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Sustainability of cooperation in dynamic games played over event trees when players have asymmetric beliefs

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Abstract: We built a time-consistent cooperative solution for the class of dynamic games played over event trees in the context where the structure of the tree is given, but the players have different beliefs about the transition probabilities between nodes. Our three-step approach is as follows. First, we consider three alternative methods for aggregating the players probability distributions, and assume that the players agree to adopt one of them if they decide to cooperate. Second, we determine the Nash bargaining outcomes for the whole duration of the game. Finally, to insure sustainability of cooperation throughout the whole duration of the game, we propose two time-consistent decompositions over nodes of each player's cooperative share, namely, a proportion-consistent and a node-consistent allocation. We illustrate our results with a simple Cournot oligopoly with capacity constraints.

Keywords: Stochastic games, event tree, S-adapted strategies, asymmetric information, belief pooling, Nash bargaining solution

Résumé : Dans cet article, nous construisons une solution coopérative cohérente dans le temps pour la classe de jeux dynamiques définis sur des arbres d'événements. La structure de l'arbre est donnée, mais les joueurs ont des croyances différentes sur les probabilités de transition entre les nœuds. Notre approche en trois étapes est la suivante. Premièrement, nous considérons trois méthodes alternatives pour agréger les distributions de probabilité des joueurs et supposons que les joueurs acceptent d'adopter l'une d'elles s'ils décident de coopérer. Deuxièmement, nous adoptons la procédure d'arbitrage de Nash pour déterminer les gains coopératifs dans tout sous-jeu. Enfin, pour assurer la durabilité de la coopération pendant toute la durée du jeu, nous proposons deux décompositions des gains, à savoir, une allocation proportionnelle et une qui est cohérente à travers les nœuds de l'arbre. Nous illustrons nos résultats avec un oligopole de Cournot avec des contraintes de capacité.

Mots clés : Jeux stochastiques, arbre d'événement, stratégies S-adaptées, information asymétrique, agrégation des croyances, procédure d'arbitrage de Nash

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Introduction

The objective of this paper is to determine a sustainable cooperative solution for the class of dynamic games played over event trees (DGPET) when the structure of the tree is given, but the players have different beliefs about the transition probabilities between the nodes. To achieve our objective, we proceed in three steps. First, assuming that the players agree on a way for pooling their beliefs, we compute the expected joint optimization (or cooperative) solution. Second, we allocate to the players the resulting optimal payoff using the Nash bargaining solution (Nash, 1950). The advantage of using the Nash bargaining solution (NBS) comparatively to other cooperative game solutions is above all computational. Indeed, to compute an NBS, we must only determine the status quo and a Pareto solution, whereas the implementation of another solution, e.g., Shapley value, core, would require the computation of the characteristic function values for all possible coalitions (Avrachenkov et al. (2015)). Third, we decompose over the nodes of the event tree each player's NBS payoff-to-go such that no player finds it individually rational to switch to her noncooperative strategy in any subgame.

There are different approaches in decision theory to describe how experts aggregate their beliefs (or subjective probabilities) about the occurrence of future events, assuming that each is willing to share her information and to account for others' opinions. The result of an aggregation process is a unique probability distribution or the set of imprecise probabilities (see Stewart and Quintana (2018) for discussion). In this paper, we assume that the initial beliefs are independent, and the pooled probability distribution is constructed by taking: (i) the weighted arithmetic mean (Stone (1961)), (ii) the weighted geometric mean (Madansky (1964)), or (iii) the product of the individual probability distributions (Dietrich (2010)). The last approach does not require to define a weight for each player, but uses instead a so-called calibration probability function.[1] Methods of combining probability distributions including Bayesian and behavioral approaches are described in Clemen and Winkler (1999). In the absence of relevant information on the accuracy (in forecasting) or the power (in bargaining) of the different players, retaining equal weights could be a sensible option. For instance, in a U.S. Nuclear Regulatory Commission study of the frequency of nuclear reactor accidents, equal weights were used by Ouchi (2004). Note that although the three aggregation approaches can be easily implemented, we note that a weighted linear combination of the players' opinions preserves convexity, which may be an important feature in some contexts.

When determining a sustainable Nash bargaining solution, we must compare for each player, at each node, her cooperative payoff-to-go to her noncooperative payoff-to-go. At initial node, the assumption is that the status quo of the NBS is given by one Nash equilibrium. In any subsequent node, we consider two approaches for determining the status quo of the NBS in the subgame starting at that node, namely: (i) proportion consistency, and (ii) node consistency. By proportion consistency, we mean that the share received by the player in any subgame is equal to her share in the whole game. Proportional time-consistent solutions in cooperative differential game are considered by Yeung and Petrosyan (2001, 2012). Under node consistency, the status quo of the NBS in a subgame, corresponds to the vector of Nash equilibrium outcomes in that subgame. The proportional-consistency method dispenses from recomputing Nash equilibria in all subgames. However, node consistency takes into account the fact that, along the cooperative state trajectory, the status quo in the subgame does not coincide with the continuation of the one computed at initial node. Therefore, an updating of the status quo is more realistic when such test is conducted.

This paper belongs to the literature in cooperative dynamic games, where the main focus is on designing time-consistent (or sustainable) solutions. In differential games, the idea of time consistency, initially termed dynamic individual rationality, is due to Petrosyan (1977) and Petrosyan and Danilov (1979). Since then, a large literature has developed. For a review of the main concepts and their applications in different areas, see the surveys in Petrosjan and Zaccour (2018), Yeung and Petrosjan (2012, 2018) and Zaccour (2008). Considering the specific class of DGPET, Reddy et al. (2013) defined a node-consistent Shapley value, and their result was extended by Parilina and Zaccour (2017)

[1] In this paper, we do not consider imprecise probabilities and refer the interested reader to You and Tonon (2012) for a discussion on their use in event trees.

to the case where the game has a random duration. We focus on constructing a sustainable cooperative solution in DGPET with incomplete information about the transition probabilities between the nodes of the event tree. The first main contribution with respect to the literature is in using belief pooling techniques to construct a cooperative solution under asymmetric players' beliefs. Second, we determine the Nash bargaining solution and derive sufficient conditions for its existence for the class of DGPET with asymmetric beliefs. See Gromova and Plekhanova (2018) for time-consistent solutions for dynamic games with random duration. Parilina and Zaccour (2015a, 2015b) determined a node-consistent core and constructed approximated cooperative equilibria for the class of DGPET. For a tutorial on sustainability of cooperation in DGPET, see Zaccour (2017). In comparison with existing literature on DGPET, we propose a new imputation distribution procedure that satisfies the property of maintaining the players' proportion payoffs the same in any subgame starting from an intermediate tree node. This property is called proportion consistency. The proportion-consistent distribution procedure defines different payments comparing with node-consistent distribution procedure defined in Reddy et al. (2013) and Parilina and Zaccour (2015b) for a class of DGPET. Therefore, a proportion-consistent distribution procedure of a cooperative solution is an alternative approach to node-consistent distribution procedure to make cooperation sustainable over time.

In Section 1, we describe the ingredients of the game played over event tree when the structure of the tree is a common knowledge and transition probabilities are subjective. In Section 2, we recall the three retained methods for pooling probability distributions. In Section 3, we determine the Nash bargaining solution. The methods of decomposition of the Nash bargaining solution over event tree are proposed in Section 4. An example is presented in Section 5. Section 6 briefly concludes.

1 Model

In this section, we recall the ingredients of dynamic games played over event trees (DGPET). We use the notations and definitions given in Haurie et al. (2012), with some adaptations dictated by the fact that here the players have different beliefs about probability transitions.

Before defining formally the game, we give a brief overview of the setup. In a DGPET, we have a finite number of players (e.g., firms) who interact strategically and repeatedly over a finite number of time periods. Players' controls at each time (e.g., advertising budget, investment in R&D) affects both their payoff functions and the evolution of the state variables (e.g., brand's reputation, stock of knowledge). The parameters are stochastic and take their values, at each period, in a given set. For instance, the next period's demand for a product can be low or high, whereas in the period to follow, it can be low, medium or high. This stochastic process is modeled as a tree, with the nodes corresponding to the possible values of the parameters and the transition between a node at a period and the successors is described by a probability distribution. One main difference between this paper and the literature is that here the players have different transition probability distributions (or beliefs). The game can be played cooperatively, that is, the players coordinate their strategies in view of optimizing a joint payoff, or noncooperatively. If the players decide to cooperate, then a crucial issue is how to pool the transition probabilities. This issue is central in the paper.

Denote by $M = \{1, \ldots, m\}$ the set of players, and let $\mathcal{T} = \{0, 1, \ldots, T\}$ be the set of periods. The stochastic process is represented by an event tree, which has a root node n^0 in period 0 and a set of nodes $\mathcal{N}^t = \left\{n_1^t, \ldots, n_{N_t}^t\right\}$ in period $t = 1, \ldots, T$. Denote by $a(n_l^t) \in \mathcal{N}^{t-1}$ the unique predecessor of node $n_l^t \in \mathcal{N}^t$ on the event-tree graph, and by $\mathcal{S}(n_l^t) \subset \mathcal{N}^{t+1}$ the set of all possible direct successors of node $n_l^t \in \mathcal{N}^t$ for $t = 0, \ldots, T-1$. A path from the root node n^0 to a terminal node n_l^T is called a *scenario*. Each scenario has a probability and the probabilities of all scenarios sum up to 1. The tree graph is known to all players. We denote by $p(n_l^t)$ the conditional probability of passing through node n_l^t if node $a(n_l^t)$ has been realized. The root node is always realized, i.e., $p(n^0) = 1$. For each node $a(n_l^t) \in \mathcal{N}^t$, $t = 1, \ldots, T$, we let $\{p(n_l^t)\}_{n_l^t \in \mathcal{S}(a(n_l^t))}$ be the probability distribution over the set of nodes $\mathcal{S}(a(n_l^t))$. This distribution is unknown to the players, and Player $j \in M$ has her own beliefs, which we denote by $\{p_j(n_l^t)\}$, $n_l^t \in \mathcal{N}^t$, $t = 1, \ldots, T$. We assume that players' initial beliefs are independent.

The probabilities of passing through a particular node are computed recurrently using the above distributions. In particular, Player $j \in M$ has a belief that the probability $\pi_j(n_l^t)$ of passing through node $n_l^t \in \mathcal{N}^t$, $t = 1, \ldots, T$, is given by

$$\pi_j(n_l^t) = p_j(n_l^t)\pi_j(a(n_l^t)) \tag{1}$$

with initial condition $\pi_j(n^0) = 1$. An example of calculation of belief $\pi_j(\cdot)$ is given in Section 5 for the event tree in Figure 1.

For each Player $j \in M$, we define a vector of decision variables indexed over the set of nodes. Denote by $u_j(n_l^t) \in \mathbb{R}^{m_j}$ the decision variables of Player j at node n_l^t, and let $u(n_l^t) = (u_1(n_l^t), \ldots, u_m(n_l^t))$. Let $X \subset \mathbb{R}^p$, with $p \in \mathbb{N}_+$, be a state set. For each node $n_l^t \in \mathcal{N}^t$, $t = 0, 1, \ldots, T-1$, let $U_j^{n_l^t} \subset \mathbb{R}^{\mu_j^{n_l^t}}$, with $\mu_j^{n_l^t} \in \mathbb{N}_+$, be the control set of Player j. Denote by $U^{n_l^t} = U_1^{n_l^t} \times \cdots \times U_j^{n_l^t} \times \cdots \times U_m^{n_l^t}$ the product control sets. A transition function $f^{n_l^t}(\cdot,\cdot) : X \times U^{n_l^t} \mapsto X$ is associated with each node n_l^t. The state equations are given by

$$x(n_l^t) = f^{a(n_l^t)}\left(x\left(a\left(n_l^t\right)\right), u\left(a\left(n_l^t\right)\right)\right), \tag{2}$$

$$u\left(a\left(n_l^t\right)\right) \in U^{a(n_l^t)},\ n_l^t \in \mathcal{N}^t, t = 1, \ldots, T. \tag{3}$$

At each node n_l^t, $t = 0, \ldots, T-1$, the reward to Player j is a function of the state and the controls of all players, given by $\phi_j^{n_l^t}(x(n_l^t), u(n_l^t))$. At a terminal node n_l^T, the reward to Player j is given by the function $\Phi_j^{n_l^T}(x(n_l^T))$.

We assume that Player $j \in M$ maximizes her expected stream of payoffs discounted at rate $\lambda \in (0,1)$. The state equations and the reward functions define the following multistage game, where we let

$$\mathbf{x} = \{x(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T\},$$
$$\mathbf{u} = \{u(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1\},$$

and $J_j(\mathbf{x}, \mathbf{u})$ be the payoff to Player j, that is,

$$J_j(\mathbf{x},\mathbf{u}) = \sum_{t=0}^{T-1} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \pi_j(n_l^t)\phi_j^{n_l^t}(x(n_l^t), u(n_l^t)) + \lambda^T \sum_{n_l^T \in \mathcal{N}^T} \pi_j(n_l^T)\Phi_j^{n_l^T}(x(n_l^T)),\ j \in M, \tag{4}$$

s.t.

$$x(n_l^t) = f^{a(n_l^t)}(x(a(n_l^t)), u(a(n_l^t))),\quad x(n^0) = x^0, \tag{5}$$
$$u(a(n_l^t)) \in U^{a(n_l^t)},\quad n_l^t \in\ \mathcal{N}^t, t = 1, \ldots, T.$$

Definition 1 *An admissible* S*-adapted strategy of Player* j *is a vector* $\mathbf{u}_j = \{u_j(n_l^t) \in U^{n_l^t} : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1\}$*, that is, a plan of actions adapted to the history of the random process represented by the event tree.*

The S-adapted strategy vector of the m players is $\mathbf{u} = (\mathbf{u}_j : j \in M)$. (We call the strategy S-adapted to highlight that at each period we have a *Sample* of events corresponding to the nodes.) We can thus define a game in normal form, with payoffs $W_j(\mathbf{u}, x^0) = J_j(\mathbf{x}, \mathbf{u})$, $j \in M$, where $\mathbf{x}$ is obtained from $\mathbf{u}$ as the unique solution of the state equations that emanate from the initial state x^0.

If the game is played noncooperatively, then the players will seek a Nash equilibrium in S-adapted strategies defined as follows:

Definition 2 *An* S*-adapted Nash equilibrium is an admissible* S*-adapted strategy profile* $\mathbf{u}^N$ *such that for every player* $j \in M$ *the following condition holds:*

$$W_j(\mathbf{u}^N, x^0) \geqslant W_j([\mathbf{u}_j, \mathbf{u}_{-j}^N], x^0),$$

where $[\mathbf{u}_j, \mathbf{u}_{-j}^N]$ *is the* S*-adapted strategy profile when all players* $i \neq j$, $i \in M$*, use their Nash equilibrium policies.*

Assumption 1 *The Nash equilibrium in any subgame is unique.*

Remark 1 *The uniqueness for the joint-optimization solution requires, as usual, strict concavity of the objective function and the control set must be compact and convex. For uniqueness of S-adapted Nash equilibrium, we observe that the multistage game has a normal form representation, and therefore the conditions for uniqueness are the same as for games with continuous payoffs with constraints as established in Rosen (1965).*

We point out that there is a subtle, but important, difference between open-loop and S-adapted information structures (and equilibria). Indeed, whereas in an open-loop information structure, the controls and the state equations are defined over time, they are defined (indexed) over the set of nodes of the event tree in an S-adapted information structure.

2 Pooling transition probabilities

If the players who have asymmetric beliefs on transition probabilities decide to cooperate, then their first task is to agree on how to pool their beliefs about these probabilities. In this section, we recall three commonly used methods to aggregate beliefs[2] that have been proposed in the literature (see, e.g., Clemen and Winkler (1999), Dietrich and List (2014) and French (1985) for an overview of belief pooling methods).

Denote by $\omega = (\omega_1, \ldots, \omega_m)$ the vector of players' weights, where $\omega_j \geqslant 0$ represents a non-negative "opinion power" of Player j, and $\sum_{j\in M} \omega_j = 1$. Let $\hat{p}(n_l^t)$ be the pooled transition probability from node $a(n_l^t)$ to node n_l^t. The value of $\hat{p}(n_l^t)$ can be determined by, e.g., one of the following methods:

Linear belief pooling: $\hat{p}(n_l^t) = \sum\limits_{j\in M} \omega_j p_j(n_l^t)$.

Geometric belief pooling: $\hat{p}(n_l^t) = c \prod\limits_{j\in M} (p_j(n_l^t))^{\omega_j}$, where $c = \frac{1}{\sum\limits_{n_k^t \in \mathcal{S}(a(n_l^t))} \prod\limits_{j\in M} (p_j(n_k^t))^{\omega_j}}$ is a normalization factor. To avoid division by zero we assume that $p_j(n_k^t) > 0$ for all $j \in M$ and any node $n_k^t \in \mathcal{N}^t$, $t = 0, \ldots, T$.

Multiplicative belief pooling: $\hat{p}(n_l^t) = c' \prod\limits_{j\in M\cup\{0\}} p_j(n_l^t)$, where $p_0(n_l^t)$ is a fixed "calibrating" probability function, and $c' = \frac{1}{\sum\limits_{n_k^t \in \mathcal{S}(a(n_l^t))} \prod\limits_{j\in M\cup\{0\}} p_j(n_k^t)}$ is a normalization factor. As for geometric pooling functions, here we also suppose that $p_j(n_k^t) > 0$ for any $j \in M$ and any node n_k^t. For a discussion and interpretation of $p_0(n_l^t)$, see, e.g., Dietrich and List (2014).

We propose the following algorithm for the updating of the players' weights.[3]

[2]We use term "belief pooling" in Section 2 when describing the methods of aggregating probability distributions, whereas in the literature the term "opinion pooling" is used (Dietrich and List (2014), Stewart and Quintana (2018), Stone (1961)). To avoid confusion, we use the "opinion pooling" technique to aggregate players' beliefs on transition probabilities but use the term "belief pooling" to be in line with the theory of dynamic games with imperfect information.

[3]Updating the weights may be considered only in the cases of linear and geometric belief pools. In multiplicative belief pooling the weights are not used. The "calibrating" probability function plays the role of weights.

1. Node n^0. The weights are $\omega_j(n^0), j \in M$ and $\hat{p}(n^0) = 1$. Let the stochastic process transit to node $n_l^1 \in \mathcal{S}(n^0)$.
2. Node n_l^1. The weights are updated using Bayes' rule, that is,

$$\omega_j(n_l^1) = \frac{\omega_j(n^0) p_j(n_l^1)}{\sum_{i \in M} \omega_i(n^0) p_i(n_l^1)}.$$

   Compute $\hat{p}(n_l^1)$ using one of the three methods described above and weights $\omega_j(n_l^1)$ if needed (for linear and geometric belief pooling).
3. Let in period t, the stochastic process be in node $n_l^t \in \mathcal{S}(n_k^{t-1})$. The weights of the players are updated using Bayes' rule, i.e.,

$$\omega_j(n_l^t) = \frac{\omega_j(n_k^{t-1}) p_j(n_l^t)}{\sum_{i \in M} \omega_i(n_k^{t-1}) p_i(n_l^t)}. \tag{6}$$

   Compute $\hat{p}(n_l^t)$ using one of three methods described above and weights $\omega_j(n_l^t)$ if needed.
4. Repeat Step 3 till terminal nodes $n_l^T \in \mathcal{N}^T$.

Example 1 *To illustrate the difference in belief pooling methods, we provide a simple example. Consider three players who want to coordinate their strategies to maximize their joint payoff, which depends on the inflation rate. Each player has her own beliefs about this rate. Suppose that the set of possible rates is $\{L, M, H\}$ corresponding to a low, medium and high inflation rate, respectively. Player j has a belief $p_j = (p_j(L), p_j(M), p_j(H))$, with $p_1 = (0.3, 0.1, 0.6)$, $p_2 = (0.7, 0.2, 0.1)$, $p_3 = (0.3, 0.4, 0.3)$. Assume that the players' weights are $\omega = (\omega_1, \omega_2, \omega_3) = (0.6, 0.2, 0.2)$ representing the "opinion power" of each player. Here, Player 1 is considered to be the best forecaster and the players agree on ω.*

The pooled probabilities of low, medium and high levels of the inflation rate are as follows:

1. *If the players use a linear belief pooling, then*

$$(\hat{p}(L), \hat{p}(M), \hat{p}(H)) = (0.38, 0.18, 0.44).$$

2. *If the players use a geometric belief pooling, then the pooled probabilities are*

$$c = \frac{1}{(0.3)^{0.6}(0.7)^{0.2}(0.3)^{0.2} + (0.1)^{0.6}(0.2)^{0.2}(0.4)^{0.2} + (0.6)^{0.6}(0.1)^{0.2}(0.3)^{0.2}} = 1.147,$$

   and

$$(\hat{p}(L), \hat{p}(M), \hat{p}(H)) = (0.408, 0.174, 0.418).$$

3. *If a multiplicative belief pooling is used, and calibrating probability function is $p_0 = (p_0(L), p_0(M), p_0(H)) = (0.25, 0.5, 0.25)$, which may correspond to the belief of an independent expert (e.g., a specialized economic magazine), we obtain*

$$c' = \frac{1}{0.3 \cdot 0.7 \cdot 0.3 \cdot 0.25 + 0.1 \cdot 0.2 \cdot 0.4 \cdot 0.5 + 0.6 \cdot 0.1 \cdot 0.3 \cdot 0.25} = 41.237,$$

   and

$$(\hat{p}(L), \hat{p}(M), \hat{p}(H)) = (0.649, 0.165, 0.186).$$

Note that the multiplicative belief pooling gives a very different result than the linear and geometric approaches. The reasons lie in the facts that the players' weights are not used in the computations and in the dependence on the calibration probability function.

If cooperation is considered after one of the events from the set $\{L, M, H\}$ is realized, then we use the formula in (6) to recalculate the players' weights. If the inflation rate turns to be low (event "L"), then the players' weights become

$$\omega_1(L) = \frac{0.6 \cdot 0.3}{0.6 \cdot 0.3 + 0.2 \cdot 0.7 + 0.2 \cdot 0.3} = 0.474,$$
$$\omega_2(L) = \frac{0.2 \cdot 0.7}{0.6 \cdot 0.3 + 0.2 \cdot 0.7 + 0.2 \cdot 0.3} = 0.368,$$

$$\omega_3(L) = \frac{0.2 \cdot 0.3}{0.6 \cdot 0.3 + 0.2 \cdot 0.7 + 0.2 \cdot 0.3} = 0.158.$$

If the realization is medium (event "M") or high (event "H"), then the players' weights become

$$\omega(M) = (\omega_1(M), \omega_2(M), \omega_3(M)) = (0.333, 0.222, 0.445),$$
$$\omega(H) = (\omega_1(H), \omega_2(H), \omega_3(H)) = (0.818, 0.046, 0.136).$$

The new weights may be used in the linear and geometric belief pooling methods in the next period following Algorithm 1.

We make the following assumptions regarding the updating of beliefs and their pooling by cooperating players.

Assumption 2 *The players do not aggregate at initial time the probabilities for the whole tree, but account down the road for the accuracy of players' predictions (beliefs) along the realized scenario.*

Assumption 3 *Cooperating players agree on a pooling method and its parameter values.*

When aggregating the probabilities, we consider two possibilities for the players' weights, namely, they remain the same along any scenario, or they are updated along the realized scenario according to Bayes' rule. In this second case, we take into account the quality of each player's beliefs and increase or decrease her weight accordingly. In either case, we determine the event tree with aggregated probabilities $\hat{p}(n_l^t)$, $n_l^t \in \mathcal{N}^t$, $t = 0, 1, \ldots, T$. Next, we define the probability of passing through node $n_l^t, n_l^t \in \mathcal{N}^t, t = 1, \ldots, T$ recurrently as follows:

$$\hat{\pi}(n_l^t) = \hat{p}(n_l^t)\hat{\pi}(a(n_l^t)), \tag{7}$$

with initial condition $\hat{\pi}(n^0) = 1$. The probability $\hat{\pi}(n_l^t)$ will replace the individual probabilities $\pi_j(n_l^t)$ for all players $j \in M$.

3 Cooperative solution of the game

If the players agree to cooperate and form the grand coalition M, then they adopt the aggregated probabilities $\hat{\pi}(n_l^t)$, and play the game over an event tree with probabilities $\hat{\pi}(n_l^t)$, $n_l^t \in \mathcal{N}^t$, $t = 0, 1, \ldots, T$. Let $\hat{J}_j(\mathbf{x}, \mathbf{u})$ be the payoff to Player j, that is,

$$\hat{J}_j(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \hat{\pi}(n_l^t)\phi_j^{n_l^t}(x(n_l^t), u(n_l^t)) + \lambda^T \sum_{n_l^T \in \mathcal{N}^T} \hat{\pi}(n_l^T)\Phi_j^{n_l^T}(x(n_l^T)),\ j \in M,$$

s.t. (5).

Now the game in a normal form is determined by payoffs $\hat{W}_j(\mathbf{u}, x^0) = \hat{J}_j(\mathbf{x}, \mathbf{u})$, $j \in M$, where $\mathbf{x}$ is obtained from $\mathbf{u}$ as the unique solution of the state equations that emanate from the initial state x^0.

The players maximize the sum of their discounted payoffs over the entire horizon, i.e.,

$$\max_{\mathbf{u}_j : j \in M} \sum_{j \in M} \hat{W}_j\left(\mathbf{u}, x^0\right).$$

Denote the resulting vector of cooperative controls by $\mathbf{u}^*$, which is found for the game starting from node n^0 and state x^0:

$$\mathbf{u}^* = \arg\max_{\mathbf{u}_j : j \in M} \sum_{j \in M} \hat{W}_j\left(\mathbf{u}, x^0\right). \tag{8}$$

Further, denote by $\mathbf{x}^* = \{x^*(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, 1, \ldots, T\}$ the cooperative state trajectory generated by the cooperative controls $\mathbf{u}^*$.

For later use, we also need to determine the subgame of the cooperative game starting from state $x^*(n_l^t)$ at node $n_l^t \in \mathcal{N}^t$, $t = 1, \ldots, T-1$. This subgame takes place on a tree subgraph $\Gamma(n_l^t)$ of the initial graph. The payoff of Player $j \in M$ in this subgame is given as follows:

$$\begin{aligned} \hat{W}_j\left(\mathbf{u}(n_l^t), x^*\left(n_l^t\right)\right) = &\sum_{\theta=t}^{T-1} \lambda^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \hat{\pi}(n_l^\theta | n_l^t) \phi_j^{n_l^\theta}(x^*(n_l^\theta), u(n_l^\theta)) \\ &+ \lambda^{T-t} \sum_{n_l^T \in \mathcal{N}_\Gamma^T} \hat{\pi}(n_l^T | n_l^t) \Phi_j^{n_l^T}(x^*(n_l^T)), \end{aligned} \tag{9}$$

where $\mathcal{N}_\Gamma^\theta = \mathcal{N}^\theta \cap \Gamma(n_l^t)$, $\mathbf{u}(n_l^t) = (\mathbf{u}_j(n_l^t) : j \in M)$ is an S-adapted strategy profile and $\mathbf{u}_j(n_l^t) = \{u_j(n_l^\theta) : n_l^\theta \in \Gamma(n_l^t)\}$ is an admissible S-adapted strategy of Player j in the subgame starting from node n_l^t, with initial state $x^*(n_l^t)$. The term $\hat{\pi}(n_l^\theta | n_l^t)$ is the conditional probability that node n_l^θ will be realized if the subgame starts from node n_l^t. This probability is given by $\hat{\pi}(n_l^\theta | n_l^t) = \hat{\pi}(n_l^\theta)/\hat{\pi}(n_l^t)$ if $\hat{\pi}(n_l^t) \neq 0$; otherwise, the subgame starting from node n_l^t cannot materialize with beliefs of coalition M.

In the cooperative subgame starting from state $x^*(n_l^t)$, the players maximize the sum of their total discounted payoffs, i.e.,

$$\max_{\mathbf{u}_j(n_l^t): j \in M} \sum_{j \in M} \hat{W}_j\left(\mathbf{u}(n_l^t), x^*\left(n_l^t\right)\right),$$

and the cooperative controls are given by

$$\mathbf{u}^*(n_l^t) = \arg \max_{\mathbf{u}_j(n_l^t): j \in M} \sum_{j \in M} \hat{W}_j\left(\mathbf{u}(n_l^t), x^*\left(n_l^t\right)\right). \tag{10}$$

Therefore, the payoff of Player j in the cooperative subgame starting from node n_l^t, with initial state $x^*(n_l^t)$, $n_l^t \in \mathcal{N}^t$, is equal to $\hat{W}_j(\mathbf{u}^*(n_l^t), x^*(n_l^t))$, $t = 1, \ldots, T$.

If the players switch to a noncooperative mode of play at node n_l^t, then they implement the Nash equilibrium in Definition 2, that is, $\mathbf{u}^N(x^*(n_l^t)) = (\mathbf{u}_j^N(x^*(n_l^t)) : j \in M)$. The corresponding S-adapted equilibrium payoff of Player j is $W_j(\mathbf{u}^N(x^*(n_l^t)), x^*(n_l^t))$. We make two comments: First, we have assumed that cooperation has prevailed from node n^0 till n_l^t. Second, we suppose that from node n_l^t and in state $x^*(n_l^t)$, the players use their own initial probability distributions in the payoff function without aggregating procedure.

We make the following

Assumption 4 *The joint-optimization solution in any subgame is unique.*

We suppose that the players use the Nash bargaining procedure to share the total cooperative, with the status quo being given by the Nash equilibrium. (The conditions for a uniqueness of the Nash equilibrium are discussed in Remark 1.)

Theorem 1 *If in the dynamic game played over event tree the inequality* $\sum_{j \in M} W_j(\mathbf{u}^N, x^0) \leqslant \sum_{j \in M} \hat{W}_j(\mathbf{u}^*, x^0)$ *holds true, then the unique individually rational Nash bargaining solution is* $\alpha(n^0) = (\alpha_j(n^0) : j \in M)$, *where*

$$\alpha_j(n^0) = W_j(\mathbf{u}^N, x^0) + \frac{1}{m}\left(\sum_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0) - \sum_{i \in M} W_i(\mathbf{u}^N, x^0)\right). \tag{11}$$

Proof. To find the Nash bargaining solution $(\alpha_j(n^0) : j \in M)$ with a status quo point $(W_j(\mathbf{u}^N, x^0) : j \in M)$ we need to solve the following optimization problem with constraints:

$$\max_{\alpha_j(n^0)} \prod_{j \in M} (\alpha_j(n^0) - W_j(\mathbf{u}^N, x^0)), \tag{12}$$

s.t.

$$\sum_{j\in M}\alpha_j(n^0)=\sum_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0), \tag{13}$$

$$\alpha_j(n^0)-W_j(\mathbf{u}^N,x^0)\geqslant 0,\quad j\in M. \tag{14}$$

If $\sum\limits_{j\in M}W_j(\mathbf{u}^N,x^0)>\sum\limits_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0)$, then the solution of (12)–(14) does not exist. If the $\sum\limits_{j\in M}W_j(\mathbf{u}^N,x^0)=\sum\limits_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0)$, the unique solution of the problem (12)–(14) is a vector $(W_j(\mathbf{u}^N,x^0): j\in M)$ and the maximal value of (12) is null. Now consider the case when $\sum\limits_{j\in M}W_j(\mathbf{u}^N,x^0)<\sum\limits_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0)$. The Lagrangian for the problem is

$$L=\prod_{j\in M}(\alpha_j(n^0)-W_j(\mathbf{u}^N,x^0))+\sum_{j\in M}\mu_j(\alpha_j(n^0)-W_j(\mathbf{u}^N,x^0))+\mu_0\left(\sum_{j\in M}\alpha_j(n^0)-\sum_{j\in M}W_j(\mathbf{u}^N,x^0)\right). \tag{15}$$

The Karush-Kuhn-Tucker conditions for the problem (12)–(14) are

$$\frac{\partial L}{\partial\alpha_j(n^0)}=\prod_{\substack{i\in M\\ i\neq j}}(\alpha_i(n^0)-W_i(\mathbf{u}^N,x^0))+\mu_j+\mu_0=0,\qquad j\in M, \tag{16}$$

$$\mu_j\geqslant 0,\qquad \mu_j(\alpha_j(n^0)-W_j(\mathbf{u}^N,x^0))=0,\qquad j\in M, \tag{17}$$

with constrains (13) and (14). To have a positive solution of problem (12)–(14) we obtain that $\alpha_j(n^0)>W_j(\mathbf{u}^N,x^0)$ for any $j\in M$, then from (17) we have $\mu_j=0$ for any $j\in M$. From (16) we obtain that $\mu_0=-\prod\limits_{\substack{i\in M\\ i\neq j}}(\alpha_i(n^0)-W_i(\mathbf{u}^N,x^0))$ for any $j\in M$, which is true only if $\prod\limits_{\substack{i\in M\\ i\neq j}}(\alpha_i(n^0)-W_i(\mathbf{u}^N,x^0))=\prod\limits_{\substack{i\in M\\ i\neq k}}(\alpha_i(n^0)-W_i(\mathbf{u}^N,x^0))$ for any j and k such that $j\neq k$. Therefore, the unique solution of the problem is when the surplus payoff $\sum\limits_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0)-\sum\limits_{j\in M}W_j(\mathbf{u}^N,x^0)$ is divided equally between the players, i.e.,

$$\alpha_j(n^0)=W_j(\mathbf{u}^N,x^0)+\frac{1}{m}\left(\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)-\sum_{i\in M}W_i(\mathbf{u}^N,x^0)\right),$$

which proves the theorem. □

The above theorem carries two interesting results: (i) it defines individually rational Nash bargaining solution, and (ii) it states that joint optimization might not yield a total payoff that is lower than the sum of individual noncooperative outcomes, i.e., $\sum\limits_{j\in M}W_j(u^N,x^0)>\sum\limits_{j\in M}\hat{W}_j(u^*,x^0)$. It is difficult to write down the conditions under which this inequality is satisfied because it depends not only on the payoff functions, but also on the initially given transition probabilities, i.e., players' beliefs. Moreover, the satisfaction of this inequality also depends on the belief pooling method chosen to construct the cooperative game. We notice that when computing the Nash equilibrium, the players do not aggregate their beliefs in the payoff functions $J_j(x,u)$, whereas they pool them in $\hat{J}_j(x,u)$. Consequently, the individual payoff functions are not the same in the two modes of play. Therefore, the inequality in Theorem 1 must be satisfied for cooperation to be implementable. An interesting open question is then what to do if this condition is not satisfied? If the three pooling techniques are the only possible ones, then the short answer is that there is no room for cooperation. Otherwise, the players should define and agree on another pooling approach, if it exists, that yields a total cooperative outcome that is larger than the total noncooperative outcome.

4 Decomposition of the Nash bargaining solution

In Section 3, we determined the joint optimization solution for the whole game starting at node n^0 and determined each player's share in the total payoff based on the Nash bargaining solution. When players redistribute the total payoff according to this solution, they need to be sure that summarized expected payments-to-go coincide with the components of the solution. In general case, if in the nodes realized in the game the players are paid according to the initially defined payoff functions, i.e., in node n_l^t Player j obtains $\phi_j^{n_l^t}(x^*(n_l^t), u^*(n_l^t))$, then the expected payoff will be $\hat{W}_j(\mathbf{u}^*, x^0)$ and will not coincide with $\alpha_j(n^0)$, given by the Nash bargaining solution. The way to solve this problem is to redistribute the total payoff of coalition M among the coalition members in any node of the event tree to get in total the payoff $\alpha_j(n^0)$ to Player $j \in M$. Therefore, we use the imputation distribution procedure which defines payments to the players in any node of the tree and allows players to get in total the components of the Nash bargaining solution.

Definition 3 *We call $\left(\beta_j(n_l^t, x^*(n_l^t)) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T, j \in M\right)$ an imputation distribution procedure of the imputation $(\alpha_j(n^0) : j \in M)$ if it satisfies the following properties:*

1. *The expected payoff of Player $j \in M$ along the cooperative trajectory in the event tree is equal to her component in imputation $\alpha(n^0)$, i.e.,*

$$\alpha_j(n^0) = \sum_{t=0}^{T} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \hat{\pi}(n_l^t) \beta_j(n_l^t, x^*(n_l^t)). \tag{18}$$

2. *At any node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$, the sum of payments to all players is equal to the sum of cooperative rewards, that is,*

$$\sum_{j \in M} \beta_j(n_l^t, x^*(n_l^t)) = \sum_{j \in M} \phi_j^{n_l^t}(x^*(n_l^t), u^*(n_l^t)), \quad n_l^t \in \mathcal{N}^t, \quad t = 0, \ldots, T-1, \tag{19}$$

$$\sum_{j \in M} \beta_j(n_l^T, x^*(n_l^T)) = \sum_{j \in M} \Phi_j^{n_l^T}(x^*(n_l^T)), \quad n_l^T \in \mathcal{N}^T. \tag{20}$$

The imputation distribution procedure (IDP) of $\alpha(n^0)$ guarantees that each player receives in total her share $\alpha_j(n^0)$ in the cooperative game. The second item insures that the budget is balanced at each node of the event tree.

We note that Definition 3 does not lead to a unique distribution of the cooperative outcome along the event tree. Next, we introduce additional properties to induce uniqueness. (One can think of these additional properties as a refinement approach of the IDP.) We shall define two alternative unique allocations, that is, a proportion-consistent Nash bargaining solution and a node-consistent Nash bargaining solution. In the proportion-consistent allocation, each player gets the same share of the total expected payoff in any subgame. In the node-consistent approach, each player gets, at each node, her share in the Nash bargaining solution computed for the subgame starting at that node. Put differently, we keep the same solution concept in place as the game evolves over the event tree. Each of the two additional properties, i.e., proportion consistency and node consistency, leads to a formula that uniquely allocate over the event tree the players' components in the Nash bargaining solution, that is, $\alpha_j(n^0), j \in M$.

4.1 Proportion-consistent Nash bargaining solution

Definition 4 *The Nash bargaining solution $\alpha(n^0)$ and its distribution procedure $\left(\beta_j(n_l^t, x^*(n_l^t)) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T, j \in M\right)$ is called proportion-consistent if, for any player $j \in M$ and subgame starting at any node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$, the following property holds:*

$$\frac{B_j(n_l^t, x^*(n_l^t))}{\sum_{i \in M} \hat{W}_i\left(\mathbf{u}^*(n_l^t), x^*\left(n_l^t\right)\right)} = \frac{\alpha_j(n^0)}{\sum_{i \in M} \alpha_i(n_0)} \equiv \gamma_j, \tag{21}$$

where

$$B_j(n_l^t, x^*(n_l^t)) = \sum_{\theta=t}^{T} \lambda^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \hat{\pi}(n_l^\theta | n_l^t) \beta_j(n_l^\theta, x^*(n_l^\theta)),$$

is the expected sum of payments to Player j according to the IDP in the subgame starting at node $n_l^t \in \mathcal{N}^t$, $t = 0, \dots, T$ in state $x^*(n_l^\theta)$.

If the imputation distribution procedure is proportion-consistent, then a player obtains the same share of the total players' expected payoff in any subgame. For example, if in the Nash bargaining solution, Player j's share is 20% of the total payoff $\sum_{i\in M} \hat{W}_i(\mathbf{u}^*, x^0) = \sum_{i\in M} \alpha_i(n^0)$, i.e., $\dfrac{\alpha_j(n^0)}{\sum_{i\in M} \alpha_i(n^0)} = 0.2$, then in any subgame she will get 20% of the total expected players' payoff-to-go.

Theorem 2 *The Nash bargaining solution $\alpha(n^0)$ is proportion-consistent if*

$$\beta_j(n_l^t, x^*(n_l^t)) = \gamma_j \sum_{i\in M} \phi_i^{n_l^t}(x^*(n_l^t), u^*(n_l^t)), \quad n_l^t \in \mathcal{N}^t, \quad t = 0, \dots, T-1, \tag{22}$$

$$\beta_j(n_l^T, x^*(n_l^T)) = \gamma_j \sum_{i\in M} \Phi_i^{n_l^T}(x^*(n_l^T)), \quad n_l^T \in \mathcal{N}^T, \tag{23}$$

for any player $j \in M$.

Proof. First, we prove that the set of $\beta_j(n_l^t, x^*(n_l^t))$ determined by (22) and (23) is a distribution procedure of the Nash bargaining solution $\alpha(n^0)$. The first property of an IDP is satisfied because

$$\begin{aligned}\sum_{t=0}^{T} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \hat{\pi}(n_l^t) \beta_j(n_l^t, x^*(n_l^t)) =& \sum_{t=0}^{T-1} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \hat{\pi}(n_l^t) \gamma_j \sum_{i\in M} \phi_i^{n_l^t}(x^*(n_l^t), u^*(n_l^t)) \\ &+ \lambda^T \sum_{n_l^T \in \mathcal{N}^T} \hat{\pi}(n_l^T) \gamma_j \sum_{i\in M} \Phi_i^{n_l^T}(x^*(n_l^T)) \\ =& \gamma_j \sum_{i\in M} \hat{W}_i(\mathbf{u}^*, x^0) = \gamma_j \sum_{i\in M} \alpha_i(n^0) = \alpha_j(n^0).\end{aligned}$$

The second property is also satisfied, in particular, (19) is true because

$$\sum_{j\in M} \beta_j(n_l^t, x^*(n_l^t)) = \sum_{j\in M} \gamma_j \sum_{i\in M} \phi_i^{n_l^t}(x^*(n_l^t), u^*(n_l^t)) = \sum_{i\in M} \phi_i^{n_l^t}(x^*(n_l^t), u^*(n_l^t)),$$

and Equation (20) is proved in the same way.

Finally, we prove the proportion consistency of the Nash bargaining solution $\alpha(n^0)$ if its distribution procedure is defined by (22) and (23). Consider any node $n_l^t \in \mathcal{N}^t$, $t = 0, \dots, T$ and subgame starting at this node. Calculate the ratio from Definition 4:

$$\begin{aligned}\frac{B_j(n_l^t, x^*(n_l^t))}{\sum\limits_{i\in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))} =& \frac{\sum\limits_{\theta=t}^{T} \lambda^{\theta-t} \sum\limits_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \hat{\pi}(n_l^\theta | n_l^t) \beta_j(n_l^\theta, x^*(n_l^\theta))}{\sum\limits_{i\in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))} \\ =& \frac{\sum\limits_{\theta=t}^{T-1} \lambda^{\theta-t} \sum\limits_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \hat{\pi}(n_l^\theta | n_l^t) \gamma_j \sum\limits_{i\in M} \phi_i^{n_l^\theta}(x^*(n_l^\theta), u^*(n_l^\theta))}{\sum\limits_{i\in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))} \\ &+ \frac{\lambda^{T-t} \sum\limits_{n_l^T \in \mathcal{N}_\Gamma^T} \hat{\pi}(n_l^T | n_l^t) \gamma_j \sum\limits_{i\in M} \Phi_i^{n_l^T}(x^*(n_l^T))}{\sum\limits_{i\in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))} = \gamma_j,\end{aligned}$$

which completes the proof. □

In the next theorem, we show that the proportion-consistent distribution procedure given by (22) and (23) has the nice property, namely, at any node, the payments-to-go correspond to the Nash bargaining payoffs in the subgame starting at that node.

Theorem 3 *Let the Nash bargaining solution distribution procedure be given by* (22) *and* (23)*, and suppose that the two inequalities* $\sum_{j\in M} W_j(\mathbf{u}^N,x^0)\leqslant\sum_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0)$*, and* $\sum_{j\in M}\hat{W}_j(\mathbf{u}^*,x^0)>0$ *are true. Then, in the subgame starting at any node* $n_l^t\in\mathcal{N}^t$*,* $t=0,\dots,T$*, the vector of payments-to-go* $(B_j(n_l^t,x^*(n_l^t)):j\in M)$ *is the Nash bargaining solution of this subgame with the status quo point being given by*

$$\left(W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}:j\in M\right). \tag{24}$$

Proof. Consider the subgame starting at any node $n_l^t\in\mathcal{N}^t$, $t=0,\dots,T$. Let the Nash bargaining solution of the subgame with status quo point (24) be $\alpha(n_l^t)=\big(\alpha_j(n_l^t):j\in M\big)$. We have to prove that $\alpha_j(n_l^t)$ is equal to $B_j(n_l^t,x^*(n_l^t))$.

To find the Nash bargaining solution $\alpha(n_l^t)$, we need to solve the following constrained-optimization problem:

$$\max_{\alpha_j(n_l^t)}\prod_{j\in M}\left(\alpha_j(n_l^t)-W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}\right) \tag{25}$$

s.t.

$$\sum_{j\in M}\alpha_j(n_l^t)=\sum_{j\in M}\hat{W}_j(\mathbf{u}^*(n_l^t),x^*(n_l^t)), \tag{26}$$

$$\alpha_j(n_l^t)-W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}\geqslant 0,\quad j\in M. \tag{27}$$

The solution of (25)–(27) exists when $\sum_{j\in M}W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}\leqslant\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))$, which is always true under the theorem assumptions.

The Lagrangian of the problem is

$$\begin{aligned}L=&\prod_{j\in M}\left[\alpha_j(n_l^t)-W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}\right]\\&+\sum_{j\in M}\mu_j\left[\alpha_j(n_l^t)-W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}\right]\\&+\mu_0\left[\sum_{j\in M}\alpha_j(n_l^t)-\sum_{j\in M}W_j(\mathbf{u}^N,x^0)\frac{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{i\in M}\hat{W}_i(\mathbf{u}^*,x^0)}\right].\end{aligned}$$

The Karush-Kuhn-Tucker conditions for problem (25)–(27) are

$$\frac{\partial L}{\partial\alpha_j(n_l^t)}=\prod_{\substack{i\in M\\i\neq j}}\left[\alpha_i(n_l^t)-W_i(\mathbf{u}^N,x^0)\frac{\sum_{k\in M}\hat{W}_k(\mathbf{u}^*(n_l^t),x^*(n_l^t))}{\sum_{k\in M}\hat{W}_k(\mathbf{u}^*,x^0)}\right]+\mu_j+\mu_0=0,\qquad j\in M, \tag{28}$$

$$\mu_j \geqslant 0, \quad \mu_j \left[\alpha_j(n_l^t) - W_j(\mathbf{u}^N, x^0) \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)} \right] = 0, \quad j \in M, \tag{29}$$

with the constraints (26) and (27). To have a solution of problem (25)–(27) we obtain that

$$\alpha_j(n_l^t) > W_j(\mathbf{u}^N, x^0) \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)}$$

for any $j \in M$, then from (29) we have $\mu_j = 0$ for any $j \in M$. From (28) we have

$$\mu_0 = -\prod_{\substack{i \in M \\ i \neq j}} \left[\alpha_i(n_l^t) - W_i(\mathbf{u}^N, x^0) \frac{\sum\limits_{k \in M} \hat{W}_k(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{k \in M} \hat{W}_k(\mathbf{u}^*, x^0)} \right]$$

for any $j \in M$, which is true only if

$$\prod_{\substack{i \in M \\ i \neq j}} \left[\alpha_i(n_l^t) - W_i(\mathbf{u}^N, x^0) \frac{\sum\limits_{k \in M} \hat{W}_k(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{k \in M} \hat{W}_k(\mathbf{u}^*, x^0)} \right] = \prod_{\substack{i \in M \\ i \neq k}} \left[\alpha_i(n_l^t) - W_i(\mathbf{u}^N, x^0) \frac{\sum\limits_{k \in M} \hat{W}_k(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{k \in M} \hat{W}_k(\mathbf{u}^*, x^0)} \right],$$

for any j and k such that $j \neq k$. Therefore, the unique solution of the problem is when the surplus payoff $\sum_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t)) - \sum_{j \in M} W_j(\mathbf{u}^N, x^0) \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)}$ is equally divided between the players, that is,

$$\begin{aligned}
\alpha_j(n_l^t) =& W_j(\mathbf{u}^N, x^0) \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)} \\
&+ \frac{1}{m} \left[\sum_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t)) - \sum_{i \in M} W_i(\mathbf{u}^N, x^0) \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)} \right] \\
=& \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)} \left[W_j(\mathbf{u}^N, x^0) + \frac{1}{m} \left(\sum_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0) - \sum_{i \in M} W_i(\mathbf{u}^N, x^0) \right) \right] \\
=& \frac{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t))}{\sum\limits_{i \in M} \hat{W}_i(\mathbf{u}^*, x^0)} \alpha_j(n^0) = \gamma_j \sum_{i \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t)) = B_j(n_l^t, x^*(n_l^t)).
\end{aligned}$$

The last equality follows from Theorem 2, i.e., from proportion consistency of the Nash bargaining solution of $\alpha(n^0)$. □

The distribution procedure defined in Theorem 2 allocates to the players their components of the Nash bargaining solution in any subgame, but the status quo vector does not coincide with the Nash equilibrium in the subgame. The status quo vector given in Theorem 3 for the subgame starting at node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$, is the Nash equilibrium in the whole game multiplied by the ratio of the total cooperative payoff in the subgame divided by the total cooperative payoff in the whole game. By choosing the status quo in this way, we avoid computing the Nash equilibria in all subgames along the cooperative state trajectory.

4.2 Node-consistent Nash bargaining solution

In this section, we define a node-consistent decomposition of players' shares in the whole game based on the idea that, at each node $n_l^t \in \mathcal{N}^t, t = 0, \ldots, T$, the cooperative payoffs-to-go are given by the Nash bargaining solution in the subgame, with the status quo vector being the Nash equilibrium outcomes in the subgame starting in node n_l^t and state $x^*(n_l^t)$.

Definition 5 *The IDP of the Nash bargaining solution $\alpha(n^0)$, that is,*

$$\big(\beta_j(n_l^t, x^*(n_l^t)) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T, j \in M\big)$$

is called node-consistent, if for any subgame starting in node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$, we have

$$B(n_l^t, x^*(n_l^t)) = \bar{\alpha}(n_l^t), \tag{30}$$

where $B(n_l^t, x^*(n_l^t)) = \big(B_j(n_l^t, x^*(n_l^t)) : j \in M\big)$, and $\bar{\alpha}(n_l^t)$ is the Nash bargaining solution of this subgame, with the status-quo vector given by the Nash equilibrium expected payoffs-to-go in the same subgame.

Remark 2 *In Section 4.1, we use $\alpha(n_l^t)$ to denote the Nash bargaining solution of the subgame starting in n_l^t, whereas in Definition 5, we use $\bar{\alpha}(n_l^t)$. The reason is that these two values differ in the definition of the status quo. They coincide only at root node n^0, i.e., $\alpha(n^0) = \bar{\alpha}(n^0)$.*

The following theorem gives the formulas for determining a node-consistent Nash bargaining solution in any subgame along the cooperative state trajectory.

Theorem 4 *The node-consistent IDP of the Nash bargaining solution $\alpha(n^0)$ is given by*

$$\beta_j(n_l^t, x^*(n_l^t)) = \bar{\alpha}_j(n_l^t) - \lambda \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \hat{\pi}(n_k^{t+1}|n_l^t)\bar{\alpha}_j\left(n_k^{t+1}\right), \quad n_l^t \in \mathcal{N}^t, \quad t = 0, \ldots, T-1, \tag{31}$$

$$\beta_j(n_l^T, x^*(n_l^T)) = \Phi_j^{n_l^T}(x^*(n_l^T)), \quad n_l^T \in \mathcal{N}^T \tag{32}$$

for any $j \in M$.

Proof. First, we prove that $\beta_j(n_l^t, x^*(n_l^t))$ determined by (31) and (32) is a distribution procedure of the Nash bargaining solution $\alpha(n^0)$. The first property is satisfied, that is,

$$\begin{aligned}\sum_{t=0}^{T} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \hat{\pi}(n_l^t)\beta_j(n_l^t, x^*(n_l^t)) &= \sum_{t=0}^{T-1} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \hat{\pi}(n_l^t)\big[\bar{\alpha}_j(n_l^t) - \lambda \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \hat{\pi}(n_k^{t+1}|n_l^t)\bar{\alpha}_j\left(n_k^{t+1}\right)\big] \\ &+ \lambda^T \sum_{n_l^T \in \mathcal{N}^T} \hat{\pi}(n_l^T)\Phi_j^{n_l^T}(x^*(n_l^T)) = \bar{\alpha}_j(n^0) = \alpha_j(n^0),\end{aligned}$$

because $\hat{\pi}(n_l^t)\hat{\pi}(n_k^{t+1}|n_l^t) = \hat{\pi}(n_k^{t+1})$ for any $n_k^{t+1} \in \mathcal{N}^t, t = 0, \ldots, T$.

The second property is obviously satisfied for terminal nodes, and also for non-terminal nodes. Indeed, we have

$$\begin{aligned}\sum_{j \in M} \beta_j(n_l^t, x^*(n_l^t)) &= \sum_{j \in M} \hat{W}_i(\mathbf{u}^*(n_l^t), x^*(n_l^t)) - \lambda \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \hat{\pi}(n_k^{t+1}|n_l^t) \sum_{j \in M} \hat{W}_j(\mathbf{u}^*(n_k^{t+1}), x^*(n_k^{t+1})) \\ &= \sum_{j \in M} \phi_j^{n_l^t}(x^*(n_l^t), u^*(n_l^t))\end{aligned}$$

for any node $n_l^t \in \mathcal{N}^t, t = 0, \ldots, T$.

Finally, we prove the node consistency of the Nash bargaining solution $\alpha(n^0)$ if its distribution procedure is defined by (31) and (32). For terminal nodes the property is obviously satisfied. Consider any non-terminal node $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T-1$ and subgame starting at this node. Calculate the expected sum of payments-to-go $B(n_l^t, x^*(n_l^t))$, that is

$$\begin{aligned} B_j(n_l^t, x^*(n_l^t)) &= \sum_{\theta=t}^{T} \lambda^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \hat{\pi}(n_l^\theta | n_l^t) \beta_j(n_l^\theta, x^*(n_l^\theta)) \\ &= \sum_{\theta=t}^{T} \lambda^{\theta-t} \sum_{n_l^\theta \in \mathcal{N}_\Gamma^\theta} \hat{\pi}(n_l^\theta | n_l^t) \left[\bar{\alpha}_j(n_l^\theta) - \lambda \sum_{n_k^{\theta+1} \in \mathcal{S}(n_l^\theta)} \hat{\pi}(n_k^{\theta+1} | n_l^t) \bar{\alpha}_j \left(n_k^{\theta+1}\right) \right] \\ &= \bar{\alpha}_j(n_l^t) - \lambda \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \hat{\pi}(n_k^{t+1} | n_l^t) \bar{\alpha}_j \left(n_k^{t+1}\right) + \lambda \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \hat{\pi}(n_k^{t+1} | n_l^t) \bar{\alpha}_j \left(n_k^{t+1}\right) - \ldots \\ &= \bar{\alpha}_j(n_l^t), \end{aligned}$$

which completes the proof. □

5 Illustrative example

To illustrate our results, we consider a 3-player Cournot competition game, with capacity constraints. The notations are borrowed from Haurie et. al (pp. 316-318), Pineau et. al (2011), Reddy and Zaccour (2019). The game is played over four periods, that is, $\mathcal{T} = \{0, 1, 2, 3\}$. Denote by $M = \{1, 2, 3\}$ the set of players and by $q_j(n_l^t)$ the quantity produced by Player j at node $n_l^t \in \mathcal{N}^t$, $t \in \mathcal{T}$, and by $q(n_l^t) = \sum_j q_{j \in M}(n_l^t)$ the total quantity at that node. The demand is stochastic and the event tree is described in Figure 1.

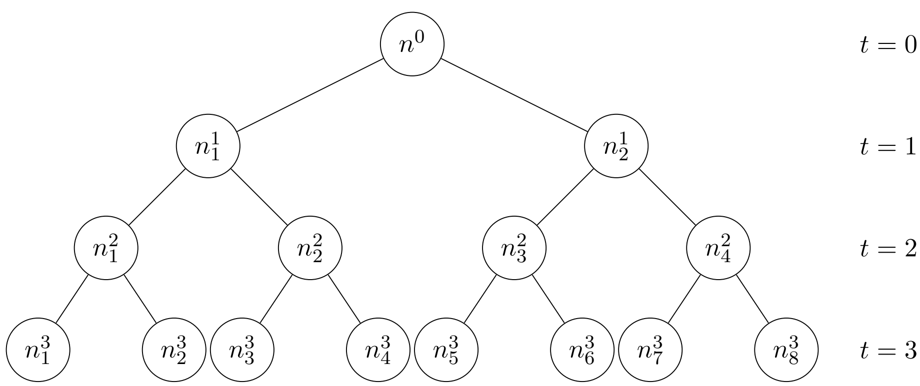

Figure 1: Event tree representing the uncertain demand.

We suppose that consumer's demand is stochastic. The inverse demand is assumed to be given by the following linear function:

$$P(q(n_l^t)) = A(n_l^t) - bq(n_l^t), \quad n_l^t \in \mathcal{N}^t, \quad t \in \mathcal{T}, \tag{33}$$

where $A(n_l^t)$ and b are positive parameters. Note that the randomness does not affect the slope of the demand, but only the choke price (or maximum willingness-to-pay).

Let $K_j(n_l^t)$ be the available production capacity and $I_j(n_l^t)$ the investment in this capacity made by Player j at node n_l^t, $t \in \mathcal{T}$. A vector of production capacities $(K_j(n_l^t) : j \in M, n_l^t \in \mathcal{N}^t, t \in \mathcal{T})$ represents the state variables. Assuming a one-period lag before an investment becomes productive, the evolution of the production capacity of Player j is then described by the following state equation:

$$K_j(n_l^t) = (1-\delta) K_j(a(n_l^t)) + I_j(a(n_l^t)), \qquad K_j(n^0) = k_j^0, \tag{34}$$

where k_j^0 denotes the initial capacity of Player j, and δ, $0 \le \delta < 1$, is the depreciation rate of capacity. The quantity produced is subject to available capacity, i.e.,

$$q_j(n_l^t) \leqslant K_j(n_l^t), \qquad n_l^t \in \mathcal{N}^t, \quad t \in \mathcal{T}. \tag{35}$$

The strategy of Player j at node $n_l^t \in \mathcal{N}^t$, $t = 0, 1, 2$, is a vector $(q_j(n_l^t), I_j(n_l^t))$ satisfying (35).

The production and investment costs are given by the following quadratic functions:

$$C_j(q_j(n_l^t)) = \frac{c_j}{2}\left(q_j(n_l^t)\right)^2, \quad c_j > 0, \tag{36}$$

$$D_j(I_j) = \frac{d_j}{2}\left(I_j(n_l^t)\right)^2, \quad d_j > 0. \tag{37}$$

Denote by $S_j\big(K_j(n^T)\big)$ the salvage value of production capacity of Player j at terminal nodes $n^T \in \mathcal{N}^T$ in time $T = 3$, given by

$$S_j\big(K_j(n^T)\big) = \frac{v_j}{2}\big(K_j(n^T)\big)^2, \quad v_j > 0.$$

Function $S_j(\cdot)$ is a Player j's payoff function defined in terminal nodes. Her payoff function in non-terminal node n_l^t is $q_j(n_l^t)P(q(n_l^t)) - C_j(q_j(n_l^t)) - D_j(I_j(n_l^t))$, where linear inverse demand $P(q(n_l^t))$ is defined by (33), production costs $C_j(q_j(n_l^t))$ and investment costs $D_j(I_j(n_l^t))$ are given by (36) and (37) respectively.

Player $j \in M$ faces the following optimization problem:

$$\begin{aligned} \max J_j = & \sum_{t=0}^{T-1} \lambda^t \sum_{n_l^t \in \mathcal{N}^t} \pi_j\left(n_l^t\right)\left(q_j\left(n_l^t\right) P\left(q\left(n_l^t\right)\right) - C_j\left(q_j\left(n_l^t\right)\right) - D_j\left(I_j\left(n_l^t\right)\right)\right) \\ & + \lambda^T \sum_{n_l^T \in \mathcal{N}^T} \pi_j\left(n_l^T\right) S_j\left(K_j\left(n_l^T\right)\right) \\ & \text{s.t. (34), (35), and} \\ \text{Non-negativity constraints}: & \; I_j\left(n_l^t\right) \geqslant 0,\; q_j\left(n_l^t\right) \geqslant 0,\; n_l^t \in \mathcal{N}^t, t = 0, \ldots, 3, \end{aligned}$$

where the contingent inverse demand laws are given by (33) and $I_j\left(n_l^T\right) \equiv 0$, $\forall\, n_l^T \in \mathcal{N}^T$.

For the numerical illustration, we adopt the following values for the parameters:

$$\begin{aligned} &c_1 = 2.5, && c_2 = 3, && c_3 = 3.25, \\ &d_1 = 20, && d_2 = 19, && d_3 = 16, \\ &k_1^0 = 15, && k_2^0 = 18, && k_3^0 = 16, \\ &v_1 = 0.45, && v_2 = 0.5, && v_3 = 0.35, \\ &\delta = 0.2, && \lambda = 0.8, && b = 3, \end{aligned}$$

and

Node	n^0	n_1^1	n_2^1	n_1^2	n_2^2	n_3^2	n_4^2
$A(n^t)$	100	120	110	90	100	130	100

The beliefs of the players or the probabilities $p_j(n_l^t)$, $j \in M$, $n_l^t \in \mathcal{N}^t$, $t = 1, 2, 3$, defining the event tree are given in Table 1, where $p_0(n_l^t)$ is a "calibration" probability function. Now, we explain the difference between probabilities $p_j(n_l^t)$ and $\pi_j(n_l^t)$. Consider node $n_1^3 \in \mathcal{N}^3$, with given probabilities $p_j(n_1^3)$, $j = 1, 2, 3$, where $p_j(n_1^3)$ is the belief of Player j on the probability of realization node n_1^3 if node $n_1^2 = a(n_1^3)$ has been realized. Let us compute the belief of Player 1 on the probability of passing through this node $\pi_1(n_1^3)$ when the game has not yet started (being at node n^0). Following Equation (1), we find the belief $\pi_1(n_1^3)$ recurrently. Having $\pi_1(n^0) = 1$, we have $\pi_1(n_1^1) = \pi_1(n^0)p_1(n_1^1) = 0.5$. Then, we find belief $\pi_1(n_1^2)$, which is equal to $\pi_1(n_1^1)p_1(n_1^2) = 0.5 \cdot 0.3 = 0.15$. Finally, the belief $\pi_1(n_1^3)$ is $\pi_1(n_1^2)p_1(n_1^3) = 0.15 \cdot 0.5 = 0.075$, which corresponds to the belief on probability of the scenario terminating at node n_1^3.

Table 1: Beliefs of the players about probabilities.

Node	n_1^1	n_2^1	n_1^2	n_2^2	n_3^2	n_4^2	n_1^3	n_2^3	n_3^3	n_4^3	n_5^3	n_6^3	n_7^3	n_8^3
$p_0(n_l^t)$	0.5	0.5	0.3	0.7	0.5	0.5	0.3	0.7	0.2	0.8	0.5	0.5	0.2	0.8
$p_1(n_l^t)$	0.5	0.5	0.3	0.7	0.4	0.6	0.5	0.5	0.1	0.9	0.6	0.4	0.2	0.8
$p_2(n_l^t)$	0.6	0.4	0.3	0.7	0.5	0.5	0.2	0.8	0.1	0.9	0.5	0.5	0.1	0.9
$p_3(n_l^t)$	0.4	0.6	0.3	0.7	0.4	0.6	0.2	0.8	0.3	0.7	0.5	0.5	0.3	0.7

Using these probabilities one computes the probabilities $\pi_j(n_l^t)$ of passing through the nodes by formula (1). The weights of the players are $(\omega_1, \omega_2, \omega_3) = (0.5, 0.3, 0.2)$. We obtain the aggregated probabilities $\hat{p}(n_l^t)$ for coalition M using three methods (linear (LBP), geometric (GBP) and multiplicative belief pool (MBP)) represented in Table 2. Notice that the aggregated beliefs obtained by linear and geometric belief pools are very similar but the beliefs obtained by multiplication belief pool is much more different from linear and geometric belief pools. It may be explained by the influence of the calibration probability function.

Table 2: Aggregated beliefs.

Method	n_1^1	n_2^1	n_1^2	n_2^2	n_3^2	n_4^2	n_1^3	n_2^3	n_3^3	n_4^3	n_5^3	n_6^3	n_7^3	n_8^3
LBP	0.51	0.49	0.3	0.7	0.43	0.57	0.35	0.65	0.14	0.86	0.55	0.45	0.19	0.81
GBP	0.51	0.49	0.3	0.7	0.43	0.57	0.33	0.67	0.13	0.87	0.55	0.45	0.18	0.82
MBP	0.5	0.5	0.03	0.97	0.31	0.69	0.03	0.97	0.001	0.999	0.6	0.4	0.003	0.997

Table 3 represents the results of aggregating beliefs using linear and geometric belief pool in case we update players' weights using Bayes rule.

Table 3: Aggregated beliefs applying the Bayes rule.

Method	n_1^1	n_2^1	n_1^2	n_2^2	n_3^2	n_4^2	n_1^3	n_2^3	n_3^3	n_4^3	n_5^3	n_6^3	n_7^3	n_8^3
LBP	0.51	0.49	0.3	0.7	0.42	0.58	0.35	0.65	0.13	0.87	0.55	0.45	0.20	0.80
GBP	0.51	0.49	0.3	0.7	0.42	0.58	0.33	0.67	0.12	0.88	0.55	0.45	0.19	0.81

The Nash bargaining solution for the three belief pooling methods are given in Table 4. One may notice that the solutions are close because the aggregated beliefs are very similar. The condition of Theorems 2 and 4 are satisfied, then we may calculate the proportion-consistent and node-consistent distribution procedures of the Nash bargaining solution, which is obtained in the cooperative game played over event tree with beliefs aggregated with a linear belief pool. The same procedures can be calculated for the games with geometric and multiplicative belief pools. The shares are $\gamma_1 = 0.36$, $\gamma_2 = 0.33$, $\gamma_3 = 0.31$. The cooperative state trajectory for this case is presented in Table 5.

Table 4: Nash bargaining solutions.

Method	Player 1	Player 2	Player 3
LBP	725.060	664.923	628.820
GBP	725.033	664.897	628.793
MBP	722.587	662.451	626.347

We determine the components of proportion-consistent distribution procedure of the NBS by formulas (22) and (23). The proportion-consistent distribution procedure of the NBS is represented on Figure 2.

Table 5: Cooperative state trajectory (linear belief pool).

Time period	$t=0$	$t=1$		$t=2$			
Node	n^0	n^1_1	n^1_2	n^2_1	n^2_2	n^2_3	n^2_4
K_1^*	15.000	12.059	12.059	9.738	9.738	9.738	9.738
K_2^*	18.000	14.483	14.483	11.715	11.715	11.715	11.715
K_3^*	16.000	12.861	12.861	10.383	10.383	10.383	10.383
	$t=3$							
Node	n^3_1	n^3_2	n^3_3	n^3_4	n^3_5	n^3_6	n^3_7	n^3_8
K_1^*	7.933	7.933	7.933	7.933	7.933	7.933	7.933	7.933
K_2^*	9.574	9.574	9.574	9.574	9.574	9.574	9.574	9.574
K_3^*	8.454	8.454	8.454	8.454	8.454	8.454	8.454	8.454

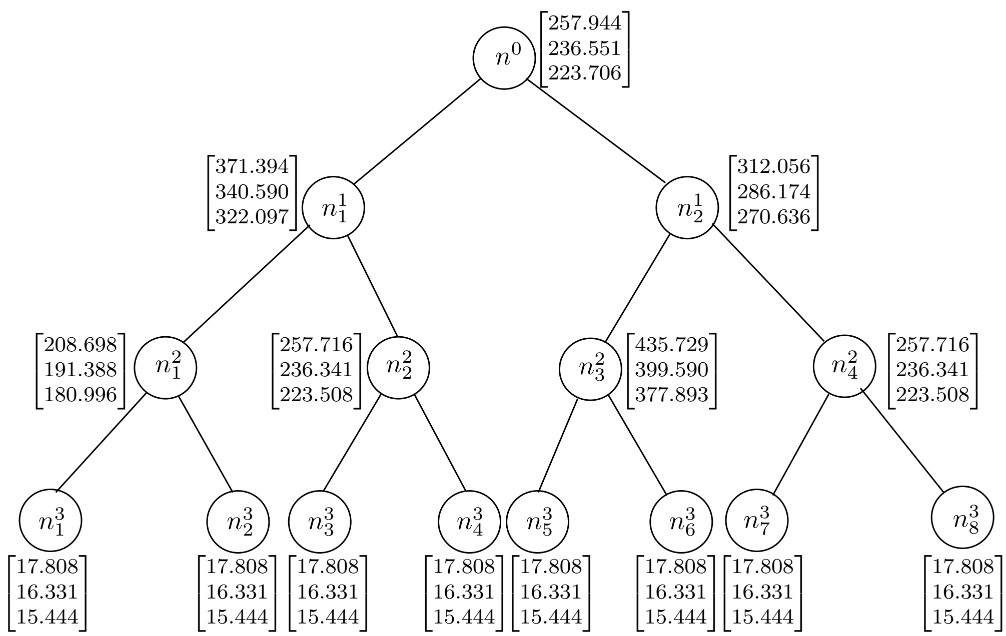

Figure 2: Proportion-consistent distribution procedure of the Nash bargaining solution.

Proportion-consistent decomposition of the Nash bargaining solution for Player 1 is as follows:

$$\begin{aligned}725.060 =&257.944 + 0.8 \cdot 0.51 \cdot 371.394 + 0.8 \cdot 0.49 \cdot 312.056 + (0.8)^2 \cdot 0.51 \cdot 0.3 \cdot 208.698\\ &+ (0.8)^2 \cdot 0.51 \cdot 0.7 \cdot 257.716 + (0.8)^2 \cdot 0.49 \cdot 0.43 \cdot 435.729 + (0.8)^2 \cdot 0.49 \cdot 0.57 \cdot 257.716\\ &+ (0.8)^3 \cdot 17.808.\end{aligned}$$

To determine node-consistent decomposition of the Nash bargaining solution one needs to calculate the Nash bargaining solutions for all subgames (see Table 6).

We determine the components of node-consistent distribution procedure of the NBS using the Nash bargaining solutions of all subgames by formulas (31) and (32). The node-consistent distribution procedure of the Nash bargaining solution is represented on Figure 3.

Node-consistent decomposition of the Nash bargaining solution for Player 1 is as follows:

$$\begin{aligned}725.060 =&260.704 + 0.8 \cdot 0.51 \cdot 369.643 + 0.8 \cdot 0.49 \cdot 309.484 + (0.8)^2 \cdot 0.51 \cdot 0.3 \cdot 209.659\\ &+ (0.8)^2 \cdot 0.51 \cdot 0.7 \cdot 258.887 + (0.8)^2 \cdot 0.49 \cdot 0.43 \cdot 437.658 + (0.8)^2 \cdot 0.49 \cdot 0.57 \cdot 258.887\\ &+ (0.8)^3 \cdot 14.161.\end{aligned}$$

Comparing the proportion-consistent and node-consistent procedures of the Nash bargaining solution, we notice that the difference between the payments to the players at initial node is not so high. In

Table 6: Nash bargaining solutions for all subgames.

Time period		$t=0$	$t=1$		$t=2$			
Node		n^0	n_1^1	n_2^1	n_1^2	n_2^2	n_3^2	n_4^2
$\bar{\alpha}_1$		725.060	574.000	587.154	220.988	270.215	448.987	270.215
$\bar{\alpha}_2$		664.923	530.225	549.154	208.128	252.738	414.747	252.738
$\bar{\alpha}_3$		628.820	502.884	508.846	191.633	234.277	389.145	234.277
	$t=3$							
Node	n_1^3	n_2^3	n_3^3	n_4^3	n_5^3	n_6^3	n_7^3	n_8^3
$\bar{\alpha}_1$	14.161	14.161	14.161	14.161	14.161	14.161	14.161	14.161
$\bar{\alpha}_2$	22.913	22.913	22.913	22.913	22.913	22.913	22.913	22.913
$\bar{\alpha}_3$	12.509	12.509	12.509	12.509	12.509	12.509	12.509	12.509

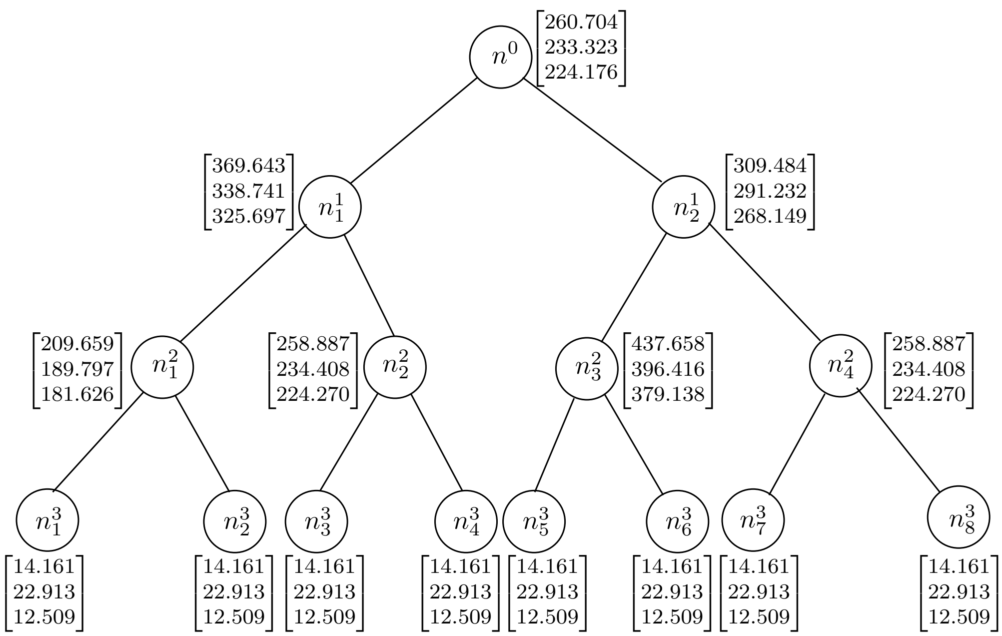

Figure 3: Node-consistent distribution procedure of the Nash bargaining solution.

particular, according to the proportion-consistent distribution procedure the payments are 257.944, 236.551 and 223.706 to Players 1, 2 and 3, respectively and 260.704, 233.323 and 224.176 according to the node-consistent distribution. However, the difference in payments at terminal nodes is more pronounced. Indeed, the proportion-consistent distribution procedure gives 17.808, 16.331 and 15.444 to Players 1, 2 and 3, respectively, while the node-consistent distribution procedure allocates 14.161, 22.913 and 12.509 to Players 1, 2 and 3, respectively.

6 Conclusions

In this paper, we tackled the problem of sustaining cooperation in games played over event trees, when the transition probabilities differ among the players. We showed how to construct an imputation distribution procedure of the Nash bargaining solution under different schemes for aggregating the players' beliefs.

Two extensions of this work can be envisioned. First, it would be interesting to adapt our formalism to a case where the duration of the game is random, that is, at each non-terminal node there is a positive probability that the game ends. In our development here, we had to pool the information for only the grand coalition. A second extension would be to consider other cooperative game solutions, e.g., Shapley value and core, where the pooling of information has to be done for each possible coalition.

Although the aggregation process is not complicated in itself, cooperation may not deliver a better outcome for some coalitions in some subgames, which might render the existence of a non-empty core an issue.

The class of dynamic games played over event trees has been used to model and predict equilibrium strategies in energy markets; see, e.g., Zaccour (1987), Haurie et al. (1990), Genc et al. (2007), Genc and Sen (2008) and Pineau et al. (2011). In these applications, the transition probabilities are given and are the same for all players. We believe that the assumption that the players may differ in their assessments of these probabilities is more realistic and open the door to applications in areas where this asymmetry is a fact. It seems that DGPET with asymmetric beliefs are well-suited to analyze global environmental problems. In particular, such class of games offer a natural paradigm to design an international environmental agreement to control pollution emissions. Indeed, one of the difficulties surrounding the negotiation for achieving such an agreement is related to the differences in countries' beliefs regarding future states of nature.

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