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Intra-brand competition in a differentiated oligopoly

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Abstract: In this paper we consider a differentiated oligopoly with two product varieties that are supplied by two groups of firms. We assume that firms can change the variety they produce, thus affecting the intensity of intra-brand competition.

We compute the Cournot solution of the game and analyze its sensitivity to the model parameters. We also study the impact of intra-brand competition on the equilibrium of the game and of the industry composition.

Keywords: Differentiated oligopoly, Cournot solution, intra-brand competition, equilibrium

Résumé : Nous considérons un oligopole différencié produisant deux variétés substituables d'un bien, qui sont mises en marché par deux groupes de firmes. Nous supposons que les firmes peuvent décider de changer la variété qu'elles produisent, affectant ainsi la concurrence intra-marque.

Nous obtenons la solution d'équilibre de Cournot du jeu entre les firmes et analysons sa sensibilité aux paramètres du modèle. Nous analysons également l'impact de la concurrence intra-marque sur l'équilibre et sur la composition de l'industrie.

Mots clés : Oligopose différencié, équilibre de Cournot, concurrence intra-marque

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1 Introduction

In response to consumers' increasing concern for the environment and interest in making greener choices,[1] firms have started investing in production practices that allow them to receive a label certifying their compliance with certain set standards (e.g. organic, bio, sustainable, fair trade). Firms that earn a "green" label diversify their product offering from a conventional one. For example, in the fishery industry, when a company adopting a specific production technique receives the Marine Stewardship Council (MSC) label following a certification process, it diversifies its product offering and competes against companies selling non MSC-certified products. Moreover, certified firms may also compete against each other. In this paper, we use the term *inter-brand competition* to designate competition among firms selling different but interchangeable products (substitute goods), and *intra-brand competition* to designate competition among firms selling the same (homogenous) product.

Inter-brand and intra-brand competition have been mainly studied in the contexts of vertical agreements or of supply channels. In the legal literature, issues consist of assessing whether vertical agreements (e.g. between manufacturers and retailers) may prevent or restrict competition. In the economics literature, vertical agreements in oligopolistic markets have been analyzed under a strategic perspective, for instance to determine the equilibrium contract offered by the upstream firms and the resulting impact on downstream competition (see for instance Saggi & Vettas 2002). A large body of papers in the marketing channel literature define intra-brand competition as the competition between retailers of a same product, while inter-brand competition occurs between manufacturers of substitutable products. This literature is primarily concerned with the coordination of marketing channels, using pricing and/or marketing mix variables (see Cai et al. 2019 for a survey).

Unlike the above-cited literature, we do not assume coordination, integration or agreements between firms. Using the emergence of green production practices as a motivating example, our objective in this paper is to analyze the consequences of introducing intra-brand competition in a context where inter-brand competition is present.

We extend the duopoly model of Singh & Vives (1984) by assuming that two varieties of a product are supplied by two groups of homogeneous firms that compete in quantity. This is different than having N firms producing one variety each, as in Vives (1985), Häckner (2000) or Amir & Jin (2001), since it accounts for both inter-brand competition and intra-brand competition. Furthermore, in our model, each firm pays a fixed cost related to the product variety supplied, which is a *sine qua non* condition to access different markets. Finally, for a given industry size, we allow firms to revise their decision about which product variety they will supply. This decision is based on the different groups' relative economic performance.

We first analyze the differentiated oligopoly model for the Cournot solution corresponding to a given industry size and composition. We perform a complete sensitivity analysis of the equilibrium solution to the model parameters, with particular emphasis on the degree of product substitutability. We then focus on intra-brand competition. Its inclusion raises two questions: the first is about its effects on the equilibrium solution of the game; and the second is related to the equilibrium configuration of the industry. Finally, to derive further insights, we perform some numerical simulations in which, rather than adopting simplifying assumptions on the parameter values, we make conjectures about their relative values, based on what we would see in an industry providing a certified "green" product and a conventional "brown" one.

[1] https://www.nielsen.com/eu/en/insights/article/2018/global-consumers-seek-companies-that-care-about-environmental-issues/;
https://www.forbes.com/sites/forbesnycouncil/2018/11/21/do-customers-really-care-about-your-environmental-impact/#6a24ad41240d;
https://www.unilever.com/news/press-releases/2017/report-shows-a-third-of-consumers-prefer-sustainable-brands.html;
https://www.independent.co.uk/life-style/food-and-drink/environment-meat-bottled-drinks-plastic-waste-consumer-customer-buy-spend-money-a9099446.html

The first part of our analysis is related to what is done in Saggi & Vettas (2002) and in Dou & Ye (2017). Both papers use a special case of the differentiated duopoly model where the two markets for the substitutable varieties are identical. In Dou & Ye (2017), the cost structure of the firms is absent, so that the two groups of firms only differ in terms of size, and results are expressed in terms of market composition only. In Saggi & Vettas (2002), the costs (fixed and variable) and group sizes result from upstream decisions.

In our model we adopt an asymmetric context that encompasses both the Dou & Ye (2017) and Saggi & Vettas (2002) models: firms sustain different production costs, and markets exhibit different features, so that all the model parameters are asymmetric. This allows us to express our results not only in terms of market composition, but also of market conditions and firms' characteristics. This broad asymmetric context is encapsulated in one parameter called "weighted relative efficiency" (WRE), expressed as the two types of firms showing different WRE levels (e.g. small and large). Symmetric results can be retrieved from ours as a very special case, when firms in both groups show the same WRE advantage, which can still be replicated in our model by firms and markets characterized by different parameter values. This is because, in our differentiated oligopoly, it is the relationship among the parameters that matters rather than their individual values.

In terms of findings, one of our main contributions regards the impact of product substitutability on the individual equilibrium quantities. We find that a stronger horizontal product competition can have a positive effect on a firm's output only if we have competing groups of firms, that is, only in the presence of intra-brand competition. This beneficial effect on a firm's output is related to the interaction between the degree of product substitutability and the (negative) variation of the other product-type quantity. For each group of firms we are able to identify the relative WRE values and the market composition that may result in a positive impact of product substitutability on the individual equilibrium quantities.

A second main result is on the impact of intra-brand competition (in terms of the number of firms in a given group) on the equilibrium supply of individual firms in the two groups. An intensification of intra-brand competition in the smaller group always has a negative effect on the individual quantity produced in this group, no matter the size of its WRE advantage. However, an intensification of intra-brand competition in the larger group can have positive consequences on the individual quantities produced in the two groups in some asymmetric scenarios, due to the interplay between the degree of product substitutability and the (positive) impact of inter-brand competition.

When we turn to the equilibrium analysis of the industry composition in a short term situation, we find that all configurations are possible (single-variety market of any type or market with two varieties), according to the values of the model parameters.

Numerical simulations indicate that a significant presence of firms producing a specific product variety is achievable only if their WRE is larger than the one of firms producing the rival variety; that a specific equilibrium combination can be reached under various combinations of horizontal product competition and WRE distribution; and, finally, that the composition of the industry is more sensitive to changes in fixed costs when the intra-brand competition is weak.

The paper is organized as follows. Section 2 presents the differentiated oligopoly model with two varieties produced by N firms. Section 3 and Section 4 analyze the impact of the demand function parameters and industry composition, respectively, on the equilibrium solution of the game. Section 5 and Section 6 investigate the consequences of intra-brand competition on firms' equilibrium output and on the equilibrium market composition, respectively. Section 7 presents some numerical illustrations and Section 8 concludes the paper.

2 A general oligopoly with intra- and inter-brand competition.

Consider an industry populated by N firms. Producers are divided into two groups of similar types, and members of the same group use the same technology to produce a homogeneous product (e.g.

"high" and "low" efficiency technologies). Let $k \in \{H, L\}$ index the product type and n_k denote the number of producers within group k, with $n_H + n_L = N$. Accordingly, assuming a linear cost function, the total production cost of a quantity q_{ki} of product $k \in \{H, L\}$ by producer $i \in \{1, ..., n_k\}$ is given by

$$C_{ki} = f_k + m_k q_{ki}$$

where $m_k \geq 0$ and $f_k \geq 0$ are, respectively, the marginal and fixed production costs.

Since goods produced by firms of a given type are homogeneous, consumers are offered two product varieties. Following Singh and Vives (1984), we assume that the representative consumer has a taste for variety, and that her quadratic utility function is strictly concave and described by

$$U(Q_H, Q_L) = A_H Q_H + A_L Q_L - \frac{1}{2}\left(F_H Q_H^2 + 2S Q_H Q_L + F_L Q_L^2\right),$$

where Q_H and Q_L are the total production of the firms of type H and L, respectively, and where $F_k > 0$, $A_k > 0$ and $S \geq 0$ for $k \in \{H, L\}$. In the same way as in Häckner (2000), the parameters A_k can be interpreted as the quality (vertical) differentiation between product varieties. The parameter $S \geq 0$ is the symmetric degree of substitutability between any pair of varieties. When $S = 0$, products H and L are completely independent, and each group of producers of a given type becomes an independent oligopoly selling a homogeneous product (pure intra-brand competition). The strict concavity of the representative consumer's utility function assumption implies that

$$S^2 < F_H F_L. \tag{1}$$

In addition, we assume that the maximum utility of the consumer is achieved in the positive quadrant, which corresponds to

$$S < \min\left\{F_H \frac{A_L}{A_H}, F_L \frac{A_H}{A_L}\right\}. \tag{2}$$

The representative consumer's utility-maximisation problem is then

$$\max\{U(Q_H, Q_L)\} \text{ s.t. } (P_H Q_H + P_L Q_L) \leq I,$$

where I is the total budget. Consequently, the inverse demand functions faced by producers of each type are given by

$$\begin{aligned} P_H =& A_H - F_H Q_H - S Q_L \\ P_L =& A_L - F_L Q_L - S Q_H. \end{aligned}$$

We denote by E_k, with $k \in \{H, L\}$, the quantity $A_k - m_k$, which is assumed to be strictly positive. The parameter E_k depends on quality and cost parameters and can be interpreted as an indicator of efficiency. For example, if $E_k > E_j$, then type-k firms are more efficient than type-j firms; this greater efficiency can result from a better quality and/or from a production cost advantage. Producers compete in quantities, both within each group, by selling a homogeneous product (intra-brand competition), and with producers of the other group, by offering a different variety (inter-brand competition).

The optimization problem of a single representative producer i of group k with $k \in \{H, L\}$ is given by

$$\max_{q_{ki} \geq 0} \{\pi_{ki} \equiv P_k q_{ki} - C_{ki}\}.$$

This is a convex optimization problem and, since producers in the same group have identical parameters, we can derive, from the first-order conditions, the reaction functions of each type of producer as

$$q_k = \frac{E_k}{F_k(n_k + 1)} - \frac{S n_j}{F_k(n_k + 1)} q_j \text{ with } k, j \in \{L, H\}, k \neq j.$$

The equilibrium output of the oligopoly game is then given by

$$q_H = \frac{F_L E_H (n_L + 1) - S E_L n_L}{\Omega} \tag{3}$$

$$q_L = \frac{F_H E_L (n_H + 1) - S E_H n_H}{\Omega} \tag{4}$$

$$\Omega = F_H F_L (n_H + 1)(n_L + 1) - S^2 n_H n_L > 0. \tag{5}$$

The corresponding equilibrium net marginal price for a producer of type k is given by

$$P_k - m_k = F_k q_k$$

and the equilibrium profit is

$$\pi_k = F_k q_k^2 - f_k.$$

For $j, k \in \{H, L\}$, $j \neq k$, define the ratio

$$\gamma_k \equiv F_j \frac{E_k}{E_j}.$$

The ratio γ_k is a weighted relative efficiency (WRE) parameter, where the weight is given by the market's sensitivity to the price of the other variety. Note that, for a given efficiency level and price sensitivity of the firms in the rival group, a better product quality and/or a lower production cost of firm k increases its WRE, while a higher price sensitivity in market k increases the WRE of the firms in the rival group.

We assume that

$$S < \min\{\gamma_L, \gamma_H\} \tag{6}$$

and that $f_k < F_k q_k^2$ for $k \in \{H, L\}$, which ensures that both types of producers participate in the market at equilibrium, for any possible value of the number of producers in each group. Without loss of generality, we assume in the sequel that $\gamma_L \leq \gamma_H$, so that a type-L firm is associated with the smallest WRE advantage and a type-H firm is associated with the largest WRE advantage. Since $F_H F_L = \gamma_H \gamma_L$, conditions (1) and (6) reduce to

$$S < \gamma_L \leq \gamma_H. \tag{7}$$

The equilibrium outputs (3)–(4) are computed for a general context that includes any market compositions, market conditions and firms' cost structures. In this way they broaden both the Cournot solution found in Singh and Vives (1984) to a framework with intra-brand competition, and the equilibrium quantities derived in Saggi & Vettas (2002) and Dou and Ye (2017) to an asymmetric setting.

3 Impact of demand function parameters

In this section we investigate how the equilibrium quantities respond to changes in the parameters related to market demand.

It is easy to see that, for a given set of prices, an increase in consumers' sensitivity to the price of product k contracts their demand for that product, while an increase in their sensitivity to the price of the alternative variety j expands their demand for product k:

$$Q_k = \frac{F_j (A_k - P_k) - S (A_j - P_j)}{F_k F_j - S^2}.$$

As a result, there is a negative relationship between the equilibrium output of a firm and the slope of its inverse demand function, while there is a positive relationship between the equilibrium output of a firm and the slope of the inverse demand function of the other variety:

$$\frac{\partial q_k}{\partial F_k} = -F_j q_k (n_k + 1) \frac{n_j + 1}{\Omega} < 0$$

$$\frac{\partial q_k}{\partial F_j} = S n_j \frac{n_j + 1}{\Omega} q_j > 0, \; k, j \in \{L, H\}, \; k \neq j.$$

The relationship between the *degree of substitutability* S and the individual equilibrium quantities is more complex and depends both on the relative values of the WREs and on the industry composition. An increase in S, that is, an increase in the degree of the products' substitutability, can be interpreted as a more intense horizontal (inter-brand) product competition.

Proposition 1 **i)** *If* $\frac{\gamma_L}{\gamma_H} < \frac{(n_L+1)(n_H-1)}{n_L n_H} \leq 1$, *then* $\frac{\partial q_L}{\partial S} < 0$, *and there exists an admissible*

$$S_H = \alpha_L\gamma_L - \sqrt{\alpha_L\gamma_H\left(\alpha_L\gamma_H - \alpha_H\gamma_L\right)} \in (0, \gamma_L)$$

such that

$$\frac{\partial q_H}{\partial S} \leq 0 \text{ for } S \leq S_H$$
$$\frac{\partial q_H}{\partial S} > 0 \text{ for } S > S_H.$$

ii) *If* $\frac{(n_L+1)(n_H+1)}{n_L(n_H+2)} < \frac{\gamma_L}{\gamma_H} < 1$, *then* $\frac{\partial q_H}{\partial S} < 0$, *and there exists an admissible*

$$S_L = \alpha_H\gamma_L - \sqrt{\alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right)} \in (0, \gamma_L)$$

such that

$$\frac{\partial q_L}{\partial S} \leq 0 \text{ for } S \leq S_L$$
$$\frac{\partial q_L}{\partial S} > 0 \text{ for } S > S_L.$$

iii) *In all other cases,* $\frac{\partial q_L}{\partial S} \leq 0$ *and* $\frac{\partial q_H}{\partial S} \leq 0$.

Proof. See Appendix 8.1. □

Proposition 1 provides the market conditions and composition and the firms' characteristics that allow us to predict the effect of an increase in the degree of product substitutability on the equilibrium quantities.

Note that the first condition in Proposition 1 can only be satisfied if $1 < n_H \leq n_L + 1$ and that the second condition requires $n_H < n_L - 1$. This means that a stronger horizontal product competition can have a positive effect on the individual equilibrium quantity only if the size of the H group is small with respect to the size of the L group. This result is the specific contribution of intra-brand competition, as it necessitates competing groups of different sizes. Indeed, if the model is simplified to a differentiated duopoly (pure inter-brand competition), then the impact of product substitutability on the equilibrium quantities can never be positive.

A second result from Proposition 1 relates to the importance of the WRE advantage, which makes it possible to observe a positive impact of product substitutability on the individual equilibrium quantities. In particular, for a type-H firm, the difference between the WREs has to be large; whereas, for an type-L firm, this difference has to be small. As a result, it is not possible to observe a positive impact in both markets.

The positive impact of greater horizontal product competition on the equilibrium quantity is due to the interplay between the degree of product substitutability and the variation in the rival product's quantity. If, for example,

$$\frac{\gamma_L}{\gamma_H} < \frac{(n_H-1)(n_L+1)}{n_L n_H} \leq 1,$$

then we know that $\frac{\partial q_L}{\partial S} < 0$, and this decrease is increasing with S. To have $\frac{\partial q_H}{\partial S} > 0$, given the reaction function $q_H = \frac{E_H}{F_H(n_H+1)} - S\frac{n_L}{F_H(n_H+1)}q_L$, the reduction of q_L has to outweigh the increase of S, which happens when S is large.

Finally, Proposition 1 also suggests that, for a given WRE ratio (from market conditions and firms' characteristics), a stronger product competition is more likely to have a positive impact on the firms' quantities the larger is its group size. Figure 1 is a representation of the three possible scenarios for the relative values of the WREs in Proposition 1. Case i) corresponds to the area below the solid blue line; Case ii) corresponds to the area above the solid red line; and finally, Case iii), where both derivatives are negative, corresponds to the area between the blue and red solid lines. A larger value for n_H rotates both the blue line and the red line counterclockwise (see Panel a), enlarging the area for a possible positive impact of S on q_H. A larger value for n_L has less impact; it rotates both the blue line and the red line clockwise (see Panel b), slightly enlarging the area for a possible positive impact of S on q_L.

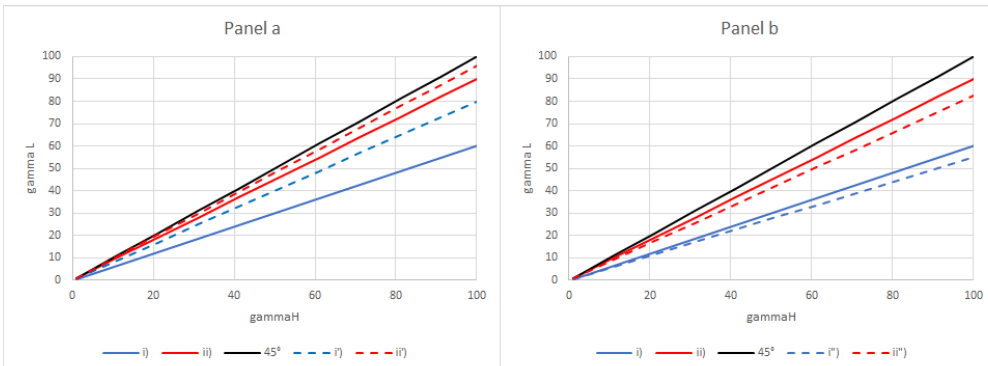

Figure 1: Regions corresponding to the three cases in Proposition 1. The base case (solid lines) is $n_H = 2$ and $n_L = 5$. In Panel a, the dashed lines correspond to $n_H = 3$ and $n_L = 5$. In Panel b, the dashed lines correspond to $n_H = 2$ and $n_L = 10$.

4 Impact of industry composition

We now assess how the equilibrium quantities respond to changes in the industry composition. More specifically, we focus on the degree of intra-brand competition when an increase in the number of firms in one group does not change the number of firms in the other group, so that the total number of firms in the industry increases. This scenario can be assimilated to long-term structural changes in the industry size and composition.

It is straightforward to check that a unilateral increase in the number of firms in a given group has a negative impact on the individual output of all the firms in the industry:

$$\frac{dq_k}{dn_k} = -\, q_k \frac{\left(F_L F_H - S^2\right) n_j + F_j F_k}{\Omega} \leq 0$$
$$\frac{dq_j}{dn_k} = -\, S F_k \frac{q_k}{\Omega} \leq 0.$$

This is due to a general intensification of the competition, specifically intra-brand competition in the expanding group and inter-brand competition with firms in the rival group.

However, the impact of a unilateral increase in the number of firms in a given group on the total output in each market is different: the total equilibrium quantity of the group that experiences a growth in size increases (due to the greater size), but the total equilibrium quantity of the competing group decreases (due to increased inter-group competition).

$$\frac{\partial Q_k}{\partial n_k} = \frac{F_k F_j \left(n_j + 1\right)}{\Omega} q_k > 0$$
$$\frac{\partial Q_j}{\partial n_k} = -\, \frac{S F_k n_j}{\Omega} q_k < 0$$

When we look at the total industry quantity Q, the consequence of an increase in the size of group k when the size of group j does not change depends on the degree of substitutability S.

$$\frac{\partial Q}{\partial n_k} = F_k q_k \frac{F_j\left(n_j+1\right) - S n_j}{\Omega}$$

If the degree of substitutability is relatively low ($S < \frac{F_j(n_j+1)}{n_j}$), that is, if the markets for the two varieties are relatively independent, then the increase of the total quantity in the group that becomes larger more than compensates for the decrease of the total quantity in the group whose size does not change, so that the total industry quantity increases. The reverse is true when the degree of substitutability is relatively high ($S > \frac{F_j(n_j+1)}{n_j}$), that is, when the markets for the two varieties are more interconnected. Note that the threshold value of S at which the impact on the total output changes is decreasing in n_j, the size of the group that does not change.

5 Impact of intra-brand competition

We now assume that the total number of firms in the industry is fixed, so that no firm can enter or leave the industry. However, we allow a firm to change the choice of its production technique; this could happen, for instance, for profit considerations. In this case, an increase in the number of firms in one group is compensated for by a decrease in the number of firms in the other group. The group that experiences an increase in size experiences stronger intra-brand competition but reduced inter-brand competition, while the reverse happens in the rival group.

If N is assumed constant, then the impact of an increase in the number of firms within group $k \in \{H, L\}$ on the total quantity produced by the group is given by

$$\begin{aligned}\frac{dQ_k}{dn_k} &= -\frac{dQ_k}{dn_j} \\ &= \gamma_k \frac{E_j}{\Omega^2}\left(-n_k^2 S^2 + \gamma_j\left(N+1\right)\left(-N+2n_k\right)S + \gamma_j\gamma_k\left(N-n_k+1\right)^2\right),\end{aligned}$$

which is positive for any feasible S (see Appendix 8.2). Note that, when N is constant, the impact of an increase in n_k is equal to the impact of a decrease in n_j.

Increasing the size of a group decreases the inter-brand competition it experiences: it increases the total quantity produced by this group and decreases the total quantity produced by the other. However, these quantities are divided among a different number of firms, so that individual quantities could increase or decrease.

The impact of an increase in the number of firms within group $k \in \{H, L\}$ on the individual equilibrium quantity is given by

$$\begin{aligned}\frac{dq_k}{dn_k} &= -\frac{dq_k}{dn_j} \\ &= \frac{\partial q_k}{\partial n_k} - \frac{\partial q_k}{\partial n_j} \text{ with } j \in \{H, L\} \text{ and } j \neq k,\end{aligned}$$

where $\frac{\partial q_k}{\partial n_k}$ represents the marginal impact of an increase in intra-group competition and $\frac{\partial q_k}{\partial n_j}$ represents the marginal impact of an increase of inter-group competition.

Proposition 2 **i)** *If $n_L > n_H$ and $\frac{\gamma_L}{\gamma_H} < 1$, then there exists an admissible $\overline{S} \in [0, \gamma_L)$ such that*

$$\begin{aligned}\frac{dq_L}{dn_L} &< 0 \text{ for } 0 \leq S < \overline{S} \\ \frac{dq_L}{dn_L} &> 0 \text{ for } \overline{S} < S < \gamma_L.\end{aligned}$$

ii) *If* $\frac{3n_L(n_L+1)+1-n_H}{3n_L^2} < \frac{\gamma_L}{\gamma_H} < 1$, *then, if* n_H *is sufficiently large and* γ_H *and* γ_L *are sufficiently close, there exists an admissible range* $(S_1, S_2) \in [0, \gamma_L)$ *such that*

$$\frac{dq_H}{dn_H} < 0 \text{ for } 0 \leq S < S_1$$
$$\frac{dq_H}{dn_H} > 0 \text{ for } S_1 < S < S_2$$
$$\frac{dq_H}{dn_H} < 0 \text{ for } S_2 < S < \gamma_L.$$

iii) *If* $\gamma_L = \gamma_H = \gamma$ *and* $n_k > \frac{3N+1}{4}$, *then*

$$\frac{dq_k}{dn_k} < 0 \text{ for } 0 \leq S < S_k$$
$$\frac{dq_k}{dn_k} > 0 \text{ for } S_k < S < \gamma_L$$

where

$$S_k = \frac{\gamma}{2n_j^2}\left(n_j - n_k + \sqrt{4n_j^2\left(n_j+1\right)^2 + \left(n_j - n_k\right)^2}\right).$$

iv) *In all other cases,* $\frac{dq_k}{dn_k} < 0$ *for all* $S \in [0, \gamma_L)$ *and* $j \in \{L, H\}$.

Proof. See Appendix 8.3. □

Proposition 2 allows us to anticipate the consequences of a shift in the number of firms on the individual quantity of a member firm when asymmetry is about group size, market conditions and firms' characteristics.

A first result following from Proposition 2 is that, regardless of the market conditions and the firms' characteristics, when the smaller group increases in size, the individual equilibrium quantity of each firm in this group decreases. In other words, the negative marginal impact of intra-group competition is always greater than the positive marginal impact of inter-group competition. This is due to the fact that when the size of the smaller group increases, this bridges the gap between the two group sizes, strengthening the impact of intra-brand competition and weakening the impact of inter-brand competition.

Furthermore, Proposition 2 shows that an increase in intra-brand competition in one group may have a positive impact on the equilibrium individual number of firms in both groups. More precisely, the impact is positive in both groups when

- the increasing group is the largest, its WRE is the lowest and products are highly substitutable (Case i);
- the increasing group is much larger than the other, the two groups have the same WRE and products are highly substitutable (Case iii);
- the increasing group is even much larger than the other, its WRE is slightly higher than the other and S is inside a range of feasible values (neither high nor low) (Case ii).

The first result (Case i) can be explained by examining the reaction function of an L firm:

$$q_L = \frac{E_L}{F_L\left(n_L+1\right)} - \frac{Sn_H}{F_L\left(n_L+1\right)}q_H$$

(an illustrative numerical example is provided in Table 1 and corresponding Figure 2). An increase in the L-group's size reduces both the intercept $(E_L/F_L\left(n_L+1\right))$ and the variable part of the reaction function $(Sn_Hq_H/F_L\left(n_L+1\right))$; in order to obtain an increase in quantity, the variable part of the reaction function has to decrease more. This variable part depends on both S and q_H. An increase in n_L increases q_H (due to the positive marginal effect of inter-brand competition); however, this increase is decreasing in S (24.97% at $S = 0.1$ and 9.69% at $S = 2.9$). Accordingly, a large enough S is able to compensate for the smaller impact of inter-brand competition, making the reduction of the variable part larger than the reduction of the intercept.

Table 1: Numerical illustration of the impact of intra-brand competition in the L group.

	$\frac{E_L}{F_L(n_L+1)}$	$\frac{Sn_Hq_H}{F_L(n_L+1)}$	q_H	q_L
$S = 0.1; n_L = 11$	0.07813	0.00203	0.38884	0.07610
$S = 0.1; n_L = 12$	0.07212	0.00175	0.48593	0.07036
% change	-7.69%	-13.48%	24.97%	-7.54%
$S = 2.9; n_L = 11$	0.07813	0.02860	0.18936	0.04952
$S = 2.9; n_L = 12$	0.07212	0.02172	0.20770	0.05040
% change	-7.69%	-24.06%	9.69%	1.76%

Parameter values are $N = 15$, $E_H = 3$, $E_L = 6$, $F_H = 1.5$, $F_L = 6.4$, $\gamma_H = 3.2$, $\gamma_L = 3$.

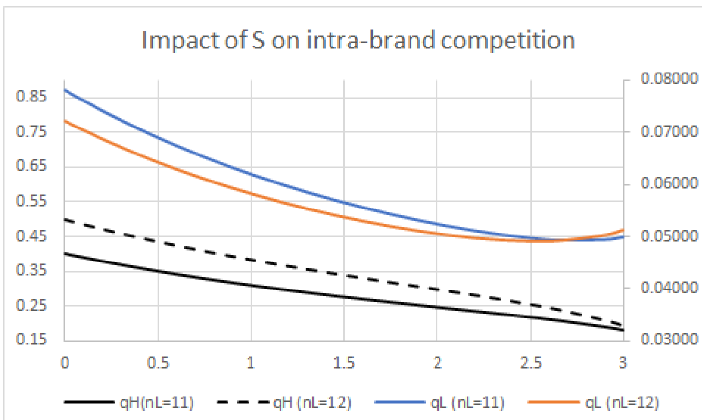

Figure 2: Impact on individual quantities of one firm switching from the H group to the L group, as a function of S. Parameter values are as in Table 1.

If, on the other hand, the H group is the largest, the degree of product substitutability can, as in the previous scenario, strengthen or soften the positive impact of inter-brand competition, which, in turn, depends negatively on S. However, the positive effect on the equilibrium quantity occurs for a range of values of S that are neither too small nor too large.

A numerical illustration of the third result (Case ii) is provided in Table 2 and corresponding Figure 3. Note that a very small degree of product substitutability ($S = 0.1$) is able to outweigh a large marginal effect of inter-brand competition (49.95%), generating a combined effect that is smaller than the negative marginal impact of intra-brand competition. This can also occur when a very high degree of product substitutability ($S = 2.9$) is outweighed by a small marginal effect of inter-brand competition (10.91%). Only for intermediate values of product differentiation (between 1.56 and 2.88) do we see a positive effect of intra-brand competition on the individual quantities.

The second result (Case iii) can be interpreted as the limit of the previous scenario, where $S_2 \to \gamma_L$ as the WREs become closer.

6 Equilibrium market composition

In this section, we keep the same assumptions as in the previous one, that is, the total number of firms in the industry is fixed. However, we now are interested in characterizing the equilibrium of the

Table 2: Numerical illustration of the impact of intra-brand competition in the H group.

	$\frac{E_H}{F_H(n_H+1)}$	$\frac{Sn_Lq_L}{F_H(n_H+1)}$	q_L	q_H
$S = 0.1; n_H = 13$	0.14286	0.00289	0.30302	0.13997
$S = 0.1; n_H = 14$	0.13333	0.00135	0.45439	0.13131
$\%\Delta$	-6.67%	-53.33%	49.95%	-6.19%
$S = 2.5; n_H = 13$	0.14286	0.07215	0.11840	0.11467
$S = 2.5; n_H = 14$	0.13333	0.03367	0.14963	0.11671
$\%\Delta$	-6.67%	-53.33%	26.37%	1.78%
$S = 2.9; n_H = 13$	0.14286	0.08369	0.0699	0.12355
$S = 2.9; n_H = 14$	0.13333	0.03906	0.07753	0.12334
$\%\Delta$	-6.67%	-53.33%	10.91%	-0.17%

Parameter values are $N = 15$, $E_H = 3$, $E_L = 6$, $F_H = 1.5$, $F_L = 6.4$, $\gamma_H = 3.2$, $\gamma_L = 3$.

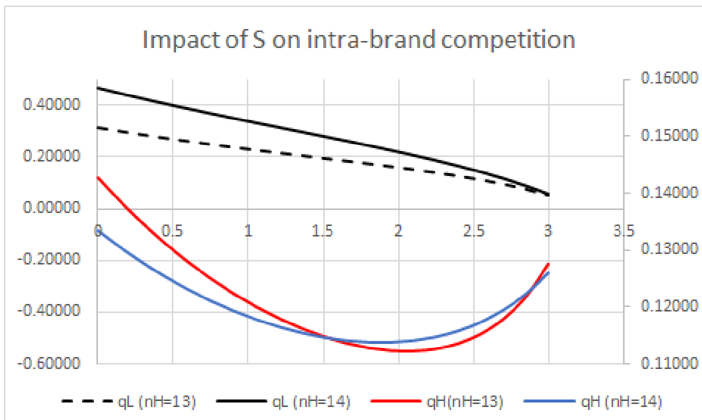

Figure 3: Impact on individual quantities of one firm switching from the L group to the H group, as a function of S. Parameter values are as in Table 1.

industry, assuming that producers switch from making one variety to the other because of the relative profits of both industries.

Note that when the fixed cost does not differ across producer types, the highest profit in the Cournot competition is achieved by the players with the highest $F_k q_k^2$. This is no longer the case when the production of different varieties generates different fixed production costs; then, comparing the profits of the groups of players is a more complex problem.

The equilibrium quantities and, therefore, the profit of both kinds of producers depend on the composition of the industry, which, for a fixed N, can be characterized by $n \equiv n_H$. Assuming that N is sufficiently large, we define the continuous extensions $\pi_k : [0, N] \to \mathbb{R}$, $k \in \{H, L\}$ of the equilibrium profit of both kinds of producers as a function of n.

An equilibrium market composition is a number n^* such that

$$\pi_H(n^*) = \pi_L(n^*), \tag{8}$$

that is,

$$E_L E_H \left(\gamma_L \left(\gamma_H + (N - n^*)(\gamma_H - S)\right)^2 - \gamma_H \left(\gamma_L + n^* (\gamma_L - S)\right)^2\right) \\ = \delta \left(\gamma_H \gamma_L (N - n^* + 1)(n^* + 1) - S^2 n^* (N - n^*)\right)^2, \tag{9}$$

where $\delta = f_H - f_L$.

We analytically derive conditions, on the fixed-cost difference δ, under which different compositions of the industry arise in equilibrium, as stated in the following proposition.

Proposition 3 *Define*

$$\lambda_1 \equiv E_H E_L \frac{\gamma_H \gamma_L - (\gamma_L (N+1) - NS)^2}{\gamma_H \gamma_L^2 (N+1)^2}$$

$$\lambda_2 \equiv E_H E_L \frac{-\gamma_H \gamma_L + (\gamma_H (N+1) - NS)^2}{\gamma_H^2 \gamma_L (N+1)^2}.$$

At equilibrium,

i) *if* $0 < \lambda_2 \leq \delta$*, then the industry consists only of type-*H *firms;;*

ii) *if* $\lambda_1 < \delta < \lambda_2$*, then the two types of firms coexist in the industry;*

iii) *otherwise, the industry consists only of type-*L *firms.*

Proof. See Appendix 8.4. □

Note that the equilibrium solution when the industry is populated by a single type of firm is readily obtained by setting $n_H = N$ or $n_L = N$ in (4)–(3). This solution yields a positive quantity and a positive profit for the firms that are not present in the market; these are interpreted as the limit quantity and profit when the number of firms of one type vanishes.

From the results of Proposition 3, comparative statics allow us to derive the following properties (see Appendix 8.5.1).

P1 Starting from an industry with only type-H firms, type-L firms appear with increases in λ_2 and decreases in δ, that is:

1. decreases in S (less substitutable products);
2. increases in $\frac{\gamma_H}{\gamma_L}$ (larger WRE advantage for type-H firms);
3. smaller (*resp. larger*) fixed cost for producers of type H (*resp. of type* L).

P2 Starting from an industry with only type-L firms, type-H firms appear with decreases in λ_1 and increases in δ, that is:

1. decreases in S (less substitutable products);
2. decreases in $\frac{\gamma_H}{\gamma_L}$ (smaller WRE advantage for the type-H firms);
3. larger (*resp. smaller*) fixed cost producers of type H (*resp. of type* L).

Moreover, we derive the following property by studying the effects of parameter changes on the stability condition (9) (see Appendix 8.5.2).

P3 Starting from a mixed industry, more type-L firms appear with:

1. increases in E_H (*resp. decreases in* E_L) (changes in the quality and/or the cost efficiency);
2. decreases in δ,

while changes in other parameter values impact the industry composition in an ambiguous way.

7 Application: Green and brown products

In this section, we investigate whether or not the results found for the general oligopoly with two varieties can be further refined. Instead of adopting simplifying assumptions on the parameters, we make conjectures about their relative values, and our assumptions are inspired by what we would expect in an industry with a certified "green" product and a conventional "brown" one. Examples of such green products are those accredited by Fairtrade, the Rainforest Alliance, the Forest Stewardship Council (FSC) and the Marine Stewardship Council (MSC).

The adoption of a production practice adequate for green certification is captured by the marginal cost m_G and the fixed cost f_G. We assume that

$$m_G > m_B,$$

that is, that the technology implemented to produce a certified green product is more expensive than the one adopted to make a conventional one. Moreover, the certification process generates a fixed cost that adds up to any other fixed cost borne by a conventional brown producer; so we assume that

$$f_G > f_B.$$

On the demand side, we make no prior assumptions on the relative sensitivity of consumers to the price of each product variety, that is,

$$F_G <=> F_B.$$

However, we assume that consumers are willing to pay a premium price for a labelled green product. We model this *green premium* by assuming that when a quantity $Q_k + Q_j$ is produced, the price of the green variety product is higher, for any feasible Q_k and Q_j:

$$A_G - F_G Q_k - S Q_j > A_B - F_B Q_k - S Q_j,$$

which translates into a higher choke price, that is,

$$A_G > A_B,$$

with the additional condition

$$A_G F_B > A_B F_G,$$

which is always satisfied if $F_B \geq F_G$. The green premium is then defined as the difference $A_G - A_B$.

Given these assumptions on the parameters, the stability condition (11) and the comparative statics results remain valid, and we can find numerical examples satisfying all the possibilities listed in Proposition 3 for the steady-state industry composition.

We now provide some numerical illustrations that represent various industry compositions. For comparison purposes and without loss of generality, we normalize the values of parameters A_B, F_B and f_B in all numerical experiments, so that $A_B = 200$, $F_B = 1$ and $f_B = 0$.

Figure 4 illustrates the case where the firms' and the market conditions are adverse to the entry of green firms, so that the equilibrium composition of the industry corresponds to a brown one: the WRE of brown firms is 1.04% higher than that of green firms, products are highly substitutable, the marginal cost of production is 33% higher in the green industry, and the green premium is only 0.5% of the choke price.

From Proposition 3 we know that a decrease in the product substitutability and/or an increase in the green WRE would help green firms enter the market.

Figure 5 is obtained from Figure 4 by increasing the green premium, which now accounts for 0.72% of the choke price, and translates into a reduction of the brown firm's WRE advantage (now the WRE of brown firms is 0.57% higher than that of green firms, compared to the previous 1.04%). In this example, we have an industry where green firms have spread moderately. At equilibrium, 21 green firms produce 21% of the total quantity, the price of the green product is 26% higher than that of a conventional one, and each green firm produces 1.2% more in terms of quantity than does a brown firm. The additional fixed costs borne by the certified green producers ($f_H = 0.08$) amount to the 2.2% of the equilibrium profit.

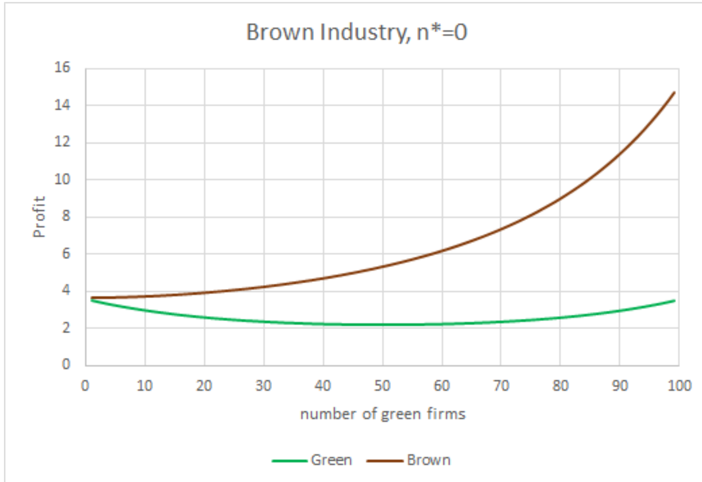

Figure 4: Brown industry. Parameter values are $N = 100$, $S = 0.99483$, $F_G = 1$, $AG = 201, m_G = 8$, $m_B = 6$, $f_G = 0.08$, **yielding** $EB = 194$, $EG = 193$; $\gamma_B = 1.00518 > \gamma_G = 0.99485$. **The equilibrium prices and quantities are** $P_B = 7.92$, $q_B = 1.92$.

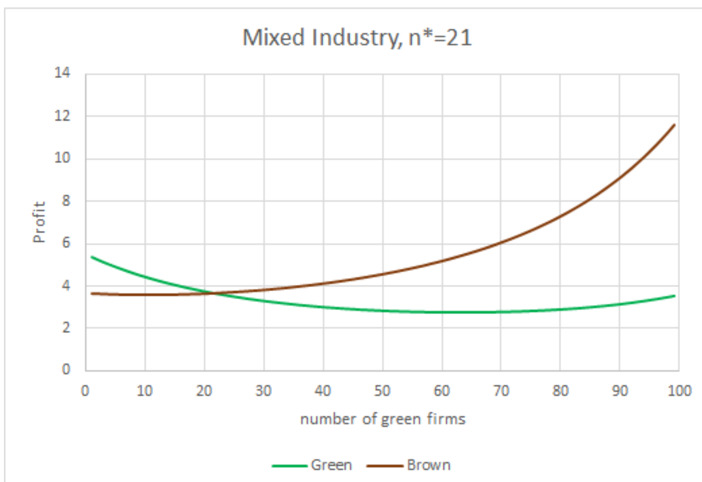

Figure 5: Mixed industry. Parameter values are $N = 100$, $S = 0.99483$, $F_G = 1$, $AG = 201.45, m_G = 8$, $m_B = 6$, $f_G = 0.08$, **yielding** $EB = 194$, $EG = 193.45$; $\gamma_B = 1.00284 > \gamma_G = 0.99716$. **The equilibrium prices and quantities are** $P_G = 9.94$, $P_B = 7.92$, $q_G = 1.94$, $q_B = 1.92$.

Figure 6 illustrates the impact of a lower substitutability between the two product varieties. This could be the result of some (exogenous) investment to make consumers more aware of the difference between the two products, like a more distinctive label or an advertising campaign focused on the green production practices. In this example, all parameter values are the same as in Figure 5, except that $S = 0.8$.[2] Even if the green WRE is smaller than the brown one, the less aggressive horizontal competition between the two product varieties gives the green products more room in the market. This results in a larger number of green firms ($n^* = 49$) as well as a higher profit, selling price and production quantity for both kinds of producers. In this industry, green firms produce 49.3% of the total quantity, the price of the green product is 25% higher than that of the conventional one, and each green firm produces 1.3% more in terms of quantity than does a brown firm. The fixed green certification cost accounts for 1.8% of the equilibrium profit.

To have a more substantial presence of green firms in the industry, we need the green WRE to be higher than the brown one. This is shown by the next two numerical examples where a high green penetration occurs, regardless of the level of horizontal product competition.

[2]Note that this example falls within Statement P3 of Proposition 3 in Section 6. It illustrates the "ambiguous effect" that this parameter may have.

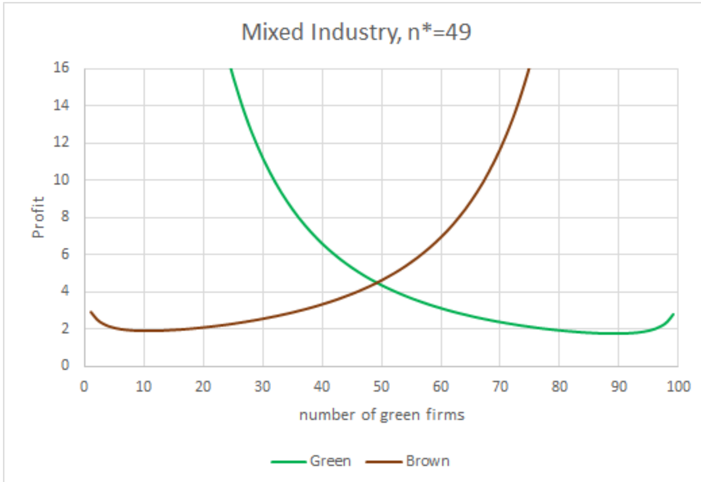

Figure 6: Mixed industry. Parameter values are $N = 100$, $S = 0.8$, $F_G = 1$, $AG = 201.45, m_G = 8$, $m_B = 6$, $f_G = 0.08$, **yielding** $EB = 194$, $EG = 193.45$; $\gamma_B = 1.00284 > \gamma_G = 0.99716$. **The equilibrium prices and quantities are** $P_G = 10.14$, $P_B = 8.12$, $q_G = 2.143$, $q_B = 2.115$.

In Figure 7, green firms have a high WRE advantage (65% more than brown firms) and the horizontal product competition is weak. This translates into market conditions and a green firm cost structure favorable to a large presence of green firms.[3] Another way to explain the large presence of green firms in the market is to look at the poor performance of the brown firms, which almost nullifies the favorable presence of weak horizontal product competition: even if the two markets are weakly linked, the conditions in the brown market are so adverse that only a few brown firms can survive.

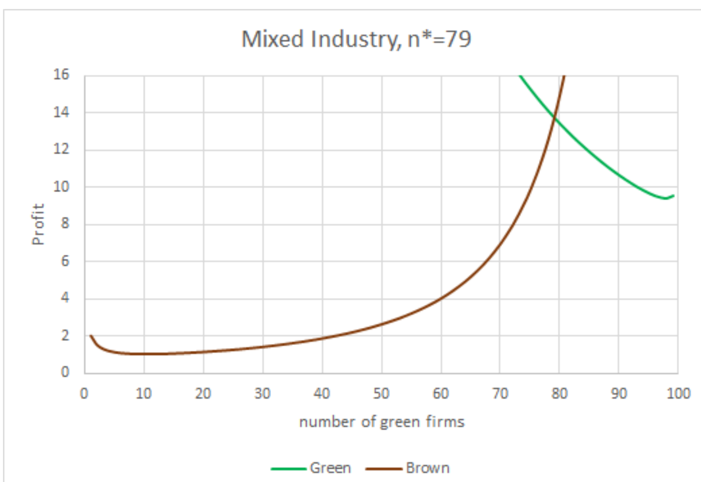

Figure 7: Mixed industry. Parameter values are $N = 100$, $S = 0.3333$, $F_G = 1/1.3$, $AG = 297, m_G = 6$, $m_B = 5$, $f_G = 0.5$, **yielding** $EB = 195$, $EG = 291$; $\gamma_B = 0.51546 < \gamma_G = 1.49231$. **The equilibrium prices and quantities are** $P_G = 9.31$, $P_B = 8.71$, $q_G = 4.307$, $q_B = 3.709$.

The high green WRE advantage comes from a high green premium, which amounts to 48.5% of the choke price. As a result, the equilibrium price of the green product is 6.9% higher than that of the brown product. At equilibrium, green firms produce 81.4% of the total quantity, each green firm produces 16.1% more than a brown firm and pays a fixed certification cost that amounts to 3.6% of the equilibrium profit.

[3] This is what we would expect from P1 of Proposition 3.

In the last case, illustrated in Figure 8, green firms have a relatively high WRE advantage (19% more than a brown firm), and the horizontal product competition is strong. The large presence of green firms is a consequence of the strong product substitutability: in this case, highly connected markets do not leave much room for inefficient firms. In Figure 8, 79 green firms produce 80.8% of the total quantity, the price of the green product is 14.6% higher than that of the brown one, with a green premium that accounts for 6.25% of the choke price. Each green firm produces 12% more than does a brown firm, and the certification costs correspond to 11% of the equilibrium profit.

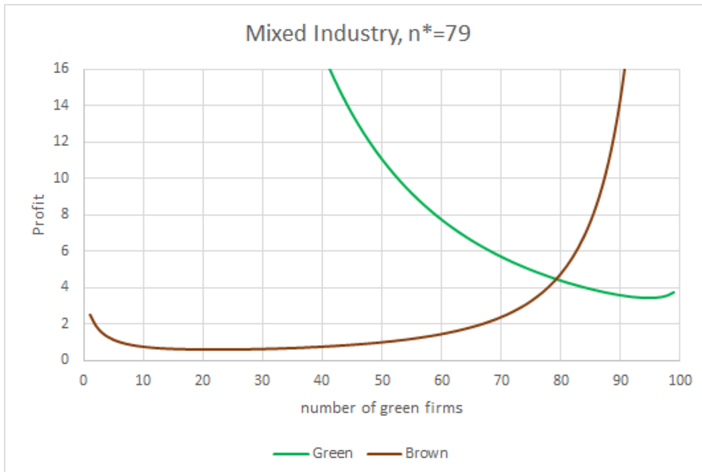

Figure 8: Mixed industry. Parameter values are $N = 100$, $S = 0.8$, $F_G = 1/1.1$, $AG = 212.5, m_G = 6$, $m_B = 5$, $f_G = 0.5$, **yielding** $EB = 195$, $EG = 206.5$; $\gamma_B = 0.85846 < \gamma_G = 1.05897$. **The equilibrium prices and quantities are** $P_G = 8.14$, $P_B = 7.10$, $q_G = 2.354$, $q_B = 2.101$.

From the previous numerical experiments, we can draw some insights related to the equilibrium industry composition. Firstly, a significant presence of firms producing a specific product variety is achievable only if their WRE is greater than that of firms producing the rival variety. Secondly, a specific equilibrium combination can be reached under different combinations of horizontal product competition and WRE advantage. Finally, changes in the fixed certification costs result in a vertical shift of the green firms' profit function, so that the composition of the industry is more sensitive to changes in the fixed certification costs when the participation of green firms is low.[4]

8 Conclusion

In this paper, we propose a differentiated oligopoly model with two product varieties made by two groups of firms. This means introducing intra-brand competition in a context where inter-brand competition is present. Markets and firms are fully asymmetric, creating a very general framework. The asymmetry is encapsulated in a parameter called the weighted relative efficiency (WRE), with relative values symbolizing the advantage of a specific group of firms.

After characterizing the equilibrium solution of the Cournot oligopoly, we analyze its response to all the model parameters, with a particular focus on the degree of product substitutability (horizontal product differentiation).

We also study the consequences of changes in intra-brand competition, both on the equilibrium solution of the game and on the equilibrium market configuration. Such changes are to be understood as the possibility for a firm to adjust its production practice and join the group producing the alternative product variety when the total number of players in the industry is fixed.

[4] For instance, to reduce the participation of green firms in the industry, the fixed cost needs to be at least 2.4% of the equilibrium profit in Figure 4, whereas it needs to be at least 59% of the equilibrium profit in Figure 6.

All the results depend on the WRE levels of the two types of firms, as well as on the market composition, making these results very general and encompassing previous results found in the literature. Finally, numerical simulations are provided to illustrate theoretical results.

Appendix

8.1 Proof of Proposition 1

Define $\alpha_k \equiv \frac{n_k+1}{n_k}$, $k \in \{H, L\}$. The impact of S on the equilibrium quantity of a firm in group k is

$$\begin{aligned}\frac{dq_k}{dS} &= n_j \frac{-E_j n_k n_j S^2 + 2F_j E_k n_k (n_j+1) S - F_k F_j E_j (n_j+1)(n_k+1)}{\Omega^2} \\ &= (n_j+1)^2 (n_k+1) E_j \frac{-S^2 + 2\alpha_j \gamma_k S - \alpha_j \alpha_k \gamma_k \gamma_j}{\alpha_j^2 \alpha_k \Omega^2}, \quad k, j \in \{L, H\}, \ k \neq j\end{aligned}$$

where $S \in [0, \gamma_L)$.

The numerator is a concave parabola in S with a negative intercept and roots

$$\begin{aligned}&\alpha_j \gamma_k \pm \sqrt{\Delta_k} \\ \Delta_k =& \alpha_j \gamma_k (\alpha_j \gamma_k - \alpha_k \gamma_j), \ k, j \in \{L, H\}, k \neq j.\end{aligned}$$

When S vanishes (products are independent), both derivatives are negative.

For $S > 0$, if $\gamma_H \alpha_L = \gamma_L \alpha_H$, then $\Delta_H = \Delta_L = 0$, so that $\frac{dq_k}{dS} < 0$ for all feasible values of S, since $S < \gamma_L < \alpha_H \gamma_L$ and $S < \gamma_H < \alpha_L \gamma_H$.

If $\gamma_L \alpha_H < \gamma_H \alpha_L$, $\Delta_H > 0$ and $\Delta_L < 0$. In that case, $\frac{dq_L}{dS} < 0$ and $\frac{dq_H}{dS} \lesseqgtr 0$, depending on the value of S. The roots of the numerator of $\frac{dq_H}{dS}$ are

$$\alpha_L \gamma_H \pm \sqrt{\alpha_L \gamma_H (\alpha_L \gamma_H - \alpha_H \gamma_L)}.$$

It is straightforward to check that

$$\begin{aligned}0 &< \alpha_L \gamma_H - \sqrt{\alpha_L \gamma_H (\alpha_L \gamma_H - \alpha_H \gamma_L)} \\ \gamma_L &< \alpha_L \gamma_H + \sqrt{\alpha_L \gamma_H (\alpha_L \gamma_H - \alpha_H \gamma_L)}.\end{aligned}$$

Assume that $\frac{\gamma_L}{\gamma_H} < \frac{(n_L+1)(n_H-1)}{n_L n_H} = \alpha_L (2 - \alpha_H) \leq 1$. Note that this condition can only be satisfied for $1 < n_H \leq n_L + 1$. We then have

$$\begin{aligned}\alpha_L \gamma_H - \alpha_H \gamma_L &> \alpha_L \gamma_H - \alpha_H (\alpha_L \gamma_H (2 - \alpha_H)) \\ &= \alpha_L \gamma_H (\alpha_H - 1)^2 > 0.\end{aligned}$$

Define $S_H = \alpha_L \gamma_H - \sqrt{\alpha_L \gamma_H (\alpha_L \gamma_H - \alpha_H \gamma_L)} > 0$. Note that

$$\gamma_L - \alpha_L \gamma_H (2 - \alpha_H) < 0$$

is equivalent to

$$(\alpha_L \gamma_H - \gamma_L)^2 < \alpha_L \gamma_H (\alpha_L \gamma_H - \alpha_H \gamma_L).$$

Taking positive roots on both sides yields

$$(\alpha_L \gamma_H - \gamma_L) < \sqrt{\alpha_L \gamma_H (\alpha_L \gamma_H - \alpha_H \gamma_L)}.$$

Consequently, the root S_H of the numerator of $\frac{dq_H}{dS}$ is strictly inside the feasible interval, $0 < S_H < \gamma_L$, and the derivative $\frac{dq_H}{dS} > 0$ for all $S \in (S_H, \gamma_L)$.

If, on the other hand, $\gamma_L\alpha_H > \gamma_H\alpha_L$, $\Delta_H < 0$ and $\Delta_L > 0$. In that case, $\frac{dq_H}{dS} < 0$ and $\frac{dq_L}{dS} \lesseqgtr 0$, depending on the value of S. The roots of the numerator of $\frac{dq_L}{dS}$ are

$$\alpha_H\gamma_L \pm \sqrt{\alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right)}.$$

It is straightforward to check that

$$\begin{aligned}0 &< \alpha_H\gamma_L - \sqrt{\alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right)}\\ \gamma_L &< \alpha_H\gamma_L + \sqrt{\alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right)}.\end{aligned}$$

Assume that $\frac{(n_L+1)(n_H+1)}{n_L(n_H+2)} = \frac{\alpha_L\alpha_H}{2\alpha_H-1} < \frac{\gamma_L}{\gamma_H} < 1$. Note that this condition can only be satisfied for $n_H < n_L - 1$. We then have

$$\begin{aligned}\alpha_H\gamma_L - \alpha_L\gamma_H &> \alpha_H\gamma_H\frac{\alpha_L\alpha_H}{2\alpha_H-1} - \alpha_L\gamma_H\\ &= \alpha_L\gamma_H\frac{\left(\alpha_H-1\right)^2}{2\alpha_H-1} > 0.\end{aligned}$$

Define $S_L = \alpha_H\gamma_L - \sqrt{\alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right)} > 0$. Note that

$$\alpha_L\gamma_H\alpha_H - \gamma_L\left(2\alpha_H - 1\right) < 0$$

is equivalent to

$$\left(\alpha_H\gamma_L - \gamma_L\right)^2 < \alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right).$$

Taking positive roots on both sides yields

$$\alpha_H\gamma_L - \gamma_L < \sqrt{\alpha_H\gamma_L\left(\alpha_H\gamma_L - \alpha_L\gamma_H\right)}.$$

Consequently, the root S_L of the numerator of $\frac{dq_L}{dS}$ is strictly inside the feasible interval, $0 < S_L < \gamma_L$, and the derivative $\frac{dq_L}{dS} > 0$ for all $S \in (S_L, \gamma_L)$.

When $\gamma_L = \gamma_H = \gamma$, the roots are given by

$$\begin{aligned}&\alpha_j\gamma \pm \sqrt{\Delta_k}\\ \Delta_k &= \gamma^2\frac{\left(n_j+1\right)\left(n_k-n_j\right)}{n_j^2 n_k},\ \ k, j \in \{L, H\}, k \neq j.\end{aligned}$$

The determinant is positive for the players in the largest group. The derivative $\frac{dq_k}{dS}$ can only be positive for $n_k > n_j$ and $\left(\frac{n_j+1}{n_j} - \sqrt{\left(n_k-n_j\right)\frac{n_j+1}{n_j^2 n_k}}\right)\gamma < S < \gamma$, which can only happen for $n_k > n_j + 1$. When $n_k = n_j + 1$, the root is at $S = \gamma$.

8.2 Impact of intra-brand competition on total quantities in each group

The total quantity produced by group k is given by

$$Q_k = E_j n_k \frac{\gamma_k\left(N - n_k + 1\right) - S\left(N - n_k\right)}{\gamma_j\gamma_k\left(N - n_k + 1\right)\left(n_k + 1\right) - S^2\left(N - n_k\right)n_k}.$$

The impact of an intensification of intra-brand competition on the total quantity produced by the group can be expressed as

$$\begin{aligned}\frac{dQ_k}{dn_k} &= \gamma_k\frac{E_j}{\Omega^2}\left(-n_k^2S^2 + \gamma_j\left(N+1\right)\left(-N+2n_k\right)S + \gamma_j\gamma_k\left(N-n_k+1\right)^2\right) \qquad (10)\\ &= \gamma_k\frac{E_j}{\Omega^2}Z(S).\end{aligned}$$

The function Z is concave quadratic in S, positive at $S = 0$.

Assume $(-N + 2n_k) \leq 0$, which implies that Z is decreasing at $S = 0$. At $S = \gamma_L$, we have

$$\begin{aligned} Z(\gamma_L) &= -\gamma_L^2 n_k^2 - \gamma_L \gamma_j (N+1)(N - 2n_k) + \gamma_j \gamma_k (N - n_k + 1)^2 \\ &> -\gamma_j \gamma_k n_k^2 + \gamma_k \gamma_j (N+1)(-N + 2n_k) + \gamma_j \gamma_k (N - n_k + 1)^2 \\ &= \gamma_j \gamma_k (N+1) > 0. \end{aligned}$$

Consequently, $Z > 0$ for all $S \in [0, \gamma_L)$ when $n_k \leq \frac{N}{2}$.

Now assume $(-N + 2n_k) > 0$. We then have

$$\begin{aligned} Z(S) &= -S^2 n_k^2 + S\gamma_j (N+1)(-N + 2n_k) + \gamma_j \gamma_k (N - n_k + 1)^2 \\ &> -S^2 n_k^2 + S^2 (N+1)(-N + 2n_k) + S^2 (N - n_k + 1)^2 \\ &= S^2 (N+1) > 0. \end{aligned}$$

Consequently, $Z > 0$ for all $S \in [0, \gamma_L)$ when $n_k > \frac{N}{2}$. The total quantity in the expanding group is always increasing.

Finally, note that

$$\frac{dQ_j}{dn_k} = -\frac{dQ_j}{dn_j} < 0,$$

so that the total quantity in the rival group is decreasing.

8.3 Proof of Proposition 2

For $k, j \in \{H, L\}$ and $k \neq j$,

$$\begin{aligned} \frac{dq_k}{dn_k} &= E_j \frac{-n_j^2 S^3 + S^2 \gamma_k \left(n_j - n_k + n_j^2\right) + S\gamma_k \gamma_j \left(n_j + n_k + n_j^2 + 1\right) - \gamma_k^2 \gamma_j (n_j + 1)^2}{\Omega^2} \\ &= -\frac{dq_k}{dn_j}. \end{aligned}$$

Define

$$\begin{aligned} Y_k &\equiv -n_j^2 S^3 + \gamma_k \left(n_j - n_k + n_j^2\right) S^2 + \gamma_k \gamma_j \left(n_j + n_k + n_j^2 + 1\right) S - \gamma_k^2 \gamma_j (n_j + 1)^2 \\ &= \frac{\Omega^2}{E_j} \frac{dq_k}{dn_k}. \end{aligned}$$

Y_k is a third-degree polynomial of S, negative and increasing in S at $S = 0$, which admits two or no positive roots by Descartes' rule of sign.

Note that, if $n_j \geq n_k$,

$$\gamma_k \left(n_j - n_k + n_j^2\right) - S n_j^2 \geq n_j^2 (\gamma_k - S) > 0,$$

so that, using $S^2 < \gamma_k \gamma_j$,

$$\begin{aligned} Y_k &= S^2 \left(\gamma_k \left(n_j - n_k + n_j^2\right) - S n_j^2\right) + S\gamma_k \gamma_j \left(n_j + n_k + n_j^2 + 1\right) - \gamma_k^2 \gamma_j (n_j + 1)^2 \\ &< \gamma_k \gamma_j \left(\gamma_k \left(n_j - n_k + n_j^2\right) - S n_j^2\right) + S\gamma_k \gamma_j \left(n_j + n_k + n_j^2 + 1\right) - \gamma_k^2 \gamma_j (n_j + 1)^2 \\ &= \gamma_k \gamma_j (n_j + n_k + 1)(S - \gamma_k) < 0. \end{aligned}$$

We conclude that $Y_k < 0$ for all feasible S when $n_k \leq n_j$.

The symmetric case

If $\gamma_k = \gamma_j$,

$$Y_k = (\gamma - S)\left(S^2 n_j^2 - S\gamma\,(n_j - n_k) - \gamma^2\,(n_j+1)^2\right).$$

This polynomial has two positive roots, γ and

$$S_k = \frac{\gamma}{2n_j^2}\left(n_j - n_k + \sqrt{4n_j^2\,(n_j+1)^2 + (n_j - n_k)^2}\right).$$

At $S = \gamma$,

$$\frac{dY_k}{dS} = \gamma^2\,(3n_j - n_k + 1)$$

If $3n_j - n_k + 1 \geq 0$, then $Y_k < 0$ for all $S < \gamma$.Otherwise, $Y_k > 0$ for $S \in (S_k, \gamma)$. Consequently:

If $\frac{N-1}{4} \leq n_k \leq \frac{3N+1}{4}$, then $Y_k < 0$ and $Y_j < 0$ for $S \in (0, \gamma)$.

If $n_k < \frac{N+1}{4}$, then $Y_k < 0$ for $S \in (0, \gamma)$, $Y_j < 0$ for $S \in (0, S_j)$ and $Y_j > 0$ for $S \in (S_j, \gamma)$.

If $n_k > \frac{3N+1}{4}$, then $Y_j < 0$ for $S \in (0, \gamma)$, $Y_k < 0$ for $S \in (0, S_k)$ and $Y_k > 0$ for $S \in (S_k, \gamma)$.

The asymmetric case with $n_j \geq n_k$

Scenario 1: $n_L > n_H$ and $\gamma_L < \gamma_H$

At $S = \gamma_L$,

$$\begin{aligned} Y_L &= S^2\left(\gamma_L\left(n_H - n_L + n_H^2\right) - S n_H^2\right) + S\gamma_L\gamma_H\left(n_H + n_L + n_H^2 + 1\right) - \gamma_L^2\gamma_H\,(n_H+1)^2 \\ &= \gamma_L^2\,(n_L - n_H)\,(\gamma_H - \gamma_L) > 0. \end{aligned}$$

Therefore, $Y_H < 0$ for all $S \in [0, \gamma_L)$ and $Y_L > 0$ when S is large enough.

Scenario 2: $n_L \leq n_H$ and $\gamma_L < \gamma_H$

$Y_L < 0$ for all $S \in [0, \gamma_L)$.

At $S = \gamma_L$,

$$\begin{aligned} Y_H &= S^2\left(\gamma_H\left(n_L - n_H + n_L^2\right) - S n_L^2\right) + S\gamma_H\gamma_L\left(n_L + n_H + n_L^2 + 1\right) - \gamma_H^2\gamma_L\,(n_L+1)^2 \\ &= -\gamma_L\,(\gamma_H - \gamma_L)\left(\gamma_H + 2\gamma_H n_L + n_L^2\,(\gamma_H - \gamma_L)\right) < 0. \end{aligned}$$

and

$$\begin{aligned} Y_H' &= S n_L^2\,(-3S + 2\gamma_H) + 2S\gamma_H\,(n_L - n_H) + \gamma_H\gamma_L\left(n_L + n_H + n_L^2 + 1\right) \\ &= \gamma_L\left(\gamma_H\,(3n_L - n_H + 1) + 3n_L^2\,(\gamma_H - \gamma_L)\right). \end{aligned}$$

If $\gamma_L \leq \gamma_H \frac{3n_L(n_L+1)+1-n_H}{3n_L^2}$, $Y_H' \geq 0$ and, consequently, $Y_H < 0$ for all $S \in [0, \gamma_L)$. Otherwise Y_H has either no roots or two positive roots in $[0, \gamma_L)$. Y_H has two positive roots that are smaller than γ_L if n_H is sufficiently large and γ_H and γ_L are sufficiently close. Note that this can only happen if $n_H > \frac{3N+1}{4}$.

8.4 Proof of Proposition 6

We first show that $\lambda_2 > 0$ and $\lambda_2 > \lambda_1$.

$$\begin{aligned}\lambda_2 &= E_H E_L \frac{-\gamma_H\gamma_L + \left(\gamma_H\left(N+1\right) - NS\right)^2}{\gamma_H^2\gamma_L\left(N+1\right)^2} \\ &\geq E_H E_L \frac{-\gamma_H\gamma_L + \left(\gamma_H + N\left(\gamma_L - S\right)\right)^2}{\gamma_H^2\gamma_L\left(N+1\right)^2} \\ &> E_H E_L \frac{\gamma_H\left(\gamma_H - \gamma_L\right)}{\gamma_H^2\gamma_L\left(N+1\right)^2} \geq 0.\end{aligned}$$

$$\lambda_2 - \lambda_1 = N E_L E_H \frac{h(S)}{\gamma_H^2\gamma_L^2\left(N+1\right)^2},$$

where

$$h(S) = S^2 N\left(\gamma_H + \gamma_L\right) - 4S\gamma_H\gamma_L\left(N+1\right) + \gamma_H\gamma_L\left(\gamma_H + \gamma_L\right)\left(N+2\right)$$

is a quadratic convex function of S, minimized at

$$S^* = 2\gamma_H\gamma_L \frac{N+1}{N\left(\gamma_H + \gamma_L\right)} > \gamma_L.$$

Consequently, for $S < \gamma_L$,

$$\begin{aligned}h(S) &> h(\gamma_L) \\ &= \gamma_L\left(\gamma_H - \gamma_L\right)\left(2\gamma_H + N\left(\gamma_H - \gamma_L\right)\right) \\ &\geq 0\end{aligned}$$

and $\lambda_2 > \lambda_1$.

The stability condition (8) can be rewritten as

$$L(n^*) = R(n^*) \tag{11}$$

where

$$L(n) = E_L E_H\left(\gamma_L\left(\gamma_H + \left(N-n\right)\left(\gamma_H - S\right)\right)^2 - \gamma_H\left(\gamma_L + n\left(\gamma_L - S\right)\right)^2\right) \tag{12}$$

$$R(n) = \delta\left(\gamma_H\gamma_L\left(N-n+1\right)\left(n+1\right) - S^2 n\left(N-n\right)\right)^2. \tag{13}$$

Compute

$$\begin{aligned}R(0) &= R(N) = \delta\gamma_H^2\gamma_L^2\left(N+1\right)^2 \\ R'(n) &= 2\delta\left(N-2n\right)\left(\gamma_H\gamma_L - S^2\right)\left(n\left(N-n\right)\left(\gamma_H\gamma_L - S^2\right) + \gamma_H\gamma_L\left(N+1\right)\right).\end{aligned}$$

If $\delta \neq 0$, according to the sign of δ, the function $R(n)$ is a strictly positive (*resp. negative*) fourth-degree polynomial, symmetric w.r.t. $\frac{N}{2}$, increasing (*resp. decreasing*) over $[0, \frac{N}{2})$ and decreasing (*resp. increasing*) over $(\frac{N}{2}, N]$.

The function $L(n)$ is a quadratic function of n, with $L(0) > 0$. Compute

$$\begin{aligned}L'(n) &= -E_L E_H\left(2\left(N-n\right)\gamma_L\left(\gamma_H - S\right)^2 + 2n\gamma_H\left(\gamma_L - S\right)^2 + 2\gamma_H\gamma_L\left(\gamma_H + \gamma_L - 2S\right)\right) < 0 \\ L''(n) &= 2E_L E_H\left(\gamma_H\gamma_L - S^2\right)\left(\gamma_H - \gamma_L\right) \geq 0.\end{aligned}$$

This shows that L is a convex, strictly decreasing function of n, with $L(0) > L(N)$. An equilibrium market composition n^* is defined by the intersection of a strictly decreasing function with the symmetric function $R(n)$. Compute

$$\begin{aligned}
L(N) - R(N) &= E_L E_H \left(\gamma_L \gamma_H^2 - \gamma_H \left(\gamma_L + N\left(\gamma_L - S\right)\right)^2\right) - \delta \gamma_H^2 \gamma_L^2 \left(N+1\right)^2 \\
&= \gamma_H^2 \gamma_L^2 \left(N+1\right)^2 \left(\lambda_1 - \delta\right) \\
L(0) - R(0) &= E_L E_H \left(\gamma_L \left(\gamma_H + N\left(\gamma_H - S\right)\right)^2 - \gamma_H \gamma_L^2\right) - \delta \gamma_H^2 \gamma_L^2 \left(N+1\right)^2 \\
&= \gamma_H^2 \gamma_L^2 \left(N+1\right)^2 \left(\lambda_2 - \delta\right)
\end{aligned}$$

We distinguish the following four cases.

Case 1: $0 < \lambda_2 \leq \delta$

In this case, $L(0) - R(0) \leq 0$. Since $L(n) < L(0) \leq R(0) < R(n)$ for all $n \in (0, N)$, the profit of firms of type H is smaller than that of type-L firms for any industry composition, so that there are only firms of type L in the industry at equilibrium.

Case 2: $\lambda_1 < \delta < \lambda_2$

In this case, $L(0) > R(0)$ and $L(N) < R(N)$. Since L and R are continuous functions, there exists at least one $n^* \in (0, N)$ where $L(n^*) = R(n^*)$; since L is strictly decreasing, if $\delta = 0$, the intersection point is unique. Otherwise, there are at most two intersection points, and at most one in the region where R is increasing. This implies that if $\delta > 0$, since $L(N) < R(N)$, it is not possible to have two intersection points in $(0, N)$. This is also the case if $\delta < 0$, since $L(0) > R(0)$. Consequently, there is a unique equilibrium market composition value n^* where both types of firms coexist.

Case 3: $\delta \leq \lambda_1$ and $\delta \leq 0$

In this case, $L(N) \geq R(N)$ and $R(n) < R(N) \leq L(N) < L(n)$ for all $n \in (0, N)$. The profit of firms of type H is larger than that of type-L firms for any industry composition, so that there are only firms of type H in the industry at equilibrium.

Case 4: $0 < \delta \leq \lambda_1$

In this case, $L(N) - R(N) \geq 0$. We will show that $L(n) - R(n)$ is strictly decreasing on $[0, N]$, which implies that $L(n) - R(n) > 0$ for all $n \in [0, N)$ and, consequently, there are only firms of type H in the industry at equilibrium.

We have

$$\begin{aligned}
L'(n) - R'(n) = &- E_L E_H \left(2\left(N-n\right)\gamma_L \left(\gamma_H - S\right)^2 + 2n\gamma_H \left(\gamma_L - S\right)^2 + 2\gamma_H \gamma_L \left(\gamma_H + \gamma_L - 2S\right)\right) \\
&- \left(2\delta \left(N - 2n\right)\left(\gamma_H \gamma_L - S^2\right)\left(n\left(N-n\right)\left(\gamma_H \gamma_L - S^2\right) + \gamma_H \gamma_L \left(N+1\right)\right)\right).
\end{aligned}$$

If $(N-2n) \geq 0$, then $L'(n) - R'(n) < 0$. Assume $N - 2n < 0$, so that

$$\begin{aligned}
L'(n) - R'(n) \leq &- E_L E_H \left(2\left(N-n\right)\gamma_L \left(\gamma_H - S\right)^2 + 2n\gamma_H \left(\gamma_L - S\right)^2 + 2\gamma_H \gamma_L \left(\gamma_H + \gamma_L - 2S\right)\right) \\
&+ \left(2\lambda_1 \left(2n - N\right)\left(\gamma_H \gamma_L - S^2\right)\left(n\left(N-n\right)\left(\gamma_H \gamma_L - S^2\right) + \gamma_H \gamma_L \left(N+1\right)\right)\right).
\end{aligned}$$

Define $w = \gamma_H - \gamma_L \geq 0$. Using

$$\begin{aligned}
\lambda_1 &= E_L E_H \frac{\gamma_H \gamma_L - \left(\gamma_L \left(N+1\right) - NS\right)^2}{\gamma_H \gamma_L^2 \left(N+1\right)^2} \\
&< E_L E_H \frac{\gamma_H \gamma_L - \left(\gamma_L \left(N+1\right) - N\gamma_L\right)^2}{\gamma_H \gamma_L^2 \left(N+1\right)^2} \\
&= \frac{E_L E_H w}{\gamma_H \gamma_L \left(N+1\right)^2},
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{L'(n)-R'(n)}{2E_LE_H} <& -(N-n)\,\gamma_L\left(w+\gamma_L-S\right)^2-n\gamma_H\left(\gamma_L-S\right)^2-\gamma_H\gamma_L\left(w+2\left(\gamma_L-S\right)\right)\\
&+\frac{w\left(2n-N\right)\left(\gamma_H\gamma_L-S^2\right)}{\gamma_H\gamma_L\left(N+1\right)^2}\left(n\left(N-n\right)\left(\gamma_H\gamma_L-S^2\right)+\gamma_H\gamma_L\left(N+1\right)\right)\\
=& -\left(\gamma_L-S\right)^2\left(N\gamma_L+nw\right)+\gamma_L\left(-2\left(\gamma_L-S\right)-w\right)\left(\gamma_H+w\left(N-n\right)\right)\\
&+\frac{w\left(2n-N\right)\left(\gamma_H\gamma_L-S^2\right)}{\gamma_H\gamma_L\left(N+1\right)^2}\left(n\left(N-n\right)\left(\gamma_H\gamma_L-S^2\right)+\gamma_H\gamma_L\left(N+1\right)\right)\\
<& -w\gamma_L\left(\gamma_H+w\left(N-n\right)\right)\\
&+\frac{w\left(2n-N\right)\left(\gamma_H\gamma_L-S^2\right)}{\gamma_H\gamma_L\left(N+1\right)^2}\left(n\left(N-n\right)\left(\gamma_H\gamma_L-S^2\right)+\gamma_H\gamma_L\left(N+1\right)\right)\\
=& -\left(N-n\right)w^2\gamma_L+\frac{\left(2n-N\right)n\left(N-n\right)w\left(\gamma_H\gamma_L-S^2\right)^2}{\gamma_H\gamma_L\left(N+1\right)^2}\\
&-w\gamma_H\gamma_L+\frac{w\left(2n-N\right)\left(\gamma_H\gamma_L-S^2\right)}{N+1}
\end{aligned}$$

where

$$\begin{aligned}
&\left(2n-N\right)n\left(N-n\right)w\left(\gamma_H\gamma_L-S^2\right)^2-\left(N-n\right)w^2\gamma_L\gamma_H\gamma_L\left(N+1\right)^2\\
&<\left(2n-N\right)n\left(N-n\right)\gamma_H\left(\gamma_H\gamma_L-\gamma_L^2\right)^2-\left(N-n\right)w^2\gamma_L\gamma_H\gamma_L\left(N+1\right)^2\\
&=-w^2\gamma_H\gamma_L^2\left(N-n\right)\left(2N-n\left(2n-N\right)+N^2+1\right)\\
&<-w^2\gamma_H\gamma_L^2\left(N-n\right)\left(2N-N\left(2N-N\right)+N^2+1\right)\\
&=-w^2\gamma_H\gamma_L^2\left(2N+1\right)\left(N-n\right)<0
\end{aligned}$$

and

$$\begin{aligned}
-\gamma_H\gamma_L\left(N+1\right)+\left(2n-N\right)\left(\gamma_H\gamma_L-S^2\right)&\le-\gamma_H\gamma_L\left(N+1\right)+N\left(\gamma_H\gamma_L-S^2\right)\\
&=-\left(\gamma_H\gamma_L+NS^2\right)<0,
\end{aligned}$$

which shows that $L(n)-R(n)$ is strictly decreasing.

8.5 Comparative statics

8.5.1 Sensitivity of λ_1 and λ_2 to the model's parameters

Compute

$$\begin{aligned}
\frac{d\lambda_1}{dE_L}&=-2E_H\frac{\gamma_L+N\left(\gamma_L-S\right)}{\gamma_H\gamma_L\left(N+1\right)}<0\\
\frac{d\lambda_1}{dE_H}&=2E_L\frac{\gamma_L\left(\gamma_H+NS\right)+N^2S\left(\gamma_L-S\right)}{\gamma_H\gamma_L^2\left(N+1\right)^2}>0\\
\frac{d\lambda_1}{dF_L}&=E_H^2\frac{\left(\gamma_L+N\left(\gamma_L-S\right)\right)^2}{\gamma_H^2\gamma_L^2\left(N+1\right)^2}>0\\
\frac{d\lambda_1}{dF_H}&=-E_L^2\frac{\gamma_L\left(\gamma_H+2NS\right)+2N^2S\left(\gamma_L-S\right)}{\gamma_H\gamma_L^3\left(N+1\right)^2}<0\\
\frac{d\lambda_1}{dS}&=2NE_LE_H\frac{\gamma_L+N\left(\gamma_L-S\right)}{\gamma_H\gamma_L^2\left(N+1\right)^2}>0.
\end{aligned}$$

The value of λ_1 increases with S and with $\frac{\gamma_H}{\gamma_L}=\frac{E_H^2F_L}{E_L^2F_H}$.

In the same way, compute

$$\begin{aligned}
\frac{d\lambda_2}{dE_L} &= -2E_H \frac{\gamma_H (\gamma_L + NS) + N^2 S (\gamma_H - S)}{\gamma_H^2 \gamma_L (N+1)^2} < 0 \\
\frac{d\lambda_2}{dE_H} &= 2E_L \frac{\gamma_H + N(\gamma_H - S)}{\gamma_H \gamma_L (N+1)} > 0 \\
\frac{d\lambda_2}{dF_L} &= E_H^2 \frac{\gamma_H (\gamma_L + 2NS) + 2N^2 S (\gamma_H - S)}{\gamma_H^3 \gamma_L (N+1)^2} > 0 \\
\frac{d\lambda_2}{dF_H} &= -E_L^2 \frac{(\gamma_H + N(\gamma_H - S))^2}{\gamma_H^2 \gamma_L^2 (N+1)^2} < 0 \\
\frac{d\lambda_2}{dS} &= -2NE_H E_L \frac{\gamma_H + N(\gamma_H - S)}{\gamma_H^2 \gamma_L (N+1)^2} < 0.
\end{aligned}$$

The value of λ_2 decreases with S and increases with $\frac{\gamma_H}{\gamma_L} = \frac{E_H^2 F_L}{E_L^2 F_H}$.

8.5.2 Effects of changes in parameters on a mixed industry

Recall the functions

$$\begin{aligned}
L(n) &= E_L E_H \left(\gamma_L (\gamma_H + (N-n)(\gamma_H - S))^2 - \gamma_H (\gamma_L + n(\gamma_L - S))^2 \right) \\
&= E_L F_H (E_H F_L - (N-n)(SE_L - E_H F_L))^2 - E_H F_L (-E_L F_H + n(SE_H - E_L F_H))^2 \\
R(n) &= \delta \left(\gamma_H \gamma_L (N-n+1)(n+1) - S^2 n (N-n) \right)^2 \\
&= \delta \left(F_H F_L (N-n+1)(n+1) - S^2 n (N-n) \right)^2
\end{aligned}$$

used to characterize an equilibrium where profits are equal for both types of producers.

Note that a change in E_k has no impact on the function R, while

$$\begin{aligned}
\frac{dL}{dE_L} &= 2S^2 \gamma_L E_H (N-n)^2 - 2S\gamma_H \gamma_L E_H (N-2n)(N+1) - 2\gamma_H \gamma_L^2 E_H (n+1)^2 \\
&< 2S\gamma_H \gamma_L E_H (N-n)^2 - 2S\gamma_H \gamma_L E_H (N-2n)(N+1) - 2\gamma_H \gamma_L^2 E_H (n+1)^2 \\
&= -2S\gamma_H \gamma_L E_H (N-n) + 2S\gamma_H \gamma_L E_H n (n+1) - 2\gamma_H \gamma_L^2 E_H (n+1)^2 \\
&< -2S\gamma_H \gamma_L E_H (N-n) + 2\gamma_H \gamma_L^2 E_H n (n+1) - 2\gamma_H \gamma_L^2 E_H (n+1)^2 \\
&= -2S\gamma_H \gamma_L E_H (N-n) - 2\gamma_H \gamma_L^2 E_H (n+1) < 0.
\end{aligned}$$

This means that a reduction in E_L shifts the $L(n)$ function upward for all n and has no effect on $R(n)$; as a result, the intersection of the two functions happens at a greater value of n.

In the same way, $\frac{dL}{dE_H} > 0$, so that an increase in the value of E_H results in a lower value of n at equilibrium.

On the other hand, a change in the value of δ has no impact on function L, while

$$\frac{dR}{d\delta} = \left(\gamma_H \gamma_L (N-n+1)(n+1) - S^2 n (N-n) \right)^2 > 0.$$

This means that a reduction in δ shifts the $R(n)$ function downward for all n and has no effect on $L(n)$; as a result, the intersection of the two functions happens at a greater value of n.

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