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Abstract: Tropical algebra is the algebra constructed over the tropical semifield $\mathbb{R}_{max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. We show here that every m-dimensional tropical module M over $\mathbb{R}_{max}$, given by a $m \times p$ matrix A can be embedded into $\mathbb{R}^n_{max}$, iff n of its rows are independent. This result yields a significant improvement to the Whitney embedding for tropical torsion modules published earlier.

Keywords: Idempotent semiring module, tropical module, embedding

Résumé : L'algèbre tropicale est construite sur le semi-corps $\mathbb{R}_{max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. On démontre ici que tout module tropical M de dimension m sur $\mathbb{R}_{max}$, donné par une matrice A de taille $n \times p$ peut être plongé dans $\mathbb{R}^n_{max}$ ssi n lignes de la matrice A sont indépendantes. Ce résultat présente une amélioration significative du théorème de plongement de Whithney pour les modules de torsion publié précédemment.

Mots clés : Module sur un anneau idempotent, plongement d'un module tropical

1 Introduction

Idempotent and tropical mathematics arose from applications. Basically, from the modelling and analysis of man-made systems, and from mathematical physics, in particular – as far as man-made systems are concerned – computers, and production systems.

After the cerebrated paper by Kleene [7], idempotent semigroups have been used in language theory [13], as well as idempotent semirings in network routing problems [3]. From the mathematical point of view, these idempotent structures have been widely investigated by Cuninghame-Green [5]. Applications to control and optimization of production systems have been developed (e.g. [1], [4], to mention only a few).

In mathematical physics, the dequantization point of view on idempotent mathematics was founded in the 1980's by V.P. Maslov and his school. This approach consists in an asymptotic view of traditional mathematics over the numerical fields making the Planck constant $\hbar$ tend to zero, taking imaginary values (cf [9]).

Independently, O. Viro [15], constructed a piecewise linear geometry of a special kind of polyhedra in finite dimensional Euclidean space.

Subsequently, the tropical approach arouse an increased interest in the algebraic geometry community ([6], [10], [12], [14]). A more complete list of references can be found in [8].

The aim of the paper is to investigate for tropical systems (max-plus or min-plus linear algebra) the equivalent of what is known as the Whitney embedding theorem for differentiable manifolds. After exhibiting the cerebrated example of an infinite dimensional tropical module embedded in the 3-dimensional tropical module $\mathbb{R}^3_{\max}$) [17], showing that any two-dimensional tropical module defined in $\mathbb{R}^n_{\max}$) can be embedded in $\mathbb{R}^2_{\max}$) [20], and provide an upper bound for the embedding of torsion tropical modules [19], we answer here the following question:

What is the minimal dimension n required for the embedding of an m-dimensional dimensional tropical module in $\mathbb{R}^n_{\max}$) ? More precisely, we show that a tropical module generated by the independent columns of a matrix A with entries in the tropical semifield $\mathbb{R}_{\max}$ can be embedded in $\mathbb{R}^n_{\max}$ iff n rows of A are independent.

The paper is organised as follows. In Section 2 below, we recall the basic properties of tropical modules. In Section 3, we revisit the classification of two-dimensional tropical modules state and prove the classification theorem for general tropical modules. This section is enriched by an example showing that the necessary invariants defned by torsion are not sufficient to characterize the isomorphy class of a tropical module. Two examples are then provided in Section 4.

2 Idempotent semirings and semiring modules

The tropical semifield $S = \mathbb{R}_{max} = (\underline{\mathbb{R}}, \vee, \cdot, \underline{\mathbf{0}}, \mathbb{1})$ is defined as follows:

- $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, with $(\underline{\mathbb{R}}, \vee, \underline{\mathbf{0}})$ a commutative monoid, where $\vee$ stands for the max operator, with neutral $\underline{\mathbf{0}} = \{-\infty\}$.
- $\cdot$ stands for usual addition, with $\mathbb{1}$ as neutral (the real number 0)
- $\cdot$ distributes over $\vee$, and $\underline{\mathbf{0}}$ is also absorbing for $\cdot$, i.e.
- $\forall \sigma \in \underline{\mathbb{R}},\ \underline{\mathbf{0}} \cdot \sigma = \sigma \cdot \underline{\mathbf{0}} = \underline{\mathbf{0}}$ ($-\infty$ is absorbing for addition)
- Since $(\mathbb{R}, \cdot, \mathbb{1}) = (\mathbb{R}, +, 0)$ is a group, this makes $\mathbb{R}_{max}$ a semifield.
  (Note that S is endowed with an order relation defined by $a \le b \iff a \vee b = b$. Since $\underline{\mathbf{0}}$ is the neutral element of $\vee$, it follows that $\underline{\mathbf{0}}$ is the bottom element of S, i.e $\forall a \in S\ ,\ \underline{\mathbf{0}} \le a$.

2.1 Notation

In the literature on semirings and semiring modules, the notation + or $\oplus$ is often used for either max or min composition laws. However, since idempotent semirings are at the intersection of linear algebra and ordered structures, we use the lattice and ordered structures notation (i.e. $\vee$ for max and $\wedge$ for min [whenever appropriate]). Note also that, unless necessary, the notation $\cdot$ (for the usual addition) will usually be omitted.

Matrix multiplication : Let A, B be two matrices of appropriate sizes with entries $(A)_{ik}$ – written a_{ik} – (resp $(B)_{kj}$ –written b_{kj}–) in S.

Define $(A \cdot B)_{ij} = \bigvee_k a_{ik} b_{kj}$, and $(A \star B)_{ij} = \bigwedge_k a_{ik} b_{kj}$.

Also, we write A^t for the transpose of A, A^- for the matrix with entries a_{ij}^{-1}, and A^{-t} for $(A^t)^- = (A^-)^t$, where a^{-1} is the multiplicative inverse of $a \in S \setminus \{\underline{\mathbf{0}}\}$.

2.2 Semimodules over an idempotent semiring

Left (right) $\vee$-semimodule over a semiring is defined similarly as module over a ring:

1. $(M, \vee)$ is a monoid with neutral $\underline{\mathbf{0}}$
2. There is a map $S \times M \to M$, called exterior multiplication, satisfying :

$$(\sigma\ ,\ x) \mapsto \sigma x.$$

   i) $(\sigma \vee \mu, x) = (\sigma x \vee \mu x)$,
   ii) $(\sigma\ ,\ x \vee y) = (\sigma x \vee \sigma y)$
   iii) $(\underline{\mathbf{0}},\ x) = (\sigma\ ,\ \underline{\mathbf{0}}) = \underline{\mathbf{0}}$.

If the semiring (semifield) is idempotent, then so is the semimodule, since $x \vee x = \mathbb{1}x \vee \mathbb{1}x = (\mathbb{1} \vee \mathbb{1})\ x = \mathbb{1}\ x = x$.

The first composition law $\vee$ in S extend to vector and matrices in a natural way. Also exterior multiplication by a scalar $\lambda \in S$ is defined componentwise (resp. entrywise) for vectors (matrices). This makes S^n and the set of matrices with entries in S, left (or right) $\vee$-semimodules over S.

2.3 Independence

Let M be a S semimodule, and $X = (x_i)_{i \in I} \subset M$. We say that $M_X = \{ \bigvee_{i \in I} \lambda_i x_i | x_i \in X\ ,\ \lambda_i \in S\ ,\ \lambda_i = \underline{\mathbf{0}}$ except for a finite number of them$\}$ is the semimodule generated by X, and that X is the set of generators of M.

In [16], (see also [20]) we considered the following concepts of independence for $X \subset S^n$.

1. $\forall Y, Z \subset X\ M_Y \bigcap M_Z = M_{Y \cap Z}$ (strong independence)
2. $\forall Y, Z \subset X\ ,\ Y \bigcap Z = \emptyset \Rightarrow M_Y \bigcap M_Z = \{\underline{\mathbf{0}}\}$ (Gondran-Minoux independence)
3. $\forall x \in X\ ,\ x \notin M_{X \setminus \{x\}}$ (independene).

Note that $1 \Rightarrow 2 \Rightarrow 3$, while the converse does not hold, although they are equivalent in vector spaces.

In [16] (see also [11]), the proof that every finitely generated semimodule has generating set satisfying 3 (called weak independence there), and that this set is unique up to a homothetic transformation $x_i \mapsto \lambda_i x_i\ ,\ x_i \in X\ ,\ \lambda_i \in S$ is given.

Let $A \in \mathrm{Hom(S^m, S^p)}$, i.e. A is a rectangular matrix of size $p \times m$ with entries in S. Clearly, the columns of A generate a finite dimensional semimodule over S. We write M_A for this subsemimodule of S^p. Also, if the columns of A are independent in the sense of 3 above, then $\dim M_A = m$. From the existence and uniqueness theorem mentioned above, it follows that for any diagonal and permutation matrices of appropriate sizes D_1, D_2, P_1, P_2, the matrices A and $B = D_1 P_1 A P_2 D_2$ generate isomorphic semimodules. We write in this case $A \sim B$.

The problem we address here is to find the minimal n such that M_A is isomorphic to a subsemimodule of S^n ? In [19], we addressed this problem for semimodules over $S = \mathbb{R}_{\max}$ with finite entries (i.e. $\neq \underline{\mathbf{0}}$) only.

3 The embedding theorem

3.1 The 2-dimensional case revisited

In [18], using the order relation in M, we showed that 2-dimensional semimodules can be classified by a 1-parameter family. More precisely, the order in M induces an order on the set of generators $X = \{x_1, x_2\}$ of M. Thus X is either an antichain, a chain, or else, we have $x_1 \le x_2 \le \lambda x_1$ for some $\lambda(> \mathbb{1}) \in M$. It follows that, representing each generator as a column vector in S^2, we necessarily have $X \in \left\{ \begin{bmatrix} \mathbb{1} & \underline{0} \\ \underline{0} & \mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \underline{0} & \mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \lambda \end{bmatrix} \right\}$, where $\lambda \in R_{\max}$ (which yields a 1-parameter family).

As an introduction to our classification result below, we revisit the two generators case. Assuming each generator to be given by a column vector of a $n \times 2$ matrix A, with $n \ge 2$, we will show that only two of the rows of A generate all the other rows.

Let $X = \{x, y\}$, with $x_i, y_i \in S^n$. We consider the following cases:

1. $\exists\, i \ne j$ s.t. $x_i = y_j = \underline{0}$ (the case $i = j$ is omitted, since then $x, y \in S^{n-1}$).
2. $\exists\, i$ s.t. $x_i = \underline{0}$, while $\forall j\,,\ y_j > \underline{0}$.
3. $\forall i\,, j\,,\ x_i\,,\, y_j > \underline{0}$.

The generators will be represented as the columns of a $n \times 2$ matrix A.

Case 1

$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \cdots & \cdots \\ x_n & y_n \end{bmatrix}$. Up to a permutation of the rows of A we may assume that $x_2 = \underline{0}\,,\ y_1 = \underline{0}$.

Let $D_1 = diag \left[x_1^{-1} \bigvee_{i=1}^{n} x_i \;\; y_2^{-1} \bigvee_{i=1}^{n} y_i \;\; \mathbb{1} \;\; \ldots \;\; \mathbb{1} \right]$, $D_2 = diag \left[(\bigvee_{i=1}^{n} x_i)^{-1} \;\; (\bigvee_{i=1}^{n} y_i)^{-1} \right]$.

We have $D_1 A D_2 = \begin{bmatrix} \mathbb{1} & \underline{0} \\ \underline{0} & \mathbb{1} \\ x_3 (\bigvee_{i=1}^{n} x_i)^{-1} & y_3 (\bigvee_{i=1}^{n} y_i)^{-1} \\ \cdots & \cdots \\ x_n (\bigvee_{i=1}^{n} x_i)^{-1} & y_n (\bigvee_{i=1}^{n} y_i)^{-1} \end{bmatrix} \sim A$, which we may rewrite as $B = [\, a \; b \,] = \begin{bmatrix} \mathbb{1} & \underline{0} \\ \underline{0} & \mathbb{1} \\ a_3 & b_3 \\ \cdots & \cdots \\ a_n & b_n \end{bmatrix}$, with $a_i, b_i \le \mathbb{1}\,,\ i = 3, \ldots, n$.

Now for any row $r_k = [\, a_k \; b_k \,]$ $(2 < k \le n)$, we have $r_k = a_k r_1 \vee b_k r_2$. Hence, the projection $P\colon S^n \to S^2\,,\ a \mapsto \begin{bmatrix} \mathbb{1} \\ \underline{0} \end{bmatrix}$ $b \mapsto \begin{bmatrix} \underline{0} \\ \mathbb{1} \end{bmatrix}$ restricted to M_B is an isomorphism.

Case 2

$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \cdots & \cdots \\ x_n & y_n \end{bmatrix}$, with $y_i \ne \underline{0}\,,\ 1 \le i \le n$. We may assume that $x_1 = \underline{0}$.

Let $D = diag(y_1^{-1} \; y_2^{-1} \; \ldots \; y_n^{-1})$, then $D_1 A = [\, c \; d \,] = \begin{bmatrix} \underline{0} & \mathbb{1} \\ x_2 y_2^{-1} & \mathbb{1} \\ \cdots & \cdots \\ x_n y_n^{-1} & \mathbb{1} \end{bmatrix}$

Up to a permutation of the rows, we may assume that $x_i y_i^{-1} \le x_{i+1} y_{i+1}^{-1}\,,\ i = 2 \ldots, n-1$.

Then, for $i = 2, \ldots, n-1$, we have $r_i = r_1 \vee x_i y_i^{-1} x_n^{-1} y_n r_n$.

We conclude as in case 1 above.

Case 3

We first consider the case $n = 2$. Let $A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\,,\ D = diag(x_1^{-1} \; x_2^{-1})$. Then $DA = \begin{bmatrix} \mathbb{1} & x_1^{-1} y_1 \\ \mathbb{1} & x_2^{-1} y_2 \end{bmatrix}$. Muliplying column 2 by $x_1 y_1^{-1}$, we get the equivalent matrix $B = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & x_1 y_2 (x_2 y_1)^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \tau \end{bmatrix}$, with $\tau = x_1 y_2 (x_2 y_1)^{-1}$.

Note that if $\tau < 1\!\!1$, then multiplying row 2 of B by τ^{-1}, followed by the pemutation of the two columns of B yields an equivalent matrix with $\tau^{-1} > 1\!\!1$.

Another point of view is that of torsion (cf [17], [19]), which can be defined as follows. Let $\lambda_{12} = \bigwedge\{\xi \in S | x_i \leq \xi_i y_i \;, i = 1, 2\}$, and $\lambda_{21} = \bigwedge\{\xi \in S | y_i \leq \xi_i x_i \;, i = 1, 2\}$. Note that the matrix $\Lambda_A = A^t \cdot A^- = \left[\begin{smallmatrix} 1\!\!1 & \lambda_{12} \\ \lambda_{21} & 1\!\!1 \end{smallmatrix}\right]$, has the property $\lambda_{12}\lambda_{21} = \tau$, which we call the torsion of M_A. This is an intrincic invariant of M_A.

Note also that $\tau = x_1 y_2 (y_1 x_2)^{-1}$ shows some similarities with the determinant of A, hence, we may call it the semi-determinant of A. In addition, for (say) $\tau > 1\!\!1$, we have $x_1 y_2 > y_1 x_2$, hence $x_1 y_2 \vee y_1 x_2 = x_1 y_2$.

For $n > 2$, let $A = \left[\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \cdots & \cdots \\ x_n & y_n \end{smallmatrix}\right]$, with $\forall i,\; x_i\,, y_i > \underline{\mathbf{0}}$. We get $\Lambda_A = \left[\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1^{-1} & y_1^{-1} \\ x_2^{-1} & y_2^{-1} \\ \cdots & \cdots \\ x_n^{-1} & y_n^{-1} \end{smallmatrix}\right] = \left[\begin{matrix} 1\!\!1 & \bigvee_{i=1}^{n} x_i y_i^{-1} \\ \bigvee_{i=1}^{n} x_i^{-1} y_i & 1\!\!1 \end{matrix}\right] = \left[\begin{matrix} 1\!\!1 & \lambda_{12} \\ \lambda_{21} & 1\!\!1 \end{matrix}\right]$.

Note that $\tau = \lambda_{12}\lambda_{21} = \bigvee_{1 \leq i,j \leq n} x_i y_j (x_j y_i)^{-1}$ corresponds to the maximum of the semi-determinants of the $n(n-1)$ square submatrices of size two of A.

Right multiplication of A by the diagonal matrix $(x_1^{-1} \;\ldots\; x_n^{-1})$, yields the equivalent matrix $\left[\begin{smallmatrix} 1\!\!1 & x_1^{-1} y_1 \\ 1\!\!1 & x_2^{-1} y_2 \\ \cdots & \cdots \\ 1\!\!1 & x_n^{-1} y_n \end{smallmatrix}\right]$. As above, up to a permutation of the rows of this matrix, we may assume that $x_i^{-1} y_i \leq x_{i+1}^{-1} y_{i+1}\,, i = 1 \ldots n{-}1$. Right multiplication of this matrix by $diag(1\!\!1 \; x_1 y_1^{-1})$ yields $B = \left[\begin{smallmatrix} 1\!\!1 & 1\!\!1 \\ 1\!\!1 & z_2 \\ \cdots & \cdots \\ 1\!\!1 & \tau \end{smallmatrix}\right] \sim A$, (for some z $[1\!\!1 \leq z_i \leq \tau]$).

As above, it is easy to show that for $k = 2, \ldots, n-1$ we have $r_k = r_1 \vee z_k \tau^{-1} r_n$, and conclude that $M_A \sim M_C$, with $C = \left[\begin{smallmatrix} 1\!\!1 & 1\!\!1 \\ 1\!\!1 & \tau \end{smallmatrix}\right]$.

3.2 The Whitney embedding theorem for tropical modules

In [19] we prove an upper bound for the embedding of a torsion tropical module. Recall that an embedding is an injective map. The following statement both improves and geneneralizes this result to arbitrary tropical modules.

Theorem 1 *Let $X = \{c_1\,, c_2\,, \ldots, c_m\}$ be set of (independent) generators of a tropical module, where $c_j = \left[\begin{smallmatrix} a_{1j} \\ a_{2j} \\ \cdots \\ a_{pj} \end{smallmatrix}\right] \in S^p$. Then M can be embedded in S^n iff $n \leq p$ is the maximum number of independent rows of the matrix $A = \left[\begin{smallmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pm} \end{smallmatrix}\right]$.*

Proof. Assume that there is an embedding $\Psi\colon M_A \;\to S^n$. The m generators of $\mathrm{Im}\Psi$ may be written in matrix form as $B = \left[\begin{smallmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_n \end{smallmatrix}\right]$. Clearly, the rows of B are independent, for if not M_A could be embedded in S^q, with $q < n$. Since Ψ is an embedding, M_A is isomorphic to $M_B = \mathrm{Im}\Psi$, and p is the maximum number of independent rows of A.

Conversely, let $n \leq p$ be the maximum number of independent rows of A. For $n = p$, there si nothing to prove. Hence we may assume $n < p$. Up to a permutation of the rows of A, we may assume that its first n rows r_i $(1 \leq \imath \leq n)$ are independent. Then for any $k\,, n+1 \leq k \leq p\,$, we have $r_k = \bigvee_{i=1}^{n} \lambda_{ik} r_i \in S^n$.

Clearly, whenever $\lambda_{ik} > \mathbf{0}$, one of the rows r_i and $\lambda_{ik} r_i$ can be removed. It follows that (always choosing the removal of $\lambda_{ik} r_i$), for $n+1 \leq k \leq p\,,\; r_k$ can be dropped.

Hence, we get a matrix B as above. Clearly $M_B \subset S^n$. The map $\Psi\colon S^p \to S^n$ sending every generator of M_A to the corresponding generator of M_B is an isomorphism, and M_A is isomorphic to M_B. $\square$

3.3 Examples

Example 1 *Let* $A = \begin{bmatrix} \mathbb{1} & 1 & 6 \\ \mathbb{1} & 2 & 3 \\ \mathbb{1} & 4 & 5 \\ \mathbb{1} & 4 & 6 \\ \mathbb{1} & 7 & 1 \\ \mathbb{1} & 7 & 5 \end{bmatrix}$. *The columns of* A *are independent. Its rows are not.*

Indeed, we have $r_4 = r_1 \vee r_3$, *and* $r_6 = r_3 \vee r_5$. *Hence* $M_A = M_B$, *with* $B = \begin{bmatrix} \mathbb{1} & 1 & 6 \\ \mathbb{1} & 2 & 3 \\ \mathbb{1} & 4 & 5 \\ \mathbb{1} & 7 & 1 \end{bmatrix}$. *It is easy to see that the rows of* B *are independent, thus* M_A *can be embedded in* S^4.

Example 2 *(4.3 of [20]) Let* $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & a & a \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix}$, *with* $5 < a < 8$. *We write* M_a *(or* M_A*) for the tropical module generated by* A. *It is not difficult to see that the rows of* A *are independent, while its columns* A *are strongly independent.*

The torsion coefficients (cf [19]) of M_A *are easily computed from the matrix* $\Lambda_A = A^t A^- = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4^{-1} \\ 9 & \mathbb{1} & \mathbb{1} \\ 15 & 12 & \mathbb{1} \end{bmatrix}$, *which yields* $\tau_{12} = 9$, $\tau_{13} = 11$, $\tau_{23} = 12$. *Thus the torsion coefficients are independent of* a.

We may also ask how the isomorphy class of M_A *depends of* a. *In order to see this, let* $b \neq a$.

Are the two tropical modules M_a, M_b *isomorphic ?*

For such an isomorphism, we must have :

$A_a = diag P_1 B A_b P_2 diag C$, *where* P_1, P_2 *are permutation matrices. However it is easy to see that row* i *of* A_b *must correspond to row* i *of* A_a, $i = 1, \ldots 6$. *Hence* $P_1 = P_2 = I_6$ *(the identity matrix). Therefore we must have :*

$$\begin{bmatrix} u_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & u_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & u_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & u_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & u_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & u_6 \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & b & b \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix} \begin{bmatrix} x_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & x_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & x_3 \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & a & a \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix},$$ *which has no solution for* $a \neq b$.

Indeed, from the first column, we must have $u_i = x_1^{-1}$, $i = 1, \ldots, 6$. *From the 1st row, we get* $x_1 = x_2 = x_3$. *Hence, from the fourth row we must have* $b x_1 x_2^{-1} = a$, *i.e.* $b = a$.

Remark 1 *Example 2 shows the following.*

1. *The torsion coefficients of a tropical module* M, *although intrinsic invariants of the isomorphy class of* M *do not characterize this class.*
2. *Although the torsion coefficients are independent of* a, *this parameters also plays an important role in the characterization the isomorphy class of* M_A.
3. *Note also that* $\Gamma^A = \Lambda_A^- = \bigvee\{X | AX \leq A\}$, *and the torsion oefficients of* M_{Γ^A}, *and* M_{Λ_A}, *are the same as those of* M_A.

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