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A continuous formulation for facility layout on rows with rectilinear distance

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Abstract: The facility layout problem is concerned with finding an arrangement of non-overlapping indivisible departments within a facility so as to minimize the total expected flow cost. For typical applications of layout, this flow cost is a measure of the quantity that one wishes to optimize, and it is proportional to the rectilinear distance between each pair of departments. In this paper we consider the special case of multi-row layout in which all the departments are to be placed in two or more rows, as occurs for example in the context of flexible manufacturing and in the design of application-specific integrated circuits. We propose a new mixed integer linear optimization formulation that is continuous in both dimensions x and y, where x represents the position within rows and y returns the row assigned to each department. We prove the interesting property that under mild assumptions, the optimal solutions achieve integer values of y, even though y is a continuous variable. Our computational results show that the proposed formulation improves on earlier linear and semidefinite formulations for instances of multi-row layout formulated using the pairwise rectilinear distance.

Keywords: facility layout, row layout, global optimal solution, mixed integer linear optimization, continuous optimization

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1 Introduction

The facility layout problem (FLP) is concerned with finding an arrangement of non-overlapping indivisible departments within a facility so as to minimize the total expected cost of flows. This cost between two departments is measured as the rectilinear distance between their centroids multiplied by the projected flow between them. For typical applications of the FLP, the cost is a measure of the quantity that one wishes to optimize, for example a transportation cost for the amount of material flowing between two departments in a manufacturing line. We refer the reader to Anjos and Vieira (2017) for a recent review of the state-of-the-art in FLP.

In this paper we consider the special case of the FLP in which all the departments are to be placed in two or more rows. Such row FLPs arise in various practical contexts. One such context is in manufacturing where the machines (equivalent to departments) are to be placed in rows with a predetermined separation between the rows to accommodate movement of people and/or materials. Typically a minimum clearance between departments within each row is needed to satisfy safety and operational requirements, but generally this clearance can be included in the lengths of the departments.

Another application of row FLPs is in the design of application-specific integrated circuits for which the layout of the components is organized in rows (called base layers), the objective is to minimize the total wirelength require d to connect the components, and the separation between rows is used for the wires connecting the components.

Our contribution is a new MILO formulation that is continuous in both dimensions x and y, where x represents the position within rows and y returns the row assigned to each department. Even though y is a continuous variable, this formulation has the interesting property that optimal solutions are attained in which the values of y are integer. We report a computational comparison of this model with the ones in Sections 2.1 and 2.2.

This paper is structured as follows. In Section 2 we review the problem and the relevant literature, with a focus on previous mathematical optimization approaches. Our new MILO formulation is presented in Section 3, and its theoretical integrality properties are proved in Section 4. The computational performance of the new formulation is explored in Section 5, together with comparisons with relevant alternatives in the literature. Finally, Section 6 concludes the paper.

2 Literature review and formulations

The FLP on rows can be stated in the following general form: Given a number of rows, a set of departments represented by rectangles, each of a given length, and a non-negative weight for each pair of departments, determine an assignment of departments to rows, and the positions of the departments in each row, so that the sum of weighted center-to-center distances is minimized.

We assume without loss of generality that the rows and the departments all have the same height, that any department can be assigned to any row, and that the distances between adjacent rows are equal. Under these assumptions, solving an instance of the row FLP means resolving three questions (Anjos and Vieira , 2017):

1. Assign each department to exactly one row;
2. Express mathematically the center-to-center distance between departments (that may or may not be in the same row);
3. Handle possible empty space between departments.

The row FLP most studied in the literature is the Single-Row FLP (SRFLP). In this case, there is no need to assign each department to a row, and the non-negativity of the pairwise weights eliminates empty space between departments at optimality. Formulating the SRFLP thus focuses on expressing the center-to-center distance between departments. Because we are concerned with row FLPs with at least two rows, we do not discuss the SRFLP any further, and refer the reader to the recent survey papers of Kothari and Ghosh (2012) and Keller and Buscher (2015) for the state-of-the-art on SRFLP, including extensions, meta-heuristics, and exact approaches.

The Double-Row FLP (DRFLP) allows departments to be placed on both sides of a central corridor. It is assumed that all flows between departments employ this corridor, and hence the distance between the two rows is neglected, so that the center-to-center distance between two departments is measured parallel to the corridor. Unlike for the single-row case, it is necessary in the DRFLP to address all three questions for row FLPs. In particular, the optimal layout may involve empty spaces between adjacent departments within a row. Concerning the row assignments, because there are only two rows, it is sufficient to determine which departments are placed in one of the rows, as the remaining departments must be in the other row. This latter property is explicitly exploited within the model presented in Amaral (2013).

To the best of our knowledge, the earliest formulation of the DRFLP is a nonlinear optimization model proposed in Heragu and Kusiak (1988) and used to find locally optimal solutions. Most of the subsequent mathematical optimization approaches in the literature use either mixed-integer linear optimization (MILO) (Chung and Tanchoco, 2010; Amaral , 2013) or semidefinite optimization (SDO) (Hungerländer and Anjos, 2015). While the SDO approach only requires binary variables, the MILO approaches use a combination of binary and continuous variables, where the former represent the assignment of departments to rows and the relative position of two departments, and the latter give the absolute positions of the department centers with respect to a fixed origin. Because the formulation in Amaral (2013) cannot be easily generalized to more than two rows, in this paper we use only the formulation presented in Chung and Tanchoco (2010). This formulation is stated in Section 2.1 below).

The row FLP with more than two rows has been referred to as multi-row facility layout problem (MRFLP). The assumptions typically made are that we are given a certain number of rows to place the departments, that the departmentss all have the same height, equal to the height of the rows, that the distances between adjacent rows are equal, and that every department can be assigned to every row.

Instances of the MRFLP arise in situations where gantry robots are used, for example in flexible manufacturing systems and in pick and place applications. Gantry robots, such as illustrated in Figure 1, have linear axes of control and move up/down and left/right with the movement directions at right angles, so the total weighted sum of the center-to-center rectilinear distances is a good measure of the total displacement of such a robot to complete a given task.

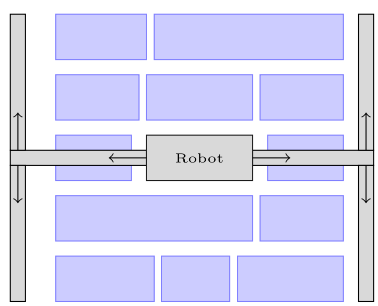

Figure 1: Operational setup of a gantry robot (Hungerländer and Anjos, 2015)

The MRFLP has received very limited attention in the operations research literature to date. Heragu and Kusiak (1988) formulated the MRFLP as a two-dimensional continuous space allocation problem, using a non-linear optimization approach. However, in many practical problems, the departments are arranged in well-defined rows because the separation between the rows is predetermined according to the features of the material-handling system; that is, this problem can be viewed as discrete in one dimension and continuous in the other. Heuristic algorithms were proposed in Heragu and Kusiak (1988), and a nonlinear formulation was given in Gen and Cheng (1997) and solved using a genetic algorithm (GA). Most recently, an SDO-based approach was introduced in Hungerländer and Anjos (2015), and to the best of our knowledge, this is the only global optimization approach for the general row FLP with more than two rows. The corresponding SDO formulation is stated in Section 2.2 below.

2.1 A MILO formulation for FLPs with arbitrary number of rows

The formulation proposed by Chung and Tanchoco (2010) (see also Zhang and Murray (2012)) uses two sets of binary variables:

$$y_{ik} = \begin{cases} 1, & \text{if department } i \text{ is assigned to row } k \\ 0, & \text{otherwise.} \end{cases}$$

$$z_{kij} = \begin{cases} 1, & \text{if department } j \text{ is placed to the right of department } i \text{ in row } k \\ 0, & \text{otherwise.} \end{cases}$$

In addition, the set of continuous variables x_{ik} represents the absolute location of department i in row k, and $x_{ik} = 0$ if department i is not assigned to row k.

Using the above variables, assuming no clearance requirements between departments, and allowing up to m rows to place the departments, the formulation is as follows:

$$\min \sum_{1 \le i < j \le n} c_{ij} \left(v_{ij}^{+} + v_{ij}^{-} \right)$$

$$\text{s.t.} \quad \sum_{k=1}^{m} x_{ik} - \sum_{k=1}^{m} x_{jk} + v_{ij}^{+} - v_{ij}^{-} = 0,\ 1 \le i < j \le n, \tag{1}$$

$$x_{ik} \le L y_{ik},\ i = 1, \ldots, n,\ k = 1, \ldots, m, \tag{2}$$

$$\sum_{k=1}^{m} y_{ik} = 1,\ i = 1, \ldots, n \tag{3}$$

$$\frac{\ell_i y_{ik} + \ell_j y_{ik}}{2} \le x_{ik} - x_{jk} + L(1 - z_{kji}),\ 1 \le i < j \le n,\ k = 1, \ldots, m, \tag{4}$$

$$\frac{\ell_i y_{ik} + \ell_j y_{ik}}{2} \le x_{jk} - x_{ik} + L(1 - z_{kij}),\ 1 \le i < j \le n,\ k = 1, \ldots, m, \tag{5}$$

$$z_{kij} + z_{kji} \le \frac{1}{2}(y_{ik} + y_{jk}),\ 1 \le i < j \le n,\ k = 1, \ldots, m, \tag{6}$$

$$z_{kij} + z_{kji} + 1 \ge y_{ik} + y_{jk},\ 1 \le i < j \le n,\ k = 1, \ldots, m, \tag{7}$$

$$\left.\begin{array}{l} x_{ik} \ge 0,\ i = 1, \ldots, n,\ k = 1, \ldots, m, \\ v_{ij}^{+}, v_{ij}^{-} \ge 0, 1 \le i < j \le n, \\ y_{ik} \in \{0,1\}, i = 1, \ldots, n,\ k = 1, \ldots, m, \\ z_{kij} \in \{0,1\}, 1 \le i,j \le n,\ i \ne j,\ k = 1, \ldots, m, \end{array}\right. \tag{8}$$

where n is the number of departments, m is the maximum number of rows allowed for the layout, ℓ_i is the length of department i, and $L = \sum_{i=1}^{n} \ell_i$.

Constraints (1) compute the distances between pairs of departments by ensuring that $v_{ij}^{+} + v_{ij}^{-} = |\sum_{k=1}^{m} x_{ik} - \sum_{k=1}^{m} x_{jk}|$, and using the fact that the optimization will force (at least) one of v_{ij}^{+}, v_{ij}^{-} to zero.

Constraints (2) set $x_{ik} = 0$ when department i is not assigned to row k. Constraints (3) ensure that a department is assigned to only one row. Constraints (4) and (5) prevent departments from overlapping if they are assigned to the same row.

Constraints (6) and (7) ensure consistency between the variables y and z as follows: If $y_{ik} = 1$ and $y_{jk} = 1$ then (6) and (7) together ensure that exactly one of z_{kij} and z_{kji} is equal to one. Otherwise, i.e., if at least one of y_{ik} and y_{jk} is equal to zero, then (6) sets both z_{kij} and z_{kji} to zero.

Constraints (7) force either z_{kij} or z_{kji} to be 1 if i and j are both in row k.

Finally, constraints (8) explicitly define the nature of the variables.

2.2 A SDO formulation for FLPs with arbitrary number of rows

The semidefinite approach proposed in Hungerländer and Anjos (2015) is based on the modeling of betweenness using products of binary variables, see e.g. (Anjos and Vieira , 2017, Section 2.1). Specifically, for any given permutation π of $1, 2, \ldots, n$, we can define binary ± 1 variables r_{ij} as

$$r_{ij} := \begin{cases} 1, & \text{if } i \text{ is to the right of } j, \\ -1, & \text{if } i \text{ is to the left of } j. \end{cases}$$

Note that $r_{ij} = -r_{ji}$.

On the other hand, given a particular assignment of ± 1 values to the r_{ij} variables, this assignment represents a permutation of $[n]$ if and only if the transitivity condition

if i is to the right of j and j is to the right of k, then i is to the right of k

is fulfilled. Equivalently, if $r_{ij} = r_{jk}$ then $r_{ik} = r_{ij}$. This necessary condition can be formulated as a set of quadratic constraints:

$$r_{ij} r_{jk} - r_{ij} r_{ik} - r_{ik} r_{jk} = -1 \text{ for all triples } 1 \leq i < j < k \leq n. \tag{9}$$

We also need a set of continuous variables to state the SDO formulation. Let z_{ij} be a continuous variables denoting the center-to-center distance between departments i and j. Using the variables r_{ij}, this distance can be expressed quadratically whether two departments are in the same row:

$$z_{ij} = \frac{1}{2}(\ell_i + \ell_j) + \sum_{\substack{k \in [n],\ k<i, \\ r(k)=r(i)}} \ell_k \frac{1 - r_{ki} r_{kj}}{2} + \sum_{\substack{k \in [n],\ i<k<j, \\ r(k)=r(i)}} \ell_k \frac{1 + r_{ik} r_{kj}}{2} + \sum_{\substack{k \in [n],\ k>j, \\ r(k)=r(i)}} \ell_k \frac{1 - r_{ik} r_{jk}}{2},\ r(i) = r(j), \tag{10a}$$

or in different rows:

$$\begin{aligned} z_{ij} = r_{ij} \Bigg[& \left(\frac{\ell_j}{2} + \sum_{\substack{k \in [n],\ k<j, \\ r(k)=r(j)}} \ell_k \frac{1 + r_{kj}}{2} + \sum_{\substack{k \in [n],\ k>j, \\ r(k)=r(j)}} \ell_k \frac{1 - r_{jk}}{2} \right) - \\ & \left(\frac{\ell_i}{2} + \sum_{\substack{k \in [n],\ k<i, \\ r(k)=r(i)}} \ell_k \frac{1 + r_{ki}}{2} + \sum_{\substack{k \in [n],\ k>i, \\ r(k)=r(i)}} \ell_k \frac{1 - r_{ik}}{2} \right) \Bigg],\ r(i) \neq r(j), \end{aligned} \tag{10b}$$

with the additional requirement that the distances between pairs of departments in non-adjacent rows have to be non-negative:

$$z_{ij} \geq 0, \qquad i, j \in [n],\ i < j,\ r(i) \neq r(j). \tag{11}$$

The above definitions lead to the following SDO relaxation:

$$\begin{aligned} \min \quad \frac{1}{2} \Bigg[& \sum_{\substack{i<j \in [n], \\ r(i)=r(j)}} c_{ij} \left(\sum_{\substack{k \in [n],\ i<k<j, \\ r(k)=r(i)}} \ell_k r_{ik} r_{kj} - \sum_{\substack{k \in [n],\ k<i, \\ r(k)=r(i)}} \ell_k r_{ki} r_{kj} - \sum_{\substack{k \in [n],\ k>j, \\ r(k)=r(i)}} \ell_k r_{ki} r_{kj} \right) \\ & + \sum_{\substack{i<j \in [n], \\ r(i) \neq r(j)}} c_{ij} r_{ij} \left(L_{r(i)} - L_{r(j)} + \sum_{\substack{k \in [n],\ k<i, \\ r(k)=r(i)}} \ell_k r_{ki} - \sum_{\substack{k \in [n],\ k>i, \\ r(k)=r(i)}} \ell_k r_{ik} - \sum_{\substack{k \in [n],\ k<j, \\ r(k)=r(j)}} \ell_k r_{kj} + \sum_{\substack{k \in [n],\ k>j, \\ r(k)=r(j)}} \ell_k r_{jk} \right) \\ & + \sum_{h=1,\ldots,m} \left[\left(\sum_{\substack{i,j \in [n],\ i<j, \\ r(i)=r(j)=h}} c_{ij} \right) \left(\sum_{\substack{i<j \in [n], \\ r(i)=r(j)=h}} \ell_i \right) \right] \Bigg] \end{aligned}$$

$$\begin{aligned}
\text{s.t.}\quad & r_{ij}r_{jk} - r_{ij}r_{ik} - r_{ik}r_{jk} = -1 \text{ for all triples } 1 \le i < j < k \le n && (12)\\
& z_{ij} + z_{ik} \ge z_{jk}, \qquad z_{ij} + z_{ik} \ge z_{jk}, \qquad z_{ik} + z_{jk} \ge z_{ij}, \qquad i < j < k \in [n] && (13)\\
& Z_{ij} + Z_{ik} + Z_{jk} \ge -1, 1 \le i < j < k \le n && (14)\\
& Z_{ij} - Z_{ik} - Z_{jk} \ge -1, 1 \le i < j \le n, k \ne i, j && (15)\\
& \operatorname{diag}(Z) = e && (16)\\
& Z \succcurlyeq 0 && (17)\\
& r_{ij} \in \{-1, 1\}, 1 \le i < j \le n, \\
& z_{ij} \ge 0, \qquad i, j \in [n],\ i < j,\ r(i) \ne r(j) && (18)
\end{aligned}$$

where the matrix variable Z has the form

$$Z := \begin{pmatrix} 1 & r^T \\ r & rr^T \end{pmatrix}, \tag{19}$$

with the vector r being a column vector of the variables r_{ij}, and L_i denotes the sum of the length of the departments on row i:

$$L_i = \sum_{\substack{k \in [n], \\ r(k) = i}} \ell_k, \quad i = 1, \ldots, m.$$

Constraints (14–17) are well known to hold for matrices Z with the structure we defined, and (13) are the triangle inequalities relating the pairwise distances between three departments. The other constraints were already discussed earlier.

3 A new MILO formulation

In this section we present our proposed new MILO formulation for row FLPs with two or more rows. Like the models reviewed in Section 2 and most other mathematical optimization models in the literature, our proposed model uses binaries variables to prevent overlap. Unlike most other models however, it uses continuous variables for the assignment of departments to rows, and we prove that these variables have integer values at optimality, so that departments are assigned to rows without the need for rounding or other similar operation.

For each department i we use the variable x_i to represent the horizontal position of department i (within the row assigned to it), and y_i to represent the vertical position of i (the row it is assigned to). For each pair of departments i and j, we use the following binary variables to encode their relative position:

$$\alpha_{ij} = \begin{cases} 1 & \text{if } i \text{ is placed to the left of } j \text{ in the same row,} \\ 0 & \text{otherwise,} \end{cases}$$
$$\beta_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are placed in different rows and } i \text{ is below } j, \\ 0 & \text{otherwise.} \end{cases}$$

We use the rectangular distance in our objective function, and let d^x_{ij} and d^y_{ij} equal the horizontal and vertical distances between i and j.

Using the above variable definitions, the proposed formulation for the double- and multi-row FLP is:

$$\min \sum_{1 \le i < j \le n} c_{ij}(d^x_{ij} + d^y_{ij}) \tag{20}$$

s.t.

$$d^x_{ij} \ge x_i - x_j,\ d^x_{ij} \ge x_j - x_i,\ 1 \le i < j \le n \tag{21}$$
$$d^y_{ij} \ge y_i - y_j,\ d^y_{ij} \ge y_j - y_i,\ 1 \le i < j \le n \tag{22}$$
$$x_j - x_i \ge \frac{1}{2}(\ell_i + \ell_j) - L(1 - \alpha_{ij}),\ 1 \le i < j \le n, \tag{23}$$

$$x_i - x_j \geq \frac{1}{2}(\ell_i + \ell_j) - L(1 - \alpha_{ji}),\ 1 \leq i < j \leq n, \tag{24}$$
$$y_j - y_i \geq d - md(1 - \beta_{ij}),\ 1 \leq i < j \leq n, \tag{25}$$
$$y_i - y_j \geq d - md(1 - \beta_{ji}),\ 1 \leq i < j \leq n, \tag{26}$$
$$\alpha_{ij} + \alpha_{ji} + \beta_{ij} + \beta_{ji} = 1,\ 1 \leq i < j \leq n, \tag{27}$$
$$\alpha_{ij} + \alpha_{jk} \leq 1 + \alpha_{ik},\ \beta_{ij} + \beta_{jk} \leq 1 + \beta_{ik},\ 1 \leq i < j \leq n \tag{28}$$
$$0 \leq y_i \leq d(m-1),\ 1 \leq i \leq n, \tag{29}$$
$$|y_i - y_j| \leq (1 - \alpha_{ij} - \alpha_{ji})(m-1)d,\ 1 \leq i < j \leq n, \tag{30}$$
$$y_1 \leq \left\lfloor \frac{m-1}{2} \right\rfloor, \tag{31}$$
$$x_p \leq x_k,\ (p,k) = \arg\min c_{ij}, \tag{32}$$
$$\frac{1}{2}\ell_i \leq x_i \leq L - \frac{1}{2}\ell_i,\ 1 \leq i \leq n. \tag{33}$$

where d is the row width, and as previously, n is the number of departments, m is the maximum number of rows allowed for the layout, ℓ_i is the length of department i, and $L = \sum_{i=1}^{n} \ell_i$.

Constraints (21)–(22) establish the horizontal and vertical distances between departments.

Constraints (23)–(24) prevent that any two departments inside the same row will overlap.

Constraints (25)–(26) avoid the overlapping of rows, and simultaneously create the rows.

Constraints (27) require the separation of i and j in only one dimension (though they may be separated in both dimensions).

Constraints (28) are triangles inequalities, a linear version of the constraints (12).

Constraints (29) restrict every feasible solution to have no more than m rows (each of width d).

Constraints (30) ensures that $y_i = y_j$ when departments i and j are placed in the same row.

Constraints (32) and (31) are symmetry-breaking constraints. Constraint (32) chooses two departments p and k based on the smallest pairwise cost and requires p to be placed to the left of k; a similar constraint was used in Amaral (2013). Constraint (31) assigns department 1 to the lower part of the layout. These constraints eliminate redundant layouts and may help reduce the computational time to solve the MILO problem. We refer the reader to Section 5 of Anjos and Vieira (2017) for a more general discussion of symmetry-breaking for layout problems.

4 Integrality properties of the model

As mentioned above, the formulation presented in Section 3 uses continuous variables to represent the assignment of departments to rows. In this Section we give the proof that these variables achieve integer values at optimality.

The first result considers the case of minimizing only the distance in the horizontal direction, as was done in Amaral (2013); Chung and Tanchoco (2010); Hungerländer and Anjos (2015); Zhang and Murray (2012). We show that the optimal solution in this case has all the components of y integer.

Theorem 1 *If d is integer, and $c_{ji} \geq 0$ for all i, j, then an optimal solution of the problem*

$$\min \sum_{1 \leq i < j \leq n} c_{ij} d^x_{ij} \tag{34}$$
$$\text{s.t.} \quad (21) - (33)$$

has all the components of y integer.

Proof. Because $c_{ji} \geq 0$ and the minimization is over the horizontal distance, an optimal solution will have as many rows as possible. Constraints (25)–(26) imply that every two values within y corresponding to departments on different rows must differ by at least d. Because of the constraints (29), this means that an optimal solution will have at most m rows. Having m different values between 0 and $d(m-1)$ that pairwise differ by at least d implies that each y_i must be one of the values $0, d, 2d, ..., (m-1)d$. The result follows by the integrality of d. □

If we minimize over the rectilinear distance, then we need an additional condition to guarantee that the components of y are integer.

Theorem 2 *If d is integer, $c_{ji} \geq 0$ for all i, j, and $\frac{\ell_i+\ell_j}{2} \geq d(m-1)$ for every pair of departments i, j, then an optimal solution (x, y) of (20)–(33) has all the components of y integer.*

Proof. Assumption $\frac{\ell_i+\ell_j}{2} \geq d(m-1)$ implies that whenever $c_{ji} > 0$, the cost of placing departments i and j side-by-side in the same row is greater than in any two different rows.

Assuming that the structure of the $c_{ij} > 0$ is such that they force an optimal solution with at least m rows, for example all $c_{ij} > 0$, then by the same arguments as in the proof of Theorem 1, having m different values between 0 and $d(m-1)$ that pairwise differ by at least d implies that each y_i must be one of the values $0, d, 2d, ..., (m-1)d$, and the result follows by the integrality of d. □

The assumption $\frac{\ell_i+\ell_j}{2} \geq d(m-1)$ is essential to obtain an optimal solution which use exactly m rows, as is shown by Example 2.

Example 1 *We consider $n = 4$ departments of length $\ell_i = 1.6$ for $i = 1, 2, 3, 4$ that are to be placed within $m = 3$ rows with width $d = 1$, and $c_{ij} = 1$ for all pairs i, j, so that $(\ell_i + \ell_j)/2 = 1.6$ and $d(m-1) = 2$, i.e., the assumption does not hold. The optimal value is 10.4, and the corresponding optimal solution uses only two rows, as shown in Figure 2.*

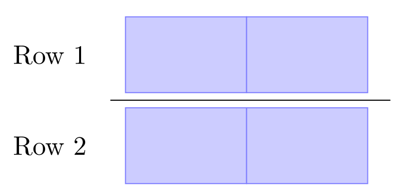

Figure 2: Optimal layout for Example 1

Recent works, such as Amaral (2013), Chung and Tanchoco (2010), Zhang and Murray (2012) and Hungerländer and Anjos (2015), consider to minimize only over the horizontal distance d^x_{ij}. This is a reasonable assumption for double-row layout because it is reasonable to suppose that all the flows take place via the corridor between the two rows. However this assumption may be less reasonable for general multi-row layout. Indeed earlier works on the MRFLP, for example Heragu and Kusiak (1991) and Gen and Cheng (1997), minimized the total (rectilinear) distance $d^x_{ij} + d^y_{ij}$. The choice of distance metric matters because the optimal solutions may differ for different metrics. We prove this by solving our formulation with the data from instance S_10 (see Section 5), $m = 4$ and $d = 1$. first minimizing the rectilinear distance, and second minimizing the horizontal distance, and observing that we obtain two different solutions. The two solutions are shown in Figures 3 and 4 respectively.

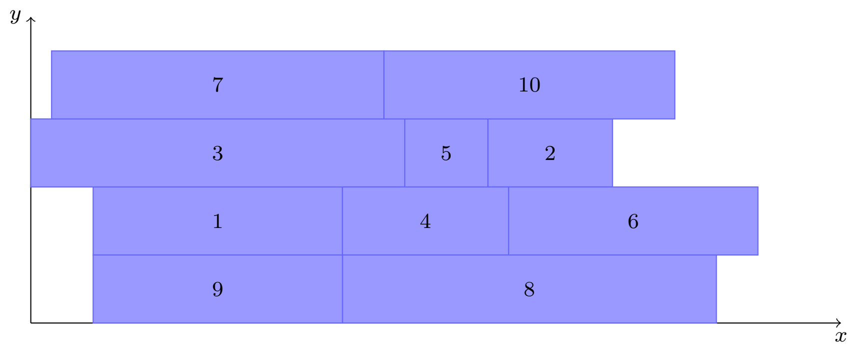

Figure 3: Optimal layout when minimizing the total distance for instance S_10

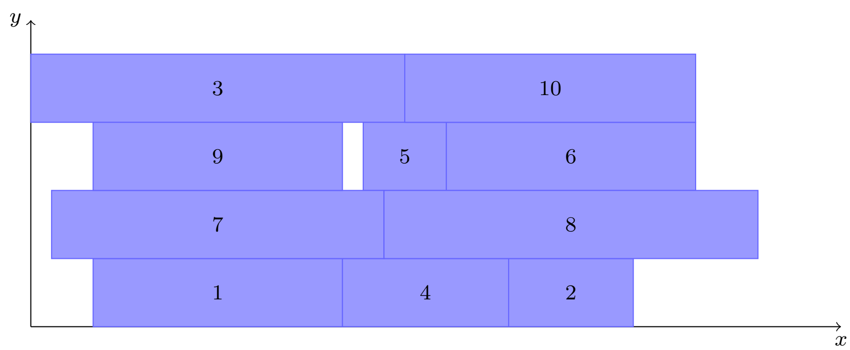

Figure 4: Optimal layout when minimizing the horizontal distance for instance S_10

5 Computational results

We implemented the formulations in Sections 2.1 and 3 using the modeling language AMPL and solved them using CPLEX (version 12.5.1.0). The computations were performed on a dual core Intel(R) Xeon(R) X5675 @ 3.07 GHz with 8 Gb of memory. All the test instances were taken from the literature, and their description and source are given in Table 1.

Table 1: Instances used for computational tests

Instance	# departments	Source
HeKu8	8	Heragu and Kusiak (1991)
HeKu12	12	Heragu and Kusiak (1991)
S_8	8	Simmons (1969)
SH_8	8	Simmons (1969)
S_9	9	Simmons (1969)
SH_9	9	Simmons (1969)
S_10	10	Simmons (1969)
S_11	11	Simmons (1969)
HuAn13	13	Hungerländer and Anjos (2012)
HuAn14	14	Hungerländer and Anjos (2012)

5.1 Computational results for minimizing horizontal distance only

In this section we provide a comparison between the model in Section 2.1 and our proposed model in Section 3.

The results in Table 2 give the results for both models, with "CT Model" denoting the model in Section 2.1, and "Proposed Model" denoting the model in Section 3. For each row, the lowest computational time is emphasized in bold. We see that our proposed model performs better for 2 rows, while the CT model has a better performance for 3 and 4 rows.

Table 2: Computational results for minimizing horizontal distance

Instance	Number of rows	Optimal value	CT Model B&B nodes	CT Model Time (s)	Proposed Model B&B nodes	Proposed Model Time (s)
HeKu8	2	2,265	8,888	3.9	2,684	**1.3**
	3	1,405	3,273	**1.1**	4,757	5.2
S_8	2	396	42,134	20.3	16,249	**14.6**
	3	241.5	20,642	**10.0**	40,368	55.2
SH_8	2	1,123	154,289	57.7	46,277	**27.4**
	3	739.5	150,175	**42.6**	145,970	109.8
S_9	2	1179	67,343	36.6	22,427	**34.6**
	3	757	109,882	**40.7**	125,930	194.4
SH_9	2	2293	1,334,963	598.7	362,224	**275.4**
	3	1413.5	933,731	**385.6**	823,176	1367.4
S_10	2	1351	216,398	187.8	29,901	**85.1**
	3	868	270,961	**130.4**	303,766	621.6
	4	578.5	712,609	**343.5**	451,158	743.2
S_11	2	3,424.5	1,289,301	788.1	322,918	**674.1**
	3	2263.5	3,004,054	**3,391.3**	5,171,917	23,368.7
	4	1689.5	4,898,443	**4,449.1**	14,311,934	51,258.8
HeKu12	2	8875	3,267,585	2,703.0	403,078	**1,278.8**
	3	5675	1,324,011	**1,825.1**	2,082,845	12,875.9
	4	4025	2,199,143	**2857.2**	11,223,022	78,870.3
HeKu13	2	1520.5	9,484,896	**8,210.03**	3,203,684	15,936.9
	3	983.5	18,504,576	**23,017.4**	No solution within 24 hours	
	4	711.5	22,783,314	**40,155.2**	Out of memory	
HeKu14	2	Out of memory				

5.2 Computational results for minimizing the total distance

In this section we compare the two MILO models using the total distance.

To make this comparison, we modify the model in Section 2.1 by:

- adding two variables $u_{ij}^+, u_{ij}^- \geq 0$ for each pair i, j of departments,
- changing the objective function to

$$\sum_{1 \leq i < j \leq n} c_{ij} \left(v_{ij}^+ + v_{ij}^- + u_{ij}^+ + u_{ij}^- \right)$$

- adding constraints to compute the distance between departments in the y direction:

$$d \sum_{k=1}^{m} k(y_{ik} - y_{jk}) + u_{ij}^+ - u_{ij}^- = 0, \ 1 \leq i < j \leq n.$$

We refer to this modified model as "Modified CT Model".

The results in Table 3 show that our proposed formulation dominates the modified CT formulation. In fact, for most of the instances, the new formulation computes an optimal solution more quickly, and it is able to solve the instance of size 13 which is beyond the ability of the modified CT model (with a timeout of 24 hours).

5.3 Comparison of the proposed model with the SDO approach

In this section we provide a comparison between our proposed model and the SDO approach to MRFLP in Hungerländer and Anjos (2015). This comparison is given in Table 4, and is done for the horizontal distance only, using the computational times reported in that paper for the SDO approach.

Note that the model we propose here is aimed at computing exact optimal solutions, while the SDO approach is aimed at computing bounds. This means that they focus on instances of different sizes, and hence the comparison is mainly for the purposes of comparing the bounds obtained using SDO to the optimal solutions obtained by the proposed MILO model.

Table 3: Computational results for minimizing the total distance

Instance	Number of rows	Optimal value	Modified CT Model B&B nodes	Modified CT Model Time (s)	Proposed Model B&B nodes	Proposed Model Time (s)
HeKu8	2	2,302	8,298	4.0	1,666	**1.4**
	3	1,473	14,689	**7.6**	5,855	9.3
S_8	2	438	34,177	21.1	10,268	**13.9**
	3	316.5	76,294	47.9	18,630	**24.1**
SH_8	2	1,220	157,916	85.0	46,637	**33.6**
	3	902.5	489,992	340.4	119,058	**121.4**
S_9	2	1,277.5	41,390	39.8	18,158	**32.9**
	3	907	253,889	227.3	120,145	**216.9**
SH_9	2	2,420	1,263,392	851.8	374,990	**311.7**
	3	1,636.5	2,815,630	2,191.1	689,665	**1203.4**
S_10	2	1,474	138,275	**141.0**	72,995	148.1
	3	1,049.5	1,275,013	1,025.1	197,480	**612.6**
	4	827.5	2,064,741	2,530.3	386,756	**1,297.7**
S_11	2	3,649.5	1,045,541	1,062.5	369,928	**917.5**
	3	2633.5	10,191,020	11,064.6	2,044,149	**10,104.6**
	4	2172.5	17,298,304	21,440.0	3,753,235	**16,223.2**
HeKu12	2	8,959	1,683,795	2,174.7	399,653	**1,502.6**
	3	5,849	14,469,132	18,733.5	2,289,118	**11,364.9**
	4	4,238	12,908,208	**20,326.9**	5,296,259	37,825.5
HuAn13	2	1639.5	7,536,912	9,756.0	2,685,457	**7,479.2**
	3	1184	no solution within 24 hours		4,608,070	**29,017.5**
	4	966	no solution within 24 hours		10,883,158	**69,481.6**
HuAn14	2		run out of memory			

Table 4: Comparison of this paper model with SDP bounds

Instance	number of rows	SDP bounds lower	SDP bounds upper	SDP bounds Time (s)	This paper optimal solution	This paper Time (s)
HeKu8	2	2,265	2,265	423.7	2,265	**1.3**
	3	1,350	1,430	3,283.1	1,405	**5.2**
S_8	2	380.5	396	409.1	396	**14.6**
	3	239	250	3,251.5	241.5	**55.2**
SH_8	2	990.5	1,125.5	406.6	1,123	**27.5**
	3	647	739.5	2,564.2	739.5	**109.8**
S_9	2	1,162	1,179	1,146.1	1,179	**34.6**
	3	757.5	770	17,025.5	757	**194.6**
S_10	2	1,314	1,353.5	3,663.3	1351	**85.1**
	3	N/A	N/A	N/A	868	621.6
	4	N/A	N/A	N/A	578.5	743.2
S_11	2	3,325.5	3,424.5	11,553.8	3,424.5	**777.4**
	3	N/A	N/A	N/A	2,263.5	25,236.3
HeKu12	2	8,385	8,875	39,927.9	8,875	**1,293.6**
	3	N/A	N/A	N/A	5,675	15,682.4

We note that for instances with up to 12 departments, the proposed model can obtain optimal solutions in a much shorter time than it takes the SDO relaxation to obtain upper and lower bounds.

6 Conclusion

In this paper we consider the special case of multi-row layout in which all the departments are to be placed in two or more rows, as happens for example in the context of flexible manufacturing and in the design of application-specific integrated circuits. We proposed a new mixed integer linear optimization formulation for the multi-row facility layout problem. This formulation has the important property that under mild assumptions, the optimal solutions achieve integer row assignments even though the corresponding variable in the model is continuous. Our computational results show that the proposed formulation improves on earlier linear and semidefinite formulations for instances of multi-row layout formulated using the pairwise rectilinear distance.

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