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Abstract: The eccentricity of a vertex $v$ in a graph $G$ is the maximum distance between $v$ and any other vertex of $G$. The diameter of a graph $G$ is the maximum eccentricity of a vertex in $G$. The eccentric connectivity index of a connected graph is the sum over all vertices of the product between eccentricity and degree. Given two integers $n$ and $D$ with $D \leq n - 1$, we characterize those graphs which have the largest eccentric connectivity index among all connected graphs of order $n$ and diameter $D$. As a corollary, we also characterize those graphs which have the largest eccentric connectivity index among all connected graphs of a given order $n$. 
1 Introduction

Let $G = (V, E)$ be a simple connected undirected graph. The distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the number of edges of a shortest path in $G$ connecting $u$ and $v$. The eccentricity $e(v)$ of a vertex $v$ is the maximum distance between $v$ and any other vertex, that is $\max\{d(v, w) \mid w \in V\}$. The diameter of $G$ is the maximum eccentricity among all vertices of $G$. The eccentric connectivity index $\xi^c(G)$ of $G$ is defined by

$$\xi^c(G) = \sum_{v \in V} \deg(v)e(v).$$

This index was introduced by Sharma et al. in [3]. Alternatively, $\xi^c$ can be computed by summing the eccentricities of the extremities of each edge:

$$\xi^c(G) = \sum_{v,w \in E} e(v) + e(w).$$

We define the weight of a vertex by $W(v) = \deg(v)e(v)$, and we thus have $\xi^c(G) = \sum_{v \in V} W(v)$. Morgan et al. [2] give the following asymptotic upper bound on $\xi^c(G)$ for a graph $G$ of order $n$ and with a given diameter $D$.

**Theorem 1 (Morgan, Mukwembi and Swart, 2011 [2])** Let $G$ be a connected graph of order $n$ and diameter $D$. Then,

$$\xi^c(G) \leq D(n-D)^2 + O(n^2).$$

In what follows, we write $G \simeq H$ if $G$ and $H$ are two isomorphic graphs, and we let $K_n$ and $P_n$ be the complete graph and the path of order $n$, respectively. We refer to Diestel [1] for basic notions of graph theory that are not defined here. A lollipop $L_{n,D}$ is a graph obtained from a path $P_D$ by joining an end vertex of this path to $K_{n-D}$. Morgan et al. [2] state that the above asymptotic bound is best possible by showing that $\xi^c(L_{n,D}) = D(n-D)^2 + O(n^2)$. The aim of this paper is to give a precise upper bound on $\xi^c(G)$ in terms of $n$ and $D$, and to completely characterize those graphs than attain the bound. As a result, we will observe that there are graphs $G$ of order $n$ and diameter $D$ such that $\xi^c(G)$ is strictly larger than $\xi^c(L_{n,D})$.

Morgan et al. [2] also give an asymptotic upper bound on $\xi^c(G)$ for graphs $G$ of order $n$ (but without a fixed diameter), and show that this bound is sharp by observing that it is attained by $L_{n, \frac{n}{2}}$.

**Theorem 2 (Morgan, Mukwembi and Swart, 2011 [2])** Let $G$ be a connected graph of order $n$. Then,

$$\xi^c(G) \leq \frac{4}{27}n^3 + O(n^2).$$

We give a precise upper bound on $\xi^c(G)$ for graphs $G$ of order $n$, and characterize those graphs that reach the bound. As a corollary, we show that for every lollipop, there is another graph $G$ of same order, but with a strictly larger eccentric connectivity index.

2 Results for a fixed order and a fixed diameter

The only graph with diameter 1 is the clique, and clearly, $\xi^c(K_n) = n(n-1)$. Also, the only connected graph with 3 vertices and diameter 2 is $P_3$, and $\xi^c(P_3) = \xi^c(K_3) = 6$. The next theorem characterizes the graphs with maximum eccentric connectivity index among those with $n \geq 4$ vertices and diameter 2. Let $M_n$ be the graph obtained from $K_n$ by removing a maximum matching (i.e., $\lceil \frac{n}{2} \rceil$ disjoint edges) and, if $n$ is odd, an additional edge adjacent to the unique vertex that still has degree $n-1$. In other words, all vertices in $M_n$ have degree $n-2$, except possibly one that has degree $n-3$. For illustration, $M_6$ and $M_7$ are drawn in Figure 1.

**Theorem 3** Let $G$ be a connected graph of order $n \geq 4$ and diameter 2. Then,

$$\xi^c(G) \leq 2n^2 - 4n - 2(n \mod 2)$$

with equality if and only if $G \simeq M_n$ or $n = 5$ and $G \simeq H_1$ (see Figure 1).
Proof. Let $G$ be a graph of order $n$ and diameter 2, and let $x$ be the number of vertices of degree $n - 1$ in $G$. Clearly, $W(v) = n - 1$ for all vertices $v$ of degree $n - 1$, while $W(v) \leq 2(n - 2)$ for all other vertices $v$. Note that if $n - x$ is odd, then at least one vertex in $G$ has degree at most $n - 3$. Hence,

$$\xi^*(G) \leq (n - 1) + 2(n - x)(n - 2) - 2((n - x) \mod 2) = 2n^2 - 4n + x(3 - n) - 2((n - x) \mod 2).$$

For $n = 4$ or $n \geq 6$, this value is maximized with $x = 0$. For $n = 5$, both $x = 1$ (i.e., $G \simeq H_1$) and $x = 0$ (i.e., $G \simeq M_5$) give the maximum value $28 = 2n^2 - 4n + (3 - n) - 2((n - 1) \mod 2) = 2n^2 - 4n - 2(n \mod 2)$. □

Before giving a similar result for graphs with diameter $D \geq 3$, we prove the following useful property.

**Lemma 1** Let $G$ be a connected graph of order $n \geq 4$ and diameter $D \geq 3$. Let $P$ be a shortest path in $G$ between two vertices at distance $D$, and assume there is a vertex $u$ on $P$ such that $\epsilon(u)$ is strictly larger than the longest distance $L$ from $u$ to an extremity of $P$. Finally, let $v$ be a vertex in $G$ such that $d(v, u) = \epsilon(u)$ and let $v = w_1 - w_2 - \ldots - w_{\epsilon(u)-1} = u$ be a path of length $\epsilon(u)$ linking $v$ to $u$ in $G$. Then

- vertices $w_1, \ldots, w_{\epsilon(u)-L}$ do not belong to $P$;
- vertex $w_{\epsilon(u)-L}$ has either no neighbor on $P$, or its unique neighbor on $P$ is an extremity at distance $L$ from $u$;
- if $\epsilon(u) - L > 1$ then vertices $w_1, \ldots, w_{\epsilon(u)-L-1}$ have no neighbor on $P$.

Proof. No vertex $w_i$ with $1 \leq i \leq \epsilon(u) - L$ is on $P$, since this would imply $d(u, w_i) \leq L$, and hence $d(u, v) = d(u, w_1) \leq L + i - 1 \leq \epsilon(u) - 1$. Similarly, no vertex $w_i$ with $1 \leq i \leq \epsilon(u) - L - 1$ has a neighbor on $P$, since this would imply $d(u, w_i) \leq L + 1$, and hence $d(u, v) = d(u, w_1) \leq L + 1 + i - 1 \leq \epsilon(u) - 1$. If vertex $w_{\epsilon(u)-L}$ has at least one neighbor on $P$, then this neighbor is necessarily an extremity of $P$ at distance $L$ from $u$, else we would have $d(u, w_{\epsilon(u)-L}) \leq L$, which would imply $d(u, v) = d(u, w_1) \leq L + (\epsilon(u) - L - 1) = \epsilon(u) - 1$. We conclude the proof by observing that if both extremities of $P$ are at distance $L$ from $u$, then $w_{\epsilon(u)-L}$ is adjacent to at most one of them since $D \geq 3$. □

Let $n, D$ and $k$ be integers such that $n \geq 4$, $3 \leq D \leq n - 1$ and $0 \leq k \leq n - D - 1$, and let $E_{n,D,k}$ be the graph (of order $n$ and diameter $D$) constructed from a path $u_0 - u_1 - \ldots - u_D$ by joining each vertex of a clique $K_{n-D-1}$ to $u_0$ and $u_1$, and $k$ vertices of the clique to $u_2$ (see Figure 1). Observe that $E_{n,D,0}$ is the lollipop $L_{n,D}$ and that $E_{n,D,n-D-1}$ can be viewed as a lollipop with a missing edge between $u_0$ and $u_2$. Also, if $D = n - 1$, then $k = 0$ and $E_{n,n-1,0} \simeq P_n$. 

![Figure 1: Graphs $H_1, H_2, H_3, M_6, M_7$ and $E_{8,4,k}$](image-url)
Lemma 2 Let \( n, D \) and \( k \) be integers such that \( n \geq 4, 3 \leq D \leq n - 1 \) and \( 0 \leq k \leq n - D - 1 \), then
\[
\xi^c(E_{n,D,k}) = 2 \sum_{i=0}^{D-1} \max\{i, D-i\} + (n-D-1)(2D-1 + D(n-D)) + k(2D-n-1 + \max\{2, D-2\}).
\]

Proof. The sum of the weights of the vertices outside \( P \) is
\[
\sum_{v \in V \setminus V(P)} W(v) = k(n-D+1)(D-1) + (n-D-1-k)(n-D)D,
\]
\[
= k(2D-n-1) + (n-D-1)(n-D)D.
\]
We now consider the weights of the vertices in \( P \). The weight of \( u_0 \) is \( D(n-D) \), the weight of \( u_1 \) is \( (D-1)(n-D+1) \), and the weight of \( u_2 \) is \( (k+2)\max\{2, D-2\} \). The weight of \( u_i \) for \( i = 3, \ldots, D-1 \) is \( 2\max\{i, D-i\} \), and the weight of \( u_D \) is \( D \). Hence, the total weight of the vertices on \( P \) is
\[
(n-D)D + (n-D+1)(D-1) + (k+2)\max\{2, D-2\} + 2 \sum_{i=3}^{D-1} \max\{i, D-i\} + D
\]
\[
= \left( (n-D-1)D + D \right) + \left( (n-D-1)(D-1) + 2(D-1) \right)
\]
\[
+ \left( k \max\{2, D-2\} + 2\max\{2, D-2\} \right) + 2 \sum_{i=3}^{D-1} \max\{i, D-i\} + D
\]
\[
= 2 \sum_{i=0}^{D-1} \max\{i, D-i\} + (n-D-1)(2D-1) + k\max\{2, D-2\}
\]
By summing up all weight in \( G \), we obtain the desired result. \( \square \)

In what follows, we denote \( f(n, D) = \frac{n-D-1}{\max_{k=0}^{D-1}} \xi^c(E_{n,D,k}) \). It follows from the above lemma that
\[
f(n, D) = \begin{cases} 
14 + \left( n - 4 \right) \left( 3n - 4 + \max\{0, 2D-n+1\} \right) & \text{if } D = 3; \\
2 \sum_{i=0}^{D-1} \max\{i, D-i\} + \left( n-D-1 \right) \left( 2D-1 + D(n-D) + \max\{0, 3D-n-3\} \right) & \text{if } D \geq 4.
\end{cases}
\]
Lemma 2 allows to know for which values of \( k \) we have \( \xi^c(E_{n,D,k}) = f(n, D) \).

Corollary 1 Let \( n \) and \( k \) be integers such that \( n \geq 4, 0 \leq k \leq n-4 \).

- If \( n < 7 \), then \( \xi^c(E_{n,3,k}) \leq f(n,3) = 2n^2-5n+2 \), with equality if and only if \( k = n-4 \).
- If \( n > 7 \), then \( \xi^c(E_{n,3,k}) \leq f(n,3) = 3n^2-16n+30 \) with equality if and only if \( k = 0 \).
- If \( n = 7 \), then all \( \xi^c(E_{n,3,k}) \) are equal to 65 for \( k = 0, \ldots, n-4 \).

Corollary 2 Let \( n, D \) and \( k \) be integers such that \( n \geq 5, 4 \leq D \leq n-1 \) and \( 0 \leq k \leq n-D-1 \).

- If \( n < 3(D-1) \), then \( \xi^c(E_{n,D,k}) = f(n, D) \) if and only if \( k = n-D-1 \).
- If \( n > 3(D-1) \), then \( \xi^c(E_{n,D,k}) = f(n, D) \) if and only if \( k = 0 \).
- If \( n = 3(D-1) \), then \( \xi^c(E_{n,D,k}) = f(n, D) \) if and only if \( k \in \{0, \ldots, n-D-1\} \).
Theorem 4
and $n$ connectiviy index, we will prove that they reach the more precise upper bound only if $G$ is isomorphic to $E_7,3,k$, while $\xi^*(H_3) = f(7, 3) = 65$. In what follows, we prove that all graphs $G$ of order $n$ and diameter $D \geq 3$ have $\xi^*(G) \leq f(n, D)$. Moreover, we show that if $G$ is not isomorphic to a $E_{n,D,k}$, then equality can only occur if $G \simeq H_2$ or $G \simeq H_3$. So, for every $n \geq 4$ and $3 \leq D \leq n - 1$, let us consider the following graph class $C_n^D$:

$$
C_n^D = \begin{cases}
\{E_{n,3,n-4}\} & \text{if } n = 4, 5 \text{ and } D = 3; \\
\{E_{n,3,2}, H_2\} & \text{if } n = 6 \text{ and } D = 3; \\
\{E_{n,3,0}, \ldots, E_{n,3,3}, H_3\} & \text{if } n = 7 \text{ and } D = 3; \\
\{E_{n,3,0}\} & \text{if } n > 7 \text{ and } D = 3; \\
\{E_{n,D,n-D-1}\} & \text{if } n < 3(D - 1) \text{ and } D \geq 4; \\
\{E_{n,D,0}, \ldots, E_{n,D,n-D-1}\} & \text{if } n = 3(D - 1) \text{ and } D \geq 4; \\
\{E_{n,D,0}\} & \text{if } n > 3(D - 1) \text{ and } D \geq 4.
\end{cases}
$$

Note that while Morgan et al. [2] state that the lollipops reach the asymptotic upper bound of the eccentric connectivity index, we will prove that they reach the more precise upper bound only if $D = n - 1$, $D = 3$ and $n \geq 7$, or $D \geq 4$ and $n \geq 3(D - 1)$.

**Theorem 4** Let $G$ be a connected graph of order $n \geq 4$ and diameter $3 \leq D \leq n - 1$. Then $\xi^*(G) \leq f(n, D)$, with equality if and only if $G$ belongs to $C_n^D$.

**Proof.** We have already observed that all graphs $G$ in $C_n^D$ have $\xi^*(G) = f(n, D)$. So let $G$ be a graph of order $n$, diameter $D$ such that $\xi^*(G) \geq f(n, D)$. It remains to prove that $G$ belongs to $C_n^D$.

Let $P = u_0 - u_1 - \cdots - u_D$ be a shortest path in $G$ that connects two vertices $u_0$ and $u_D$ at distance $D$ from each other. In what follows, we use the following notations for all $i = 0, \ldots, D$:

- $o_i$ is the number of vertices outside $P$ and adjacent to $u_i$;
- $\delta_i = \max\{i, D - i\}$;
- $r_i = \epsilon(u_i) - \delta_i$.

Also, let $r^* = \max_{i=1}^{D-1} r_i$. Note that $\delta_i \geq 2$ and $r_i \leq \lfloor \frac{D}{2} \rfloor$ for all $i$, and $r_0 = r_D = 0$ since $\epsilon(u_0) = \epsilon(u_D) = \delta_0 = \delta_D = D$. Since $P$ is a shortest path linking $u_0$ to $u_D$, no vertex outside $P$ can have more than three neighbors in $P$. We consider the following partition of the vertices outside $P$ in 4 disjoint sets $V_0, V_{1,2}, V_3^{D-1}, V_3^D$, and denote by $n_0, n_{1,2}, n_3^{D-1}, n_3^D$ their respective size:

- $V_0$ is the set of vertices outside $P$ with no neighbor on $P$;
- $V_{1,2}$ is the set of vertices outside $P$ with one or two neighbors in $P$;
- $V_3^{D-1}$ is the set of vertices $v$ outside $P$ with three neighbors in $P$ and $\epsilon(v) \leq D - 1$;
- $V_3^D$ is the set of vertices $v$ outside $P$ with three neighbors in $P$ and $\epsilon(v) = D$.

Clearly, all vertices $v$ outside $P$ can have $\epsilon(v) = D$ except those in $V_3^{D-1}$. The maximum degree of a vertex in $V_0$ is $n - D - 2$, while it is $n - D$ for those in $V_{1,2}$ and $n - D + 1$ for those in $V_3^{D-1} \cup V_3^D$. For a vertex $v \in V_{1,2} \cup V_3^{D-1} \cup V_3^D$, let

$$
\rho(v) = \max\{r_i | u_i \text{ is adjacent to } v\}
$$

Hence, $r^* \geq \rho^*$. We first show that the total weight of the vertices in $V_0 \cup V_{1,2}$ is at most

$$
D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D) - 2Dr^* + D \min\{1, \rho^*\}.
$$

- If $r^* = 0$, then the largest possible weight of the vertices in $V_0 \cup V_{1,2}$ occurs when all of them have two neighbors in $P$ (i.e., $n_0 = 0$ and no vertex in $V_{1,2}$ has one neighbor on $P$). In such a case, $n_0 + n_{1,2} = n - D - 1 - n_3^{D-1} - n_3^D$, and all these vertices have degree $n - D$. Hence, their total weight is at most $D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D)$. 

• If \( r^* > 0 \) and \( \rho^* > 0 \), then let \( i \) be such that \( r_i = r^* \). It follows from Lemma 1 that there is a path \( w_1 - \ldots - w_{v(u_i)+1} \) such that \( w_1, \ldots, w_{r_i-1} \) have no neighbor on \( P \) and \( w_{r_i} \) has at most one neighbor on \( P \). Hence, the largest possible weight of the vertices in \( V_0 \cup V_{1,2} \) occurs when \( r^* - 1 \) vertices have 0 neighbor on \( P \), one vertex has one neighbor on \( P \), and \( n - D - 1 - n_3^{D-1} - n_3^D - r^* \) vertices have 2 neighbors in \( P \). Hence, the largest possible weight for the vertices in \( V_0 \cup V_{1,2} \) is
\[
D(n - D - 2)(r^* - 1) + D(n - D - 1) + D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D - r^*)
= D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D - r^*) - 2Dr^* + D.
\]

• If \( r^* > 0 \) and \( \rho^* = 0 \), then consider the same path \( w_1 - \ldots - w_{v(u_i)+1} \) as in the above case. If \( w_{r_i} \) has no neighbor on \( P \), then there are at least \( r^* \) vertices with no neighbor on \( P \) and the largest possible weight for the vertices in \( V_0 \cup V_{1,2} \) is
\[
D(n - D - 2)(r^* + 1) + D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D - r^*)
= D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D - r^*) - 2Dr^*.
\]

Also, if there are at least two vertices in \( V_{1,2} \) with only one neighbor on \( P \), then the largest possible weight for the vertices in \( V_0 \cup V_{1,2} \) is
\[
D(n - D - 2)(r^* - 1) + 2D(n - D - 1) + D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D - r^* - 1)
= D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D - r^*) - 2Dr^*.
\]

So assume \( w_{r_i} \) is the only vertex in \( V_{1,2} \) with only one neighbor on \( P \). We thus have \( d(u_i, w_{r_i}) = \delta_i + 1 \). We now show that this case is impossible. We know from Lemma 1 that \( w_{r_i} \) is adjacent to \( u_0 \) or (exclusive) to \( u_D \). Since \( \rho(v) = 0 \) for all vertices \( v \) outside \( P \), we know that \( u_i \) has no neighbor outside \( P \). Hence, \( w_{\epsilon(u_i)} = u_i-1 \) or \( u_i+1 \), say \( u_i+1 \) (the other case is similar). Then \( w_{r_i} \) is not adjacent to \( u_0 \) else there is \( j \) with \( r^* + 1 \leq j \leq \epsilon(u_i) - 1 \) such that \( w_j \) is outside \( P \) and has \( w_{j+1} \) as neighbor on \( P \), and since \( w_j \) must have a second neighbor \( u_\ell \) on \( P \) with \( \ell \geq i + 2 \), we would have
\[
i + 2 \leq \ell = d(u_0, w_{r_i}) \leq d(w_{r_i}, w_j) + 2 \leq d(w_{r_i}, u_i) - 2 + 2 = i + 1.
\]

Hence, \( w_{r_i} \) is adjacent to \( u_D \). Then there is also a path linking \( u_i \) to \( w_1 \) going through \( u_i-1 \) else \( d(u_0, w_1) = d(u_0, u_i) + d(u_i, w_1) > i + \delta_i \geq D \). Let \( Q \) be such a path of minimum length. Clearly, \( Q \) has length at least equal to \( \epsilon(u_i) \). So let \( w'_1 - \ldots - w'_{\epsilon(u_i)+1} \) be the subpath of \( Q \) of length \( \epsilon(u_i) \) and having \( u_i \) as extremity (i.e., \( w'_{\epsilon(u_i)} = u_i-1 \) and \( w'_{\epsilon(u_i)+1} = u_i \)). Applying the same argument to \( w'_{r_i} \), as was done for \( w_{r_i} \), we conclude that \( w'_{r_i} \) has \( n_0 \) as unique neighbor on \( P \). We thus have two vertices in \( V_{1,2} \) with a unique neighbor on \( P \), a contradiction.

The total weight of the vertices in \( V_3^{D-1} \cup V_3^D \) is at most \( (n - D + 1)(D - 1)n_3^{D-1} + Dn_3^D \), which gives the following upper bound \( B \) on the total weight of the vertices outside \( P \):
\[
B = D(n - D)(n - D - 1 - n_3^{D-1} - n_3^D) + (n - D + 1)(D - 1)n_3^{D-1} + Dn_3^D - 2Dr^* + D \min\{1, \rho^*\}
= (n - D - 1)D(n - D) + n_3^{D-1}(2D - n - 1) + Dn_3^D - 2Dr^* + D \min\{1, \rho^*\}.
\]

This bound can only be reached if all vertices outside \( P \) are pairwise adjacent. But Lemma 1 shows that this cannot happen if \( \rho^* > 0 \). Indeed, consider a vertex \( v \) in \( V_{1,2} \cup V_3^D \cup V_3^{D-1} \) with \( \rho(v) > 0 \). There is a vertex \( u_i \) in \( P \) adjacent to \( v \) such that \( \rho(v) = r_i = \epsilon(u_i) - \delta_i > 0 \). We know from Lemma 1 that there is a shortest path \( w_1 - w_2 - \ldots - w_{\rho(v)+1} = u_i \), linking \( u_i \) to a vertex \( w_1 \) with \( d(u_i, w_1) = \epsilon(u_i) \) and such that \( w_1, \ldots, w_{\rho(v)} \) do not belong to \( P \). In what follows, we denote \( Q^v \) such a path. If \( v \) is adjacent to \( w_j \) with \( 1 \leq j \leq \rho(v) \), then the path \( u_i - v - w_j - \ldots - w_{\rho(v)} \) links \( u_i \) to \( w_j \) and has length at most \( \rho(v) + 1 < r_i + \delta_i = \epsilon(u_i) \), a contradiction. Hence \( v \) has at least \( \rho(v) \) non-neighbors outside \( P \). Also, as shown in Lemma 1, \( w_1, \ldots, w_{\rho(v)-1} \) belong to \( V_0 \), while \( w_{\rho(v)} \) belongs to \( V_0 \cup V_{1,2} \). In the upper bound \( B \), we have assumed that \( \epsilon(u_1) = \ldots = \epsilon(w_{\rho(v)}) = D \). Hence, if \( v \in V_{1,2} \cup V_3^D \), we can gain 2\( D \) units on \( B \) for every \( w_j, j = 1, \ldots, \rho(v) \) (\( D \) for \( v \) and \( D \) for \( w_j \)), while the gain is \( 2D - 1 \) (\( D - 1 \) for \( v \) and \( D \) for \( w_j \)) if \( v \in V_3^{D-1} \).
We can gain an additional $2D$ for every $v \in V_3^D$. Indeed, consider such a vertex $v$ and let $w^*$ be a vertex at distance $D$ from $v$. Note that $w^*$ is not on $P$ and has at most one neighbor on $P$ else $d(v, w^*) \leq D - 1$. Hence, if $\rho(v) = 0$, we can gain $2D$ (one $D$ for $v$ and one $D$ for $w$) in the above upper bound. So assume $\rho(v) > 0$, and consider again the shortest path $Q^v = w_1 - w_2 - \ldots - w_{\epsilon(u_i)+1} = u_i$, with $\rho(v) = r_i$. Also, let $W = \{w_1, \ldots, w_{\rho(v)}\}$. To gain an additional $2D$, it is sufficient to determine a vertex in $V_0 \cup V_{1,2} \setminus W$ which is not adjacent to $v$. So assume no such vertex exists, and let us prove that such a situation cannot occur. Note that $w^* \notin V_3^D \cup V_3^{D-1}$ (since it has at most one neighbor on $P$), which implies $w^* \in W$.

- If a vertex $w_j \in W$ has a neighbor $x \in V_0 \cup V_{1,2}$ outside $W$, then $v$ is adjacent to $x$, and the path $v - x - w_j - \ldots - w^*$ has length at most $1 + \rho(v) \leq 1 + \left\lfloor \frac{D}{2} \right\rfloor < D$, a contradiction.
- If a vertex $w_j \in W$ has a neighbor $x \in V_3^D \cup V_3^{D-1}$, then $d(u_i, x) \leq d(u_i, x) + d(x, w_1) \leq \delta_i - 1 + r_i < \epsilon(u_i)$, a contradiction.

Since $G$ is connected and $w_1, \ldots, w_{\rho(v)-1}$ have no neighbors outside $Q^v$, we know that $w_{\rho(v)}$ is adjacent to the extremity of $P$ at distance $\delta_i$ from $u_i$ (and to no other vertex on $P$). Hence, the vertices on $P$ and those in $W$ induce a path of length $D + \rho(v) > D$ in $G$, a contradiction.

In summary, the following value is a more precise upper bound on the total weight of the vertices outside $P$:

$$B - \sum_{v \in V_3^D \cup V_3^{D-1}} 2D\rho(v) - \sum_{v \in V_3^{D-1}} (2D - 1)\rho(v) - 2Dn_3^D$$

$$\leq (n - D - 1)D(n - D) - n_3^{D-1}(2D - n - 1) - Dn_3^D - 2Dr^* + D \min\{1, \rho^*\}$$

$$- \sum_{v \in V_3^D \cup V_3^{D-1}} (2D - 1)\rho(v).$$

Let us now consider the vertices on $P$. We have $W(u_0) = D(1 + o_0)$, $W(u_D) = D(1 + o_D)$, and $W(u_i) = \epsilon(u_i)(2 + o_i)$ for $i = 1, \ldots, D - 1$. Since $\epsilon(u_i) = \delta_i + r_i$, the total weight of the vertices on $P$ is

$$2D + D(o_0 + o_D) + \sum_{i=1}^{D-1} (\delta_i + r_i)(2 + o_i)$$

$$= 2 \sum_{i=0}^{D-1} \delta_i + 2 \sum_{i=1}^{D-1} r_i + \sum_{i=1}^{D-1} \delta_i o_i + \sum_{i=0}^{D-1} \delta_i o_i.$$

Each edge that links a vertex $v$ outside $P$ to a vertex $u_i$ in $P$ contributes for $r_i \leq \rho(v)$ in the sum $\sum_{i=1}^{D-1} r_i o_i$. Hence,

$$\sum_{i=1}^{D-1} r_i o_i \leq \sum_{v \in V_3^D \cup V_3^{D-1}} 2\rho(v) + \sum_{v \in V_3^D \cup V_3^{D-1}} 3\rho(v) \leq \sum_{v \in V_3^D \cup V_3^{D-1}} 3\rho(v).$$

Since $2 \sum_{i=1}^{D-1} r_i \leq 2r^*(D - 1)$, we get the following valid upper bound on the total weight of the vertices on $P$:

$$2 \sum_{i=0}^{D-1} \delta_i + \sum_{i=1}^{D-1} \delta_i o_i + 2r^*(D - 1) + \sum_{v \in V_3^D \cup V_3^{D-1}} 3\rho(v).$$

Summing up the bounds for the vertices outside $P$ with those on $P$, we get the following upper bound for the total weight of the vertices in $G$:

$$(n - D - 1)D(n - D) - n_3^{D-1}(2D - n - 1) - Dn_3^D + 2 \sum_{i=0}^{D-1} \delta_i + \sum_{i=0}^{D-1} \delta_i o_i$$

$$- \sum_{v \in V_3^D \cup V_3^{D-1}} (2D - 4)\rho(v) - 2r^* + D \min\{1, \rho^*\}.$$
Let us decompose this bound into two parts $A_1 + A_2$ with $A_1$ being equal to the sum of the first terms of the above upper bound, and $A_2$ being equal to the sum of the last ones:

$$A_1 = (n - D - 1)D(n - D) + n_3^{D-1}(2D - n - 1) - Dn_3^D + 2 \sum_{i=0}^{D-1} \delta_i + \sum_{i=0}^{D} \delta_i o_i$$

$$A_2 = - \sum_{v \in V_1,2 \cup V_2^D \cup V_3^{D-1}} (2D - 4) \rho(v) - 2r^* + D \min\{1, \rho^*\}.$$ 

- If $r^* = 0$, then $A_2 = 0$, which implies $A_1 + A_2 = A_1$.
- If $\rho^* > 0$, then $A_2 \leq 4 - 2D - 2r^* + D = 4 - D - 2r^* < 0$, which implies $A_1 + A_2 < A_1$.
- If $r^* > 0$ and $\rho^* = 0$, then $A_2 = -2r^* < 0$, which implies $A_1 + A_2 < A_1$.

In summary, the best possible upper bound is $A_1$ and is attained only if $r^* = 0$, $n_0 = 0$, $\epsilon(v) = D$ for all vertices in $V_{1,2}$, and the vertices outside $P$ are pairwise adjacent. We now have to compare $A_1$ with $f(n, D)$.

Let us start with $D = 3$. In that case, we have $f(n, 3) = 14 + (n - 4)(3n - 4 + \max\{0, 7 - n\})$, while $A_1 = (n - 4)3(n - 3) + n_3^2(5 - n) - 3n_3^3 + 14 + \sum_{i=0}^{D} \delta_i o_i$. Hence, the difference is:

$$f(n, 3) - A_1 = (n - 4)(5 + \max\{0, 7 - n\}) - n_3^2(5 - n) + 3n_3^3 - \sum_{i=0}^{D} \delta_i o_i.$$ 

We have

$$\sum_{i=0}^{3} o_i \leq 3(n_3^2 + n_3^3) + 2(n - 4 - n_3^2 - n_3^3) = 2(n - 4) + n_3^2 + n_3^3.$$ 

Since $o_0 + o_3 \leq n - 4$ to avoid a path of length 2 joining $u_0$ to $u_3$, we have

$$\sum_{i=0}^{3} \delta_i o_i \leq 3(n - 4) + 2(n - 4 + n_3^2 + n_3^3).$$ 

Hence,

$$f(n, 3) - A_1 \geq (n - 4) \max\{0, 7 - n\} - n_3^2(7 - n) + n_3^3.$$ 

This difference is minimized if and only if $n_3^3 = 0$, while $n_3^2 = 0$ if $n > 7$, $n_3 = 0, 1, 2$ or 3 if $n = 7$, and $n_3 = n - 4$ if $n < 7$. In all such cases, we get $f(n, 3) - A_1 = 0$.

- If $n = 4$, there is no vertex outside $P$, and $G \simeq E_{1,3,0}$ which is the unique graph in $C_3^3$.
- If $n = 5$, $n_3^2 = 1$, which means that the unique vertex outside $P$ is adjacent to 3 consecutive vertices on $P$. Hence, $G \simeq E_{5,3,1}$ which is the unique graph in $C_3^3$.
- If $n = 6$, $n_3^2 = 2$, which means that both vertices outside $P$ are adjacent to 3 consecutive vertices on $P$. If one of them is adjacent to $u_0, u_1, u_2$, while the other is adjacent to $u_1, u_2, u_3$, we have $G \simeq H_2$. Otherwise, we have $G \simeq E_{6,3,2}$. 
- If $n = 7$, $n_3^2 \in \{0, 1, 2, 3\}$ and $n_1,2 = 3 - n_3^2$. If $n_1,2 > 0$ then the vertices in $V_{1,2}$ are all adjacent to $u_0$ and $u_1$ or all to $u_2$ and $u_3$, since they are pairwise adjacent, and they all have eccentricity 3. So assume without loss of generality, they are all adjacent to $u_0$ and $u_1$. Then the vertices in $V_3^2$ are all adjacent to $u_0, u_1, u_2$, else the vertices in $V_{1,2}$ would have eccentricity 2. But $G$ is then equal to $E_{7,3,0}, E_{7,3,1}$ or $E_{7,3,2}$. If $n_1,2 = 0$, then the three vertices outside $P$ are all adjacent to three consecutive vertices on $P$. If they are all adjacent to $u_0, u_1, u_2$, or all to $u_1, u_2, u_3$, then $G \simeq E_{7,3,3}$, else $G \simeq H_3$.
- If $n > 7$, all vertices outside $P$ are adjacent to $u_0, u_1$, or to $u_2, u_3$ (so that they all have eccentricity 3). Hence, $G \simeq E_{n,3,0}$.

Assume now $D \geq 4$. We have

$$f(n, D) = 2 \sum_{i=0}^{D-1} \delta_i + \left(n - D - 1\right)\left(2D - 1 + D(n - D) + \max\{0, 3D - n - 3\}\right)$$


Hence, the difference is:

\[
f(n, D) - A_1 = (n - D - 1)(2D - 1 + \max\{0, 3D - n - 3\}) - n_3^{D-1}(2D - n - 1) + Dn_3^D - \sum_{i=0}^{D} \delta_i o_i.
\]

We have

\[
\sum_{i=0}^{D} o_i \leq 3(n_3^{D-1} + n_3^D) + 2(n - D - 1 - n_3^{D-1} - n_3^D) = 2(n - D - 1) + n_3^{D-1} + n_3^D.
\]

Let \( p \) be the number of vertices linked to both \( u_1 \) and \( u_{D-1} \).

- If \( D \geq 5 \), then \( p = 0 \), else \( d(u_0, u_D) \leq 4 < D \).
- If \( D = 4 \), then no vertex outside \( P \) linked to \( u_1 \) and \( u_{D-1} \) can also be linked to \( u_0 \) or to \( u_D \) since \( d(u_0, u_D) \) would be strictly smaller than 4. Since no vertex outside \( P \) can be linked to both \( u_0 \) and \( u_D \) (else \( d(u_0, u_D) < 3 \)) we have \( o_0 + o_D \leq n - D - 1 - p \) and \( o_1 + o_{D-1} \leq n - D - 1 + p \). Hence, \( o_2 \leq n_3^{D-1} + n_3^D \). So,

\[
\sum_{i=0}^{D} \delta_i o_i \leq D(n - D - 1 - p) + (D - 1)(n - D - 1 + p) + (D - 2)(n_3^{D-1} + n_3^D)
\]

\[
= (n - D - 1)(2D - 1) + (D - 2)(n_3^{D-1} + n_3^D) - p.
\]

This value is maximized for \( p = 0 \).

Hence, in all cases, we have

\[
\sum_{i=0}^{D} \delta_i o_i \leq (n - D - 1)(2D - 1) + (D - 2)(n_3^{D-1} + n_3^D).
\]

Hence,

\[
f(n, D) - A_1 \geq (n - D - 1) \max\{0, 3D - n - 3\} - n_3^{D-1}(3D - n - 3) + 2n_3^D.
\]

This difference is minimized if and only if \( n_3^D = 0 \), while \( n_3^{D-1} = 0 \) if \( n > 3(D-1) \), \( n_3^{D-1} \in \{0, \ldots, n-D-1\} \) if \( n = 3(D-1) \), and \( n_3^{D-1} = n - D - 1 \) if \( n < 3(D-1) \). In all such cases, we get \( f(n, D) - A_1 = 0 \).

- If \( n < 3(D-1) \), then all vertices outside \( P \) are adjacent to 3 consecutive vertices on \( P \). They are all adjacent to \( u_0, u_1, u_2 \), or all adjacent to \( u_{D-2}, u_{D-1}, u_D \), else \( d(u_0, u_D) \leq 3 < D \). Hence, we have \( G \simeq E_{n,D,n-D-1} \).
- If \( n = 3(D-1) \), \( n_3^{D-1} \in \{0, \ldots, n-D-1\} \) and \( n_{1,2} = 2D - 2 - n_3^{D-1} \). If \( n_{1,2} > 0 \) then the vertices in \( V_{1,2} \) are all adjacent to \( u_0 \) or all to \( u_{D-1} \) and \( u_D \), since they are pairwise adjacent, and they all have eccentricity \( D \). So assume without loss of generality, they are all adjacent to \( u_0 \) and \( u_1 \). Then the vertices in \( V_3^{D-1} \) are all adjacent to \( u_0, u_1, u_2 \), else \( d(u_0, u_D) \leq 3 < D \). But \( G \) is then equal to \( E_{n,D,n_D} \). If \( n_{1,2} = 0 \), then all vertices outside \( P \) are adjacent to \( u_0, u_1, u_2 \), or all of them are adjacent to \( u_{D-2}, u_{D-1}, u_D \), else \( d(u_0, u_D) \leq 3 < D \). Hence, \( G \simeq E_{n,D,n-D-1} \).
- If \( n > 3(D-1) \), all vertices outside \( P \) are adjacent to \( u_0, u_1 \), or to \( u_2, u_3 \) (so that they all have eccentricity \( D \)). Hence, \( G \simeq E_{n,D,0} \).
3 Results for a fixed order and no fixed diameter

We now determine the connected graphs that maximize the eccentric connectivity index when the order $n$ of the graph is given, while there is no fixed diameter. Clearly, $K_3$ and $P_3$ are the only connected graphs of order $n = 3$ and $\xi^e(K_3) = \xi^e(P_3) = 6$. For $n > 3$, $\xi^e(M_n) = 2n^2 - 4n - 2(n \mod 2) > n^2 - n = \xi^e(K_n)$, which means that the optimal diameter is not $D = 1$.

- If $n = 4$, $f(4, 3) = 14 < \xi^e(M_4) = 16$, which means that $M_4$ has maximum eccentric connectivity among all connected graphs with 4 vertices.
- If $n = 5$, $f(5, 3) = 27$, $f(5, 4) = 24$ and $\xi^e(M_5) = 30$, which means that $M_5$ and $H_1$ have maximum eccentric connectivity index among all connected graphs with 5 vertices.
- If $n = 6$, $f(6, 3) = 44$, $f(6, 4) = 42$, $f(6, 5) = 38$ and $\xi^e(M_6) = 48$, which means that $M_6$ has maximum eccentric connectivity index among all connected graphs with 6 vertices.

Assume now $n \geq 7$. We first show that lollipops are not optimal. Indeed, consider a lollipop $E_{n,D,0}$ of order $n$ and diameter $D$.

- If $D = n - 1$, then $G \simeq P_n$ which implies
  \[
  \xi^e(E_{n,n-1,0}) = \sum_{i=1}^{D-1} 2 \max\{i, D - i\} + 2D = \frac{3D^2 + D \mod 2}{2} \\
  \leq \frac{3D^2 + 1}{2} = \frac{3n^2}{2} - 3n + 2 < 2n^2 - 4n - 2 \leq \xi^e(M_n).
  \]

- If $D < n - 1$ then either $n < 3(D - 1)$, and we know from Corollary 2 that $\xi^e(E_{n,D,n-D-1}) > \xi^e(E_{n,D,0})$, or $n \geq 3(D - 1)$, in which case we show that $\xi^e(E_{n,D+1,n-D-2}) > \xi^e(E_{n,D,0})$. Since $2\sum_{i=0}^{D-1} \max\{i, D - i\} = 3D^2 + D \mod 2$, we know from Lemma 2 that
  \[
  \xi^e(E_{n,D+1,n-D-2}) = 2\sum_{i=0}^{D} \max\{i, D + 1 - i\} \\
  + \left(n - D - 2\right) \left(2(D + 1) - 1 + (D + 1)(n - D - 1)\right) \\
  + \left(n - D - 2\right) \left(2(D + 1) - n - 1 + (D + 1) - 2\right) \\
  = \frac{3(D + 1)^2 + (D + 1) \mod 2}{2} + \left(n - D - 2\right) \left(3D + D(n - D)\right)
  \]

and
  \[
  \xi^e(E_{n,D,0}) = 2\sum_{i=0}^{D-1} \max\{i, D - i\} + \left(n - D - 1\right) \left(2D - 1 + D(n - D)\right) \\
  = \frac{3D^2 + D \mod 2}{2} + \left(n - D - 1\right) \left(2D - 1 + D(n - D)\right).
  \]

Simple calculations lead to
  \[
  \xi^e(E_{n,D+1,n-D-2}) - \xi^e(E_{n,D,0}) = n - 2D + (D - 1) \mod 2 \geq n - 2\left(\frac{n}{3} + 1\right) = \frac{n}{3} - 2 > 0.
  \]

Hence, the remaining candidates to maximize the eccentric connectivity index when $n \geq 7$ are $M_n$ and $E_{n,D,n-D-1}$. Let
  \[
  g(n) = \max_{D=[\frac{n}{3}+2]} \xi^e(E_{n,D,n-D-1}).
  \]

We can rewrite $\xi^e(E_{n,D,n-D-1})$ as follows:
  \[
  \xi^e(E_{n,D,n-D-1}) = D^3 - D^2(n + \frac{5}{2}) + D(n^2 + 5n - 1) - n^2 - 3n + 4 + D \mod 2.
  \]
It is then not difficult to show that \( g(n) = \xi^c(E_{n,D^*,n-D^*-1}) \) with \( D^* = \lceil \frac{n+1}{3} \rceil + 1 \), and simple calculations lead to

\[
g(n) = \frac{1}{54}(8n^3 + 21n^2 - 36n + 6n + 1) \quad \begin{cases} 
0 & \text{if } n \mod 6 = 0 \\
6n + 1 & \text{if } n \mod 6 = 1 \\
32 & \text{if } n \mod 6 = 2 \\
27 & \text{if } n \mod 6 = 3 \\
6n + 28 & \text{if } n \mod 6 = 4 \\
59 & \text{if } n \mod 6 = 5 
\end{cases}
\]

We then have \( g(7) = 66 < 68 = \xi^c(M_7) \), which means that \( M_7 \) has the largest eccentric connectivity among all graphs with 7 vertices. Also, \( g(8) = 96 = \xi^c(M_8) \), which means that both \( E_{8,4,3} \) and \( M_8 \) have the largest eccentric connectivity index among all graphs with 8 vertices. For graphs of order \( n \geq 9 \), we have \( \frac{8n^3 + 21n^2 - 36n}{54} > 2n^2 - 4n \), which means that \( E_{n,D^*,n-D^*-1} \) is the unique graph with largest eccentric connectivity index among all graphs with \( n \) vertices. These results are summarized in Table 1, where \( \xi_n^c \) stands for the largest eccentric connectivity index among all graphs with \( n \) vertices.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \xi_n^c )</th>
<th>optimal graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>( K_3 ) and ( P_3 )</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>( M_4 )</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>( M_5 ) and ( H_1 )</td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>( M_6 )</td>
</tr>
<tr>
<td>7</td>
<td>68</td>
<td>( M_7 )</td>
</tr>
<tr>
<td>8</td>
<td>96</td>
<td>( M_8 ) and ( E_{8,4,3} )</td>
</tr>
<tr>
<td>( \geq 9 )</td>
<td>( g(n) )</td>
<td>( E_{n,\lceil \frac{n+1}{3} \rceil + 1,n-\lceil \frac{n+1}{3} \rceil - 2} )</td>
</tr>
</tbody>
</table>

Note finally that Tavakoli et al. [4] state that \( g(n) = \xi^c(E_{n,D,n-D-1}) \) with \( D = \lceil \frac{n}{3} \rceil + 1 \) while we have shown that the best diameter for a given \( n \) is \( D = \lceil \frac{n+1}{3} \rceil + 1 \). Hence for all \( n \geq 9 \) with \( n \mod 3 = 0 \), we get a better result. For example, for \( n = 9 \), they consider \( E_{9,4,4} \) which has an eccentric connectivity index equal to 132 while \( g(9) = 134 \).

## 4 Conclusion

We have characterized the graphs with largest eccentric connectivity index among those of fixed order \( n \) and fixed or non-fixed diameter \( D \). It would also be interesting to get such a characterization for graphs with a given order \( n \) and a given size \( m \). We propose the following conjecture which is more precise than the one proposed in [5].

**Conjecture.** Let \( n \) and \( m \) be two integers such that \( n \geq 4 \) and \( m \leq \binom{n-1}{2} \). Also, let

\[
D = \left\lceil \frac{2n+1 - \sqrt{17 + 8(m-n)}}{2} \right\rceil \quad \text{and} \quad k = m - \left( \frac{n-D+1}{2} \right) - D + 1
\]

Then, the largest eccentric connectivity index among all graphs of order \( n \) and size \( m \) is attained with \( E_{n,D,k} \). Moreover,

- if \( D > 3 \) then \( \xi^c(G) < \xi^c(E_{n,D,k}) \) for all other graphs \( G \) of order \( n \) and size \( m \).
- if \( D = 3 \) and \( k = n-4 \), then the only other graphs \( G \) with \( \xi^c(G) = \xi^c(E_{n,D,k}) \) are those obtained by considering a path \( u_0 - u_1 - u_2 - u_3 \), and by joining \( 1 \leq i \leq n-3 \) vertices of a clique \( K_{n-4} \) to \( u_0, u_1, u_2 \) and the \( n-4-i \) other vertices of \( K_{n-4} \) to \( u_1, u_2, u_3 \).
References


