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A dynamic program under Lévy processes for valuing corporate securities

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Abstract: Most structural models for valuing corporate securities assume a geometric-Brownian motion to describe the firm's assets value. However, this does not reflect market-stylized features; the default is more often conducted by sudden informations and shocks, which are not captured by the Gaussian model assumption. To remedy this, we propose a dynamic program for valuing corporate securities under various Lévy processes. Specifically, we study two jump diffusions and a pure-jump process. Under these settings, we build and experiment with a flexible framework, which accommodates the balance-sheet equality, arbitrary corporate debts, multiple seniority classes, tax benefits, and bankruptcy costs. While our approach applies to several Lévy processes, we compute and detail the equity's, debt's, and firm's total values, as well as the debt's credit-spreads under Gaussian, double exponential, and variance-gamma-jump models.

Keywords: Credit risk, corporate securities, credit spreads, Lévy processes, dynamic programming, finite elements

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1 Introduction

According to the Securities Industry and Financial Markets Association (SIFMA), the size of the U.S. corporate debt issuance reached \$1.25 trillion at the end of 2016. Public corporate debt is traded through a dealer-based system. Market signals are partially inaccurate since part of public corporate debt is not heavily traded, and there is a complete lack of information for private corporate debt. Hence, there is a great need for a rational modeling framework for valuing corporate securities.

The structural approach enables the valuation of various corporate securities, including corporate debt portfolio, related by the balance-sheet equality. Corporate securities are seen as derivatives on the firm's assets value and interpreted as the state process. Default is triggered when that state variable crosses a default barrier. Despite their great success, several empirical studies have been critical of the structural approach developed under geometric-Brownian motion. The major drawbacks of such an approach are that it does not reflect data from the bond market; instead, it over-evaluates the debt, and thus, under-evaluates credit spreads. It generates a zero credit spread when the maturity of the debt approaches zero. However, several money-market debt instruments, such as commercial paper, trade at significant spreads above zero. Moreover, an ongoing issue has been to explain the credit spreads of bonds belonging to high investment-grade firms. Huang and Huang (2012) state that even firms with low default risk still have credit spreads that are sizable and positive. Tauchen and Zhou (2011) conclude that firms are exposed to large sudden movements and unexpected information, which confirms the existence of jumps in financial markets. Geske and Delianedis (2001) have also confirmed this claim and sate that jumps are one of the most important components of credit risk.

Our aim in this paper is to build and experiment with a flexible structural model where the value of the firm's assets is described by a Lévy process. We propose this as a solution to some of the major drawbacks under the geometric-Brownian motion model. Specifically, our model allows for significant credit spreads over short maturities. Lévy processes effectively model the firm's assets value, offering realistic financial features and enriching the structural modeling framework. While it initially allows for the leptokurtic feature observed in the financial market, it also introduces unexpected components that allow for low-leverage companies to default, even over short time intervals, without compromising economic intuition. In fact, we combine the unpredictability of the default event generated by reduced-form models with the economic background of structural models.

Merton (1974) pioneered the structural approach by considering a firm that has a trivial capital structure (zero coupon) with a possibility of defaulting only upon maturity of the debt. The proposed model is based on options theory and links to the evaluation of corporate debt when the firm's assets follow geometric-Brownian motion. Many other researchers have worked on extending Merton's (1974) model to achieve higher levels of realism, either by proposing a more complex capital structure, by incorporating frictions, or by experimenting with alternative Markov stochastic processes. Under geometric-Brownian motion modeling, some researchers have proposed exogenous-barrier models (Black and Cox, 1976, Ericsson and Reneby, 1998, Collin-Dufresne et al., 2001), while others have opted for endogenous barriers (Geske, 1977, Leland, 1994, Leland and Toft, 1996, Anderson and Sundaresan, 1996, François and Morellec, 2004, Nivorozhkin, 2005).

Several researchers have developed work on credit modeling under Lévy processes; however, some shortcomings still remain. Zhou (1997) generalizes Merton's (1974) work by considering Gaussian jumps for the firm's assets. Hilberink and Rogers (2002) work under a perpetual debt structure and use a negative spectral Lévy process that only allows for negative jumps. Cariboni and Schoutens (2007) present a structural model and price credit-default-swap under the variance-gamma process using integro-differential equations. Le Courtois and Quittard-Pinon (2008) propose an extension of the Hilberink and Rogers (2002) model using specific stable Lévy processes that also consider upward and downward jumps. Chen and Kou (2009) use a jump-diffusion model with double-exponential-type jumps à la Kou (2002), and also generalize the models of Leland (1994) and Leland and Toft (1996). These methodologies make it possible to solve some of the structural models' problems, but they all work under restrictive hypotheses: a simple debt structure, a pre-defined, often exponential, debt-maturity structure, and fixed coupons. In addition, they do not allow for seniority classes.

We extend the flexible framework of Altieri and Vargiolu (2000) and Ayadi et al. (2016) to Lévy processes for valuing corporate securities. Our setting is based on dynamic programming coupled with finite elements and accommodates arbitrary corporate debts, market frictions, and multiple seniority classes. Moreover, our research considers three types of Lévy processes: symmetric Gaussian jumps, asymmetric double-exponential jumps, and a pure-jump process, specifically, the variance-gamma process.

Our paper is organized as follows: Section 3.2 provides a theoretical background on Lévy processes and presents two finite-activity Lévy models (Merton, 1976, Kou, 2002) and an infinite-activity Lévy process, namely Madan et al.'s (1998) variance-gamma model. Section 3.3 defines the optimal stopping problem and presents the structural model under Lévy processes. Section 3.4 describes our dynamic programming, Section 3.5 presents our numerical investigation which discusses the impact of jumps in the credit structure, and finally, Section 3.6 concludes the paper.

2 The Lévy process

We assume that the firm's assets value follows an exponential Lévy process. On the one hand, our attention is focused on finite-activity-finite-variation Lévy processes (i.e. Lévy jump-diffusions), specifically Merton's (1976) model where the jump size has normal distribution, and the double exponential jump model of Kou (2002). On the other hand, we work with a special case of infinite-activity-finite-variation Lévy process to describe the firm's assets value, which is the variance-gamma process.

We describe the asset value of the firm by a stochastic process $V = \{V_t, 0 \leq t \geq T\}$ of the form

$$V_t = V_0 \exp(L_t),$$

where $L = \{L_t, t \geq 0\}$ is an exponential Lévy process.

2.1 Finite-activity Lévy processes

Pure-jump Lévy processes of finite activity are characterized as compound Poisson processes and referred to as jump-diffusion processes. We provide some theoretical background to define finite-activity-Lévy processes in Appendix A. For our default model, we consider Merton's (1976) and Kou's (2002) models, where jumps are considered rare events so that there are a finite number of jumps in any given finite interval. Under this setting, the firm's assets value described with Lévy process L_t is

$$L_t = at + \sigma W_t + \sum_{n=1}^{N_t} J_n - t\lambda\kappa,$$

where $a \in \mathbb{R}$ is the drift , $\sigma \geq 0$ is the volatility, $W = \{W_t, t \geq 0\}$ is a standard Brownian motion, $N = \{N_t, t \geq 0\}$ is a Poisson process with parameter λ, and $J = \{J_n, n \in \mathbb{N}^*\}$ is an i.i.d sequence of random variables with probability distribution F. It describes the jumps that arrive according to a Poisson process, where $\mathbb{E}[e^{J_n} - 1] = \kappa < \infty$. The processes W, N, and J are assumed to be independent under $\mathbb{P}$. Under Merton's (1976) model, jumps are supposed to be normally distributed and thus symmetric, whereas under Kou's (2002) setting, the distribution of jumps are assumed to be double exponential and, as a result, asymmetric.

2.2 Infinite-activity Lévy processes

For infinite-activity Lévy processes, the Lévy measure has infinite mass, implying that there are an infinite number of jumps in a finite time interval. We provide some theoretical background to define infinite-activity Lévy processes in Appendix A. Many of these models can be obtained with Brownian subordination $W = \{W_{G_t}, t \geq 0\}$ where G is called a subordinator. The subordinator is an increasing Lévy process that has no diffusion component. For instance, if the subordinator G is a gamma process, then W_{G_t} leads to the variance-gamma model upon which we base our default model. The firm's assets value at time t, under the risk-neutral measure $\mathbb{Q}$, is assumed to follow a Lévy process of the variance-gamma type L_t

$$L_t = (r - q + w)\, t + X\,(t; \sigma, \nu, \theta)\,,$$

where r is the constant riskless rate, q is the dividend rate, and w is a compensator that makes the risk neutral return on V equal to $r-q$. Define $\mathbb{E}\left[\exp\left(X_t\right)\right] = \left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)^{-t/\nu}$, we set

$$w = \nu^{-1} \log\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right),$$

and $X\,(t; \sigma, \nu, \theta)$ is a variance-gamma process with parameters σ, ν and θ. The advantage behind the variance-gamma-structural-default model is that the firm's value process becomes asymmetric and that the jump structure allows for random default times.

3 Problem formulation

We consider a market with risky asset V which represents the total asset value of the firm. Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a complete probability space and $\mathcal{P} = \{t_0, t_1, \dots t_n, \dots, t_N = T\}$ be a set of payment dates. We assume that the interest rate r_n, for $n \in \{1, \dots, N+1\}$, are known from the beginning. For each $k \in \{0, \dots, N+1\}$, where k is the bankruptcy time, set the discount factor $\beta_0 = 1$ and $\beta_k = e^{-r_k}\beta_{k-1}$. The case $k = N+1$ means that the firm survives until time t_N. We assume that the stochastic process $V = (V_t)_{t\in[0,T]}$ describing the firm's asset value is modeled as a Lévy process with $V_0 > 0$. In this setting, the firm is financed by debt and equity. More precisely, the firm's capital structure is characterized by senior and junior bonds and a common stock. The firm distributes a coupon payment by which the firm receives tax benefits. The model assumes that shareholders determine the time to default by maximizing the firm's equity value which is subject to limited liability constraints. We assume herein the strict priority rule. The value functions, in terms of the bankruptcy time k, are expressed as follows:

Bankruptcy costs We assume that on default, a fraction $w \in [0,1]$ of the firm's asset value is due to cover the bankruptcy costs, at time t_n, defined by

$$BC_k^{(n)} = \begin{cases} 0, & k < n \text{ or } k = N+1, \\ w\,\frac{\beta_k}{\beta_n}\,V_k, & n \le k \le N. \end{cases}$$

Debt At each date t_n, the firm undertakes to pay a total outflow indicated by $d_n = d_n^{(jun)} + d_n^{(sen)}$ to its bondholders, where $d_n^{(sen)}$ and $d_n^{(jun)}$ refers to the senior and junior bondholders respectively. These payments include principal as well as interest payments. The latter are indicated by d_n^{int}. At the beginning of the contracts, the payment structure are known to all investors. Define the total debt by $D_k^{(n)} = DS_k^{(n)} + DJ_k^{(n)}$, where $DS_k^{(n)}$ and $DJ_k^{(n)}$ are referring to senior and junior debt evaluated at date t_n. The last payment dates for senior and junior debts are indicated by $T^{(sen)}$ and $T^{(jun)}$, with $0 \le T^{(sen)} \le T^{(jun)} = T$. For each $n \in \{0, \dots, N\}$ and $k \in \{0, \dots, N+1\}$, the senior, junior and total debts are defined as follows:

$$DS_k^{(n)} = \begin{cases} 0, & k < n, \\ \frac{\beta_k}{\beta_n} \min\left\{(1-w)V_k, d_k^{(sen)}\right\} + \frac{1}{\beta_n}\sum_{j=n}^{k-1} \beta_j d_j^{(sen)}, & n \le k \le N, \\ \frac{1}{\beta_n}\sum_{j=n}^{N} \beta_j d_j^{(sen)}, & k = N+1, \end{cases}$$

$$DJ_k^{(n)} = \begin{cases} 0, & k < n, \\ \frac{\beta_k}{\beta_n} \max\left\{(1-w)V_k - d_k^{(sen)}, 0\right\} + \frac{1}{\beta_n}\sum_{j=n}^{k-1} \beta_j d_j^{(jun)}, & n \le k \le N, \\ \frac{1}{\beta_n}\sum_{j=n}^{N} \beta_j d_j^{(jun)}, & k = N+1. \end{cases}$$

If the firm defaults at time t_n, the senior bondholders receive $\min\left\{(1-w)V_n,\, d_n^{(sen)}\right\}$, while the junior bondholders receive $\max\left\{(1-w)V_n - d_n^{(sen)}, 0\right\}$, so that the bondholders receive, in total, $(1-w)V_n$. If the firm survives at time t_n, then the senior bondholders receive $d_n^{(sen)}$ while the junior bondholders receive $d_n^{(jun)}$, for a total of $d_n^{(sen)} + d_n^{(jun)} = d_n$. and

$$D_k^{(n)} = \begin{cases} 0, & k < n, \\ \frac{1}{\beta_n}\sum_{j=n}^{N} \beta_j d_j, & k = N+1, \end{cases}$$

where $\sum_{j=n}^{n-1} = 0$ by convention.

Tax benefits The firm benefits from the tax shield from debt financing as long as it survives. In case of default, tax benefits cannot be claimed. Let $r_n^c \in [0,1]$ be the periodic corporate tax rate over time $[t_n, t_{n+1}]$ that is considered as a known constant. The tax benefit is seen as a security that pays a coupon $tb_n = r_n^c d_n^{int}$. The tax benefit is indicated by $TB_k^{(n)}$ and given by

$$TB_k^{(n)} = \begin{cases} 0, & k < n, \\ \frac{1}{\beta_n}\sum_{j=n}^{k-1} \beta_j tb_j, & n \le k \le N+1. \end{cases}$$

The total value of the firm The total value of the firm $W_k^{(n)}$ is the sum of the firm's assets V_n, and the tax shield of interest payment $TB_k^{(n)}$, minus the bankruptcy costs $BC_k^{(n)}$. That is

$$W_k^{(n)} = V_n + TB_k^{(n)} - BC_k^{(n)} = \begin{cases} 0, & k < n, \\ V_n + \frac{1}{\beta_n}\sum_{j=n}^{k-1} \beta_j tb_j - w\frac{\beta_k}{\beta_n} V_k, & n \le k \le N, \\ V_n + \frac{1}{\beta_n}\sum_{j=n}^{N} \beta_j tb_j, & k = N+1. \end{cases}$$

Equity In the case that the firm survives survival to date t_n, the stockholders receive the total value of the firm minus the total debt defined by

$$\mathcal{E}_k^{(n)} = W_k^{(n)} - D_k^{(n)}.$$

Now, let $\mathcal{T}$ be the set of stopping times with values in $\{0,\ldots,N+1\}$. As a result, for any stopping time $\tau \in \mathcal{T}$ with $\tau \ge n$, we obtain

$$E\left(\mathcal{E}_\tau^{(n)} | \mathcal{F}_n\right) = \mathcal{B}_n^{(\tau)} \mathbf{1}(\tau > n),$$

where $\mathcal{B}_N^{(\tau\vee N)} = \mathcal{B}_N = V_N + tb_N - d_N$ and

$$\mathcal{B}_n^{(\tau)} = V_n + tb_n - d_n - E\left(e^{-r_{n+1}} V_{n+1} | \mathcal{F}_n\right) + E\left(e^{-r_{n+1}} \mathcal{E}_{\tau\vee(n+1)}^{(n+1)} | \mathcal{F}_n\right),$$

for all $n \in \{0,\ldots,N-1\}$.

Since $\mathcal{B}_n^{(\tau)}$ only depends on $\tau \vee (n+1)$, it follows that the definition is sound.

Definition 1

$$\mathcal{T}_n = \left\{ \tau \in \mathcal{T}; \tau \ge n,\ \{\tau > k\} \subset \left\{ E\left(\mathcal{E}_{\tau\vee k}^{(k)} | \mathcal{F}_k\right) > 0 \right\},\ \ for\ k \ge n \right\}.$$

Finally, define $J_\tau^{(n)} = TB_\tau^{(n)} - BC_\tau^{(n)}$, and set

$$\bar{J}_n = \sup_{\tau\in\mathcal{T}_n} E\left(J_\tau^{(n)} \middle| \mathcal{F}_n \right),$$

for all $n \in \{0,\ldots,N\}$. Note that $\sup_{\tau\in\mathcal{T}_n} E\left\{ W_\tau^{(n)} \middle| \mathcal{F}_n \right\} = V_n + \bar{J}_n$.

The main aim is to find a sequence of stopping times $\tau_n^\star \in \mathcal{T}_n$, corresponding to optimal bankruptcy times, so that the total expected wealth at time n is maximized; that is $V_n + \bar{J}_n = E\left\{ W_{\tau_n^\star}^{(n)} | \mathcal{F}_n \right\}$. The solution follows.

Theorem 1 *Set*

$$\mathcal{E}_N = \max\left(V_N + tb_N - d_N, 0\right), \tag{1}$$

and for any $k \in \{0, \ldots, N-1\}$, set

$$\mathcal{E}_k = \max\left\{V_k + tb_k - d_k - E\left(e^{-r_{k+1}} V_{k+1} | \mathcal{F}_k\right) + E\left(e^{-r_{k+1}} \mathcal{E}_{k+1} | \mathcal{F}_k\right), 0\right\}. \tag{2}$$

Next, define

$$\tau_k^\star = \begin{cases} N+1, & \text{if } \mathcal{E}_j > 0 \text{ for all } j \in \{k, \ldots, N\}, \\ \min\{k \le j \le N; \mathcal{E}_j = 0\}, & \text{otherwise.} \end{cases}$$

Then

$$\bar{J}_N = E\left(J^{(N)}_{\tau_N^\star} | \mathcal{F}_N\right) = -wV_N \mathbf{1}(\mathcal{E}_N = 0) + tb_N \mathbf{1}(\mathcal{E}_N > 0),$$

and for all $k \in \{0, \ldots, N-1\}$,

$$\begin{aligned} \bar{J}_k &= E\left(J^{(k)}_{\tau_k^\star} | \mathcal{F}_k\right) \\ &= -wV_k \mathbf{1}(\mathcal{E}_k = 0) + \left\{tb_k + E\left(e^{-r_{k+1}} \bar{J}_{k+1} | \mathcal{F}_k\right)\right\} \mathbf{1}(\mathcal{E}_k > 0). \end{aligned}$$

The proof of Theorem 1 is provided in Ben Abdellatif et al. (2016).

Remark 1 *If $(\beta_k V_k)_{k=0}^N$ is a martingale, then*

$$\mathcal{E}_n = \max\left\{E\left(e^{-r_{n+1}} \mathcal{E}_{n+1} | \mathcal{F}_n\right) + tb_n - d_n, 0\right\}. \tag{3}$$

3.1 Expressions for the debts and equity

Using Theorem 1 from Ben Abdellatif et al. (2016), we have the following expressions for the debt: for any $n \in \{0, \ldots, N\}$, set $D_n = E\left(D^{(n)}_{\tau_n^\star} | \mathcal{F}_n\right)$, $DS_n = E\left(DS^{(n)}_{\tau_n^\star} | \mathcal{F}_n\right)$, $DJ_n = E\left(DJ^{(n)}_{\tau_n^\star} | \mathcal{F}_n\right)$. Then

$$\mathcal{E}_N = \max\left(V_N + tb_N - d_N, 0\right), \tag{4}$$

$$D_N = (1-w)V_N \mathbf{1}(\mathcal{E}_N = 0) + d_N \mathbf{1}(\mathcal{E}_N > 0), \tag{5}$$

$$DS_N = \min\left\{(1-w)v, d_N^{(sen)}\right\} \mathbf{1}(\mathcal{E}_N = 0) + d_N \mathbf{1}(\mathcal{E}_N > 0), \tag{6}$$

$$DJ_N = \max\left\{(1-w)V_N - d_N^{(sen)}, 0\right\} \mathbf{1}(\mathcal{E}_N = 0) + d_N^{(jun)} \mathbf{1}(\mathcal{E}_N > 0), \tag{7}$$

and for any $n \in \{0, \ldots, N-1\}$,

$$\mathcal{E}_n = \max\left\{E\left(e^{-r_{n+1}} \mathcal{E}_{n+1} | \mathcal{F}_n\right) + tb_n - d_n, 0\right\}. \tag{8}$$

$$D_n = (1-w)V_n \mathbf{1}(\mathcal{E}_n = 0) + \left\{d_n + \left(e^{-r_{n+1}} D_{n+1} | \mathcal{F}_n\right)\right\} \mathbf{1}(\mathcal{E}_n > 0), \tag{9}$$

$$DS_n = \min\left\{(1-w)V_n, d_n^{(sen)}\right\} \mathbf{1}(\mathcal{E}_n = 0) + \left\{d_n^{(sen)} + E\left(e^{-r_{n+1}} DS_{n+1} | \mathcal{F}_n\right)\right\} \mathbf{1}(\mathcal{E}_n > 0), \tag{10}$$

$$DJ_n = \max\left\{(1-w)V_n - d_n^{(sen)}, 0\right\} \mathbf{1}(\mathcal{E}_n = 0) + \left\{d_n^{(jun)} + E\left(e^{-r_{n+1}} DJ_{n+1} | \mathcal{F}_n\right)\right\} \mathbf{1}(\mathcal{E}_n > 0). \tag{11}$$

3.2 Expressions for tax benefits and bankruptcy costs

Using Theorem 1 from Ben Abdellatif et al. (2016), we have the following expressions for the tax benefits and bankruptcy costs: for any $n \in \{0, \ldots, N\}$, set $TB_n = E\left(TB^{(n)}_{\tau_n^\star} | \mathcal{F}_n\right)$ and set $BC_n = E\left(BC^{(n)}_{\tau_n^\star} | \mathcal{F}_n\right)$, and $v = V_n$. Then

$$TB_N = tb_N \mathbf{1}(\mathcal{E}_N > 0), \tag{12}$$
$$BC_N = wV_N \mathbf{1}(\mathcal{E}_N = 0), \tag{13}$$

and for any $n \in \{0, \ldots, N-1\}$,

$$TB_n = \left\{ tb_n + E\left(e^{-r_{n+1}} TB_{n+1} | \mathcal{F}_n\right) \right\} \mathbf{1}(\mathcal{E}_n > 0), \tag{14}$$
$$BC_n = wV_n \mathbf{1}(\mathcal{E}_n = 0) + E\left(e^{-r_{n+1}} BC_{n+1} | \mathcal{F}_n\right) \mathbf{1}(\mathcal{E}_n > 0). \tag{15}$$

Note that under our Markovian case and for any integrable function Ψ on $\mathbb{R} \times [0, \infty)$, one has

$$E\left[e^{-r_{n+1}} \Psi\{V(t_{n+1})\} | \mathcal{F}_n\right] = \Psi\{V(t_n)\},$$

where $V_n = V(t_n)$ and $\mathcal{F}_n = \sigma\{V(u); 0 \le u \le t_n\}$.

4 Dynamic programming approach

The implementation of the optimal stopping time problem presented in Section 3.3 is performed by using dynamic programming coupled with finite elements. Let $\mathcal{G} = \{v_1, \ldots, v_p\}$ be a mesh of grid points for the firm's assets value, where $0 = v_0 < v_1 < \ldots < v_p < +\infty$ and $\Delta v_i = v_i - v_{i-1}$, $i = 1, \ldots, p$. The grid $\mathcal{G}$ must be selected so that $\text{mesh}(\mathcal{G}) = \max_{1 \le i \le p} \Delta v_i$, $\mathbb{Q}(V_t < v_1)$ and $\mathbb{Q}(V_t > v_p)$ all converge to 0 as $p \to \infty$, for $t \in \{t_1, \ldots, t_N\}$. We use the quantile of the state process $\{V\}$ at time $t_N = T$ for grid construction; however, the optimal choice of $\mathcal{G}$ is not addressed in this paper. For simplicity, we assume constant annual interest rate r and $dt = t_{n+1} - t_n$ a positive constant. The dynamic program works as follows:

1. At maturity the value functions are known in closed form and computed using Equation (4), Equation (5), Equation (6), Equation (7), Equation (12), and Equation (13).
2. Suppose that an approximation of the value functions are available at a given future decision date t_{n+1} on the grid $\mathcal{G}$, indicated by $\widehat{\Psi}_{n+1}(v_k)$, for $k = 1, \ldots, p$. This is not a strong assumption since the value functions are known in closed form at maturity. We use a piecewise polynomial interpolation for each value function $\widehat{\Psi}_{n+1}$ at t_{n+1} from $\mathcal{G}$ to the overall state space by setting

$$\widehat{\Psi}_{n+1}(v) = \sum_{i=0}^{p} \left(\beta_i^0 + \beta_i^1\, v + \cdots + \beta_i^d\, v^d\right) \mathbb{I}(v_i \le s < v_{i+1}), \qquad \text{for } v > 0,$$

   where d is the degree of the piecewise polynomial, whose local coefficients depend on the time step t_{n+1}.
3. Approximate every expected discounted value function on $\mathcal{G}$ by

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r\,dt} \widehat{\Psi}_{n+1}\left(V_{t_{n+1}}\right) | V_{t_n} = v_k\right] = e^{-r\,dt} \sum_{i=0}^{p} \left(\beta_i^0\, T_{ki}^0 + \cdots + \beta_i^d\, T_{ki}^d\right), \tag{16}$$

   where the transition tables of order j are defined as

$$T_{ki}^j = \mathbb{E}_{\mathbb{Q}}\left[V_{t_{m+1}}^j \mathbb{I}\left(v_i \le V_{t_{m+1}} < v_{i+1}\right) | V_{t_m} = v_k\right], \qquad \text{for } j = 0, \ldots, d.$$

   For example, T_{ki}^0 is the conditional probability that the firm's assets value at t_{n+1} falls in the interval $[v_i, v_{i+1})$, given that the firm's assets value at time t_n is v_k. We present the computation of the transition tables under Merton's (1976), Kou's (2002), and Madan et al.'s (1998) models in Appendices B.1, B.2, and B.3.
4. Compute the value functions at t_n on $\mathcal{G}$ following Equation (8), Equation (9), Equation (10), Equation (11), Equation (14), and Equation (15), using Equation (16).
5. Go to step 2 and repeat until $n = 0$.

5 Numerical investigation

In this section, we study the overall impact of introducing jumps in the dynamic of the firm's assets value. For our numerical investigation, we first focus on Kou's (2002) model and then report some results for the pure-jump credit model of Madan et al. (1998). These settings lead to several scenarios for the credit structure. Under the double-exponential jump model, we concentrate on the scenario of infrequent large jumps. We omit the case of a moderate number of small jumps since it is akin to the pure-diffusion case, especially for short-maturity bonds.

Figure 1 examines the impact of jump volatility versus diffusion volatility on credit spreads when fixing the total volatility for both asset processes. We found that the large infrequent jumps scenario reduces credit spreads for long-maturity bonds while increasing credit spreads for short-maturity bonds. Thus, we overcome the major drawbacks of structural models based on pure diffusion which are incapable of producing significant credit spreads for short-maturity bonds. Hence, by including jumps, our results remain consistent with the empirical work of Sarig and Warga (1989) and Fons (1994), and are also consistent with Chen and Kou (2009). But, unlike Chen and Kou (2009), our framework does not suppose an exponential maturity profile for the debt, rather, it enables arbitrary debt structure, accommodating multiple seniority classes, for instance, senior and junior debt.

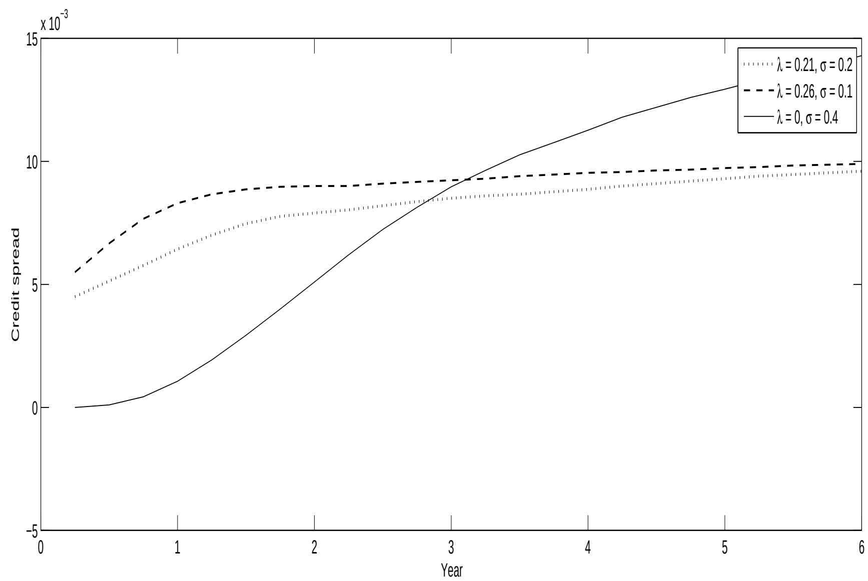

Figure 1: The impact of jump volatility versus diffusion volatility on credit spreads. The debt is a 10% bond with a maturity of 6 years. We use a leverage ratio (debt principal over firm's assets) of 30%. Set $r^c = 35\%$ (per year) and $w = 0.25$. The diffusion parameters are given by $r = 8\%$ (per year), and the jump parameters by $\eta_1 = 3$, $\eta_2 = 2$, and $p_u = 0.5$.

Figure 2 shows that our model can produce upward, humped and downward shapes for term structures of credit spreads depending on the financial situation of the firm. These credit spreads are observable in bond markets and documented in the empirical work of Sarig and Warga (1989). In fact, the authors find a downward slope for bonds rated B or C, a hump shape for rating class BB, and an upward slope for the investment grade classes. Chen and Kou (2009) argue that the credit spread is normally upward sloping and, as the firm's financial situation worsens, it becomes humped and even downward. Moreover, Fons'(1994) empirical paper suggests that speculative bonds may have humped or downward credit spread curves.

Figure 3 shows that our credit-Lévy model is able to generate an upward shape for speculative bonds with a leverage level of 50%. This feature is pointed out in the empirical work of Helwege and Turner (1999) who claim that speculative bonds can have upward-shaped. Figure 3 also shows that the pure-diffusion model is unable to produce a significant credit spread for very short maturity and tends to approach zero as maturity reaches zero. Collin-Dufresne et al. (2001) and Chen and Kou (2009) generate a similar upward credit curve for speculative bonds. With greater flexibility, our model agrees with Chen and Kou's (2009) model. However, Collin-Dufresne et al. (2001) work with diffusion models and are unable to generate non-zero credit spreads as the maturity approaches zero.

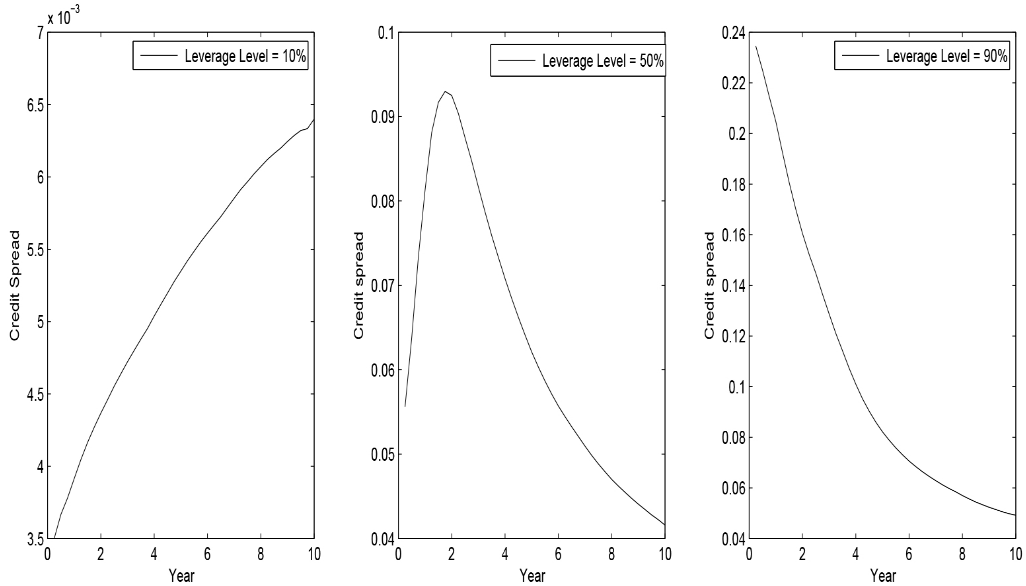

Figure 2: Several shapes of yield spread for a 20% bond for different leverage ratio. Set $V_0 = 100$, $r^c = 35\%$ (per year), and $w = 0.5$. The diffusion parameters are given by $r = 8\%$ (per year) and $\sigma = 0.2$, and the jump parameters by $\eta_1 = 3$, $\eta_2 = 2$, $\lambda = 0.2$ (per year) and with probability of upward jumps $p_u = 0.75$.

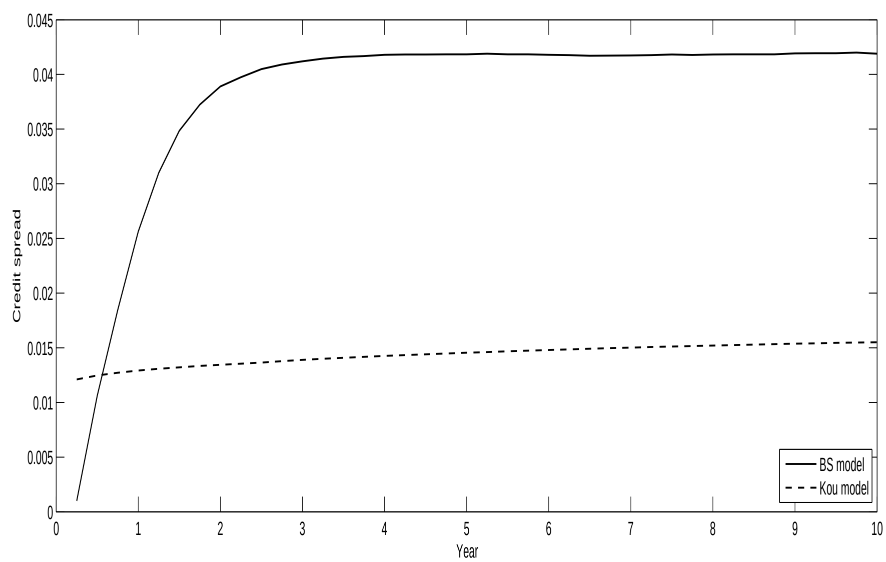

Figure 3: Upward credit spread for high-risk bonds. The debt is a 20% bond with a maturity of 10 years and a principal amount of 50\$. Set $V_0 = 100$, $r^c = 35\%$ (per year), and $w = 0.25$. The jump parameters are $\eta_1 = 3$, $\eta_2 = 2$, $p_u = 0.5$, $\lambda = 0.2062$ (per year), and with $\sigma = 0.1$, for a total volatility of 0.4. Under the pure-diffusion case, the volatility $\sigma = 0.4$.

Figure 4 shows that our setting reproduces the negative relation between the risk-free rate and the credit spreads discussed in empirical papers. Hence, it shows that as the interest rate increases, credit spreads decrease. This feature is discussed in Longstaff and Schwartz (1995) who point out that a higher spot rate increases the risk-neutral drift of the Lévy-value process. Consequently, this reduces the probability of default and further decreases the credit spread.

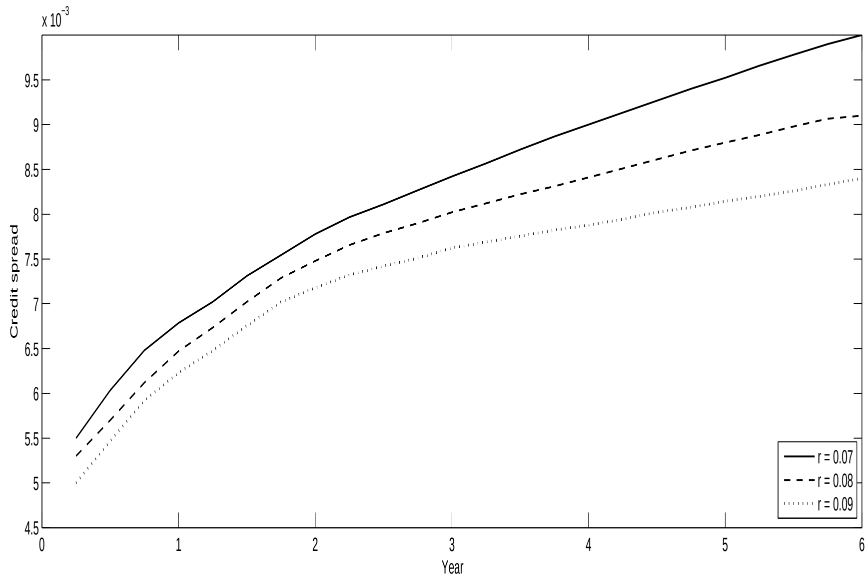

Figure 4: The effect of the risk-free rate on credit spreads. The debt is a 10% bond with a maturity of 6 years, and with a principal amount of $50\$$. **Set** $V_0 = 100$, **and** $r^c = 35\%$ **(per year), and** $w = 0.25$. **The jump parameters are given by** $\eta_1 = 3$, $\eta_2 = 2$, $p_u = 0.5$, $\lambda = 0.2$ **(per year), and with** $\sigma = 0.2$.

Figures 5 and 6 highlight the effect of diffusion volatility and of the jump's frequency on credit spreads. In fact, credit spreads increase with σ and with λ. Figure 5 shows that the effect of diffusion volatility on credit spreads increases with maturity. In other words, diffusion volatility σ has a significant impact on default for long-maturity bonds. Our findings remain consistent with Chen and Kou (2009) results on medium to long maturity bonds.

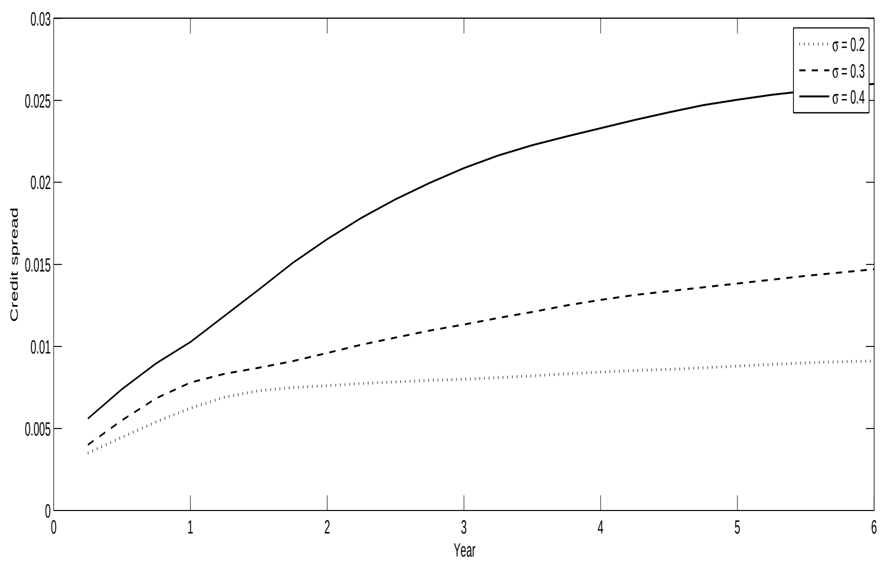

Figure 5: The effect of diffusion volatility on credit spreads. The debt is a 10% bond with a maturity of 6 years, and a principal amount of $50\$$. **Set** $V_0 = 100$, $r = 8\%$ **(per year),** $r^c = 35\%$ **(per year), and** $w = 0.25$. **The jump parameters are given by** $\eta_1 = 3$, $\eta_2 = 2$, $p_u = 0.5$, **and** $\lambda = 0.2$ **(per year).**

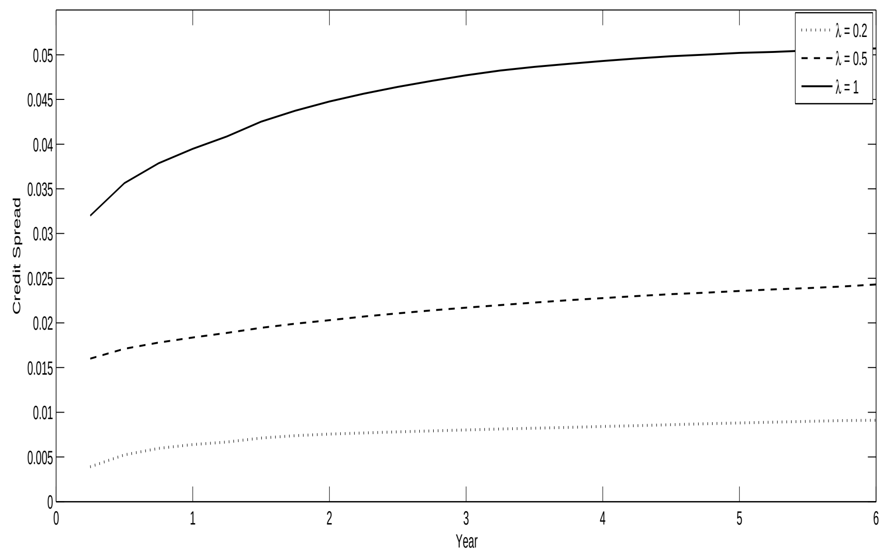

Figure 6: The effect of jump frequency on credit spreads. The debt is a 10% bond with a maturity of 6 years, a principal amount of $50\$$. **Set** $V_0 = 100$, $r = 8\%$ **(per year),** $\sigma = 0.2$, $r^c = 35\%$ **(per year), and** $w = 0.25$. **The jump parameters are given by** $\eta_1 = 3$, $\eta_2 = 2$, **and** $p_u = 0.5$.

We still work under the exponential-jump diffusion model and also compute a call-equity option and obtain the implied volatility by inverting the Black-Scholes formula. While Chen and Kou (2009) evaluates equity option by Monte Carlo simulation, we consider equity options as an additional derivative on the Lévy's assets value and use dynamic programming for valuing all these contingent claims. Thus, we suppose a numerical error, but not a statistical one. Figures 7 and 8 show how default and jumps can, together, generate significant volatility smile, and suggest that the implied volatility and credit spreads tend to be positively correlated. We illustrate the effect of diffusion volatility σ and jump frequency λ on the implied volatility of 1 year maturity call options. Essentially, the implied volatility seems to be increasing in σ and λ. These findings are in line with Chen and Kou (2009).

As a final investigation, we examine the impact of jumps on credit spreads when modeling the Lévy's assets value with the variance-gamma process. Cariboni and Schoutens (2007) calibrate the variance-gamma model to the credit-default-swaps term structure and found a set of estimated parameters for several companies. We selected a Lévy process with the following estimated parameters: $\sigma = 0.3553$, $\nu = 2.8132$, and $\theta = -0.0824$, for a total variance $\sigma_{total} = 0.3812$. Under this setting, we can see that the variance-gamma model generates a positive short-term credit spread. Conversely, for the same total volatility, the geometric-Brownian-motion model fails to capture this main feature observed in actual bond markets. Furthermore, as shown in Figure 9, the impact of jump volatility is limited on longer-term spreads.

Figure 10 reports a credit spread under the variance-gamma process with several total volatility. The applied parameters are also from Cariboni and Schoutens (2007) for two different companies with varying credit-risk ratings. Set the first Lévy's parameters as $\sigma = 0.2041$, $\nu = 0.9644$, and $\theta = -0.0851$, for a total variance of $\sigma_{total} = 0.2205$ and with rating Baa3, relying on Moody's database. Set the second estimated Lévy's parameters as $\sigma = 0.3553$, $\nu = 2.8132$, and $\theta = -0.0824$, for a total variance of $\sigma_{total} = 0.3812$, and with rating A3. From Figure 10, we see that this rating is consistent with the credit spreads generated by our structural Lévy model under the variance-gamma process.

For non-redundancy, we do not report all credit spread figures based on the variance-gamma process. We emphasize that several features discussed under Kou's (2002) model are also confirmed under the variance-gamma model. As we provide all the detailed calculations under the three-Lévy-type models, one can undertake an empirical study and work with the most suitable process by conducting a goodness-of-fit test.

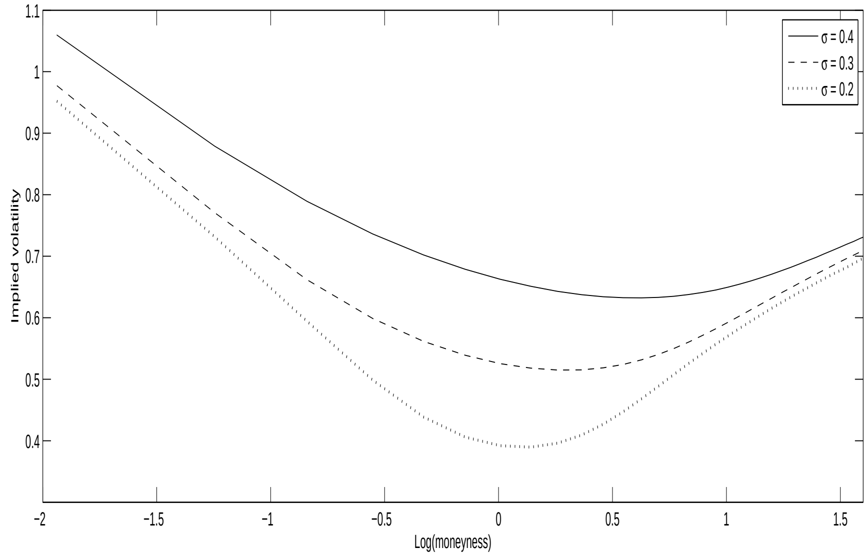

Figure 7: Implied volatility versus credit spreads. The senior debt is a 10% bond with a maturity of 1 year. We use a leverage ratio of 30%, $r = 8\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$. The jump parameters are given by $\eta_1 = 3$, $\eta_2 = 2$, $p_u = 0.5$, and $\lambda = 0.2$.

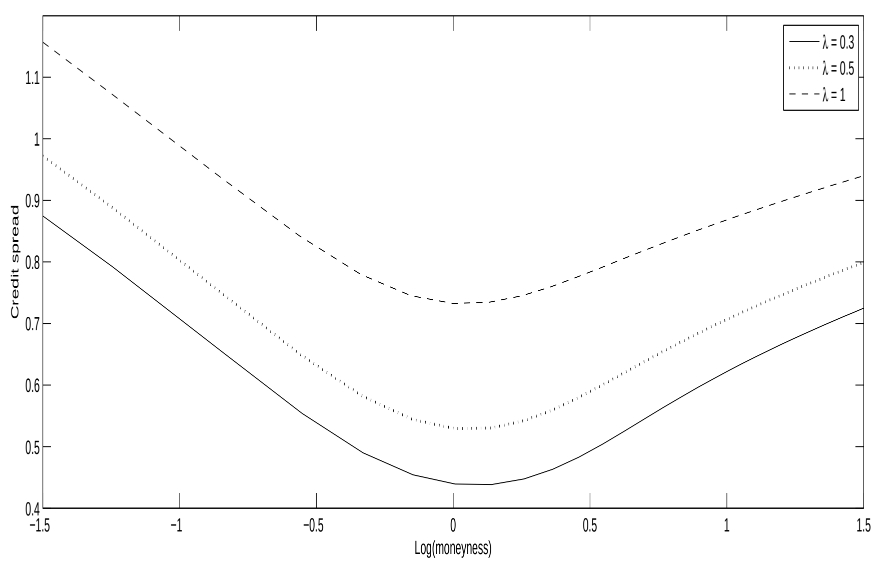

Figure 8: The effect of jump's frequency on implied volatility. The senior debt is a 10% bond with a maturity of 1 year. We use a leverage ratio of 30%, $r = 8\%$ (per year), $\sigma = 0.2$, $r^c = 35\%$ (per year), and $w = 0.25$. The jump parameters are given by $\eta_1 = 3$, $\eta_2 = 2$, and $p_u = 0.5$.

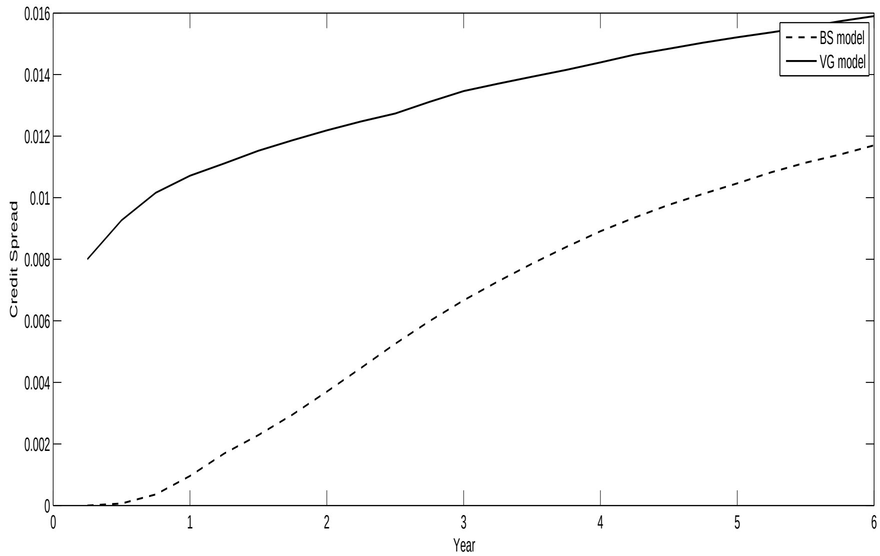

Figure 9: The impact of jump volatility versus diffusion volatility on credit spreads. The senior debt is a 10% bond with a maturity 6 years. We use a leverage ratio of 30%. Set $r = 8\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$, and with a total variance $\sigma_{total} = 0.3812$.

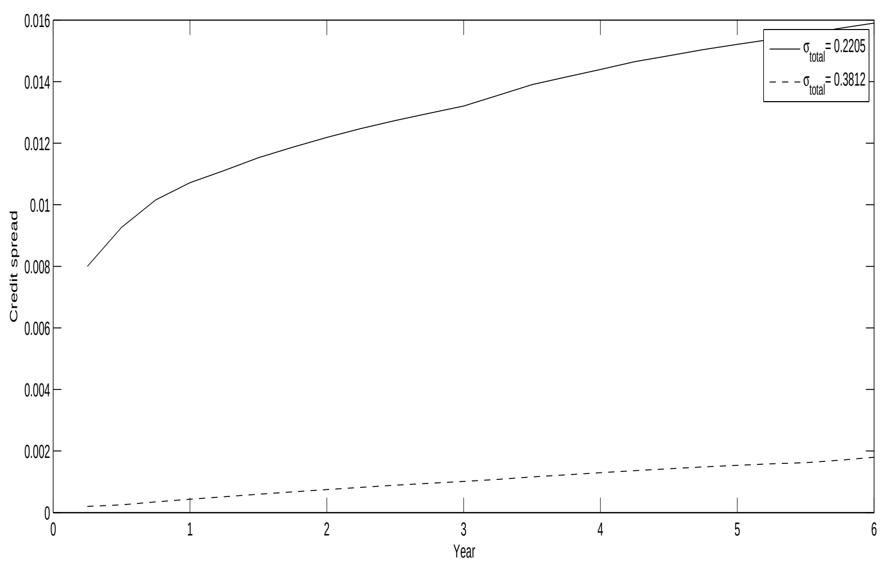

Figure 10: Credit spreads under the variance-gamma process. The debt is a 10% bond with a maturity of 6 years. We use a leverage ratio of 30%. Set $r = 8\%$ (per year), $r^c = 35\%$ (per year), and $w = 0.25$.

6 Conclusion

We propose a dynamic program that introduces Lévy-structural-credit models. Our setting accommodates arbitrary corporate debt, several classes of seniority, tax benefits, and bankruptcy costs. We focus on the contribution of jumps by correcting the shortcomings of modeling the firm's assets under geometric-Brownian motion. In fact, Lévy processes are important in financial modeling as they can mimic the stylized feature observed in bond markets discussed in empirical papers such as in Sarig and Warga (1989). By adding jumps in the firm's assets dynamic, we capture the impact of unexpected components.

Future research avenues that could be explored consist in: working under a reorganization process, valuing bonds with embedded options under Lévy processes, handling structural Lévy frameworks for multidimensional corporate securities, and finally, introducing a default framework under non-Markovian state process.

Appendix A Finite versus infinite-activity-Lévy-processes

Proposition 1 *From Rémillard (2013), let L be a Lévy process with characteristics (a, b, k), where k is the Lévy measure defined on $\mathbb{R}$ such that $k(\{0\}) = 0$ and $\int_{\mathbb{R}\setminus 0} \left(1 \wedge |x|^2\right) k(dx) \leq \infty$. In a finite time interval, the number of jumps of Lévy process can be finite or infinite, according as $k(\mathbb{R}) < \infty$ or $k(\mathbb{R}) = \infty$.*

Proposition 2 *From Rémillard (2013), let L be a Lévy process with characteristics (a, b, k), where k is the Lévy measure defined on $\mathbb{R}$ such that $k(\{0\}) = 0$ and $\int_{\mathbb{R}\setminus 0} \left(1 \wedge |x|^2\right) k(dx) \leq \infty$. A Lévy process has jumps of finite variation if and only if $b = 0$ and $\int_{|x|<1} |x| k(dx) < \infty$.*

Appendix B Transition tables

B.1 Transition tables - Merton (1976)

The transition parameters $T^{\nu}_{k,i}$, for $\nu \in \{0, 1, 2\}$, $k \in \{1, \ldots, p\}$, and $i \in \{0, \ldots, p\}$ are

$$T^{\nu}_{k,i} = \sum_{n=0}^{\infty} \mathbb{Q}\left(N_{\Delta t} = n\right) \; \eta^{\nu}_k(n) e^{c(n)^2/2} \left[\Phi\left(c_{k,i+1}(n) - c\left(n\right)\right) - \Phi\left(c_{k,i}(n) - c\left(n\right)\right)\right],$$

where $N_{\Delta t}$ is the number of jumps over $[t_m, t_{m+1}]$, $c\left(n\right) = \nu \sigma_n \sqrt{\Delta t}$, and

$$\begin{aligned}
\mathbb{Q}\left(N_{\Delta t} = n\right) &= e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^n}{n!}, \\
\sigma_n^2 &= \sigma^2 + \frac{n}{\Delta t}\delta^2, \\
\eta_k(n) &= a_k e^{\left(r - \bar{d} - \lambda\kappa - \sigma_n^2/2\right)\Delta t + n(\gamma + \delta^2/2)}, \\
c_{k,i}(n) &= \frac{\log\left(a_i/a_k\right) - \left(r - \bar{d} - \lambda\kappa - \sigma_n^2/2\right)\Delta t - n(\gamma + \delta^2/2)}{\sigma_n},
\end{aligned}$$

and $\Phi\left(\cdot\right)$ is the standard normal distribution function.

B.2 Transition tables - Kou (2002)

The transition parameters $T^{\nu}_{k,i}$, for $\nu \in \{0, 1, 2\}$, $k \in \{1, \ldots, p\}$, and $i \in \{0, \ldots, p\}$ are

$$\begin{aligned}
T^0_{k,i} &= \Upsilon(\mu_0, \sigma, \lambda, p_1, \eta_1, \eta_2, x_{i+1}, \Delta t) - \Upsilon(\mu_0, \sigma, \lambda, p_1, \eta_1, \eta_2, x_i, \Delta t), \\
T^1_{k,i} &= \rho^{-1} a_k [\Upsilon(\mu_1, \sigma, \tilde{\lambda}, \tilde{p}_1, \tilde{\eta}_1, \tilde{\eta}_2, x_{i+1}, \Delta t) - \Upsilon(\mu_1, \sigma, \tilde{\lambda}, \tilde{p}_1, \tilde{\eta}_1, \tilde{\eta}_2, x_i, \Delta t)], \\
T^2_{k,i} &= b\rho^{-2} a_k^2 [\Upsilon(\mu_2, \bar{\sigma}, \bar{\lambda}, \bar{p}_1, \bar{\eta}_1, \bar{\eta}_2, \bar{x}_{i+1}, \Delta t) - \Upsilon(\mu_2, 2\sigma, \bar{\lambda}, \bar{p}_1, \bar{\eta}_1, \bar{\eta}_2, \bar{x}_i, \Delta t)],
\end{aligned}$$

where $\mu_0 = r - \frac{1}{2}\sigma^2 - \lambda\kappa, x_i = \log(a_i/a_k), \rho = \exp(-(r-\bar{d})\Delta t), \mu_1 = r + \frac{1}{2}\sigma^2 - \lambda\kappa, \tilde{\lambda} = \lambda(1+\kappa), \tilde{p}_1 = p\eta_1/(1+\kappa)(\eta_1 - 1), \tilde{\eta}_1 = \eta_1 - 1, \tilde{\eta}_2 = \eta_2 + 1, \bar{\sigma} = 2\sigma, \bar{\kappa} = p_1(\eta_1/2\bar{\eta}_1) + (1-p_1)(\eta_2/2\bar{\eta}_2) - 1, \mu_2 = 2r + \frac{1}{2}\bar{\sigma}^2 - \lambda\bar{\kappa}, \bar{\lambda} = \lambda(1+\bar{\kappa}), \bar{\eta}_1 = \eta_1/2 - 1, \bar{\eta}_2 = \eta_2/2 + 1, b = \exp(\sigma^2 + \lambda(\bar{\kappa} - 2\kappa)\Delta t)$, and $\bar{x}_i = x_i - \log(b)$. The function $\Upsilon(.)$ is defined by

$$\begin{aligned}
\Upsilon\left(\mu,\sigma,\lambda,\eta_1,\eta_2,p_1,x_i,\Delta t\right) &= \frac{e^{(\sigma\eta_1)^2\Delta t/2}}{\sigma\sqrt{2\pi\Delta t}}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n}P_{n,k}\left(\sigma\sqrt{\Delta t}\eta_1\right)^k \\
&\times I_{k-1}\left(x_i - \mu\Delta t; -\eta_1, -\frac{1}{\sigma\sqrt{\Delta t}}, -\sigma\eta_1\sqrt{\Delta t}\right) \\
&+ \frac{e^{(\sigma\eta_2)^2\Delta t/2}}{\sigma\sqrt{2\pi\Delta t}}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n}Q_{n,k}\left(\sigma\sqrt{\Delta t}\eta_2\right)^k \\
&\times I_{k-1}\left(x_i - \mu\Delta t; \eta_2, \frac{1}{\sigma\sqrt{\Delta t}}, -\sigma\eta_2\sqrt{\Delta t}\right) \\
&+ \pi_0\Phi\left(-\frac{x_i - \mu\Delta t}{\sigma\sqrt{\Delta t}}\right),
\end{aligned}$$

and by

$$\begin{aligned}
&P_{n,k} = \textstyle\sum_{i=k}^{n-1}\begin{pmatrix} n-k-1 \\ i-k \end{pmatrix}\begin{pmatrix} n \\ i \end{pmatrix}\cdot\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i}p_1^i p_2^{n-i}, \\
&Q_{n,k} = \textstyle\sum_{i=k}^{n-1}\begin{pmatrix} n-k-1 \\ i-k \end{pmatrix}\begin{pmatrix} n \\ i \end{pmatrix}\cdot\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k}p_1^{n-i} p_2^{i}, \\
&I_n\left(c;\alpha,\beta,\delta\right) = \textstyle\int_c^{\infty} e^{\alpha x} Hh_n\left(\beta x - \delta\right)dx,
\end{aligned}$$

for arbitrary constants α, c, $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$.

B.3 Transition tables – Variance Gamma - Madan et al. (1998)

From Madan et al. (1998), we define the degenerate hypergeometric function of two variables $\Psi(a,b,\gamma)$ in terms of the modified Bessel function of the second kind $K(.)$ as

$$\begin{aligned}
\Psi(a,b,\gamma) &= \frac{c^{\gamma+\frac{1}{2}}\exp(\operatorname{sign}(a)c)(1+u)^{\gamma}}{\sqrt{(2\pi)}\Gamma(\gamma)\gamma}K_{\gamma+\frac{1}{2}}(c)\times \\
&\times \Phi(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\operatorname{sign}(a)c(1+u)) \\
&- \operatorname{sign}(a)\frac{c^{\gamma+\frac{1}{2}}\exp(\operatorname{sign}(a)c)(1+u)^{1+\gamma}}{\sqrt{(2\pi)}\Gamma(\gamma)(1+\gamma)}K_{\gamma-\frac{1}{2}}(c) \\
&\times \Phi(1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -\operatorname{sign}(a)c(1+u)) \\
&+ \operatorname{sign}(a)\frac{c^{\gamma+\frac{1}{2}}\exp(\operatorname{sign}(a)c)(1+u)^{\gamma}}{\sqrt{(2\pi)}\Gamma(\gamma)\gamma}K_{\gamma-\frac{1}{2}}(c) \\
&\times \Phi(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\operatorname{sign}(a)c(1+u)),
\end{aligned}$$

where $c = |a|\sqrt{2+b^2}$, $u = b/\sqrt{2+b^2}$, and where the degenerate hypergeometric function of two variables Φ has the integral representation

$$\Phi\left(\alpha,\beta,\gamma;x,y\right) = \frac{\Gamma\left(\gamma\right)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}e^{uy}du.$$

Let $x = \frac{1+u}{2}$, $\lambda = 2\,\text{sign}(a)c$, so that $c = |\lambda|/2$, and set

$$\begin{aligned}\Psi_1(x,\lambda,\gamma) &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma}}{2\sqrt{\pi}\Gamma(\gamma)\gamma} K_{\gamma+\frac{1}{2}}(c)\, \Phi(\gamma, 1-\gamma, 1+\gamma; x, -\lambda x) \\ &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(c) \int_0^1 z^{\gamma-1}(1-zx)^{\gamma-1} e^{-\lambda zx} dz\end{aligned}$$

$$\begin{aligned}\Psi_2(x,\lambda,\gamma) &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma+1}}{\sqrt{\pi}\Gamma(\gamma)(\gamma+1)} K_{\gamma-\frac{1}{2}}(c)\, \Phi(1+\gamma, 1-\gamma, 2+\gamma; x, -\lambda x) \\ &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma+1}}{\sqrt{\pi}\Gamma(\gamma)} K_{\gamma-\frac{1}{2}}(c) \int_0^1 z^{\gamma}(1-zx)^{\gamma-1} e^{-\lambda zx} dz\end{aligned}$$

$$\Psi_3(x,\lambda,\gamma) = \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(c) \int_0^1 z^{\gamma-1}(1-zx)^{\gamma-1} e^{-\lambda zx} dz.$$

If $\lambda > 0$, let $t = \lambda xz$, set

$$\Psi_1(x,\lambda,\gamma) = \frac{\sqrt{\lambda} e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(\lambda/2) \int_0^{\lambda x} t^{\gamma-1}\left(1-\frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt,$$

and set $I_1(x,\lambda,\gamma) = \int_0^{\lambda x} t^{\gamma-1}\left(1-\frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt$. Integrating by parts,

$$\begin{aligned}I_1(x,\lambda,\gamma) \quad = \quad & \frac{(\lambda x)^{\gamma}}{\gamma}(1-x)^{\gamma-1} e^{-\lambda x} + \frac{1}{\gamma}\int_0^{\lambda x} t^{\gamma}\left(1-\frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt \\ & + \frac{\gamma-1}{\lambda\gamma}\int_0^{\lambda x} t^{\gamma}\left(1-\frac{t}{\lambda}\right)^{\gamma-2} e^{-t} dt.\end{aligned}$$

Set $h_1(x,\lambda,\gamma) = \int_0^{\lambda x} t^{\gamma}\left(1-\frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt$ and $h_2(x,\lambda,\gamma) = \int_0^{\lambda x} t^{\gamma}\left(1-\frac{t}{\lambda}\right)^{\gamma-2} e^{-t} dt$, where h_1 and h_2 are evaluated by Gauss-Legendre quadrature. Then

$$\begin{aligned}\Psi_2(x,\lambda,\gamma) &= \frac{e^{\lambda/2}}{\sqrt{\pi\lambda}} K_{\gamma-\frac{1}{2}}(\lambda/2) \int_0^{\lambda x} t^{\gamma}\left(1-\frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt \\ &= \frac{e^{\lambda/2}}{\sqrt{\pi\lambda}} K_{\gamma-\frac{1}{2}}(\lambda/2)\, h_1(x,\lambda,\gamma),\end{aligned}$$

and $\Psi_3(x,\lambda,\gamma) = \frac{\sqrt{\lambda} e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma-\frac{1}{2}}(\lambda/2)\, I_1(x,\lambda,\gamma)$, hence

$$\Psi(x,\lambda,\gamma) = \Psi_1(x,\lambda,\gamma) - \text{sign}(a)\,\Psi_2(x,\lambda,\gamma) + \text{sign}(a)\,\Psi_3(x,\lambda,\gamma).$$

If $\lambda < 0$, let $t = -\lambda xz$, let

$$\Psi_1(x,\lambda,\gamma) = \frac{\sqrt{-\lambda} e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(-\lambda/2) \int_0^{-\lambda x} t^{\gamma-1}\left(1+\frac{t}{\lambda}\right)^{\gamma-1} e^{t} dt,$$

and set $I_2(x,\lambda,\gamma) = \int_0^{-\lambda x} t^{\gamma-1}\left(1+\frac{t}{\lambda}\right)^{\gamma-1} e^{t} dt$. Integrating by parts,

$$\begin{aligned}I_2(x,\lambda,\gamma) = & \frac{(-\lambda x)^{\gamma}}{\gamma}(1-x)^{\gamma-1} e^{-\lambda x} - \frac{1}{\gamma}\int_0^{-\lambda x} t^{\gamma}\left(1+\frac{t}{\lambda}\right)^{\gamma-1} e^{t} dt \\ & - \frac{\gamma-1}{\lambda\gamma}\int_0^{-\lambda x} t^{\gamma}\left(1+\frac{t}{\lambda}\right)^{\gamma-2} e^{t} dt.\end{aligned}$$

Set $h_3(x,\lambda,\gamma)=\int_0^{-\lambda x} t^{\gamma}\left(1+\frac{t}{\lambda}\right)^{\gamma-1}e^t dt$ and $h_4(x,\lambda,\gamma)=\int_0^{-\lambda x} t^{\gamma}\left(1+\frac{t}{\lambda}\right)^{\gamma-2}e^t dt$, where h_3 and h_4 are evaluated by Gauss-Legendre quadrature. Then

$$\Psi_2(x,\lambda,\gamma)=\frac{\sqrt{(-\lambda)}e^{\lambda/2}}{\lambda\sqrt{\pi}\Gamma(\gamma)}K_{\gamma-\frac{1}{2}}\left(-\lambda/2\right)h_3(x,\lambda,\gamma),$$
$$\Psi_3(x,\lambda,\gamma)=\frac{\sqrt{(-\lambda)}e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)}K_{\gamma-\frac{1}{2}}\left(-\lambda/2\right)I_2(x,\lambda,\gamma).$$

As a result,

$$\Psi(x,\lambda,\gamma)=\Psi_1(x,\lambda,\gamma)-\operatorname{sign}(a)\Psi_2(x,\lambda,\gamma)+\operatorname{sign}(a)\Psi_3(x,\lambda,\gamma).$$

The transition parameters $T^j_{k,i}$, for $j\in\{0,1,2\}$, $k\in\{1,\dots,p\}$, and $i\in\{0,\dots,p\}$ are

$$T^0_{k,i}=\Psi(x_0,\lambda^{(0)}_i,dt/\nu)-\Psi(x_0,\lambda^{(0)}_{i+1},dt/\nu),$$
$$T^1_{k,i}=\rho^{-1}a_k\Big[\Psi(x_1,\lambda^{(1)}_i,dt/\nu)-\Psi(x_1,\lambda^{(1)}_{i+1},dt/\nu)\Big],$$
$$T^2_{k,i}=e^{\eta_2}\rho^{-2}a_k^2\Big[\Psi(x_2,\lambda^{(2)}_i,dt/\nu)-\Psi(x_2,\lambda^{(2)}_{i+1},dt/\nu)\Big],$$

where $\rho=\exp\{-(r-q)dt\}$, $x_0=\frac{1+u_0}{2}$, $u_0=\frac{b_0}{\sqrt{2+b_0^2}}$, $b_0=\alpha\sqrt{\frac{\nu}{1-\xi_2}}$, $\xi_2=\frac{\nu\alpha^2}{2}$, $\alpha=\zeta s$, with $\zeta=\frac{\theta}{\sigma^2}$ and $s=\frac{\sigma}{\sqrt{1+\left(\frac{\theta}{\sigma}\right)^2\frac{\nu}{2}}}$. Thus $\lambda^{(0)}_i=2\operatorname{sign}(a_0)c_0$, with $c_0=|a_0|\sqrt{2+b_0^2}$, $a_0=d_i\sqrt{\frac{1-\xi_2}{\nu}}$, $\xi_1=\frac{\nu(\alpha+s)^2}{2}$, and

$$d^{(0)}_i=\frac{1}{s}\Big[\ln\left(\frac{a_k}{a_i}\right)+rdt+\frac{dt}{\nu}\ln\left(\frac{1-\xi_1}{1-\xi_2}\right)\Big].$$

Next, $x_1=\frac{1+u_1}{2}$, $u_1=\frac{b_1}{\sqrt{2+b_1^2}}$, $b_1=(\alpha+s)\sqrt{\frac{\nu}{1-\xi_1}}$, $\lambda^{(1)}_i=2\operatorname{sign}(a_1)c_1$, with $c_1=|a_1|\sqrt{2+b_1^2}$, $a_1=d^{(0)}_i\sqrt{\frac{1-\xi_1}{\nu}}$.

Further, let $r_2=2r$, $\sigma_2=2\sigma$, $\theta_2=2\theta$, $q_2=2q$, $\alpha_2=\zeta_2 s_2$, with $\zeta_2=\frac{\theta_2}{\sigma_2^2}$ and $s_2=\frac{\sigma_2}{\sqrt{1+\left(\frac{\theta_2}{\sigma_2}\right)^2\frac{\nu}{2}}}$. Then $x_2=\frac{1+u_2}{2}$, $u_2=\frac{b_2}{\sqrt{2+b_2^2}}$, $b_2=(\alpha_2+s_2)\sqrt{\frac{\nu}{1-\xi^{(2)}_1}}$, with $\xi^{(2)}_1=\frac{\nu(\alpha_2+s_2)^2}{2}$. Thus, $\lambda^{(2)}_i=2\operatorname{sign}(a_2)c_2$, with $c_2=|a_2|\sqrt{2+b_2^2}$, $a_2=d^{(2)}_i\sqrt{\frac{1-\xi^{(2)}_2}{\nu}}$, $\xi^{(2)}_2=\frac{\nu\alpha_2^2}{2}$ and

$$d^{(2)}_i=\frac{1}{s_2}\Big[2\ln\left(\frac{a_k}{a_i}\right)+r_2dt+\eta_2+\frac{dt}{\nu}\ln\left(\frac{1-\xi^{(2)}_1}{1-\xi^{(2)}_2}\right)\Big],$$

where $\eta_2=2w_1-w_2$, $w_1=\frac{dt}{\nu}\ln\left(\frac{1-\xi_1}{1-\xi_2}\right)$, and $w_2=\frac{dt}{\nu}\ln\left(\frac{1-\xi^{(2)}_1}{1-\xi^{(2)}_2}\right)$.

Thus, for $j\geq 3$

$$T^j_{k,i}=e^{\eta_j}\rho^{-j}a_k^j\Big[\Psi(x_j,\lambda^{(j)}_i,dt/\nu)-\Psi(x_j,\lambda^{(j)}_{i+1},dt/\nu)\Big],$$

where $r_j=jr$, $\sigma_j=j\sigma$, $\theta_j=j\theta$, $q_j=jq$, $\alpha_j=\zeta_j s_j$, with $\zeta_j=\frac{\theta_j}{\sigma_j^2}$ and $s_j=\frac{\sigma_j}{\sqrt{1+\left(\frac{\theta_j}{\sigma_j}\right)^2\frac{\nu}{2}}}$. Hence, $x_j=\frac{1+u_j}{2}$, $u_j=\frac{b_j}{\sqrt{2+b_j^2}}$, $b_j=\left(\alpha_j+s_j\right)\sqrt{\frac{\nu}{1-\xi^{(j)}_1}}$, with $\xi^{(j)}_1=\frac{\nu\left(\alpha_j+s_j\right)^2}{2}$, $\xi^{(j)}_2=\frac{\nu\alpha_j^2}{2}$, and $\lambda^{(j)}_i=2\operatorname{sign}\left(a_j\right)c_j$, with $c_j=\left|a_j\right|\sqrt{2+b_j^2}$, $a_j=d^{(j)}_i\sqrt{\frac{1-\xi^{(j)}_2}{\nu}}$, and

$$d^{(j)}_i=\frac{1}{s_j}\Big[j\ln\left(\frac{a_k}{a_i}\right)+r_jdt+\eta_j+\frac{dt}{\nu}\ln\left(\frac{1-\xi^{(j)}_1}{1-\xi^{(j)}_2}\right)\Big],$$

where $\eta_j=jw_1-w_j$, $w_1=\frac{dt}{\nu}\ln\left(\frac{1-\xi_1}{1-\xi_2}\right)$, and $w_j=\frac{dt}{\nu}\ln\left(\frac{1-\xi^{(j)}_1}{1-\xi^{(j)}_2}\right)$.

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