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When should a retailer invest in brand advertising?

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Abstract: We consider a dynamic marketing channel comprising of one manufacturer and one retailer, where consumer demand depends on price and on brand reputation. We investigate two scenarios. In the first one, the retailer may also invest in brand advertising, while in the second, it does not. Comparing the results of the two scenarios yields interesting insights into the impact of the retailer's brand advertising. Findings include that when the retailer invests in the brand's reputation, both players achieve higher profits and the consumer obtains a higher surplus. This gives the retailer's brand advertising a (partial) channel coordination flavor, a result that has never before been reported in the literature.

Keywords: Marketing channels, advertising, differential games, brand reputation

Résumé : On considère un jeu dynamique représentant un canal de distribution formé d'un manufacturier et d'un détaillant où la demande dépend du prix et de la réputation de la marque. On détermine et contraste les stratégies d'équilibres et gains dans deux scénarios. Dans le premier, les deux joueurs investissent en publicité pour améliorer la réputation de la marque, tandis que dans le second le détaillant ne le fait pas. On montre que l'investissement publicitaire du détaillant améliore le profit des deux joueurs ainsi que la valeur du surplus du consommateur. Ce résultat signifie que la promotion de la marque par le détaillant joue un rôle de coordination du canal.

Mots clés : Canaux de distribution, publicité, jeux différentiels, réputation de la marque

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1 Introduction

There is a sizable literature on marketing channels where advertising efforts are part of the model. This is in fact quite natural since these efforts influence demand and hence are payoff-relevant to the parties (manufacturers and retailers). Advertising activities are generally divided into brand advertising, which is typically done by the (national) manufacturer that owns the brand, and local advertising, which are activities managed by the retailers. Brand advertising aims at building a brand's reputation (brand equity or goodwill), and is a determinant of both current and future sales. Because the effect of brand advertising on demand is indirect, textbooks refer to it as pull marketing. As brand advertising has a carryover effect, it must be considered an investment and not an expenditure. Consequently, a model designed to properly account for its cost and benefit must be dynamic; otherwise, we are ignoring the future impact of brand advertising, which comes down to adopting a myopic view of what this activity is. This carryover effect is a key element in the abundant literature using dynamic optimization and dynamic games in advertising models. We refrain from reviewing this literature, given that there are recent surveys. Huang et al. (2012) give an extensive and comprehensive survey of dynamic models of advertising competition since 1994, their coverage starting where the previous survey in Feichtinger et al. (1994) stopped. Jørgensen and Zaccour (2004) cover advertising models in oligopolies (horizontal strategic interactions) and in marketing channels (vertical strategic interactions), whereas He et al. (2007) focus on Stackelberg differential game models in supply chains and marketing channels that include advertising. Finally, Aust and Buscher (2014) and Jørgensen and Zaccour (2014) concentrate on cooperative advertising in marketing channels.

Local advertising, which is often referred to as non-price promotional activities (sales displays, flyers, etc.), aims at pushing products to consumers and boosting current sales. Typically, the dynamic games literature on supply chains and marketing channels has made the assumption that the retailer's advertising, e.g., in-store displays or advertising in local newspapers, affects current sales and does not impact on manufacturer's brand reputation. To the best of our knowledge, only two papers have in some way deviated from this framework: Jørgensen et al. (2000, 2003). In Jørgensen et al. (2000), the authors consider a retailer that invests in both short- and long-term advertising in an infinite-horizon differential game, with a main focus on cooperative advertising. In Jørgensen et al. (2003), the assumption is that (too much) local advertising harms the brand image, which affects future demand.

There is ample empirical evidence that retailers (and especially large ones) run advertising campaigns that feature manufacturers' brands. Examples include Best Buy's 2015 campaign for the Samsung UHD TV,[1] Walmart's 2011 campaign for Coca-Cola[2] and Fnac's 2013 campaign for Apple products.[3] The videos used in these advertisements emphasize the product's brand name and features, together with the retailer's own information. These campaigns, which ran on national television networks, do not focus, as one might expect, on the retail chain itself, on the product categories it sells or on its special offers. These campaigns are clearly not of the local promotion/advertising variety, but rather are co-brand advertising campaigns. By endorsing the brand, the retailer is contributing to its reputation and is not seeking an immediate response, which is the effect sought by push marketing. If this reasoning is correct, then the retailer's brand advertising must be accounted for properly in terms of its influence on brand reputation and future demand. Again, this calls for the use of a dynamic model.

We consider a marketing channel with a finite planning horizon and we characterize equilibrium pricing and advertising strategies and outcomes in two scenarios: in the first one, the retailer engages in brand advertising, whereas in the second, it does not. By contrasting the results of the two scenarios, we aim to answer the following research questions:

1. How do advertising strategies and brand reputation compare in the two scenarios?
2. How do wholesale and retail prices compare in the two scenarios?

[1] Best Buy Samsung Ultra Curve UHD TV commercial, "Adam's Beta Test", retrieved from https://www.ispot.tv/ad/7XjQ/best-buy-samsung-ultra-curve-uhd-adams-beta-test on March 20, 2017.

[2] Walmart/Coke Commercial, "Stock Up on Joy," retrieved from https://www.youtube.com/watch?v=AXXPIk4v6aw on March 20, 2017.

[3] Pub Fnac - Apple 2013, "Léa veut tout le temps savoir 'pourquoi?'", retrieved from https://www.youtube.com/watch?v=VVXwv-BvdM4 on March 20, 2017.

3. Under what conditions is it optimal for the retailer to do brand advertising?
4. How does the retailer's brand advertising affect the manufacturer's payoff and the consumer's surplus?

Our results show that the price and the wholesale price are both increasing in reputation, and that the manufacturer does more brand advertising when the retailer also does, and thus, that the brand's reputation is higher in the first scenario. It follows that the price and the wholesale price are always higher when the retailer engages in brand advertising. This is also the case for the retailer's local advertising.

In terms of instantaneous profits, there is a certain time threshold for each firm until which the first scenario is less profitable. Each firm focuses on building up reputation in the short run, leading to higher costs and lower profits. In the sequel, they benefit from the higher demand achieved due to a boosted reputation. At the end of the game, each firm's total profits are higher in the first scenario. Therefore, it is always optimal for the retailer to engage in brand advertising.

Lastly, a higher reputation increases the market potential, leading to a larger number of consumers accessing the product. The positive effect of reputation on consumer surplus outweighs the negative effect of higher price. Consequently, the consumer's surplus is also higher in the first scenario.

The rest of the paper is structured as follows: In Section 2, we introduce the model, and in Section 3, we solve for the two scenarios. Section 4 analytically compares the equilibrium strategies and outcomes in the two scenarios and, Section 5 completes this comparison with some numerical results. Section 6 briefly concludes.

2 The model

Consider a marketing channel made up of one manufacturer, player M, and one retailer, player R. At each instant of time $t \in [0, T]$, where T is the planning horizon, the manufacturer decides on the wholesale price $w(t)$ and the brand advertising $b_M(t)$, whereas the retailer chooses the price to consumers $p(t)$, local advertising effort $l_R(t)$—e.g. flyers, display—and the investment in brand advertising $b_R(t)$. The reputation $r(t)$ of the manufacturer's brand evolves according to the following dynamics:

$$\frac{dr}{dt}(t) = \dot{r}(t) = \psi_M b_M(t) + \psi_R b_R(t) - \delta r(t), \qquad r(0) = r_0, \tag{1}$$

where $\psi_i \geq 0$, $i = M, R$, is player i's advertising efficiency parameter, and δ is the positive decay rate. The case where the retailer does not contribute to brand advertising is captured by setting $\psi_R = 0$.

The demand depends on the retail price, the retailer's local advertising effort and the brand's reputation. Following a long tradition in economics and marketing, we retain the following linear form:

$$D(p(t), l_R(t), r(t)) = \theta - \alpha p(t) + \beta l_R(t) + \gamma r(t), \tag{2}$$

where θ, α, β and γ are positive parameters. The parameters α, β and γ measure demand's sensitivity to a change in the value of the corresponding variable, that is, the price, local advertising and reputation, respectively. The above demand shows that the market potential is not a given constant but depends on local advertising effort and brand reputation and is given by $\theta + \beta l_R(t) + \gamma r(t)$. Whereas local advertising activities directly affect current demand, brand advertising indirectly affects the demand through reputation. We shall refer to θ as the intrinsic market size.

Following the literature—see, e.g., Huang et al. (2012) and Jørgensen and Zaccour (2004)—the advertising costs are assumed to be increasing and convex and taken quadratic for tractability, that is,

$$G_M^b(b_M) = \frac{\phi_M^b b_M^2}{2}, \quad G_R^l(l_R) = \frac{\phi_R^l l_R^2}{2}, \quad G_R^b(b_R) = \frac{\phi_R^b b_R^2}{2},$$

where ϕ_M^b, ϕ_R^l and ϕ_R^b are positive parameters.

Denote by c the manufacturer's marginal production cost. Assuming profit-maximization behavior, the manufacturer's and retailer's objective functionals are given by

$$J_M = \int_0^T \left((w(t) - c) D(p(t), l_R(t), r(t)) - G_M^b(b_M(t)) \right) dt + S_M(r(T)), \tag{3}$$

$$J_R = \int_0^T \left((p(t) - w(t)) D(p(t), l_R(t), r(t)) - G_R^l(l_R(t)) - G_R^b(b_R(t)) \right) dt + S_R(r(T)), \tag{4}$$

where $S_i(r(T))$ is the salvage value of player $i = M, R$. We suppose that this value can be well approximated by the linear function

$$S_i(r(T)) = s_i r(T), \tag{5}$$

where s_i is a nonnegative parameter.

To recapitulate, by (3), (4) and (1), we have defined a two-player finite-horizon differential game with one state variable (r) and five control variables: three for the retailer (p, l_R, b_R) and two for the manufacturer (w, b_M).

In the sequel, we shall determine and contrast the equilibrium solutions in the two scenarios, where the retailer invests and does not invest in brand advertising, respectively. In both scenarios, we follow the established tradition in marketing channels (see, e.g., the books by Ingene and Parry (2004) and Jørgensen and Zaccour (2004), and the survey in Ingene et al. (2012)), and assume that the game is played à la Stackelberg, with the manufacturer acting as the leader and the retailer as the follower. We retain a feedback-information structure, that is, the players' strategies are functions of the state variable and time. Note that with few exceptions—see, e.g., Buratto et al. (2007), Buratto and Zaccour (2009), and Buratto (2012, 2013)—the dynamic marketing channels literature has assumed an infinite horizon. Here, the game is played over a finite horizon, which may be a more realistic format given how quickly technology and consumer tastes change. In a sense, the planning horizon T can then be interpreted as a season (in the case of fashion or perishable products) or the date at which the good under investigation becomes obsolete. The by-product of having a finite horizon is that the strategies and the value functions will be time dependent and much harder to compute than in an infinite-horizon model. The reason for this is that the resulting Ricatti equations form a differential-equation system instead of an algebraic one, as is the case in an infinite-horizon model.

To determine the impact of the retailer's brand advertising on consumers, we use the following expression for the consumer surplus:

$$CS(p, l_R, r) = \int_{s=p}^{\hat{p}(l_R, r)} D(s, l_R, r) ds = \frac{(D(p, l_R, r))^2}{2\alpha},$$

where $\hat{p}(l_R, r)$ is such that $D(\hat{p}(l_R, r), l_R, r) = 0$.

3 The two scenarios

In this section, we derive the system of differential equations to be solved in both scenarios. Denote by $V_i(r, t)$ the value function of player $i = M, R$. To obtain the retailer's reaction function to the manufacturer's announcement of a wholesale price and an investment in brand advertising, we need to solve the retailer's optimization problem. We shall omit the time argument from now on, when no ambiguity may arise.

3.1 Retailer invests in brand advertising

Introduce the retailer's Hamilton-Jacobi-Bellman (HJB) equation:

$$-\frac{\partial V_R}{\partial t} = \max_{p, l_R, b_R} \Big\{ (p - w) D(p, l_R, r) - G_R^l(l_R) - G_R^b(b_R) + \frac{\partial V_R}{\partial r}(\psi_M b_M + \psi_R b_R - \delta r) \Big\}. \tag{6}$$

Assuming an interior solution, differentiating the right-hand side of (6) with respect to p, l_R and b_R, respectively, and equating to zero, we obtain the following system:

$$\begin{aligned} \theta + \beta l_R + \gamma r - 2\alpha p + \alpha w &= 0, \\ \beta\,(p - w) - \phi_R^l l_R &= 0, \\ -\phi_R^b b_R + \psi_R \frac{\partial V_R}{\partial r} &= 0. \end{aligned}$$

Solving these equations yields the price and local advertising effort as functions of the manufacturer's wholesale price and the brand's reputation, that is,

$$p^*(w,r) = \frac{\phi_R^l\,(\theta + \gamma r) + \left(\alpha\phi_R^l - \beta^2\right) w}{2\alpha\phi_R^l - \beta^2}, \tag{7}$$

$$l_R^*(w,r) = \frac{\beta\,(\theta + \gamma r - \alpha w)}{2\alpha\phi_R^l - \beta^2} \tag{8}$$

and the following retailer's brand-advertising reaction function:

$$b_R^*(r,t) = \frac{\psi_R}{\phi_R^b}\frac{\partial V_R}{\partial r}(r,t). \tag{9}$$

We can make the following comments on the above reaction functions: (i) The price $p^*(w,r)$ and the local advertising $l_R^*(w,r)$ are independent of the manufacturer's brand advertising and value function. This result is due to the fact that there is no interaction between these variables and b_M, and that they do not appear in the dynamics. Although they are state dependent, they result from a static optimization. (ii) Under the following restriction on the parameter values, the price is increasing in reputation:

$$2\alpha\phi_R^l - \beta^2 > 0. \tag{10}$$

As it is intuitively reasonable to suppose that a brand enjoying a higher reputation commands a higher retail price, we shall assume from now on that the above inequality holds. (iii) Under (10), we have that local advertising is decreasing in the wholesale price, meaning that these two variables are strategic substitutes. Further, local advertising is increasing in reputation. (iv) The strategic interaction between the retail price and the wholesale price depends on the sign of $\alpha\phi_R^l - \beta^2$. If this term is positive, then p and w are strategic complements; otherwise, they are strategic substitutes. (v) Finally, we see from (9) that for our assumption of an interior solution to hold true, the retailer's value function must be increasing in the brand's reputation.

To solve the manufacturer's optimization problem, we substitute for (7), (8) and (9) in its objective (3) and in the dynamics (1), and write the HJB as follows:

$$-\frac{\partial V_M}{\partial t} = \max_{w,b_M}\Big\{\,(w-c)\,D(p^*(w,r), l_R^*(w,r), r) - G_M^b(b_M) + \frac{\partial V_M}{\partial r}(\psi_M b_M + \psi_R b_R^* - \delta r)\Big\}. \tag{11}$$

Assuming an interior solution, differentiating the right-hand side of (11) with respect to w and b_M leads to

$$w^*(r) = \frac{1}{2\alpha}\,(\theta + \alpha c + r\gamma)\,, \tag{12}$$

$$b_M^*(r,t) = \frac{\psi_M}{\phi_M}\frac{\partial V_M}{\partial r}(r,t). \tag{13}$$

Two comments can be made regarding the above policies. First, the wholesale price is linear and increasing in brand reputation. Whereas linearity is a by-product of the game's linear-quadratic structure, the positive relationship between the wholesale price and reputation can be traced back to the fact that building reputation is costly and that higher reputation commands a higher willingness-to-pay by the consumer, with the manufacturer cashing in on part of this. Second, our assumption of an interior solution implies that the manufacturer's value function is increasing in its brand reputation, which is what we also found for the retailer. Replacing $w^*(r)$ in $p^*(w,r)$ and $l_R^*(w,r)$ leads to the following pricing and local advertising linear policies for the retailer:

$$p^*(r) = \frac{\left(3\alpha\phi_R^l - \beta^2\right)(\theta + r\gamma) + \alpha c\left(\alpha\phi_R^l - \beta^2\right)}{2\alpha\left(2\alpha\phi_R^l - \beta^2\right)}, \tag{14}$$

$$l_R^*(r) = \frac{\beta\left(\theta + \gamma r - \alpha c\right)}{2\left(2\alpha\phi_R^l - \beta^2\right)}. \tag{15}$$

Computing the difference between $p^*(r)$ and $w^*(r)$, we obtain the following retailer's margin:

$$m_R^*(r) = p^*(r) - w^*(r) = \frac{\phi_R^l}{2}\left(\frac{\theta + r\gamma - \alpha c}{2\alpha\phi_R^l - \beta^2}\right), \tag{16}$$

which is increasing in reputation. Similarly, the manufacturer's margin, which is given by

$$m_M^*(r) = w^*(r) - c = \frac{1}{2\alpha}\left(\theta - \alpha c + r\gamma\right), \tag{17}$$

is also increasing in the brand's reputation. This gives both players an incentive to invest in brand advertising to raise reputation and secure higher margins. Further, substituting for $l_R^*(r)$ and $p^*(r)$ in the demand, we obtain

$$D^*(r) = \alpha m_R^*(r), \tag{18}$$

that is, the demand is proportional to the retailer's margin. Therefore, demand also exhibits a positive relationship with the brand's reputation.

The difference of the two players' margins is given by

$$m_M^*(r) - m_R^*(r) = \frac{\left(\alpha\phi_R^l - \beta^2\right)\left(\theta + \gamma r - \alpha c\right)}{2\alpha\left(2\alpha\phi_R^l - \beta^2\right)}.$$

The sign of the above expression is the same as the sign of $\left(\alpha\phi_R^l - \beta^2\right)$, implying that the manufacturer makes a larger margin than the retailer when the retail and wholesale prices are strategic complements, that is, when $\left(\alpha\phi_R^l - \beta^2\right) > 0$. If $\left(\alpha\phi_R^l - \beta^2\right)$ is negative, then the retailer obtains a larger margin and we have a strategic substitution.

Substituting for (9), (12), (13), (14) and (15) into the HJB equations (6) and (11) and rearranging the terms leads to the following equations:

$$-\frac{\partial V_M}{\partial t}(r,t) = 2\kappa_1 + 2\kappa_2 r + 2\kappa_3 r^2 + \kappa_4 r\frac{\partial V_M}{\partial r}(r,t) + \kappa_5\left(\frac{\partial V_M}{\partial r}(r,t)\right)^2 + 2\kappa_6\frac{\partial V_M}{\partial r}(r,t)\frac{\partial V_R}{\partial r}(r,t), \tag{19}$$

$$-\frac{\partial V_R}{\partial t}(r,t) = \kappa_1 + \kappa_2 r + \kappa_3 r^2 + \kappa_4 r\frac{\partial V_R}{\partial r}(r,t) + \kappa_6\left(\frac{\partial V_R}{\partial r}(r,t)\right)^2 + 2\kappa_5\frac{\partial V_R}{\partial r}(r,t)\frac{\partial V_M}{\partial r}(r,t), \tag{20}$$

where the coefficients κ_i, $i = 1, \ldots, 6$ are given by

$$\kappa_1 = \frac{1}{8}\frac{\phi_R^l(\theta - \alpha c)^2}{2\alpha\phi_R^l - \beta^2}, \qquad \kappa_2 = \frac{1}{4}\frac{\phi_R^l\gamma(\theta - \alpha c)}{2\alpha\phi_R^l - \beta^2}, \qquad \kappa_3 = \frac{1}{8}\frac{\phi_R^l\gamma^2}{2\alpha\phi_R^l - \beta^2}, \tag{21}$$

$$\kappa_4 = -\delta, \qquad \kappa_5 = \frac{\psi_M^2}{2\phi_M}, \qquad \kappa_6 = \frac{\left(\psi_R^b\right)^2}{2\phi_R^b}. \tag{22}$$

As the game is of the linear-quadratic variety, we make the informed guess that the value functions are quadratic, that is,

$$V_M(r,t) = \mathcal{C}_1^M(t) + \mathcal{C}_2^M(t)r + \mathcal{C}_3^M(t)r^2, \tag{23}$$

$$V_R(r,t) = \mathcal{C}_1^R(t) + \mathcal{C}_2^R(t)r + \mathcal{C}_3^R(t)r^2. \tag{24}$$

Replacing for the above forms into (19)–(20) and rearranging the terms, we obtain the following six–dimensional Ricatti differential equation system in t:

$$-\frac{d\mathcal{C}_1^M(t)}{dt} = 2\kappa_1 + \kappa_5\mathcal{C}_2^M(t)^2 + 2\kappa_6\mathcal{C}_2^M(t)\mathcal{C}_2^R(t), \tag{25}$$

$$-\frac{d\mathcal{C}_2^M(t)}{dt} = 2\kappa_2 + \kappa_4\mathcal{C}_2^M(t) + 4\kappa_5\mathcal{C}_2^M(t)\mathcal{C}_3^M(t) + 4\kappa_6\mathcal{C}_3^M(t)\mathcal{C}_2^R(t) + 4\kappa_6\mathcal{C}_2^M(t)\mathcal{C}_3^R(t), \tag{26}$$

$$-\frac{d\mathcal{C}_3^M(t)}{dt} = 2\kappa_3 + 2\kappa_4\mathcal{C}_3^M(t) + 4\kappa_5\mathcal{C}_3^M(t)^2 + 8\kappa_6\mathcal{C}_3^M(t)\mathcal{C}_3^R(t), \tag{27}$$

$$-\frac{d\mathcal{C}_1^R(t)}{dt} = \kappa_1 + \kappa_6\mathcal{C}_2^R(t)^2 + 2\kappa_5\mathcal{C}_2^M(t)\mathcal{C}_2^R(t), \tag{28}$$

$$-\frac{d\mathcal{C}_2^R(t)}{dt} = \kappa_2 + \kappa_4\mathcal{C}_2^R(t) + 4\kappa_5\mathcal{C}_3^M(t)\mathcal{C}_2^R(t) + 4\kappa_5\mathcal{C}_2^M(t)\mathcal{C}_3^R(t) + 4\kappa_6\mathcal{C}_2^R(t)\mathcal{C}_3^R(t), \tag{29}$$

$$-\frac{d\mathcal{C}_3^R(t)}{dt} = \kappa_3 + 2\kappa_4\mathcal{C}_3^R(t) + 4\kappa_6\mathcal{C}_3^R(t)^2 + 8\kappa_5\mathcal{C}_3^M(t)\mathcal{C}_3^R(t). \tag{30}$$

Given our assumption on the salvage value function (see (5)), the boundary conditions on the Ricatti equation system are

$$\mathcal{C}_1^i(T) = 0, \quad \mathcal{C}_2^i(T) = s_i \geq 0, \quad \mathcal{C}_3^i(T) = 0, \quad i = M, R. \tag{31}$$

Clearly, it is not possible to have an analytical solution to the above system. To solve it numerically, we apply the differential equation solver in Mathematica (given by the built-in NDSolve function) and replace the computed solutions of $\mathcal{C}_j^i(t)$ for $t \in [0,T]$ in the value functions (23) and (24). Furthermore, using (1), (9) and (13) allows us to obtain the equilibrium trajectory of $r(t)$ for $t \in [0,T]$, which is then used to determine the equilibrium control trajectories.

3.2 Retailer does not invest in brand advertising

In this scenario, the objective functionals become

$$J_M^0 = \int_0^T \big((w(t)-c)D(p(t),l_R(t),r(t)) - G_M^b(b_M(t))\big)\,dt + S_M(r(T)), \tag{32}$$

$$J_R^0 = \int_0^T \big((p(t)-w(t))D(p(t),l_R(t),r(t)) - G_R^l(l_R(t))\big)\,dt + S_R(r(T)), \tag{33}$$

and the reputation dynamics

$$\dot{r}\,(t) = \psi_M b_M(t) - \delta r(t), \qquad r\,(0) = r_0. \tag{34}$$

To determine the equilibrium solution, we proceed in the same way as we did in the previous scenario. Starting with the retailer's problem, we have the following HJB equation:

$$-\frac{\partial V_R}{\partial t} = \max_{p,l_R}\Big\{(p-w)\,D(p,l_R,r) - G_R^l(l_R) + \frac{\partial V_R}{\partial r}(\psi_M b_M - \delta r)\Big\}. \tag{35}$$

Assuming an interior solution, differentiating the right-hand side with respect to p and l_R, respectively, equating to zero and solving yields the following reaction functions:

$$p^0(w,r) = \frac{\phi_R^l\,(\theta+\gamma r) + \left(\alpha\phi_R^l - \beta^2\right)w}{2\alpha\phi_R^l - \beta^2}, \tag{36}$$

$$l_R^0(w,r) = \frac{\beta\left(\theta + \gamma r - \alpha w\right)}{2\alpha\phi_R^l - \beta^2}, \tag{37}$$

where the superscript 0 is used to identify the solution in this scenario.

To solve the manufacturer's optimization problem, we substitute for (36) and (37) in its objective (3) and in the dynamics (1), and write the HJB as follows:

$$-\frac{\partial V_M}{\partial t} = \max_{w,b_M}\left\{ (w-c)\, D(p^0(w,r), l_R^0(w,r), r) - G_M^b(b_M) + \frac{\partial V_M}{\partial r}(\psi_M b_M - \delta r)\right\}. \tag{38}$$

Assuming an interior solution, differentiating the right-hand side of (38) with respect to w and b_M leads to

$$w^0(r) = \frac{1}{2\alpha}\left(\theta + \alpha c + r\gamma\right), \tag{39}$$

$$b_M^0(r,t) = \frac{\psi_M}{\phi_M}\frac{\partial V_M}{\partial r}(r,t). \tag{40}$$

Substituting for $w^0(r)$ in (36) and (37) yields the following pricing and advertising policies for the retailer:

$$p^0(r) = \frac{\left(3\alpha\phi_R^l - \beta^2\right)(\theta + r\gamma) + \alpha c\left(\alpha\phi_R^l - \beta^2\right)}{2\alpha\left(2\alpha\phi_R^l - \beta^2\right)}, \tag{41}$$

$$l_R^0(r) = \frac{\beta\left(\theta + \gamma r - \alpha c\right)}{2\left(2\alpha\phi_R^l - \beta^2\right)}. \tag{42}$$

As in the previous scenario, we get the following margins and demand for the players:

$$m_M^0(r) = w^0(r) - c = \frac{1}{2\alpha}\left(\theta - \alpha c + r\gamma\right), \tag{43}$$

$$m_R^0(r) = p^0(r) - w^0(r) = \frac{\phi_R^l}{2}\left(\frac{\theta + r\gamma - \alpha c}{2\alpha\phi_R^l - \beta^2}\right), \tag{44}$$

$$D^0(r) = \alpha m_R^0(r). \tag{45}$$

Here, we also obtain that both margins are increasing in brand reputation and that demand is proportional to the retailer's margin.

The Ricatti equation system to be solved in this scenario is slightly different from the previous one. Indeed, the parameters κ_i, $i \in \{1,2,3,4,5\}$ remain the same, whereas we have $\kappa_6 = 0$. Consequently, the solution in this scenario is given by the value functions (23) and (24), and the following six-dimensional Ricatti differential-equation system:

$$-\frac{d\mathcal{C}_1^M(t)}{dt} = 2\kappa_1 + \kappa_5\mathcal{C}_2^M(t)^2, \tag{46}$$

$$-\frac{d\mathcal{C}_2^M(t)}{dt} = 2\kappa_2 + \kappa_4\mathcal{C}_2^M(t) + 4\kappa_5\mathcal{C}_2^M(t)\mathcal{C}_3^M(t), \tag{47}$$

$$-\frac{d\mathcal{C}_3^M(t)}{dt} = 2\kappa_3 + 2\kappa_4\mathcal{C}_3^M(t) + 4\kappa_5\mathcal{C}_3^M(t)^2, \tag{48}$$

$$-\frac{d\mathcal{C}_1^R(t)}{dt} = \kappa_1 + 2\kappa_5\mathcal{C}_2^M(t)\mathcal{C}_2^R(t), \tag{49}$$

$$-\frac{d\mathcal{C}_2^R(t)}{dt} = \kappa_2 + \kappa_4\mathcal{C}_2^R(t) + 4\kappa_5\mathcal{C}_3^M(t)\mathcal{C}_2^R(t) + 4\kappa_5\mathcal{C}_2^M(t)\mathcal{C}_3^R(t), \tag{50}$$

$$-\frac{d\mathcal{C}_3^R(t)}{dt} = \kappa_3 + 2\kappa_4\mathcal{C}_3^R(t) + 8\kappa_5\mathcal{C}_3^M(t)\mathcal{C}_3^R(t). \tag{51}$$

4 Comparison

In this section, we compare the equilibrium policies and outcomes in the two scenarios and conduct a sensitivity analysis. As we will see, some of the comparisons can be made analytically, whereas others must be done numerically.

Proposition 1 *The wholesale and retail prices, and local advertising policies are the same in the two scenarios.*

Proof. It suffices to compare (12) to (39), (14) to (41) and (15) to (42) to get the results. □

The main reason for the above result is that the prices (retail and wholesale) and the local advertising by the retailer do not appear in the dynamics of reputation. In both scenarios, they could be obtained by static optimization. It is important to realize that this does not imply that the trajectories corresponding to the policies (that is, strategies as a function of the brand's reputation) will be the same. Indeed, the brand reputation is influenced by the players' investments in brand advertising, and clearly, these investments differ in the two scenarios. This implies that the trajectories of $r^*(t)$ and $r^0(t)$ will be different for all $t \in (0, T]$. Looking at the trajectories, our results can be summarized as follows:

Proposition 2 *In both scenarios, the evolution over time of the wholesale price, retail price and local advertising follows the evolution of the brand reputation.*

Proof. Compute the following derivatives:

$$
\begin{aligned}
\frac{dw^*}{dt} &= \frac{\gamma}{2\alpha}\frac{dr^*}{dt}, \qquad \frac{dw^0}{dt} = \frac{\gamma}{2\alpha}\frac{dr^0}{dt}, \\
\frac{dp^*}{dt} &= \frac{\gamma\left(3\alpha\phi_R^l - \beta^2\right)}{2\alpha\left(2\alpha\phi_R^l - \beta^2\right)}\frac{dr^*}{dt}, \qquad \frac{dp^0}{dt} = \frac{\gamma\left(3\alpha\phi_R^l - \beta^2\right)}{2\alpha\left(2\alpha\phi_R^l - \beta^2\right)}\frac{dr^0}{dt}, \\
\frac{dl_R^*}{dt} &= \frac{\beta\gamma}{2\left(2\alpha\phi_R^l - \beta^2\right)}\frac{dr^*}{dt}, \qquad \frac{dl_R^0}{dt} = \frac{\beta\gamma}{2\left(2\alpha\phi_R^l - \beta^2\right)}\frac{dr^0}{dt}.
\end{aligned}
$$

Clearly, all coefficients of $\frac{dr^*}{dt}$ and $\frac{dr^0}{dt}$ are positive. Consequently, the sign of the left-hand side derivative with respect to time is the same as the sign of the right-hand side derivative. □

In the following proposition, we compare the trajectories over time:

Proposition 3 *For all $t \in [0, T]$, the wholesale price, retail price and local advertising compare as follows in the two scenarios:*

$$
sign\left(w^*(t) - w^0(t)\right) = sign\left(p^*(t) - p^0(t)\right) = sign\left(l_R^*(t) - l_R^0(t)\right) = sign\left(r^*(t) - r^0(t)\right).
$$

Proof. Compute the differences:

$$
\begin{aligned}
w^*(t) - w^0(t) &= \frac{\gamma}{2\alpha}\left(r^*(t) - r^0(t)\right), \\
p^*(t) - p^0(t) &= \frac{\gamma\left(3\alpha\phi_R^l - \beta^2\right)}{2\alpha\left(2\alpha\phi_R^l - \beta^2\right)}\left(r^*(t) - r^0(t)\right), \\
l_R^*(t) - l_R^0(t) &= \frac{\beta\gamma}{2\left(2\alpha\phi_R^l - \beta^2\right)}\left(r^*(t) - r^0(t)\right).
\end{aligned}
$$

Clearly, all the coefficients of the term $\left(r^*(t) - r^0(t)\right)$ are positive; hence the result. □

Proposition 3 shows that the differences in the prices (wholesale and retail) and local advertising trajectories between the two scenarios depend solely on the difference between $r^*(t)$ and $r^0(t)$, which cannot be computed analytically. Therefore, the signs in the statement of the above proposition can only be characterized numerically, which is done in the next section.

Proposition 4 *The demand and consumer surplus compare as follows in the two scenarios:*

$$
sign\left(D^*(t) - D^0(t)\right) = sign\left(CS^*(t) - CS^0(t)\right) = sign\left(r^*(t) - r^0(t)\right), \text{ for all } t \in [0, T].
$$

Proof. It suffices to compute the following differences to get the results:

$$
\begin{aligned}
D^*(t) - D^0(t) &= \frac{\alpha\gamma\phi_R^l}{2\left(2\alpha\phi_R^l - \beta^2\right)}\left(r^*(t) - r^0(t)\right), \\
CS^*(t) - CS^0(t) &= \frac{\alpha\gamma\left(\phi_R^l\right)^2\left(\gamma\left(r^* + r^0\right) + 2\left(\theta - \alpha c\right)\right)}{8\left(2\alpha\phi_R^l - \beta^2\right)^2}\left(r^*(t) - r^0(t)\right).
\end{aligned}
$$

□

From the above two propositions we conclude that, if the reputation is higher when both players advertise, then they can secure higher margins and this will not be detrimental to consumers. Indeed, although consumers will be paying a higher price, the demand and consumer surplus will be higher in the first scenario.

Table 1 shows how the prices and local advertising strategies vary with the parameter values. Note that these strategies are independent of all other parameters that are not listed.

Table 1: Sensitivity analysis

	θ	β	γ	α	ϕ_R^l	c
$w^*(r), w^0(r)$	>0	$=0$	>0	<0	$=0$	>0
$p^*(r), p^0(r)$	>0	>0	>0	<0	<0	?
$l_R^*(r), l_R^0(r)$	>0	>0	>0	<0	<0	<0

The following comments can be made: (i) The three equilibrium strategies (w, p, l_R) are increasing in the intrinsic market size (θ) and consumer sensitivity to reputation (γ), whereas they are all decreasing in consumer sensitivity to price (α). (ii) The retailer's strategies (p and l_R) are both increasing in consumer sensitivity to local advertising (β) and decreasing in its cost (ϕ_R^l). The equilibrium wholesale price (w) is not affected by a change in these parameters. (iii) An increase in the marginal production cost (c) increases the wholesale price and decreases the retailer's level of local advertising. However, its effect on retail price is ambiguous. The derivative of $p^*(r)$ with respect to c is given by $\frac{dp^*}{dc} = \frac{\alpha\phi_R^l - \beta^2}{2(2\alpha\phi_R^l - \beta^2)}$, where the denominator is clearly positive but the nominator's sign depends on the strategic interaction between p and w. If p and w are strategic complements $(\alpha\phi_R^l - \beta^2 > 0)$, then a higher production cost leads to a higher retail price. In contrast, when there is strategic substitution between p and w, a higher production cost leads to a lower retail price.

5 Numerical results

The systems of differential equations (25)–(30) and (46)–(51) cannot be solved analytically, therefore we proceed numerically. Our model has 15 parameters (two fewer in the second scenario where the retailer does not invest in brand advertising), namely,

$$
\begin{aligned}
\text{Demand} &: \theta, \beta, \gamma, \alpha \\
\text{Cost} &: \phi_M^b, c, \phi_R^l, \phi_R^b, \\
\text{Reputation dynamics} &: \psi_M, \psi_R, \delta, r_0, \\
\text{Salvage value} &: s_M, s_R, \\
\text{Planning horizon} &: T.
\end{aligned}
$$

With such a large number of parameters, we need to organize the simulations in a comprehensive way, that is, we start from a given parameter constellation and then vary the parameter values one at a time. To account for strategic complementarity between the players, which occurs when $\alpha\phi_R^l - \beta^2 > 0$, and strategic substitutability, which happens when $\alpha\phi_R^l - \beta^2 < 0$, we in fact consider two starting parameter constellations (or benchmarks), which only differ in the values of α and β. (This is sufficient to distinguish between the two types of strategic interactions.) The following parameter values are common in both benchmark cases:

$$\begin{array}{lllllll} \theta = 1, & \gamma = 0.1, & \phi_M^b = 1.05, & c = 0.5, & \phi_R^b = 1.1, & \psi^R = 1, & \phi_R^l = 0.95, \\ \psi^M = 1, & \delta = 0.05, & r_0 = 5, & s_M = 1, & s_R = 0, & T = 12, & \end{array}$$

while α and β are given different values, that is,

$$\begin{array}{lll} \text{Strategic complementarity } \left(\alpha\phi_R^l - \beta^2 > 0\right) & \alpha = 1, & \beta = 0.9, \\ \text{Strategic substitutability } \left(\alpha\phi_R^l - \beta^2 < 0\right) & \alpha = 0.9, & \beta = 1. \end{array}$$

We ran a large number of experiments while varying the above values marginally as well as drastically, and in all simulations, we obtained the same qualitative results presented below.[4] Consequently, to save on space, we only show some representative results.

Figure 1 shows the reputation trajectories for both scenarios. In each panel, we have two graphs, each corresponding to a benchmark: one with parameter values satisfying the condition of strategic complementarity and the other of strategic substitutability. For each benchmark, denote by $r^*(\cdot)$ the brand's reputation trajectory where both players invest in brand advertising, and by $r^0(\cdot)$ its counterpart when only the manufacturer invests. Further, we have the trajectories resulting from a 10% increase in the value of one parameter, denoted by $r^*_\Delta(\cdot)$ and $r^0_\Delta(\cdot)$, respectively.

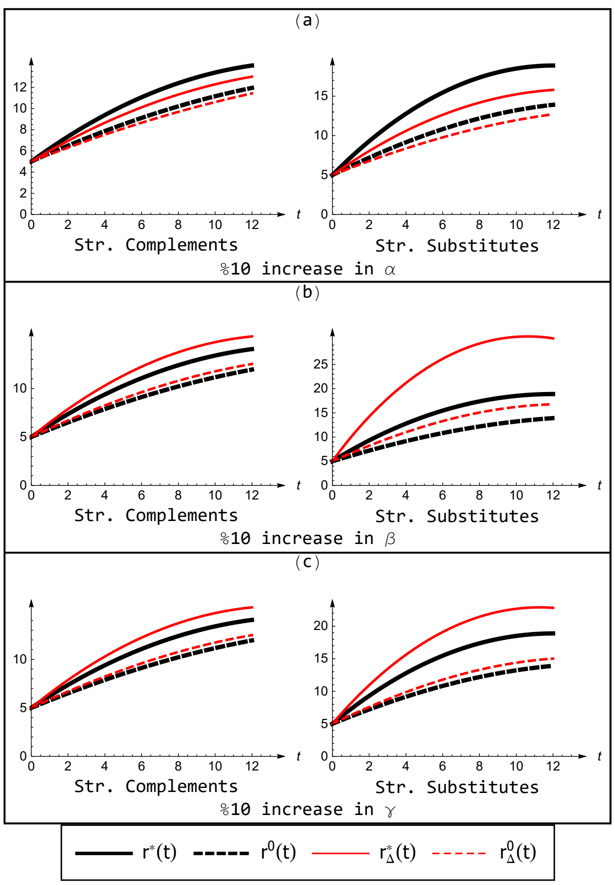

Figure 1: Sensitivity analysis of the trajectory of reputation in strategic complements and substitutes cases

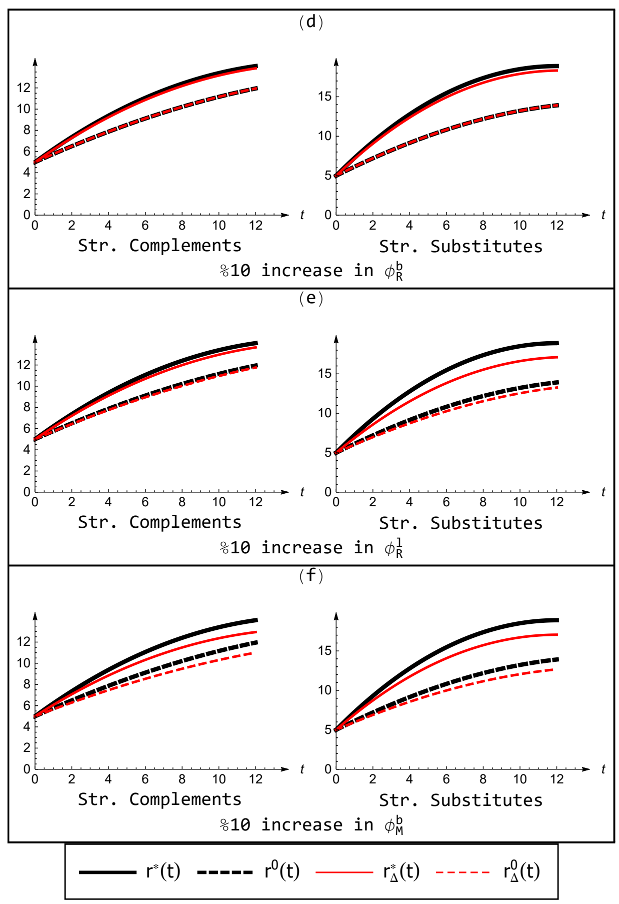

Figure 1 (cont.): Sensitivity analysis of the trajectory of reputation in strategic complements and substitutes cases

[4] Numerical results can be provided for any constellation of parameter values.

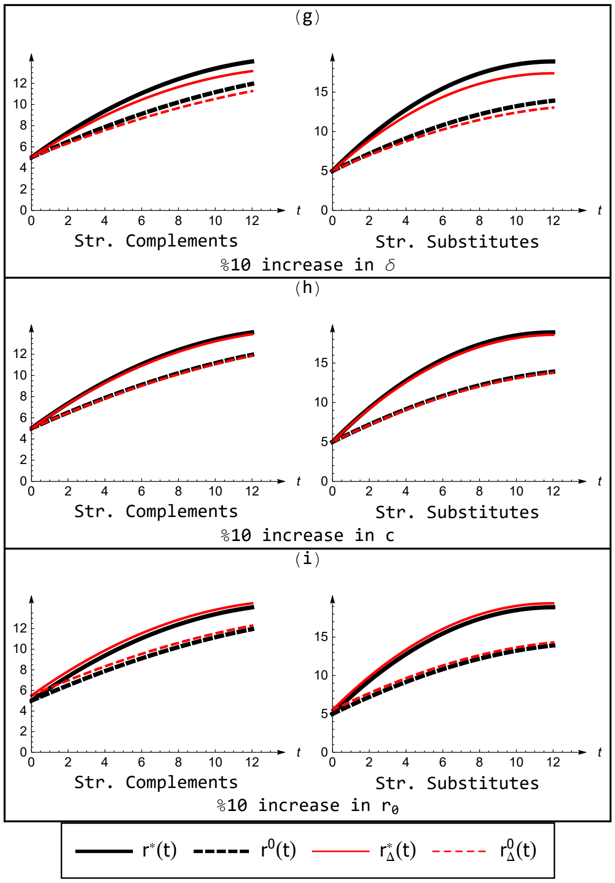

Figure 1 (cont.): Sensitivity analysis of the trajectory of reputation in strategic complements and substitutes cases

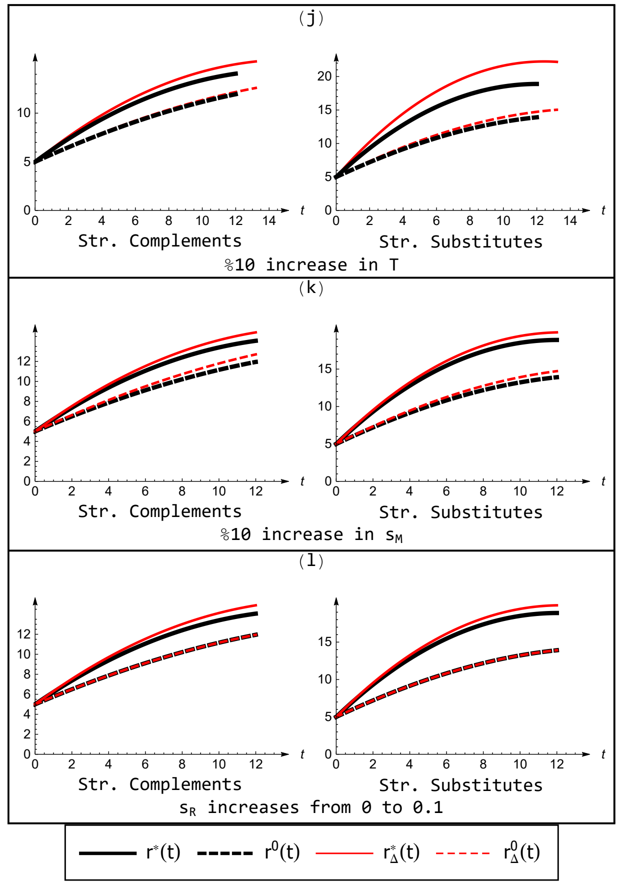

Figure 1 (cont.): Sensitivity analysis of the trajectory of reputation in strategic complements and substitutes cases

We summarize our findings in the following claims:

Claim 1 *For all parameter values,* $r^*(t) > r^0(t)$ *for all* $t \in (0, T]$.

Claim 2 *For all parameter values,* $r^*(\cdot)$ *and* $r^0(\cdot)$ *are concave functions of time.*

Claim 3 *For all parameter values, the type of strategic interaction does not affect the shape of* $r^*(\cdot)$ *and* $r^0(\cdot)$.

Claim 1 and Proposition 3 imply the following result:

Corollary 1 *For all parameter values, the wholesale price, the retail price and the local advertising effort by the retailer are higher when both players advertise the brand than when only the manufacturer does so, that is,*

$$w^*(t) > w^0(t), \quad p^*(t) > p^0(t), \quad l_R^*(t) > l_R^0(t), \text{ for all } t \in (0, T].$$

Using the results in Claim 1 and the expressions of the players' margins in the two scenarios, that is, (17)–(16) and (43)–(44), we get the following result:

Corollary 2 *For all parameter values and all* $t \in (0, T]$, *we have*

$$m_M^*(t) > m_M^0(t), \qquad m_R^*(t) > m_R^0(t).$$

Further, Claim 1 and Proposition 4 lead to

Corollary 3 *For all parameter values and all $t \in (0, T]$, we have*

$$D^*(t) > D^0(t), \qquad CS^*(t) > CS^0(t).$$

For $r^*(t)$ to be larger than $r^0(t)$ for all $t \in (0, T]$, it must hold true that the total impact of brand advertising on the reputation stock is larger when the two players invest in brand advertising than when only the manufacturer advertises, that is,

$$\psi_M b_M^*(t) + \psi_R b_R^*(t) > \psi_M b_M^0(t), \quad \text{for all } t \in [0, T],$$

which is equivalent to

$$\psi_M \left(b_M^*(t) - b_M^0(t)\right) + \psi_R b_R^*(t) > 0, \quad \text{for all } t \in [0, T].$$

Two scenarios can be envisioned. In the first one, the manufacturer reduces its advertising effort when the retailer invests in brand advertising, that is, we are in a public good situation where the manufacturer free-rides on the retailer's effort such that $\left(b_M^*(t) - b_M^0(t)\right) < 0$. In the second scenario, the manufacturer invests more when the retailer also advertises than when it is alone in supporting the brand $\left(b_M^*(t) - b_M^0(t)\right) > 0$. We state the following claim:

Claim 4 *For all parameter values, we have*

$$b_M^*(t) > b_M^0(t), \quad \text{for all } t \in [0, T),$$
$$b_M^*(T) = b_M^0(T) = \frac{\psi_M}{\phi_M} s_M.$$

The above claim, which holds true in all our simulations, indicates that the retailer's investment in brand advertising acts as an incentive for the manufacturer to increase its own advertising effort. The result is illustrated in Figure 2, where we show the manufacturer's brand-advertising trajectories for both benchmark parameter values.

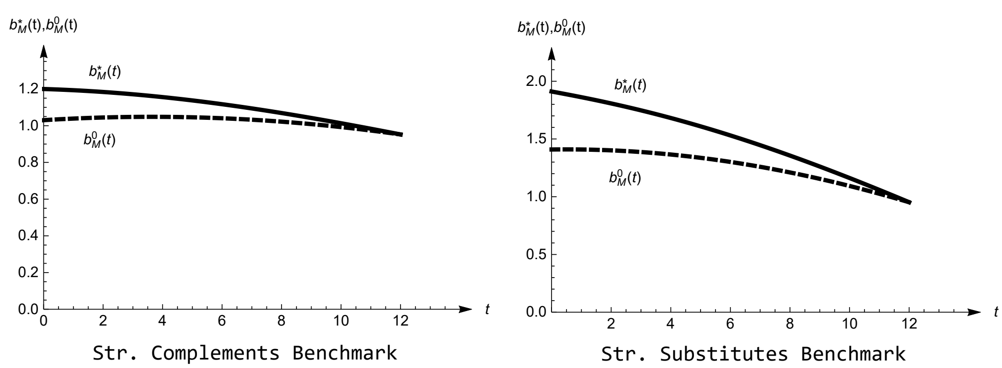

Figure 2: The manufacturer's brand advertising trajectories for both benchmarks

Finally, we compare the profits in the two scenarios.

Claim 5 *For all parameter values, $J_M^* > J_M^0$ and $J_R^* > J_R^0$.*

This claim means that the retailer's brand advertising is Pareto profit-improving, and that the total channel's profit is also higher in this scenario. This result has an interesting interpretation, namely, that the retailer's brand advertising (partially) coordinates the marketing channel. Local advertising by the retailer with cooperative advertising support from the manufacturer has been shown to improve the players' profits in a dyad—see the literature reviews in Aust and Buscher (2014) and Jørgensen and Zaccour (2014). To the best of our knowledge, this is the first time it is obtained that the retailer's brand advertising plays such a coordinating role.

Looking at the differences in the profits' trajectories (see the representative Figure 3) allows us to make three observations. First, we clearly see that there is a first period of time in which each player's profit is lower when the retailer advertises the brand. This shows that they are actually investing heavily in advertising to raise the reputation level in order to achieve higher profits later on. Second, we note that the manufacturer's instantaneous profit is negative during the first period. If we constrained this profit to be positive at each instant of time, the result would be different, but there is no valid conceptual reason to do so. Third, in all the simulations conducted, we obtained that the date when the difference in profits becomes positive depends on the value of T, but the same pattern is observed in all simulations.

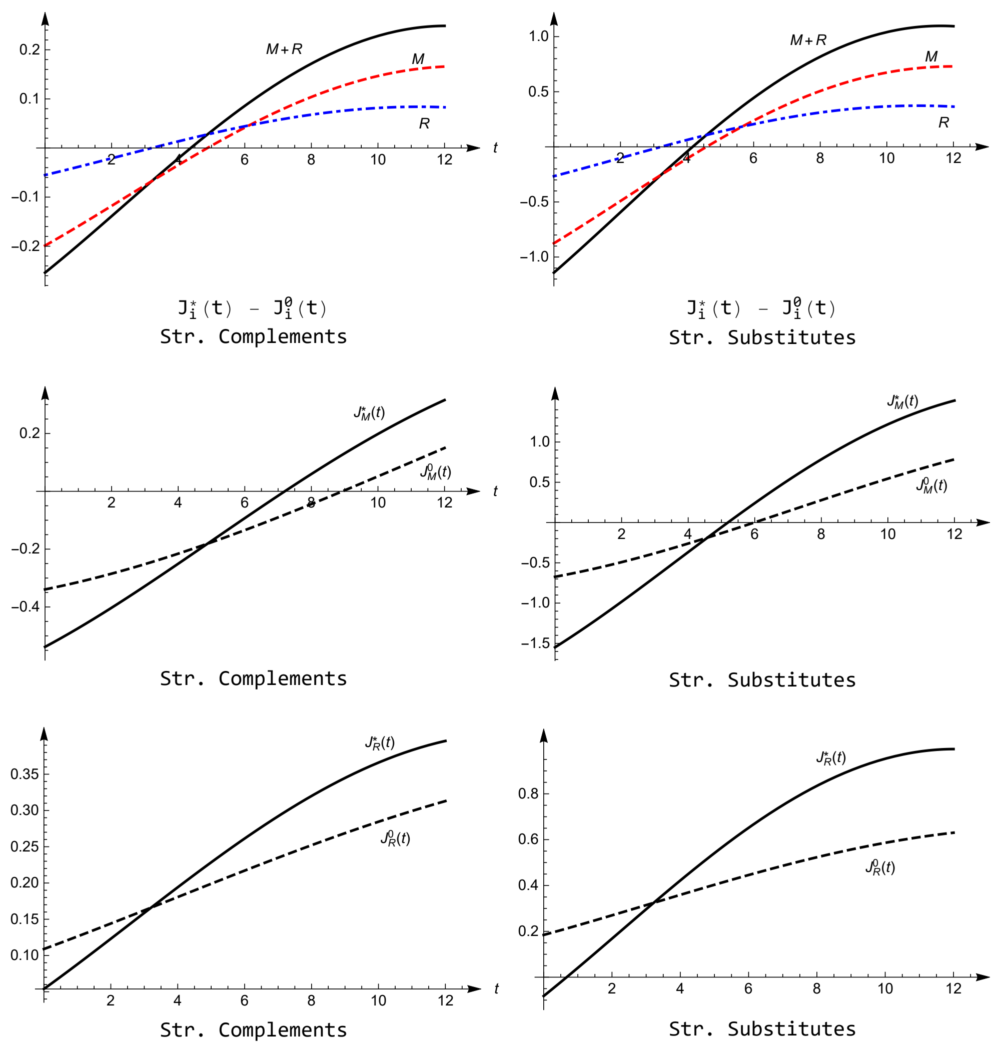

Figure 3: Trajectories of instantaneous profits and their differences

6 Conclusion

The main takeaway of this paper is that brand advertising by the retailer leads to higher profits for both players and also benefits the consumer. To the best of our knowledge, this result that retailer's brand advertising plays a partial coordinating role for the marketing channel has never before been brought out in the literature.

Our results trigger a number of interesting questions that merit future investigation. First, we assumed that the impact of the two players' advertising on the reputation stock is additive. Would our conclusion that the manufacturer does not free-ride on the retailer's brand advertising still hold if we had some interactions between the two players' investments? Second, how would the results be affected if the manufacturer supported the retailer's brand advertising? Third, we supposed that the retail price does not affect the brand's reputation dynamics, implying that this price is actually determined by static optimization (but is still state dependent). How would the results change if the price affected the evolution of the brand's reputation? This question is of interest since price affects the consumer perception of the brand image. A final question that is worth investigating is how introducing some degree of horizontal competition in retailing would affect the results.

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