2}-1\right)} = \sum_{j=1}^{k}(-1)^{j-1}\cos\left((2j-1)\frac{\pi}{n} + (-1)^{j-1}\delta\right) = \frac{\sin\left(\frac{\pi}{n} - \delta\right) + \cos\frac{2k\pi}{n}\sin\left(\delta - (-1)^k\frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}} = y_{n-k\left(\frac{n}{2}-1\right)} \tag{5b}$$

for all $k = 1, 2, \ldots, n/4$, and

$$x_{k\left(\frac{n}{2}-1\right)+\frac{n}{2}} = x_{k\left(\frac{n}{2}-1\right)} + (-1)^k\sin\frac{2k\pi}{n} = -x_{\frac{n}{2}-k\left(\frac{n}{2}-1\right)} \tag{5c}$$
$$y_{k\left(\frac{n}{2}-1\right)+\frac{n}{2}} = y_{k\left(\frac{n}{2}-1\right)} + (-1)^k\cos\frac{2k\pi}{n} = y_{\frac{n}{2}-k\left(\frac{n}{2}-1\right)} \tag{5d}$$

for all $k = 1, 2, \ldots, n/4 - 1$.

Finally, δ is chosen so that $x_{\frac{n}{4}\left(\frac{n}{2}-1\right)} = x_{\frac{3n}{4}} = -\frac{1}{2}$. It follows, from (5a),

$$\frac{\sin\left(\delta - \frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}} = -\frac{1}{2} \Rightarrow \delta = \frac{\pi}{n} - \arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right).$$

Thus, from (4), we have

$$L(\mathtt{B}_n) = 2n \sin\frac{\pi}{2n} \cos\left(\frac{\pi}{2n} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)\right),$$
$$W(\mathtt{B}_n) = \cos\left(\frac{\pi}{n} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)\right),$$

and

$$\overline{L}_n - L(\mathtt{B}_n) = \frac{\pi^7}{32n^6} + \frac{11\pi^9}{768n^8} + O\left(\frac{1}{n^{10}}\right),$$
$$\overline{W}_n - W(\mathtt{B}_n) = \frac{\pi^4}{8n^4} + \frac{11\pi^6}{192n^6} + O\left(\frac{1}{n^8}\right).$$

By construction, $\mathtt{B}_n$ is small and convex for all $n = 2^s$ and $s \geq 3$. This completes the proof of Theorem 1.

Table 1 shows the perimeters of $\mathtt{B}_n$, along with the upper bounds $\overline{L}_n$, the perimeters of $\mathtt{R}_n$, $\mathtt{R}^+_{n-1}$, $\mathtt{T}_n$, and $\mathtt{M}_n$ for $n = 2^s$ and $3 \leq s \leq 7$. As suggested by Theorem 1, when n is a power of 2, $\mathtt{B}_n$ provides a tighter lower bound on the maximal perimeter L^*_n compared to the best prior convex small n-gon $\mathtt{M}_n$. For instance, we can note that $L^*_{128} - L(\mathtt{B}_{128}) < \overline{L}_{128} - L(\mathtt{B}_{128}) < 2.15 \times 10^{-11}$. By analysing the fraction $(L(\mathtt{B}_n) - L(\mathtt{M}_n))(\overline{L}_n - L(\mathtt{M}_n))$ of the length of the interval $[L(\mathtt{M}_n), \overline{L}_n)]$ where $L(\mathtt{B}_n)$ lies, it is not surprising that $L(\mathtt{B}_n)$ approaches $\overline{L}_n$ much faster than $L(\mathtt{M}_n)$ as n increases. After all, $L(\mathtt{B}_n) - L(\mathtt{M}_n) \sim \frac{\pi^5}{16n^5}$ for large n.

Table 2 displays the widths of $\mathtt{B}_n$, along with the upper bounds $\overline{W}_n$, the widths of $\mathtt{R}_n$ and $\mathtt{R}^+_{n-1}$. Again, when $n = 2^s$, $\mathtt{B}_n$ provides a tighter lower bound for the maximal width W^*_n compared to the best prior convex small n-gon $\mathtt{R}^+_{n-1}$. We also remark $W(\mathtt{B}_n)$ approaches $\overline{W}_n$ much faster than $W(\mathtt{R}^+_{n-1})$ as n increases. It is interesting to note that $W(\mathtt{B}_8) = W^*_8$, i.e., $\mathtt{B}_8$ is an optimal solution for the maximal width problem when $n = 8$. It is then natural to ask if $\mathtt{B}_n$ is optimal for the maximal width problem when $n = 2^s$ and $s \geq 4$.

Conjecture 3 *Let $n = 2^s$ with integer $s \geq 3$. Then $\mathtt{B}_n$ is an optimal solution for the maximal width problem and $W^*_n = W(\mathtt{B}_n)$.*

Table 1: Perimeters of $\mathtt{B}_n$

n	$L(\mathtt{R}_n)$	$L(\mathtt{R}^+_{n-1})$	$L(\mathtt{T}_n)$	$L(\mathtt{M}_n)$	$L(\mathtt{B}_n)$	$\overline{L}_n$	$\frac{\overline{L}_n - L(\mathtt{B}_n)}{\overline{L}_n - L(\mathtt{M}_n)}$
8	3.0614674589	3.1181091119	3.1190543124	3.1209757852	3.1210621230	3.1214451523	0.1839
16	3.1214451523	3.1361407965	3.1364381783	3.1365320240	3.1365427675	3.1365484905	0.6524
32	3.1365484905	3.1402809876	3.1403234211	3.1403306141	3.1403310687	3.1403311570	0.8374
64	3.1403311570	3.1412710339	3.1412767980	3.1412772335	3.1412772496	3.1412772509	0.9211
128	3.1412772509	3.1415130275	3.1415137720	3.1415138006	3.1415138011	3.1415138011	0.9606

Table 2: Widths of $\mathtt{B}_n$

n	$W(\mathtt{R}_n)$	$W(\mathtt{R}^+_{n-1})$	$W(\mathtt{B}_n)$	$\overline{W}_n$	$\frac{\overline{W}_n - W(\mathtt{B}_n)}{\overline{W}_n - W(\mathtt{R}^+_{n-1})}$
8	0.9238795325	0.9749279122	0.9776087734	0.9807852804	0.4577
16	0.9807852804	0.9945218954	0.9949956687	0.9951847267	0.7148
32	0.9951847267	0.9987165072	0.9987837929	0.9987954562	0.8523
64	0.9987954562	0.9996891820	0.9996980921	0.9996988187	0.9246
128	0.9996988187	0.9999235114	0.9999246565	0.9999247018	0.9619

Proposition 1 and Proposition 2 highlight some interesting properties of $\mathtt{B}_n$.

Proposition 1 *Let $n = 2^s$ with integer $s \geq 3$.*

1. *The coordinates of $\mathtt{v}_{\frac{3n}{4}}$ are $(-1/2, 1/2)$.*

2. *For all* $k = 0, 1, \ldots, n/4 - 1$*, the pendant edge* $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}}$ *passes through the point* $\mathtt{u} = (0, 1/2)$*.*

Proof. Let $n = 2^s$ with integer $s \geq 3$. Let $\delta = \frac{\pi}{n} - \arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)$.

1. We have, from (5a),

$$x_{\frac{3n}{4}} = x_{\frac{n}{4}(\frac{n}{2}-1)} = \frac{\sin\left(\delta - \frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}} = -\frac{1}{2},$$
$$y_{\frac{3n}{4}} = y_{\frac{n}{4}(\frac{n}{2}-1)} = \frac{\sin\left(\frac{\pi}{n} - \delta\right)}{\sin\frac{2\pi}{n}} = \frac{1}{2}.$$

2. For all $k = 0, 1, \ldots, n/4 - 1$, coordinates (x_i, y_i) in (5) are

$$x_{k(\frac{n}{2}-1)} = \frac{\sin\frac{2k\pi}{n}\sin\left(\delta - (-1)^k\frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}},$$
$$x_{k(\frac{n}{2}-1)+\frac{n}{2}} = x_{k(\frac{n}{2}-1)} + (-1)^k\sin\frac{2k\pi}{n},$$
$$y_{k(\frac{n}{2}-1)} = \frac{1}{2} + \frac{\cos\frac{2k\pi}{n}\sin\left(\delta - (-1)^k\frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}},$$
$$y_{k(\frac{n}{2}-1)+\frac{n}{2}} = y_{k(\frac{n}{2}-1)} + (-1)^k\cos\frac{2k\pi}{n}.$$

It follows that, for all $k = 0, 1, \ldots, n/4 - 1$,

$$\frac{x_{k(\frac{n}{2}-1)+\frac{n}{2}} - x_{k(\frac{n}{2}-1)}}{y_{k(\frac{n}{2}-1)+\frac{n}{2}} - y_{k(\frac{n}{2}-1)}} = \tan\frac{2k\pi}{n} = \frac{x_{k(\frac{n}{2}-1)}}{y_{k(\frac{n}{2}-1)} - \frac{1}{2}},$$

i.e., the pendant edge $\mathtt{v}_{k(\frac{n}{2}-1)} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}}$ passes through the point $\mathtt{u} = (0, 1/2)$.

□

Proposition 2 *Let* $n = 2^s$ *with integer* $s \geq 3$*. The area of* $\mathtt{B}_n$ *is* $\frac{n}{8}\sin\frac{2\pi}{n}$*, which is the area of the regular small* n*-gon* $\mathtt{R}_n$*.*

Proof. Let $n = 2^s$ with integer $s \geq 3$. For all $k = 0, 1, \ldots, n/4 - 1$, let A_k be the area of the quadrilateral formed by the vertices $\mathtt{u} = (0, 1/2)$, $\mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}-1}$, $\mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}}$, and $\mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}+1}$. Let $A_{\frac{n}{4}}$ be the area of the triangle formed by the vertices $\mathtt{u} = (0, 1/2)$, $\mathtt{v}_{\frac{n}{4}}$, and $\mathtt{v}_{\frac{n}{4}+1}$. The area of $\mathtt{B}_n$ is given by

$$A(\mathtt{B}_n) = A_0 + 2\sum_{k=1}^{n/4-1} A_k + 2A_{\frac{n}{4}}.$$

We have

$$\begin{aligned} A_k &= \frac{1}{2}\|\mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}} - \mathtt{u}\|\|\mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}+1} - \mathtt{v}_{k(\frac{n}{2}-1)+\frac{n}{2}-1}\| \\ &= \begin{cases} \frac{1}{2}\sin\left(\frac{\pi}{n} + \delta\right) & \text{if } k \text{ is even,} \\ \frac{1}{2}\sin\frac{2\pi}{n} - \frac{1}{2}\sin\left(\frac{\pi}{n} + \delta\right) & \text{if } k \text{ is odd,} \end{cases} \end{aligned}$$

for all $k = 0, 1, \ldots, n/4 - 1$, and

$$A_{\frac{n}{4}} = \frac{1}{2}(x_{\frac{n}{4}}(y_{\frac{n}{4}+1} - 1/2) - (y_{\frac{n}{4}} - 1/2)x_{\frac{n}{4}+1}) = \frac{1}{4}\sin\left(\frac{\pi}{n} + \delta\right).$$

Thus,

$$A(\mathtt{B}_n) = \frac{n}{8}\sin\left(\frac{\pi}{n} + \delta\right) + \frac{n}{8}\left(\sin\frac{2\pi}{n} - \sin\left(\frac{\pi}{n} + \delta\right)\right) = \frac{n}{8}\sin\frac{2\pi}{n}.$$

□

4 Tight bounds on the maximal width of unit-perimeter polygons

Let $\hat{\mathtt{P}}$ denote the polygon obtained by contracting a small polygon $\mathtt{P}$ so that $L(\hat{\mathtt{P}}) = 1$. Thus, the width of the unit-perimeter polygon $\hat{\mathtt{P}}$ is given by $W(\hat{\mathtt{P}}) = W(\mathtt{P})/L(\mathtt{P})$. For a given integer $n \geq 3$,

$$W(\hat{\mathtt{R}}_n) = \begin{cases} \frac{1}{2n}\cot\frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ \frac{1}{n}\cot\frac{\pi}{n} & \text{if } n \text{ is even.} \end{cases}$$

We remark that $W(\hat{\mathtt{R}}_n) < W(\hat{\mathtt{R}}_{n-1})$ for all even $n \geq 4$. This suggests that $\hat{\mathtt{R}}_n$ does not have maximum width for any even $n \geq 4$. Indeed, when n is even, we can construct an unit-perimeter n-gon with the same width as $\hat{\mathtt{R}}_{n-1}$ by adding a vertex in the middle of a side of $\hat{\mathtt{R}}_{n-1}$.

When n has an odd factor m, one can note that

$$W(\hat{\mathtt{R}}_{m,n}) = \frac{1}{2n}\cot\frac{\pi}{2n}.$$

Theorem 4 (Audet, Hansen, and Messine [9]) *For all $n \geq 3$, let w_n^* denote the maximal width among all unit-perimeter n-gons and let $\overline{w}_n := \frac{1}{2n}\cot\frac{\pi}{2n}$.*

- *When n has an odd factor m, $w_n^* = \overline{w}_n$ is achieved by finitely many equilateral n-gons, including $\mathtt{R}_{m,n}$. The optimal n-gon $\hat{\mathtt{R}}_{m,n}$ is unique if m is prime and $n/m \leq 2$.*
- *When $n = 2^s$ with integer $s \geq 2$, $w(\hat{\mathtt{R}}_n) < \overline{w}_{n-1} \leq w_n^* < \overline{w}_n$.*

When $n = 2^s$, the maximal width w_n^* of unit-perimeter n-gons is only known for $s = 2$. Audet, Hansen, and Messine [9] showed that $w_4^* = \frac{1}{4}\sqrt{6\sqrt{3}-9} > w_3^* = \frac{1}{6}\sqrt{3}$. For $s \geq 3$, exact solutions appear to be presently out of reach. However, it is interesting to note that

$$W(\hat{\mathtt{B}}_n) = \frac{1}{2n}\left(\cot\frac{\pi}{2n} - \tan\left(\frac{\pi}{2n} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)\right)\right)$$

is a tighter lower bound compared to $\overline{w}_{n-1}$ on w_n^* when $n = 2^s$ and $s \geq 3$. Indeed, we can show that, for all $n = 2^s$ and integer $s \geq 3$,

$$\overline{w}_n - W(\hat{\mathtt{B}}_n) = \frac{1}{2n}\tan\left(\frac{\pi}{2n} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)\right) = \frac{\pi^3}{8n^4} + O\left(\frac{1}{n^6}\right),$$

while

$$\overline{w}_n - W(\hat{\mathtt{R}}_n) = \frac{\pi}{4n^2} + O\left(\frac{1}{n^4}\right),$$
$$\overline{w}_n - \overline{w}_{n-1} = \frac{\pi}{6n^3} + O\left(\frac{1}{n^4}\right)$$

for all even $n \geq 4$.

Table 3 lists the widths of $\hat{\mathtt{B}}_n$, along with the upper bounds $\overline{w}_n$, the lower bounds $\overline{w}_{n-1}$, and the widths of $\hat{\mathtt{R}}_n$ for $n = 2^s$ and $3 \leq s \leq 7$. As n increases, it is not surprising that $W(\hat{\mathtt{B}}_n)$ approaches $\overline{W}_n$ much faster than $\overline{w}_{n-1}$.

Table 3: Widths of $\hat{\mathtt{B}}_n$

n	$W(\hat{\mathtt{R}}_n)$	$\overline{w}_{n-1}$	$W(\hat{\mathtt{B}}_n)$	$\overline{w}_n$	$\frac{\overline{w}_n - W(\hat{\mathtt{B}}_n)}{\overline{w}_n - \overline{w}_{n-1}}$
8	0.3017766953	0.3129490191	0.3132295145	0.3142087183	0.2227
16	0.3142087183	0.3171454818	0.3172268776	0.3172865746	0.5769
32	0.3172865746	0.3180374156	0.3180504765	0.3180541816	0.7790
64	0.3180541816	0.3182439224	0.3182439224	0.3182459678	0.8870
128	0.3182459678	0.3182936544	0.3182936544	0.3182939071	0.9428

5 Solving the maximal perimeter problem

5.1 Nonconvex quadratically constrained quadratic optimization

For all integer $n \geq 3$, the maximal perimeter problem can be formulated as follows:

$$\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{v}} \quad \sum_{i=1}^{n} v_i \tag{6a}$$

$$\text{s.t.} \quad (x_j - x_i)^2 + (y_j - y_i)^2 \leq 1 \qquad \forall 1 \leq i < j \leq n-1, \tag{6b}$$

$$x_i^2 + y_i^2 \leq 1 \qquad \forall 1 \leq i \leq n-1, \tag{6c}$$

$$y_i \geq 0 \qquad \forall 1 \leq i \leq n-1, \tag{6d}$$

$$x_i y_{i+1} - y_i x_{i+1} \geq 0 \qquad \forall 1 \leq i \leq n-2, \tag{6e}$$

$$x_i y_{i+2} - y_i x_{i+2} \leq x_i y_{i+1} - y_i x_{i+1} + x_{i+1} y_{i+2} - y_{i+1} x_{i+2} \qquad \forall 1 \leq i \leq n-3, \tag{6f}$$

$$v_i^2 \leq (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \qquad \forall 1 \leq i \leq n, \tag{6g}$$

$$v_i \geq 0 \qquad \forall 1 \leq i \leq n. \tag{6h}$$

At optimality, for all $i = 1, 2, \ldots, n$, $v_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$, which corresponds to the length of the side $\mathtt{v}_{i-1}\mathtt{v}_i$. Constraint (6f) ensures that the feasible n-gon is convex.

Problem (6) is a nonconvex quadratically constrained quadratic optimization problem and can be reformulated as a difference-of-convex optimization (DCO) problem of the form

$$\max_{\boldsymbol{z}} \quad g_0(\boldsymbol{z}) - h_0(\boldsymbol{z}) \tag{7a}$$

$$\text{s.t.} \quad g_i(\boldsymbol{z}) - h_i(\boldsymbol{z}) \geq 0 \qquad \forall 1 \leq i \leq m, \tag{7b}$$

where $g_0, \ldots, g_m$ and $h_0, \ldots, h_m$ are convex quadratic functions. For a fixed $\boldsymbol{c}$, we have $\underline{g}_i(\boldsymbol{z}; \boldsymbol{c}) := g_i(\boldsymbol{c}) + \nabla g_i(\boldsymbol{c})^T(\boldsymbol{z} - \boldsymbol{c}) \leq g_i(\boldsymbol{z})$ for all $i = 0, 1, \ldots, m$. Then the following problem

$$\max_{\boldsymbol{z}} \quad \underline{g}_0(\boldsymbol{z}; \boldsymbol{c}) - h_0(\boldsymbol{z}) \tag{8a}$$

$$\text{s.t.} \quad \underline{g}_i(\boldsymbol{z}; \boldsymbol{c}) - h_i(\boldsymbol{z}) \geq 0 \qquad \forall 1 \leq i \leq m \tag{8b}$$

is a convex restriction of the DCO problem (7).

Nonconvex constraints (6e), (6f), and (6g) are quadratic constraints of the form $\|P\boldsymbol{z}\|^2 \leq \|Q\boldsymbol{z}\|^2$, for some matrices P and Q. Indeed, (6e) is equivalent to

$$(x_i - y_{i+1})^2 + (y_i + x_{i+1})^2 \leq (x_i + y_{i+1})^2 + (y_i - x_{i+1})^2$$

for all $i = 1, 2, \ldots, n-2$, and (6f) can be rewritten as

$$\begin{aligned} &(x_i + \sqrt{3}y_i + x_{i+1} - \sqrt{3}y_{i+1} - 2x_{i+2})^2 + (-\sqrt{3}x_i + y_i + \sqrt{3}x_{i+1} + y_{i+1} - 2y_{i+2})^2 \\ \leq\ &(x_i - \sqrt{3}y_i + x_{i+1} + \sqrt{3}y_{i+1} - 2x_{i+2})^2 + (\sqrt{3}x_i + y_i - \sqrt{3}x_{i+1} + y_{i+1} - 2y_{i+2})^2 \end{aligned}$$

for all $i = 1, 2, \ldots, n-3$. To obtain a larger convex restriction of Problem (6) around a point $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{a}, \boldsymbol{b})$, we replace (6e), (6f), and (6g) by conic constraints of the form $\|P\boldsymbol{z}\| \leq (Q\boldsymbol{c})^T Q\boldsymbol{z}/\|Q\boldsymbol{c}\|$ instead of quadratic convex constraints of the form $\|P\boldsymbol{z}\|^2 \leq 2(Q\boldsymbol{c})^T Q\boldsymbol{z} - \|Q\boldsymbol{c}\|^2$ as suggested by Proposition 3.

Proposition 3 *Let $P \in \mathbb{R}^{p\times n}$ and $Q \in \mathbb{R}^{q\times n}$. Consider the nonconvex set $\Omega := \{\boldsymbol{z} \in \mathbb{R}^n \colon \|P\boldsymbol{z}\|^2 \leq \|Q\boldsymbol{z}\|^2\}$. For $\boldsymbol{c} \in \mathbb{R}^n$ such that $Q\boldsymbol{c} \neq \boldsymbol{0}$, let*

$$\begin{aligned} \Omega_1 &:= \{\boldsymbol{z} \in \mathbb{R}^n \colon \|P\boldsymbol{z}\|^2 \leq 2(Q\boldsymbol{c})^T Q\boldsymbol{z} - \|Q\boldsymbol{c}\|^2\}, \\ \Omega_2 &:= \{\boldsymbol{z} \in \mathbb{R}^n \colon \|P\boldsymbol{z}\| \leq (Q\boldsymbol{c})^T Q\boldsymbol{z}/\|Q\boldsymbol{c}\|\}. \end{aligned}$$

Then $\Omega_1 \subset \Omega_2 \subset \Omega$.

Proof. Let $\boldsymbol{z} \in \Omega_1$. Since $f(\boldsymbol{z}) := \sqrt{2(Q\boldsymbol{c})^T Q\boldsymbol{z} - \|Q\boldsymbol{c}\|^2}$ is a concave function on its domain, we have $f(\boldsymbol{z}) \le f(\boldsymbol{c}) + \nabla f(\boldsymbol{c})^T(\boldsymbol{z}-\boldsymbol{c}) = (Q\boldsymbol{c})^T Q\boldsymbol{z}/\|Q\boldsymbol{c}\|$. Then $\boldsymbol{z} \in \Omega_2$. On the other hand, $\mathbf{0} \in \Omega_2$ but $\mathbf{0} \notin \Omega_2$, i.e., $\Omega_1 \subset \Omega_2$.

Now, let $\boldsymbol{z} \in \Omega_2$. By Cauchy-Schwarz inequality, $(Q\boldsymbol{c})^T Q\boldsymbol{z} \le \|Q\boldsymbol{c}\|\|Q\boldsymbol{z}\|$. Then $\boldsymbol{z} \in \Omega$. On the other hand, $-\boldsymbol{c} \in \Omega$ but $-\boldsymbol{c} \notin \Omega_2$, i.e., $\Omega_2 \subset \Omega$. □

We propose to solve the DCO problem (7) with a sequential convex optimization approach given in Algorithm 1, also known as concave-convex procedure. A proof of showing that a sequence $\{\boldsymbol{z}_k\}_{k=0}^{\infty}$ generated by Algorithm 1 converges to a KKT point $\boldsymbol{z}^*$ of the original DCO problem (7) can be found in [10, 11]. This algorithm was recently used in [8] to solve the maximal area problem, which consists in finding the largest small polygon.

Model 1 Sequential convex optimization

1: Initialization: choose a feasible solution $\boldsymbol{z}_0$.
2: $\boldsymbol{z}_1 := \arg\max\{\underline{g}_0(\boldsymbol{z};\boldsymbol{z}_0) - h_0(\boldsymbol{z})\colon \underline{g}_i(\boldsymbol{z};\boldsymbol{z}_0) - h_i(\boldsymbol{z}) \ge 0, i = 1,2,\dots,m\}$
3: $k := 1$
4: **while** $\frac{\|\boldsymbol{z}_k - \boldsymbol{z}_{k-1}\|}{\|\boldsymbol{z}_k\|} > \varepsilon$ **do**
5: $\quad\boldsymbol{z}_{k+1} := \arg\max\{\underline{g}_0(\boldsymbol{z};\boldsymbol{z}_k) - h_0(\boldsymbol{z})\colon \underline{g}_i(\boldsymbol{z};\boldsymbol{z}_k) - h_i(\boldsymbol{z}) \ge 0, i = 1,2,\dots,m\}$
6: $\quad k := k+1$
7: **end while**

5.2 Computational results

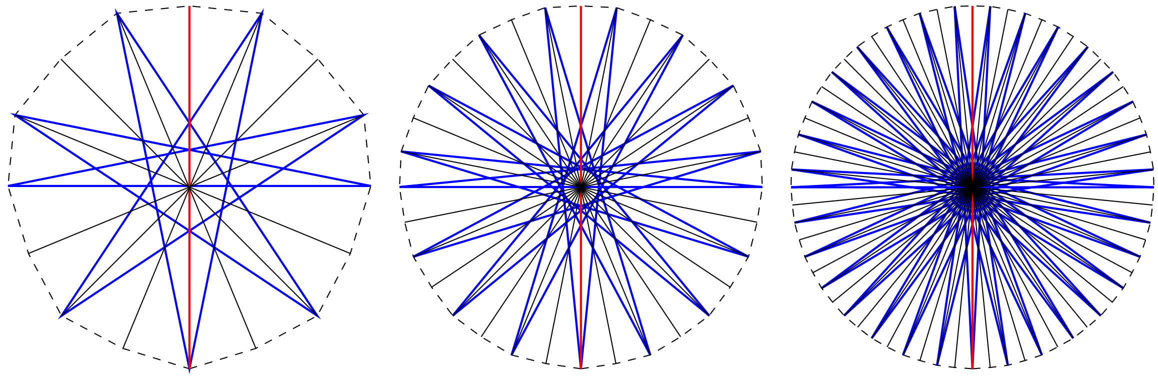
Problem (6) was solved for n power of 2 in MATLAB using CVX 2.2 with MOSEK 9.1.9 and `default precision` (tolerance $\epsilon = 1.49 \times 10^{-8}$). All the computations were carried out on an `Intel(R) Core(TM) i7-3540M CPU @ 3.00 GHz` computing platform. Algorithm 1 was implemented as a MATLAB package: OPTIGON [12], which is freely available at https://github.com/cbingane/optigon. OPTIGON requires that CVX be installed. CVX is a MATLAB-based modeling system for convex optimization, which turns MATLAB into a modeling language, allowing constraints and objectives to be specified using standard MATLAB expression syntax [13].

We chose the n-gons $\mathtt{B}_n$ given by (5) as starting points, and the stopping criteria $\varepsilon = 10^{-5}$. Table 4 shows the optimal values L_n^* of the maximal perimeter problem for $n = 2^s$ and $3 \le s \le 7$, along with the perimeters of the initial n-gons $\mathtt{B}_n$, the best lower bounds $\underline{L}_n$ found in the literature, the upper bounds $\overline{L}_n$, and the fraction $\lambda_n^* := (L_n^* - L(\mathtt{B}_n))/(\overline{L}_n - L(\mathtt{B}_n))$ of the length of the interval $[L(\mathtt{B}_n), \overline{L}_n]$ where L_n^* lies. We also report the number k of iterations in Algotithm 1 for each n. The results support the following keypoints:

1. For $n = 2^s$ and $3 \le s \le 6$, $\underline{L}_n - L_n^* \le 10^{-8}$, i.e., Algorithm 1 converges to the best known optimal solutions found in the literature.
2. By analysing λ_n^*, the maximal perimeter L_n^* appears to approach $L(\mathtt{B}_n)$ as n increases.
3. For all n, the solutions obtained with Algorithm 1 verify, within the limit of the numerical computations, Conjecture 1 and Conjecture 2. We illustrate the optimal 16-, 32- and 64-gons in Figure 8. Furthermore, we remark that both conjectures are verified by each polygon of the sequence generated by Algorithm 1.

Table 4: Maximal perimeter problem

n	$L(\mathtt{B}_n)$	$\overline{L}_n$	$\underline{L}_n$	L_n^*	λ_n^*	# ite. k
8	3.1210621230	3.1214451523	3.121147134 [4, 5]	3.1211471326	0.2219	31
16	3.1365427675	3.1365484905	3.136543956 [5]	3.1365439518	0.2069	14
32	3.1403310687	3.1403311570	3.140331086 [5]	3.1403310855	0.1907	6
64	3.1412772496	3.1412772509	3.1412772498 [14]	3.1412772498	0.1762	4
128	3.1415138011	3.1415138011	–	3.1415138011	0.1288	2



(a) $(\mathtt{V}_{16}, 3.136544)$ **(b)** $(\mathtt{V}_{32}, 3.140331)$ **(c)** $(\mathtt{V}_{64}, 3.141277)$

Figure 8: Three convex small n-gons with longest perimeter $(\mathtt{V}_n, L_n^*)$

6 Conclusion

Tigther lower bounds on the maximal perimeter and the maximal width of convex small n-gons were provided when n is a power of 2. For all $n = 2^s$ with integer $s \geq 3$, we constructed a convex small n-gon $\mathtt{B}_n$ whose the perimeter and the width cannot be improved for large n by more than $\frac{\pi^7}{32n^6}$ and $\frac{\pi^4}{8n^4}$ respectively. It is conjectured that the n-gon $\mathtt{B}_n$ is an optimal solution for the maximal width problem when n is a power of 2.

In addition, a nonconvex quadratically quadratic optimization problem was proposed for finding convex small polygons with longest perimeter and a sequential convex optimization approach to solve it was developed. This approach, also known as the concave-convex procedure, guarantees convergence to a locally optimal solution. Numerical experiments on n-gons, $n = 2^s$ and $3 \leq s \leq 7$, showed that the optimal solutions obtained are near-global and appear to approach $\mathtt{B}_n$ as n increases.

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