Les Cahiers du GERAD ISSN: 0711–2440

Spatial effects and strategic behaviour in a multiregional transboundary pollution dynamic game

J. de Frutos,
G. Martín-Herrán

G–2016–91

November 2016

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Spatial effects and strategic behaviour in a multiregional transboundary pollution dynamic game

Javier de Frutos [a,b]
Guiomar Martín-Herrán [a,b]

[a] *IMUVA, Universidad de Valladolid, 47011 Valladolid, Spain*

[b] *GERAD, HEC Montréal, Montréal (Québec) Canada, H3C 3A7*

frutos@mac.uva.es
guiomar@eco.uva.es

November 2016

Les Cahiers du GERAD
G–2016–91

Abstract: We analyze a transboundary pollution differential game where pollution control is spatially distributed among a number of agents with predetermined spatial relationships. The analysis emphasizes, first, the effects of the different geographical relationships among decision makers; and second, the strategic behaviour of the agents. The dynamic game considers a pollution stock (the state variable) distributed among one large region divided in subregions which control their own emissions of pollutants. The emissions are also represented as distributed variables. The dynamics of the pollution stock is defined by a parabolic partial differential equation. We numerically characterize the feedback Nash equilibrium of a discrete-space model that still captures the spatial interactions among agents. We evaluate the impact of the strategic and spatially dynamic behaviour of the agents on the design of equilibrium environmental policies.

Keywords: Transboundary Pollution, spatial dynamics, spatially distributed controls, differential games, parabolic differential equations

---

Acknowledgments: The authors would like to thank H. Benchekroun and participants at SURED 2016 (Banyuls sur Mer), EAERE 2016 (Zurich), GeComplexity Conference 2016 (Heraklion) and the 17th International Symposium on Dynamic Games and Applications (Urbino) for stimulating discussions. This research is partially supported by MEC under projects MTM2013-42538-P (first author) and ECO2014-52343-P (second author), co-financed by FEDER funds. The authors thank the support of European Cooperation in Science and Technology through COST Action IS1104, "The EU in the new complex geography of economic systems: models, tools and policy evaluation".

1 Introduction

The standard dynamic models used in the literature to study different types of economic and environmental problems have restricted their attention to time, ignoring space even when studying problems with an important geographic flavour. However, it seems natural to try to extend the analysis to a spatial dimension in order to describe in a more realistic manner the world's economic and environmental problems. The technical difficulties that arise when optimization takes place in spatio-temporal domains are undoubtedly the reason for the lack of abundant literature on this subject. Recently some authors have introduced in different economic contexts (for example, allocation of economic activity; technological diffusion; environmental and climate problems) this spatial dimension and analyzed finite or infinite time optimal control problems extended to infinite dimensional state space. These contributions have focussed either on the problem of a social planner or on special private optimization problems. In the first case, the social planner allocates resources in order to maximize the present value of an objective over the entire spatial domain subject to the spatio-temporal evolution of the state variable. In the second case, particular assumptions either on the economic agents' behaviour or on the diffusion process are made implying that the agents behave myopically in both the temporal and the spatial dimensions, in the sense that they do not care to take the future allocation paths into account, and as a result agents solve static problems. To the best of our knowledge there are no previous studies that consider agents who behave both dynamically and strategically. This paper fills this gap in the literature and presents a first approach to characterize the equilibrium outcomes of an intertemporal transboundary pollution dynamic game where there is a continuum of spatial sites and the pollution stock diffuses over these sites.

The previous contributions in this area can be classified in two broad groups. First, the works focussing on economic growth theory with spatial diffusion; and second, the contributions which explore the spatial dimension in environmental and resource economics. In the first group the diffusion mechanisms involved are production factor mobility or technological diffusion, while in the second group are diffusion of a pollutant or the distribution of the biomass of a natural resource. The following is a no exhaustive list of works which can be classified in the first group: Brito (2004), Boucekkine et al. (2009, 2013a, 2013b), Camacho et al. (2008), Brock & Xepapadeas (2008a), Brock et al. (2014a) and Fabbri (2016). A list of papers belonging to the second group of the literature includes: Brock & Xepapadeas (2008b, 2010), Brock et al. (2014b), Camacho & Pérez-Barahona (2015), Xepapadeas (2010). Finally, there are some recent studies (among others, Anita et al. (2013), Yamaguchi (2014), Desmet & Rossi-Hansberg (2015), and La Torre et al. (2015)) which could be classified in both groups. These works have put the focus on the analysis of problems coupling capital accumulation and pollution diffusion.

Brito (2004), Boucekkine et al. (2009, 2013a), Camacho et al. (2008) and Fabbri (2016) analyze optimal dynamic social welfare for spatial economic growth models. In all cases a policy maker maximizes consumers' well-being, capital is mobile across space and household's budget that describes the behaviour of physical capital across time and space is governed by a Parabolic Partial Differential Equation (PDE). Brock & Xepapadeas (2008a) also study the optimal spatial allocation of economic activity in a dynamic setting with capital accumulation, but unlike the aforementioned papers there is no capital mobility and the spatial component is introduced through technological diffusion, specifically assuming a spatial capital externality. All these papers use extended versions of Pontryagin's maximum principle to obtain necessary conditions for the different optimization problems at hand and usually focus on the problem of a social planner who allocates resources. Exceptions are Brock & Xepapadeas (2008a) and Brock et al. (2014a) who in addition to considering the social planner problem also study the problem where an economic agent considers certain external effects as outside his control and treats them as exogenous, and therefore, there is an incomplete internalization of the spatial externality by the optimizing agents. Desmet & Rossi-Hansberg (2010) surveyed all this research devoted to the analysis of spatial economic growth models and Boucekkine et al. (2013b) present a survey of the use of parabolic PDEs in economic growth theory.

One of the first contributions to the stream of the literature that explores the spatial dimension in environmental and resource economics is Brock & Xepapadeas (2008b) where the results on local stability analysis for infinite horizon optimal control problems adapted to their spatial context are illustrated with two applications: optimal ecosystem management model, where the ecosystems are spatially connected, and renewable resource harvesting models, where the resource itself diffuses across space. Brock & Xepapadeas (2010) apply their results on pattern formation to the management and regulation of a semi-arid system assuming two different settings. First, economic agents maximize myopic profits and ignore spillovers onto agents at other sites. Second, a social planner internalizes these spillovers. As the authors already noted due to the myopic assumptions regarding agents' behaviour, the private optimum discussed in their paper represents a series of static optimization problems defined at each point in time and space. The same assumptions on myopic agents' behaviour are considered in Xepapadeas (2010) where the author revises the tools for studying the interactions of pattern formation and agglomeration mechanisms in optimal control problems with applications to resource management. Brock et al. (2014b) revisit these methods and tools in their review of the applications of optimal control of diffusive transport processes to environmental and climate problems in economics. Camacho & Pérez-Barahona (2015) analyze optimal land use from a social planner's point of view who decides the land use activities taking into account that local actions affect the whole space because pollution flows across locations resulting on both local and global damages.

Among the different examples presented in Brock et al. (2014b), the example closest to the problem which concerns us corresponds to the pollution control in a spatial setting. The PDE describing the evolution and diffusion of pollutants in the environment in Brock et al. (2014b) is a particular specification of the PDE describing this evolution in our problem. More importantly, we have a two-dimensional equation (theirs is one dimensional) which is an important feature when the geographical or spatial aspect of the problem is taken into account. Apart from this difference in spatio-temporal variability of the stock of pollutant, the main difference comes from different settings concerning the optimizing agents. Brock et al. (2014b) consider an environmental regulator who seeks to maximize the discounted benefits net of environmental damages due to the concentration of pollutants over the entire spatial domain subject to the spatio-temporal evolution of the stock of a pollutant. Our paper studies dynamic optimization for the pollution control in a spatial setting with strategic agents and focuses on the equilibrium emission strategies in a multiregional setting. An essential difference with respect to the previous literature is that we consider that each economic agent responsible for controlling the emissions at each region takes into account the spatial transport phenomena across space when taking the emission decisions at this region in order to maximize his profits.

Our contribution two the literature is twofold. First, we ad the spatial aspect to the literature on transboundary pollution dynamic games (Ploeg and Zeeuw (1992) and Dockner and Long (1993) are seminal papers in this area, and Jørgensen et al. (2010) surveyed this literature). Second, we ad the strategic aspect to the literature on spatial economics, and in particular, to the pollution control in a spatial setting described in the previous paragraph.

Building on these two branches of the literature, the objective of this paper is to investigate the impact of the strategic and spatially dynamic behaviour of the economic agents responsible for controlling the emissions of pollutant on the design of equilibrium environmental policies. More specifically, we aim at answering the following research questions:

1. In a transboundary pollution dynamic game setting, do the environmental policies that take into account the spatial context differ from those that ignore the spatial transport phenomena ?
2. Considering the pollution control in a spatial setting, do the environmental policies that take into account the strategic behaviour of the decision agents differ from those fixed by an environmental regulator ?

In order to answer our research questions we state and analyze a problem for multiregional spatially distributed control of pollution. We state the model in continuous space and continuous time with two spatial dimensions and one time dimension. The planar region of interest is divided in J subregions and in each subregion there is a decision-maker who decides the emission level in order to maximize the present value of benefits net of environmental damages due to the concentration of pollutants over the spatial subdomain corresponding to his region subject to the spatio-temporal evolution of the stock of a pollutant. This spatio-temporal evolution is described by a Convection-Reaction-Diffusion PDE and general boundary conditions are assumed. Summarizing, our specification corresponds to a J-player differential game with one control variable for each player (the emission decision at his subregion) and one infinite-dimensional state variable (the stock of a pollutant). Each player decides his net profits maximizing level of emission at each spatial point located in his subregion and at each time taking into account the PDE describing the spatio-temporal evolution of the stock of a pollutant. It is worth noting that unlike the literature on spatial economics previously discussed seeking private equilibria where the economic agents act spatially myopically by ignoring spatial transport, in our formulation each player does not ignore the spatial aspect in his optimization problem. We characterize Markov-perfect Nash equilibria of the differential game.

As a first approach to characterize the equilibrium outcomes of the transboundary pollution dynamic game with spatial effects, and in order to overcome the difficulties arising from the infinite dimensionality of the model we consider a simplified model capturing the spatial interactions and that can be seen as a spatial discretization of the original model. This space-discretized model has J players and J state variables, each one describing the average pollution in one subregion. Each player has one control variable which is the average total emission in his subregion and when making profit maximizing emission decision at this subregion takes into account the time evolution of the J state variables described by a system of J Ordinary Differential Equations (ODEs). This space-discretized formulation fits the structure used by Mäler & Zeeuw (1998) in their analysis of the acid rain differential game. Recently, Graß & Uecker (2015) apply a similar spatial discretization approach to transform a system of PDEs into a very large system of ODEs in order to numerically treat spatially distributed optimal control problems with an infinite time horizon. As an example they analyze a shallow lake model with diffusion.

The space-discretized model is solved using a numerical algorithm adapted from De Frutos & Martín-Herrán (2015). Essentially, the numerical algorithm solves an approximate time-discrete dynamic game. Numerical experiments are presented to illustrate the results. A first important conclusion which can be derived from our results is that in a spatial context the environmental policies might be very different from the traditional policies which ignore either the spatial transport phenomena or the strategic behaviour of the decision makers.

The paper is organized as follows. In the next section we present the multiregional spatially distributed control of pollution formulated initially as a continuous-space model, and in a second step, as a discrete-space model. Section 3 presents some numerical examples highlighting some properties of the Markov-perfect Nash equilibria of the model and their main differences with the formulation either without spatial effects or without strategic interactions among the players. Section 4 is devoted to present some concluding remarks. The characterization of the Markov-perfect Nash equilibrium of the model, the derivation of the discrete-space model as well as the description of the numerical method are relegated to the Appendix.

2 The model

The model is a J-player non-cooperative differential game. Let us consider a planar region Ω with a given partition in J subdomains (subregions) Ω_j, $j = 1, \dots, J$, satisfying

$$\overline{\Omega} = \bigcup_{j=1}^{J} \overline{\Omega}_j, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j, \tag{1}$$

with $\overline{\Omega}$ the closure of Ω, and let us denote ∂_{ij} the common boundary between subdomains Ω_i and Ω_j,

$$\partial_{ij} := \partial\Omega_i \cap \partial\Omega_j = \overline{\Omega}_i \cap \overline{\Omega}_j, \quad i \neq j. \tag{2}$$

Player (region) i wishes to choose the rate of pollutant emissions in region Ω_i to maximize his own payoff.

Let us denote by $u_i(\boldsymbol{x}, t)$, $i = 1, \dots, J$, the emission rate of region i at time $t \geq 0$ at the particular point of the region $\boldsymbol{x} \in \Omega$. It is convenient to think of $u_i(\boldsymbol{x}, t)$ as a density of emission rates which are distributed along the region Ω. Also it is convenient to assume that although $u_i(\boldsymbol{x}, t)$ is defined for all $\boldsymbol{x} \in \Omega$, $u_i(\boldsymbol{x}, t) = 0$ for $\boldsymbol{x} \notin \overline{\Omega}_i$. We denote by $P(\boldsymbol{x}, t)$ the stock of pollution defined for all $\boldsymbol{x} \in \Omega$. Along the paper the symbol ∇f denotes the spatial gradient of a scalar function $f : \Omega \to \mathbb{R}$, and the symbol $\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ represents the divergence of a vectorial function $\boldsymbol{f} = [f_1, f_2] : \Omega \to \mathbb{R}^2$.

The spatio-temporal dynamics of the stock of pollution is given by the following parabolic partial differential equation:

$$\begin{aligned} \frac{\partial P}{\partial t} &= \nabla \cdot (k\nabla P) + \boldsymbol{b}^T \nabla P - cP + N(P) + F(\boldsymbol{u}), \quad \boldsymbol{x} \in \Omega, \\ P(\boldsymbol{x}, 0) &= P_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ \alpha(\boldsymbol{x})P(\boldsymbol{x}, t) + k(\boldsymbol{x})\nabla P^T(\boldsymbol{x}, t)\boldsymbol{n} &= \alpha(\boldsymbol{x})P_b(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{3}$$

Here $\boldsymbol{u} = [u_1, \ldots, u_J]^T$ is the vector of emission rates, $k = k(\boldsymbol{x})$ is a local diffusion coefficient which is assumed to be a smooth function satisfying $k_m \leq k(\boldsymbol{x}) \leq k_M$, for all $\boldsymbol{x} \in \Omega$, where $0 < k_m \leq k_M$ are two given constants. The coefficient $k = k(\boldsymbol{x})$ is a characteristic diffusion parameter that can depend on $\boldsymbol{x} \in \Omega$. It measures the velocity at which the stock of pollution is diffused away in a particular location $\boldsymbol{x}$. The vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}, t)$ is a smooth convective field which can be time dependent. The term $\boldsymbol{b}^T \nabla P$ accounts for a pure transport phenomenon. Pollution is transported in the direction of the vector field $\boldsymbol{b}$ with speed $(\boldsymbol{b}^T \boldsymbol{b})^{1/2}$. We assume, for simplicity, that $\nabla \cdot \boldsymbol{b} = 0$. The term $N(P)$ denotes a possibly non-linear reaction function and cP, $c = c(\boldsymbol{x}, t) \geq 0$, is a natural decay of the pollutant. We assume that the source term $F(\boldsymbol{u})$ is of the form

$$F(\boldsymbol{u}) = \sum_{j=1}^{J} F_j(u_j(\boldsymbol{x}, t))\mathbf{1}_{\Omega_j}, \tag{4}$$

for a given family of smooth functions F_j, $j = 1, \ldots, J$, with $\mathbf{1}_{\Omega_j}$ denoting the characteristic function of set Ω_j, that is, the function defined to be identically one on Ω_j, and zero elsewhere.

The rationale of the model is that although with the modelization assumption (4) the emission rates of region i contribute to enlarge the stock of pollution only in region i, the convection-diffusion process modelized by the state Equation (3) transfers part of the pollution to the whole region Ω. We remark that, due to the diffusive character of the state equation, the emissions in region Ω_i instantaneously affect each one of the regions Ω_j, $i \neq j$. How much the emissions of region i do affect region j depends on the time elapsed from the instant when the emissions take place and the distance between regions i and j, as well as on the direction of the convective term $\boldsymbol{b}$. In fact, given a fixed profile of emissions, in the long term, it is the average over time of the vector field $\boldsymbol{b}$ which has an influence in the distribution of the pollution stock over the region Ω. So that we assume from now on that $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x})$ is time independent. The second equation in (3) is the initial distribution of the stock of pollution along region Ω. The last ingredient of the model is the boundary condition stated in the third equation of (3). Function $\alpha(\boldsymbol{x})$ is a non-negative smooth function that appears after applying Newton's law of diffusion on the boundary of Ω, so that the third equation on (3) simply states that the flux of pollution throughout $\partial\Omega$ is proportional to the difference $P_b(\boldsymbol{x}) - P(\boldsymbol{x})$, where $P_b(\boldsymbol{x})$ is a given function representing the concentration of pollution in the exterior of Ω and $\boldsymbol{n}$ denotes the normal vector exterior to Ω.

Let us observe that the specification (3) presents non-linear reaction and source terms, unlike the linear terms in Brock et al. (2014b). Furthermore it allows for convective terms and the diffusion coefficient is variable, depending on the space itself, while in Brock et al. (2014b) is constant.

The objective of player i, $i = 1, \ldots, J$, is to maximize his payoff

$$J_i(u_1, \ldots, u_J, P_0) = \int_0^{+\infty} \int_{\Omega_i} e^{-\rho t} G_i(u_1, \ldots, u_J, P)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t, \tag{5}$$

subject to the dynamics given in Equation (3). Here $\rho > 0$ is a given and common time-discount rate. As it is standard in dynamic pollution games (see Jørgensen et al. (2010) for a survey of this literature), the instantaneous welfare of each region is given by a benefit from consumption minus the damage caused by the stock of pollution. Each region produces a single consumption good, the production of which generates emissions. The preferences of consumers and the emission-consumption trade-off functions are such that the instantaneous benefits of region is given by a function of the emission rates $B_i(u_i)$. Furthermore, the environmental damage caused by the accumulated stock of pollution is represented by function $D_i(P)$. Therefore, the net benefits from consumption have the form

$$G_i(u_1 \dots, u_J, P) = (B_i(u_i) - D_i(P))\, \mathbf{1}_{\Omega_i}, \tag{6}$$

for given smooth functions B_i and D_i. As common in the literature it is assumed that B_i and D_i are concave and convex functions of their arguments, respectively. We remark that the payoff (5) can be seen as an average over Ω_i of a density of revenue represented by the function $(B_i(u_i) - D_i(P))\, \mathbf{1}_{\Omega_i}$.

Assuming that the state variable can be observed and used for conditioning behaviour, we focus on stationary Markov-perfect Nash equilibria. Thus at any point in time and space the emission decision of an agent depends only on the state of the pollution stock at that moment and point in space. These stationary Markovian strategies do not require precommitment to a course of action over time and have been assumed to be a good description of realistic behaviour (see, for example, Haurie et al. (2012) and Jørgensen et al. (2010)). In Appendix A we collect the set of technical hypotheses, assumed from now on, needed to characterize the stationary Markov-perfect Nash equilibrium of the dynamic game defined by (3)–(6).

To concentrate our attention on our main research questions and focus us on the spatial relationships with respect to other decision makers we ad two reasonable hypotheses. First, each decision maker is indifferent about where to produce, and hence, where to emit. In other words, the decision maker j has not preference about the particular point $\boldsymbol{x} \in \Omega_j$ in which to emit. The technology used to produce, and consequently the profits derived from production are identical at each point in subregion $\boldsymbol{x} \in \Omega_j$. The second hypothesis assumes that each decision maker is indifferent to the environmental damage, in the sense that each decision maker j identically values the environmental damage throughout the whole set Ω_j. These two hypotheses justify the use of aggregated variables. Appendix B presents the details of the derivation of the discrete-space model derived using these aggregate variables. The discrete-space model can also be seen as a space discretization of the continuous-space model. It is worth highlighting that the discrete-space model still presents the three main features of the original formulation: first, the model is truly dynamic; second, the decision agents behave strategically; third, the model incorporates spatial aspects. None of the main features of the original model has been removed after the introduction of the aggregated variables.

The discrete-space model reads as follows: The objective of Player i is to maximize the space averaged payoff

$$\widetilde{J}_i(v_1, \dots, v_J, \boldsymbol{p}^0) = \int_0^\infty e^{-\rho t} \widetilde{G}_i(v_i, p_i)\, \mathrm{d}t, \tag{7}$$

subject to the dynamics of the aggregated stock of pollution in each subregion ($p_i, i = 1, \dots, J$) described by the following system of ordinary differential equations:

$$m_i \frac{\mathrm{d}p_i}{\mathrm{d}t} = \sum_{\substack{j=0 \\ j\neq i}}^{J} k_{ij}(p_j - p_i) + \sum_{\substack{j=0 \\ j\neq i}}^{J} b_{ij}\varphi(p_i, p_j) - m_i c_i p_i + m_i N(p_i) + m_i F_i(v_i), \quad i = 1, \dots, J. \tag{8}$$

The system is supplemented with the initial conditions given by

$$p_i(0) = \frac{1}{m_i} \int_{\Omega_i} P_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} := p_i^0, \quad i = 1, \dots, J, \tag{9}$$

where $P_0(\boldsymbol{x})$ is the initial data in (3), and $\boldsymbol{p}^0 = [p_1^0, \dots, p_J^0]^T$.

Remark 1 *The information concerning the spatial relationships among agents (see (15) for details) can be condensed by means of the matrix of coefficients in the first term of the right hand side of (8). More precisely, let us define the matrix K by*

$$K = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1J} \\ k_{21} & k_{22} & \dots & k_{2J} \\ \dots & \dots & \dots & \dots \\ k_{J1} & \dots & k_{JJ-1} & k_{JJ} \end{pmatrix}.$$

Note that $k_{ij} = 0$ if and only if $\partial_{ij} = \emptyset$. That is, $k_{ij} = 0$, $j \neq i$, if there is no common boundary between regions Ω_i and Ω_j and $k_{ij} \neq 0$ only if regions Ω_i and Ω_j have a common boundary $\partial_{ij} \neq \emptyset$. Furthermore,

$k_{ii} = -\sum_{j\neq i} k_{ij}$. Then, matrix K defines a graph (weighted graph) with one node for each subregion Ω_i and one edge joining nodes i and j if and only if $\partial_{ij} \neq \emptyset$, see next section for some examples. This graph constitutes a simplified description of the geography of the region Ω and put more emphasis on interaction among neighbourhoods than on particular attributes, as the size of regions or length of common boundaries, for example, features that have already been taken into account in the definition of the aggregate variables. The size of k_{ij} indicates how fast the pollution spreads across ∂_{ij} in absence of external transport phenomena.

It is worth noting that (7) and (8) define a J-player infinite horizon differential game with one decision variable for each player (the averaged emission rates in his subregion) and J state variables (the averaged stock of pollution in each subregion) with time evolution described by the system of ODEs in (8). The first term in the right hand side of this differential Equation collects the diffusion effect that tends to equilibrate the pollution between regions: the pollution entering Ω_i is proportional to the difference between the stock of pollution in the adjacent regions, the pollution moves from regions with high levels of concentration to regions with low levels of concentration (Flick's law of diffusion). The second term is a convective term that collects the pure transport phenomenon, which is unidirectional and tends to accumulate pollution in some regions. It can be interpreted as the flux of contaminants due to wind or other means of transport. The third term is pollution degradability or natural degradation of the pollution stock. The fourth term is a non-linear reaction term which represents an increment of pollution due to the very fact that pollution exists. This term is explicitly taken into account in the optimal management shallow lake problem (among other models) and is usually assumed to be S-shaped (see Section 3.2 for an example). Finally, the fifth term is the flow of emissions.

3 Numerical examples

In this section we numerically characterize the feedback Nash equilibrium of the differential game defined in the preceding section using the numerical method described in Appendix C. We analyze different examples and provide numerical results that help us understand the spatial problem of transboundary pollution. The different examples have been chosen in order to illustrate, on the one hand, the effects of the introduction of the strategic behaviour on the equilibrium environmental policies; and on the other hand, the differences with respect to the results obtained using standard dynamic game models which disregard the spatial aspect. The specifications are inspired in the literature of transboundary pollution dynamic games (Jørgensen et al. 2010)

3.1 Linear-quadratic specification

First we present different examples for a linear-quadratic model specification. This specification is inspired by the transboundary pollution game in the seminal papers by Dockner & Long (1993) and Ploeg & Zeeuw (1992). Let us consider the following functional specifications:

$$\begin{aligned}
&F_i(v_1,\ldots,v_J) := \beta_i v_i,\\
&g_i(v_1,\ldots,v_J,\boldsymbol{p}) := v_i(A_i - \frac{v_i}{2}) - \frac{\varphi_i}{2}p_i^2,\\
&\boldsymbol{p} = [p_1,\ldots,p_J]^T, \quad v_i = v_i(\boldsymbol{p}),\\
&m_i = m_j, \quad b_{ij} = 0,\ \forall i,j = 1,\ldots,J.
\end{aligned}$$

With these functional forms the problem player i is facing consists in choosing his control variable v_i in order to maximize

$$J_i(v_1,\ldots,v_J,\boldsymbol{p}_0) = J_i(v_i,\boldsymbol{p}_0) = \int_0^{+\infty} e^{-\rho t}\left(v_i(A_i - \frac{v_i}{2}) - \frac{\varphi_i}{2}p_i^2\right)\,\mathrm{d}t,$$

subject to:

$$\dot{p}_i = \sum_{j=1}^{J} k_{ij}p_j - c_i p_i + \beta_i v_i, \quad i = 1,\ldots,J.$$

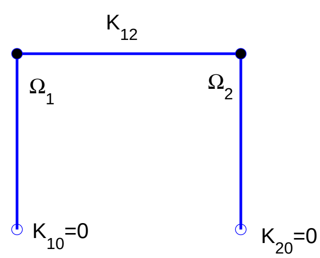
Unless otherwise indicated, the parameter values used in the numerical examples are the following:

$$\beta_i = 1, \;\; \varphi_i = 1, \;\; A_i = 0.5, \;\; \rho = 0.01, \;\; c_i = 0.5, \; i = 1, \dots, J.$$

Note that with this choice of the parameter values we are highlighting the fact that the players are completely symmetrical in every respect, except in their geographical positions described by the entries k_{ij} of matrix K (see Remark 1) (the only exception is Example 3.1.1 in which the position of both regions is completely interchangeable).

3.1.1 Example 1: 2 regions isolated from outside

Let us start with the simplest possible configuration corresponding to the case of two identical regions (players) with identical geographical position and each one controlling the averaged emissions over region Ω_i. The two regions are interconnected (there is one edge joining subregions Ω_1 and Ω_2, each subregion represented by one node) and are isolated from outside, in the sense that there is no flux of pollution neither entering nor exiting the geographical space formed by the two regions ($k_{10} = k_{20} = 0$), as the following figure shows, with K the matrix containing the spatial relationship among the two regions, and k_{12} indicating how fast pollution spreads across the common boundary between Ω_1 and Ω_2



$$K = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

The initial pollution stocks in each region are assumed to be identical in both regions and equal to 0.1. Therefore, there is no difference between the regions from a spatial (geographical) point of view.

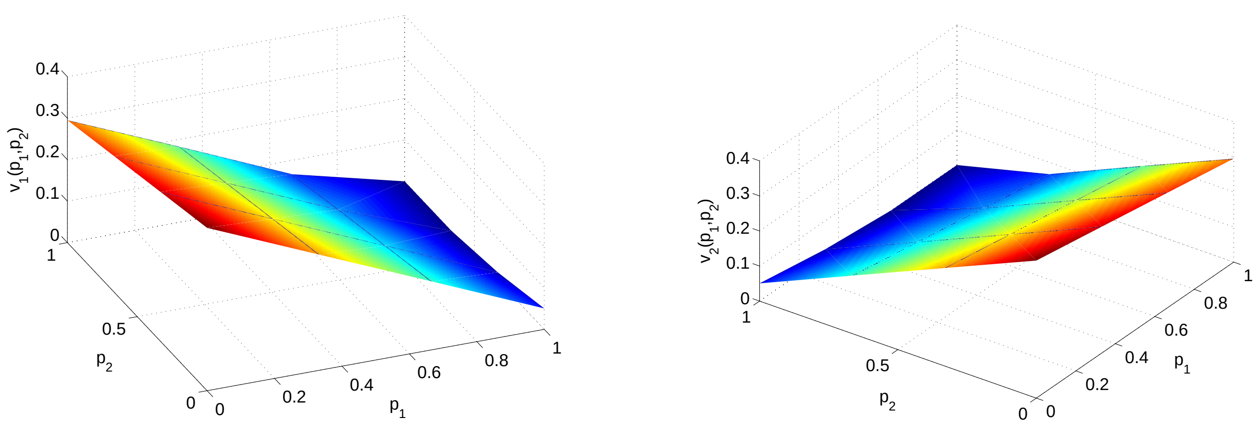


Figure 1: Example 1. Regions' feedback strategies. Region 1 (left); Region 2 (right)

Figure 1 left and right shows the feedback Nash equilibrium strategies for player 1 and 2, respectively. These strategies are the equilibrium emissions as functions of the two state variables, the pollution stocks in regions 1 and 2 (p_1 and p_2). Specifically, they are piecewise affine functions, each one obtained as the maximum between an affine function and zero. From these graphs it can be easily shown that the role played by p_i and p_j, $j \neq i$ in the optimal strategy of player i is identical to the role played by p_j and p_i, $i \neq j$ in the optimal strategy of player j. This symmetry clearly appears when drawing the time-paths of the emission levels and the pollution stock along the equilibrium strategy (Figure 2). The plot on the right shows the optimal time-paths of the pollution stocks that due to the completely symmetric example we are considering are, as expected, identical in both regions. From the initial value 0.1 along the optimal time-path the stock of pollution increases towards its steady state value. Correspondingly, the time-path of emissions along the equilibrium strategy, on the left of Figure 2, coincides for both players and, as expected, decreases with time until the steady-state value is attained.

Both the emission and pollution time-paths along the equilibrium strategies result to be equivalent to those obtained for a standard non-spatial dynamic game with one state variable, given by the mean of the pollution stock in the two regions under consideration, and in which each region contributes to the accumulation of the stock of pollution with a factor of $\beta/2$.

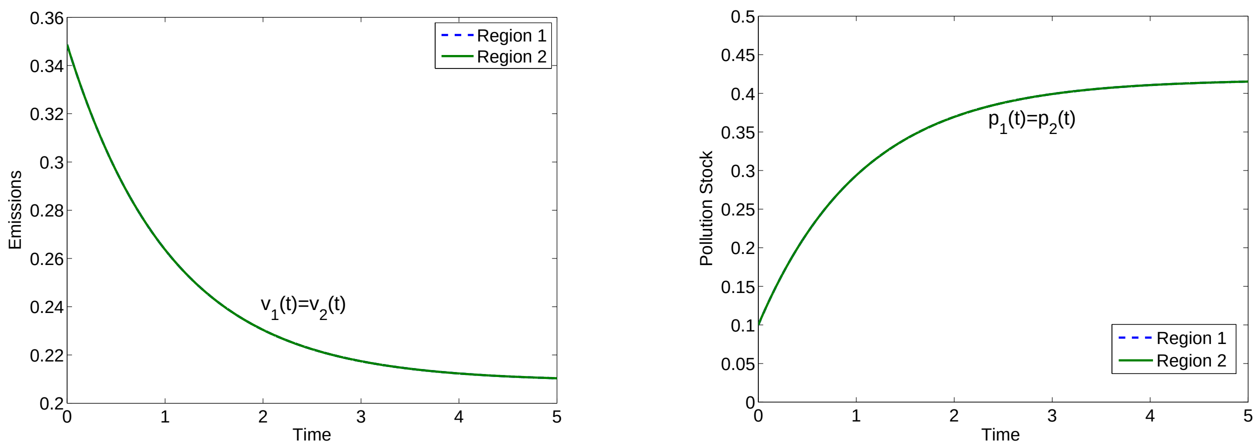
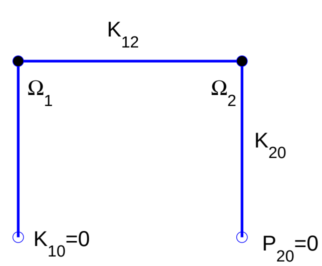


Figure 2: Example 1. Emission (left) and pollution stock (right) time-paths along the equilibrium strategy

3.1.2 Example 2: 2 regions with different geographical neighbourhoods

Let us now introduce in the 2-player setting a unique asymmetric factor related to the geographical position. The two regions are again interconnected, but unlike the previous example only region 1 is isolated from outside ($k_{10} = 0$), while region 2 shares a boundary ($k_{20} = 1$) with another region with a lower concentration of pollutants than region 2. In particular, as an extreme case, here we assume that the pollution stock in this additional region is zero ($p_{20} = 0$). For example, one could think that this region is the sea. With this geographical configuration there is a flux of pollution from region 2 to the sea, but not in the opposite direction. The following graph represents such a situation:



$$K = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

Again the initial pollution stocks in each region are assumed to be both equal to 0.1. Then, the only difference between the two regions is their geographical position. The influence of the geographical position in the feedback Nash equilibrium strategies is evident in Figures 3 and 4. On the one hand, Figure 3 shows that the feedback equilibrium strategies have lost the symmetric property presented in Example 1 (Figure 1).

We remark that because the asymmetry in the piecewise affine strategies the standard methods for linear-quadratic differential games based on the Ricatti equations cannot be easily used to find the feedback Nash equilibrium strategies of the problem at hand.

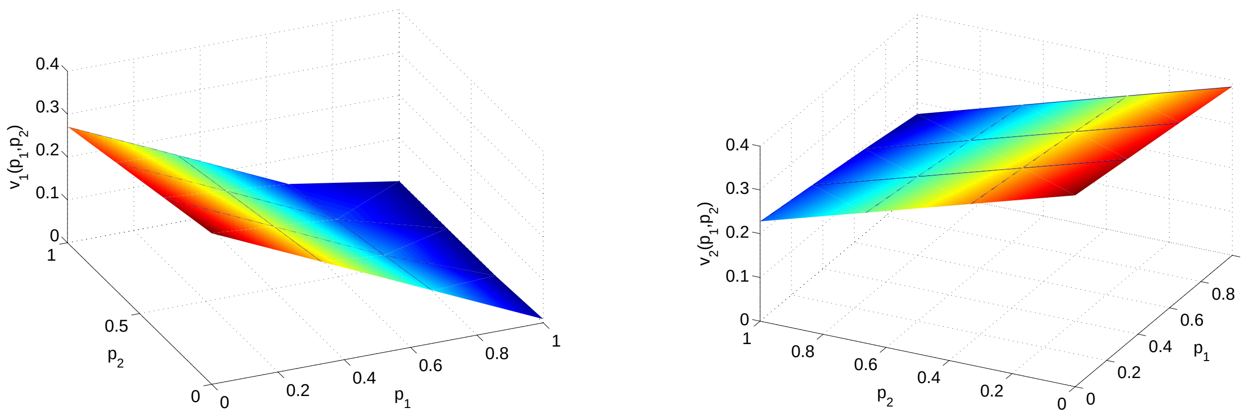


Figure 3: Example 2. Regions' feedback strategies. Region 1 (left); Region 2 (right)

On the other hand, more importantly, as Figure 4 shows, region 2 can benefit from its spatial position and emit along the equilibrium strategy above the level of emission of region 1. However, interestingly, the time path of the stock of pollution in region 2 along the equilibrium strategy is always lower than the corresponding stock of pollution in region 1. The sole reason for this behaviour is that there is an important flow of pollution exiting region 2 towards the boundary region Ω_0 (the sea), allowing region 2 to considerably increase its emissions in comparison with those of region 1 that due to its geographical position only can "exchange" pollution concentration with region 2. Because the latter is emitting at a high emission rate, its pollution concentration is high and in consequence, there is a net flow of pollution moving from region 2 to region 1. It is worth noting that the somehow counterintuitive result "the higher the emissions, the lower the pollution stock" is due exclusively to the inclusion of the spatial aspect in the model at hand, stressing the importance of taking into account this aspect. This result cannot be reproduced in a transboundary pollution dynamic game with symmetric players when the spatial transport of pollution is neglected.

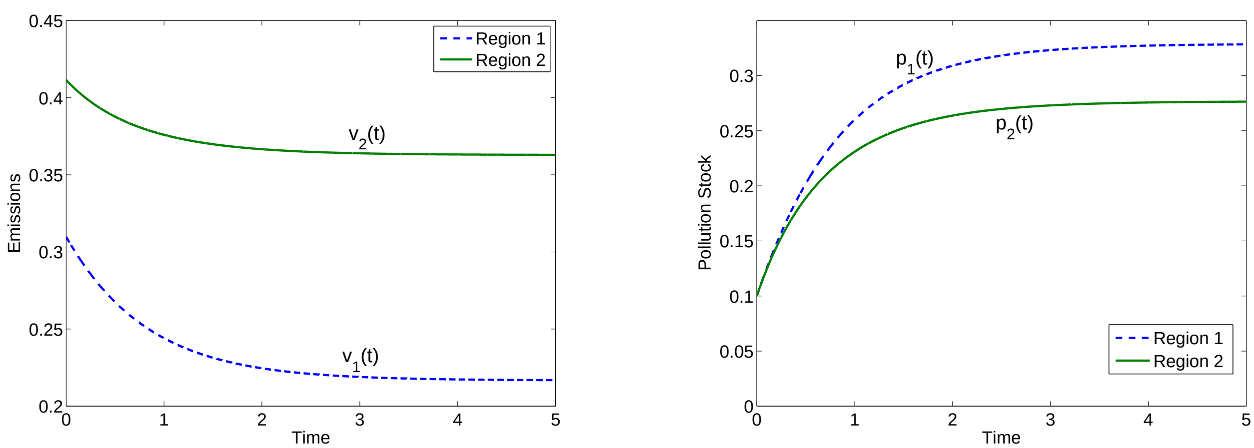
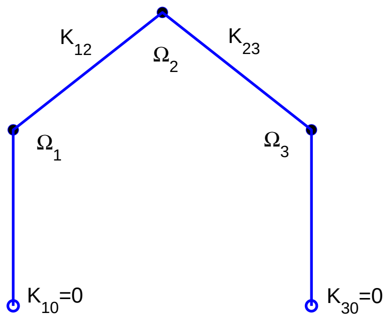


Figure 4: Example 2. Emission (left) and pollution stock (right) time-paths along the equilibrium strategy

3.1.3 Example 3: 3 regions isolated from outside

Let consider three regions and the geographical configuration described by the following graph with matrix K containing the spatial relationships among these regions.



$$K = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

As in Example 1 the three regions are isolated from outside ($k_{10} = k_{30} = 0$), but unlike Example 1 the regions are not all identical with respect to their spatial position. Specifically, regions 1 and 3 are identical and share boundary with region 2 and with the exterior of the region under consideration, in this last case without exchange neither in nor out of pollution, while region 2 has a different geographical position because it shares boundary with regions 1 and 3. We have computed the feedback Nash equilibrium strategies that, for this example, are functions of three state variables (the pollution stock in each region). We refrain from drawing these strategies, and we just draw the time-paths of emissions and pollution stock along the equilibrium strategy. Figure 5 shows only two curves in each graph, because, as expected, the level of emissions in regions 1 and 3 are equal, and consequently, the optimal time-path of the pollution stock coincides for both regions. Unlike in Example 2, in the present example the region with a higher level of emissions (region 2) presents a higher stock of pollution too. This is due to the spatial positions of the regions. Region 1 (the same argument applies to region 3) to attain its pollution steady-state value has to reduce its equilibrium emission. Because of its geographical position, region 1 only can exchange pollution with region 2, that in this case has a greater concentration of pollution because it receives from the other two regions. As a result, the pollution flows from the region with a greater concentration (region 2) to the region with a lower concentration (region 1).

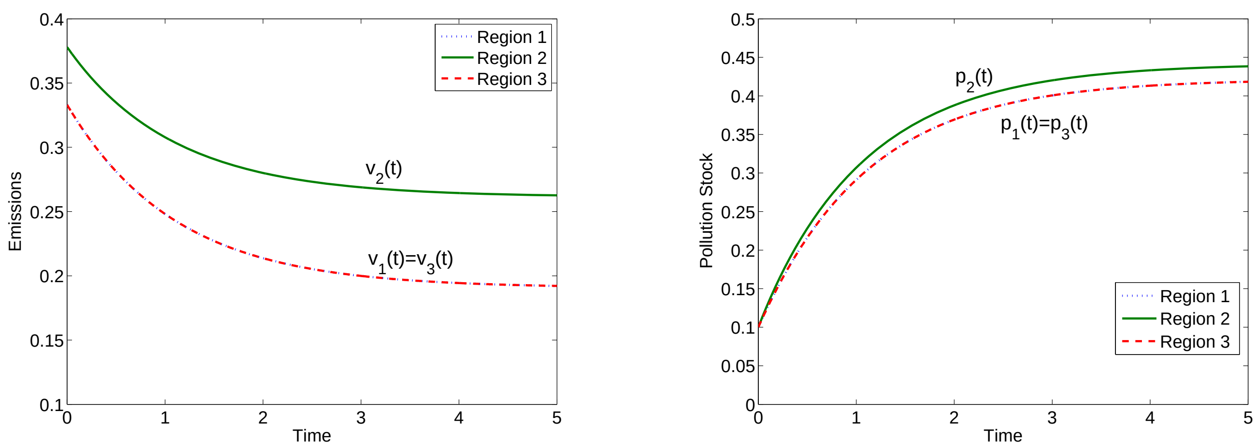


Figure 5: Example 3. $p_i(0) = 0.1$. **Emission (left) and pollution stock (right) time-paths along the equilibrium strategy**

Figure 6 presents for this example the emission and pollution stock time-paths along the equilibrium strategy when the initial pollution stocks are assumed to be equal to 1 in the three regions. As expected, with high initial pollution stocks the optimal time-paths decrease towards their steady-state levels. The initially high level of the stock of pollution prevents regions 1 and 3 emit until the level of pollution drops below a certain threshold. After this initial period of time, the emission rates increase towards their long-run values.

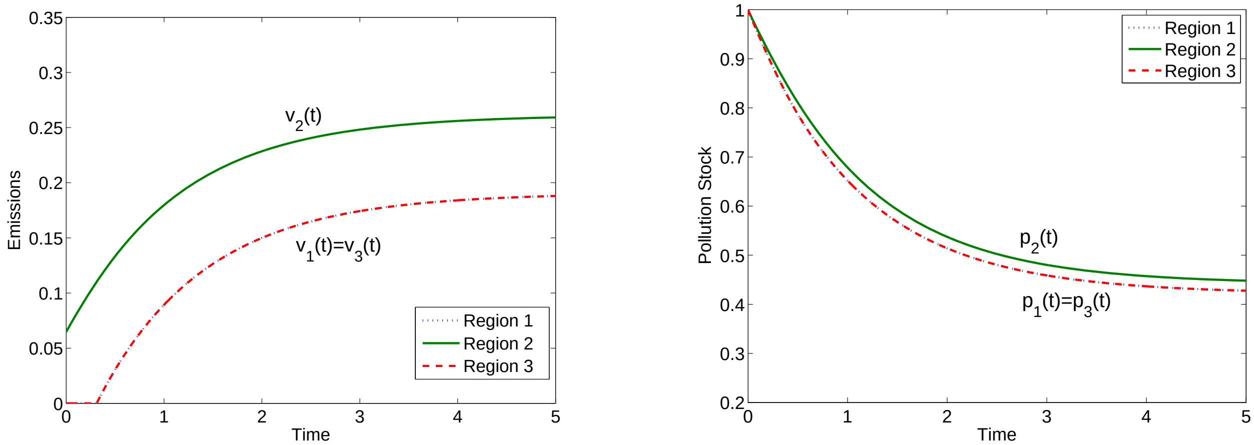
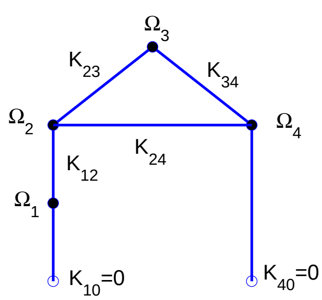


Figure 6: Example 3. $p_i(0) = 1$ **Emission (left) and pollution stock (right) time-paths along the equilibrium strategy**

3.1.4 Example 4: 4 regions isolated from outside

Let consider four regions and the geographical configuration described by the following graph, with K containing the spatial relationships among these regions.



$$K = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

As the previous graph shows, the four regions as a whole are isolated from outside ($k_{10} = k_{40} = 0$). In this spatial configuration region 1 shares boundaries with region 2 and the exterior of the whole region under consideration, Ω, but no exchange of pollution is possible. Region 2 shares boundary with the other three regions, while region 3 has a common boundary with regions 2 and 4. Finally, region 4 shares boundary with regions 2 and 3, as well as with the exterior of the whole region Ω, from which it is isolated. Therefore, in terms of the flow of pollution concentration, regions 3 and 4 are identical, while regions 1 and 2 are different from each other and different from the other two regions. It is worth noting that this example is genuinely two dimensional in the sense that the particular spatial configuration showed in this example cannot be reproduced in a one-dimensional setting.

Figure 7 clearly illustrates the geographical advantage of region 2. Because this region has three neighbours to which the pollution can flow, along the equilibrium strategy it can emit at a rate clearly greater than that of regions 3 and 4 at each instant of time (Figure 7 left), but the optimal time-path of the stock of pollution is similar for all these three regions (Figure 7 right).

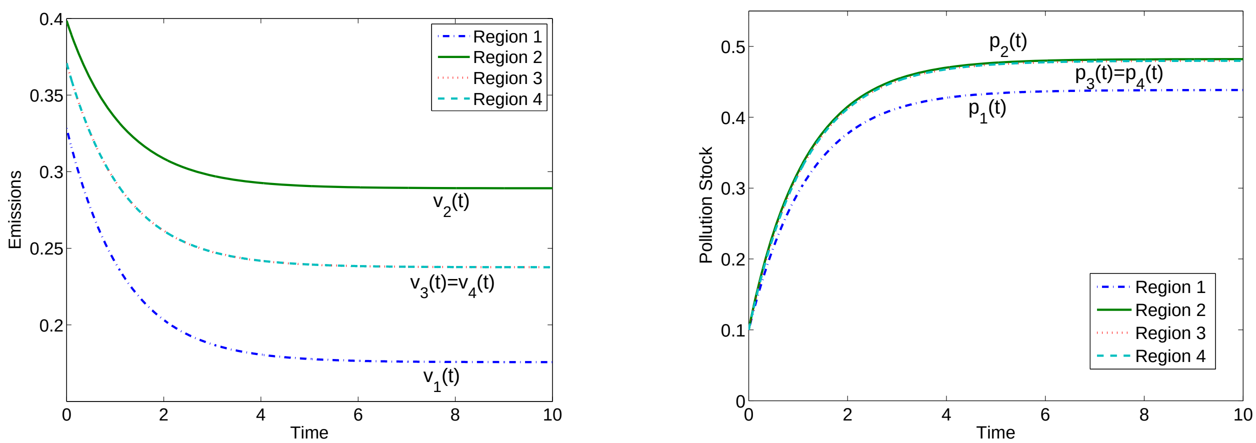
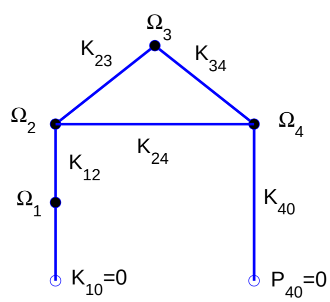


Figure 7: Example 4. Emission (left) and pollution stock (right) time-paths along the equilibrium strategy

3.1.5 Example 5: 4 regions with different geographical neighbourhoods

Let consider now an example quite similar to the previous one, but presenting two main differences, as shown in the following graph:



$$K = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

Firstly, unlike the previous example only region 1 is isolated from outside ($k_{10} = 0$), while region 4 shares a boundary with the exterior of the whole region Ω which is assumed to have a lower (zero) concentration of pollutants than region 4 ($p_{40} = 0$). Secondly, the natural regeneration rate c_i is assumed to be zero in the four regions ($c_i = 0$).

As Figure 8 shows the region emitting at the highest level (region 4) along the whole time horizon presents the lowest stock of pollution in the long run. As in Example 2 this in principle counterintuitive result comes as a result that there is a flow of pollution from region 4 to outside Ω while no flow is entering region 4 from outside Ω. Because the stock of pollution is assumed to be zero outside Ω, there is an important flow of pollution exiting region 4 towards the exterior of the whole region Ω, allowing region 4 considerably increase its emissions in comparison with those of the other countries that because of their geographical positions only can "exchange" pollution concentration with the other regions in Ω.

Figure 8 also shows that the steady-state values of the stock of pollution in regions 1, 2 and 3 are quite similar, despite the fact that region 1 emits along the whole time horizon at a much softer level than the other regions, and even after an initial period of time this region does not emit at all. The sole reason for this behaviour is that region 1 only can exchange pollution with region 2, while the other regions have two or three neighbours which can be the recipients of the flow of pollution.

It worth noting that contrary to what happens in the standard case, where a differential game without spatial interactions is analyzed, a positive natural generation of the pollution stock is not needed in order to ensure the convergence of the optimal time-path of the pollution stock towards an asymptotically stable steady state. However, the lack of natural degradation of the pollution stock could imply that one of the regions has to stop emitting during a period of time (as region 1 does in this example).

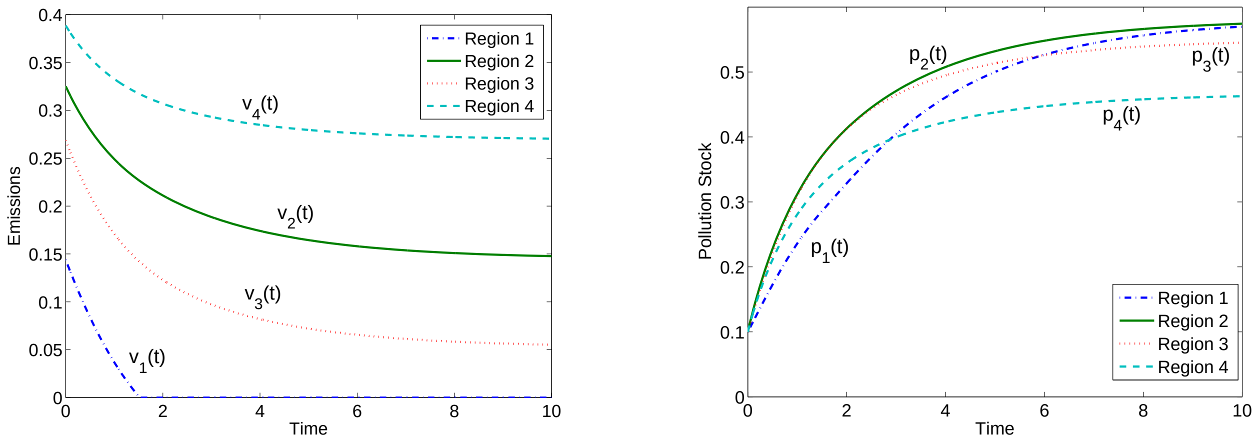


Figure 8: Example 5. Emission (left) and pollution stock (right) time-paths along the equilibrium strategy

3.2 Non-linear specification: The shallow lake model

All the previous examples correspond with a linear-quadratic model specification. We present now a non-linear specification. This specification is inspired by the shallow lake model (see, for example, Scheffer (1997), Wagener (2003) and Kossioris et al. (2008)). Let us consider the following functional specifications:

$$\begin{aligned}
&F_i(v_1,\ldots,v_J) := \beta_i v_i,\\
&g_i(v_1,\ldots,v_J,\boldsymbol{p}) := \log(v_i) - \varphi_i p_i^2,\\
&\boldsymbol{p} = [p_1,\ldots,p_J]^T, \quad v_i = v_i(\boldsymbol{p}),\\
&m_i = m_j, \quad b_{ij} = 0,\ \forall i,j = 1,\ldots,J.
\end{aligned}$$

With these functional forms the problem player i is facing consists in choosing his control variable v_i in order to maximize

$$J_i(v_1,\ldots,v_J,\boldsymbol{p}_0) = \int_0^{+\infty} e^{-\rho t}\left(\log(v_i) - \varphi_i p_i^2\right)\,\mathrm{d}t$$

subject to:

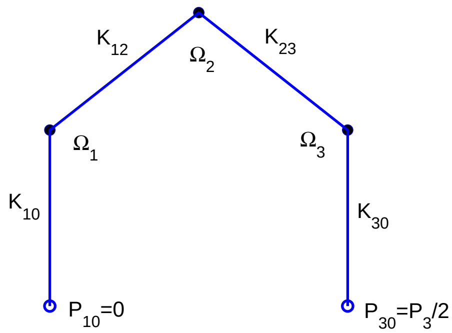
$$\dot{p}_i = \sum_{j=1}^{J} k_{ij} p_j - c_i p_i + \frac{p_i^2}{1+p_i^2} + \beta_i v_i, \quad i = 1,\ldots,J.$$

The parameter values used in the numerical experiments are the following:

$$\beta_i = 1, \ \ \rho = 0.01, \ \ c_i = 0.6, \ \ \varphi_i = 1, \quad i = 1,\ldots,J,$$

implying, as in the previous linear-quadratic examples, that the players are completely symmetric in every respect, except in their geographical positions.

We assume three regions and their respective geographical position is represented by the following graph:



$$K = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$$

Region 1 shares a boundary with region 2 and a boundary with the exterior of the whole region Ω which is assumed to have a lower (zero) concentration of pollutants than region 1 ($p_{10} = 0$). Region 3 also shares a boundary with region 2 and a boundary with the exterior of the complete region Ω which is assumed to have a lower (half of the) concentration of pollutants than region 3 ($p_{30} = p_3/2$). Therefore, the main difference between regions 1 and 3 is that although both regions are not recipients of pollution coming from outside the whole region Ω, pollution flows out from region 1 to exterior of Ω at a greater rate than it does from region 3. Finally, region 2 is isolated from outside and shares two boundary with regions 1 and 3, respectively.

The important role played by the geographical position is reflected in Figure 9 which shows the emission and pollution optimal time-paths along the equilibrium strategy. In this example, the region with the greatest level of emissions ends up with the lowest pollution stock in the long run, although this is not necessary the behaviour in the short run. Unlike in the previous example where the number of neighbours seemed to favour that a lower steady-state value of the pollution stock comes together with a greater emission level, in the present example the opposite result applies. Region 2 presents the lowest emission level, but the highest pollution stock in the long run. We can conclude that not only the number of neighbours matters, but also the type of neighbours plays an important role in determining the equilibrium environmental policies.

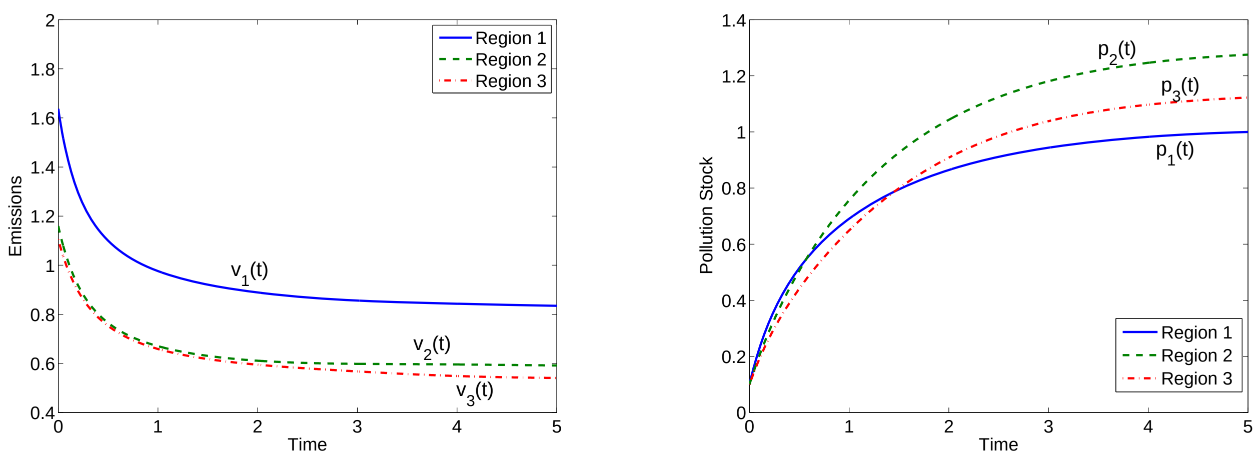


Figure 9: Non-linear example. Emission (left) and pollution stock (right) time-paths along the equilibrium strategy

3.3 Cooperative vs. non-cooperative strategies

In order to assess the impact of the strategic behaviour of the decision makers in the definition of the environmental policies we focus on the comparison of the cooperative and non-cooperative strategies of the transboundary pollution dynamic game with spatial effects. All the previous examples in this section present the non-cooperative strategies, the feedback Nash equilibrium strategies which have been obtained under the assumption that each region's emission policy is chosen to further its own interest, given the other regions' emission policies. In the cooperative scenario a unique decision maker (an environmental regulator) chooses all regions' emission policies in order to maximize the joint welfare of all regions. Therefore, the cooperative solution is obtained solving a J-state variable optimal control problem. Any cooperative outcome constitutes a first-best solution but relies on a high degree of commitment to follow the agreed-upon emission policies.

We revisit two of the linear-quadratic examples analyzed in Section 3.1, specifically Examples 3 and 5.

For Example 3, Figure 10 presents the results of the comparison of the emission (left) and pollution stock (right) time-paths along the cooperative and non-cooperative strategies. In the figure, the superscript C stands for 'cooperative'.

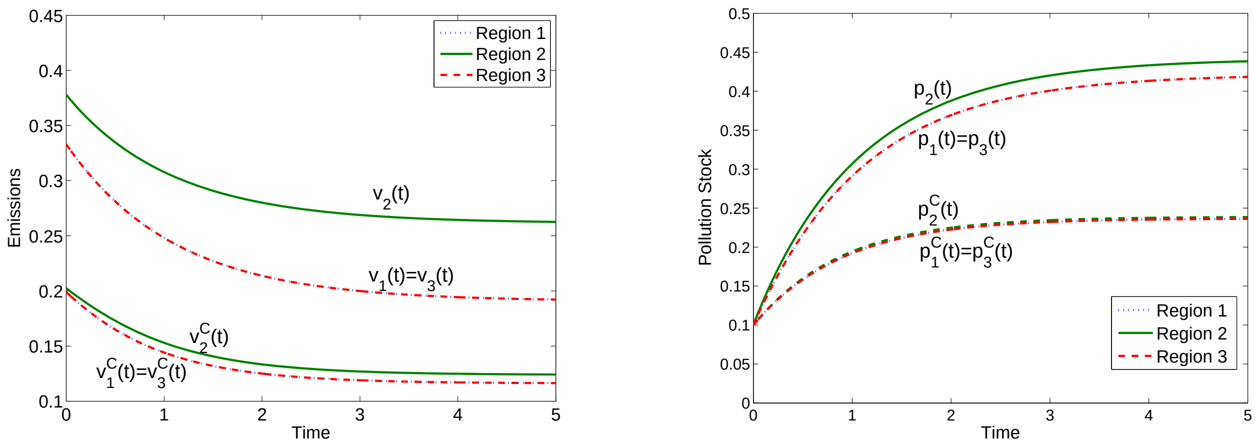


Figure 10: Example 3. Emission (left) and pollution stock (right) time-paths along the cooperative and non-cooperative strategies

As expected in a cooperative framework the three regions reduce the optimal emission levels at each time with respect to these levels in the non-cooperative scenario. Let us recall that in this example region 2 shares boundary with regions 1 and 3, while the last two (identical) regions share boundary with region 2 and with the exterior of the whole region Ω. Under cooperation, region 2, taking advantage of its geographical position, still emits over the other two regions which emit at an identical rate. However, as Figure 10 shows the difference between the emission levels is substantially greater under non-cooperation than under cooperation. The effect of the reduction of the emission levels in the cooperative scenario unequivocally leads to lower pollution stocks time-paths along the whole time horizon. In the cooperative scenario the pollution stocks of the three regions are almost identical along the optimal trajectory. The environmental regulator (the unique decision maker in the cooperative setting) chooses the emissions in the three regions in such a way that the three regions suffer a similar environmental damage because of a almost identical time path of the pollution stock along the cooperative solution.

For Example 5 and for ease of presentation we present on the left side of Figures 11 and 12 the emission and pollution stock time-paths along the non-cooperative equilibrium strategies and on the right side of these figures the time-paths along the cooperative solution. Figure 11 shows that as in the previous example, region 4 takes advantage of its geographical position (region 4 shares a boundary with the exterior of the whole region Ω which is assumed to have a zero concentration of pollutants, $p_{40} = 0$) and along the cooperative solution and at each point in time emits more than any other region. In the cooperative framework, the environmental regulator optimally chooses a smoother behaviour for the optimal time-paths of the emission level of the different regions along the cooperative solution. However, unlike the previous example, one of the regions (region 1) increases its optimal emission level with respect to its level in the non-cooperative scenario. In the non-cooperative framework region 1 in a first period of time emits at a much lower level than the other regions, and after this initial period it does not emit at all. Importantly, region 1 in the cooperative setting emits along the whole time horizon at a positive rate. The emission time-paths along the cooperative solution are very similar for regions 1, 2 and 3, although they were much different in the non-cooperative case.

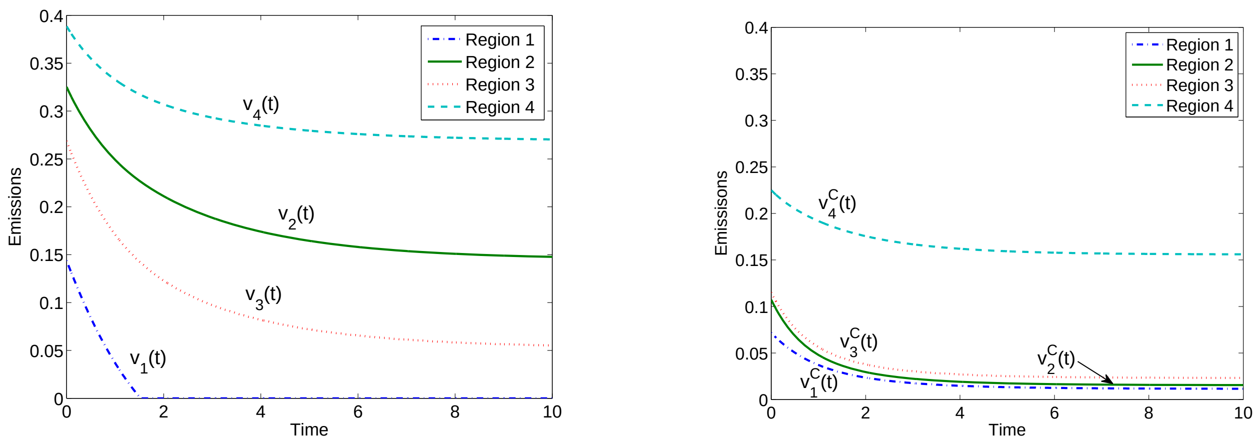


Figure 11: Example 5. Emission time-paths along the cooperative (right) and non-cooperative (left) strategies

Finally, Figure 12 illustrates that the regulator's choice of more uniform emission levels between regions leads to pollution time-paths along the cooperative solution much lower than the corresponding non-cooperative levels, and very similar for the four regions, both in the short and in the long run.

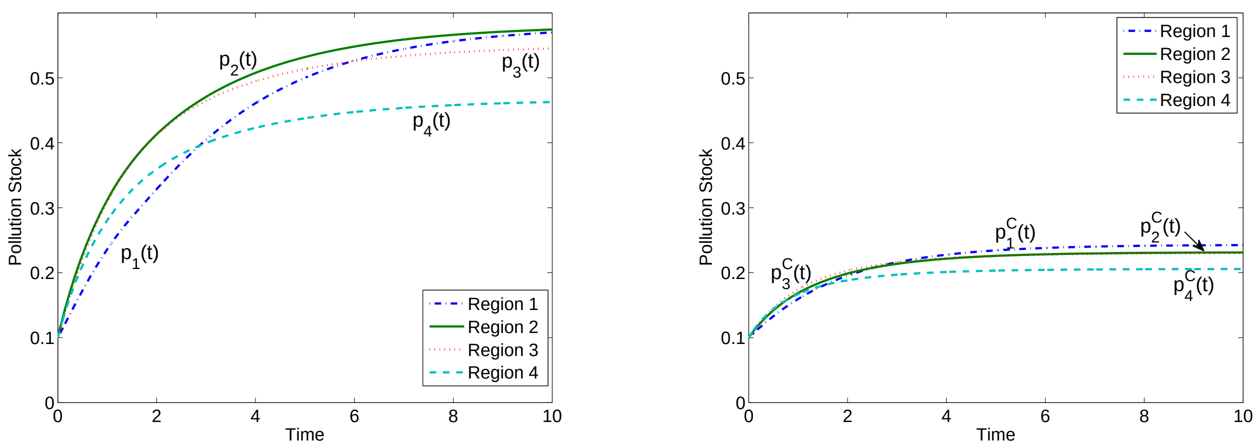


Figure 12: Example 5. Pollution stock time-paths along the cooperative (right) and non-cooperative (left) strategies

4 Concluding remarks

This paper analyzes a transboundary pollution dynamic game with spatial effects and strategic decision makers and tries to make a contribution to the quite recent literature that has added the spatial dimension to the standard (temporal) dynamic models used to study economic and environmental problems. The technical difficulties that arise when optimization takes place in spatio-temporal domains are undoubtedly the reason for the lack of abundant literature on this subject. However, recently some authors have introduced in different economic contexts this spatial dimension. As revised in the introduction of this paper, these contributions have focussed either on the problem of a social planner (who allocates resources to maximize the present value of an objective over the entire spatial domain subject to the spatio-temporal evolution of the state variable(s)) or on special private optimization problems (problems where a special structure is added in such a way that the economic agents behave myopically in both the temporal and the spatial dimensions, and therefore, agents solve static problems). Departing from these two settings, in this paper we focus on the analysis of a transboundary pollution problem with spatial diffusion in a multiregional setting, and consider agents who behave dynamically and strategically. We present a first characterization of the equilibrium outcomes of an intertemporal pollution problem where there is a continuum of spatial sites and the pollution stock diffuses over these sites.

Originally, we state the model in continuous space and continuous time with two spatial dimensions and one temporal dimension. The planar region of interest is divided in J subregions and in each subregion there is a decision-maker who decides the emission level in order to maximize the present value of benefits net of environmental damages due to the concentration of pollutants over his subregion subject to the spatio-temporal evolution of the stock of a pollutant, described by a Convection-Reaction-Diffusion PDE. An essential difference with respect to the previous literature is that each economic agent responsible for controlling the emissions at each subregion takes into account the spatial transport phenomena across space (across all the planar region of interest) when taking the emission decisions at this subregion in order to maximize his profits. Therefore, our original formulation is a J-player differential game with one control variable for each player (the emission decision at his subregion) and one infinite-dimensional state variable (the stock of a pollutant). The Markovian Nash Equilibrium of this differential game is approximated by the solution of a simplified model which still captures the spatial interactions and that can be interpreted as a spatial discretization of the original model. This space-discretized model has J players and J state variables, each one describing the average pollution in one subregion. Each player decides the average total emission in his subregion in order to maximize his profits net of environmental damages subject to the time evolution of J state variables described by a system of J ordinary differential equations (ODEs). The solutions of the space-discretized model are characterized using a numerical algorithm that solves an approximate time-discrete dynamic game. Our numerical examples show, on the one hand, how the equilibrium emission policies in a spatial context differ from those characterized ignoring the spatial dimension. On the other hand, the comparison of the equilibrium emission policies we have obtained in our differential game version and those obtained for the same model when the decision maker is an environmental regulator allows us to evaluate the impact of the strategic and spatially dynamic behaviour of the agents on the design of equilibrium environmental policies.

One of the possible extensions that we think merit exploration in more detail include the addition of a second dimension for the pollution. In the present formulation pollution has a local dimension as a direct consequence of the production of the consumption good in a particular region. However, additionally pollution produced in other regions may also harm welfare. In this case, the environmental damage function would depend on the pollution over the entire spatial domain. The double dimension (local and global) of pollution in a different framework has been already introduced in Camacho & Pérez-Barahona (2015) in their study of optimal land use and environmental degradation. This analysis is one of the subjects of our future research.

Appendix A: Technical hypotheses and definitions

Along the paper we assume the following set of technical hypotheses some of which have been already stated in the body of the manuscript and that are included here for the reader's convenience.

- H1 The regions $\Omega \subset \mathbb{R}^2$ and Ω_i, $i = 1, \ldots, J$ defined in (1), are bounded Lipschitz domains.
- H2 The diffusion coefficient in (3), $k(\boldsymbol{x})$, is continuously differentiable in Ω. There exist two constants $0 < k_m \leq k_M$ such that $k_m < k(\boldsymbol{x}) < k_M$, for all $\boldsymbol{x} \in \Omega$. The velocity field $\boldsymbol{b}(\boldsymbol{x})$ is continuously differentiable. Furthermore, $\nabla \cdot \boldsymbol{b} = 0$. Function c is continuously differentiable and non negative.
- H3 The non-linear function $N(P)$ in (3) is continuously differentiable and either monotone decreasing or globally Lipschitz continuous.
- H4 Function F in (3) is of the form (4) with F_i monotone increasing differentiable functions with Lipschitz derivative defined for $u_i \in \mathbb{U}_i$, where $\mathbb{U}_i \subset \mathbb{R}_+$ is closed and bounded.
- H5 The functions B_i, $i = 1, \ldots, J$, in (6) are two times differentiable, concave, non-decreasing functions defined in $\mathbb{U}_i$. The functions D_i, $i = 1, \ldots, J$, in (6) are two times differentiable convex functions defined in $\mathbb{R}$.
- H6 The initial pollution state $P_0(\boldsymbol{x})$ is a function in $L^2(\Omega)$, the space of square integrable functions defined in Ω.

For simplicity in the notation we will write $v(t)$ to represent a generic function $v = v(\boldsymbol{x}, t)$, such that $v(\cdot, t) \in H$ where H is a given function space. Let $\mathbb{U}_i$ the set of functions $v \in L^2(\Omega_i)$ such that $v(\boldsymbol{x}) \in \mathbb{U}_i$ for almost every $\boldsymbol{x} \in \Omega_i$. We define the set of admissible controls $\mathcal{U}_i$ as the set of functions $u_i : [0,T] \to U_i$ with $u_i \in L^2(\Omega_i \times (0,T))$.

The set of hypotheses above guarantees that for each choice of controls $u_i(\boldsymbol{x}, t)$, with $u_i \in \mathcal{U}_i$ the state Equation (3) has a unique (weak) solution $P \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$, for all $T > 0$, (see, for example, Barbu (1993), Tröltzsch (2009), Liu & Yong (1995)). Here, and in the rest of the paper, $H^r(\Omega)$ represents the Sobolev space of functions with r distributional derivatives in $L^2(\Omega)$.

We are interested in finding stationary Markov-perfect Nash equilibria of the dynamic game defined by (3)–(6) so that we look for controls $u_i \in \mathcal{U}_i$ of the form $u_i(t) = \Lambda_i(P(t))$, where the strategies $\Lambda_i : L^2(\Omega) \to U_i$, are assumed to be compatible with the dynamics (3) in the sense that the closed-loop system

$$\begin{aligned} \frac{\partial P}{\partial t} &= \nabla \cdot (k \nabla P) + \boldsymbol{b}^T \nabla P - cP + N(P) + F(\boldsymbol{\Lambda}(P(t), t)), \quad \boldsymbol{x} \in \Omega, \\ P(\boldsymbol{x}, \tau) &= P_\tau(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ \alpha P(\boldsymbol{x}, t) + k(\boldsymbol{x}) \nabla P^T(\boldsymbol{x}, t) \boldsymbol{n} &= \alpha P_b(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega, \end{aligned} \tag{10}$$

has a unique solution defined in $[\tau, \infty)$ for every $\tau \geq 0$ and every initial condition P_τ. Here, we are using the notation $\boldsymbol{\Lambda} = [\Lambda_1, \ldots, \Lambda_J]^T$.

A J-tuple of admissible strategies $\boldsymbol{\Lambda}^* = [\Lambda_1^*, \ldots, \Lambda_J^*]^T$ are a Markov-perfect Nash equilibrium if for all initial state P_0,

$$J_i(\boldsymbol{u}^*, P_0) \geq J_i([u_i, \boldsymbol{u}_{-i}^*], P_0), \quad \forall u_i \in \mathcal{U}_i,$$

where $\boldsymbol{u}^* = [u_1^*, \ldots, u_J^*]^T$, $u_j^*(t) = \Lambda_j^*(P^*(t))$, P^* is the solution of (10) with $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^*$ and $[u_i, \boldsymbol{u}_{-i}^*] = [u_1^*, \ldots, u_i, \ldots, u_J^*]$. Here, for simplicity, we are using the concept of strong optimality. When dealing with the infinite horizon problem other possibilities are also advisable, see Haurie et al. (2012).

Let us suppose that $\Lambda_i^*(P)$, $i = 1 \ldots, J$, are a stationary Markov-perfect Nash equilibrium. Then the value functions V^i, $i = 1, \ldots, J$, satisfy, for $P \in H^2(\Omega)$, the infinite-dimensional Hamilton-Jacobi-Bellman system

$$\rho V^i(P) = \sup_{u_i \in U_i} \Big\{ \mathcal{G}^i\big(P, [u_i, \boldsymbol{\Lambda}_{-i}^*]\big) + \big\langle \mathcal{F}\big(P, [u_i, \boldsymbol{\Lambda}_{-i}^*]\big), \nabla V^i \big\rangle \Big\}, \tag{11}$$

with transversality condition

$$\lim_{T \to \infty} e^{-\rho T} V^i(P(T)) = 0. \tag{12}$$

Here, we are using the notations

$$\mathcal{G}^i(P, u_1, \ldots, u_J) = \int_{\Omega_i} G_i(P, u_1, \ldots, u_J) d\boldsymbol{x}$$

and

$$\mathcal{F}(P, u_1, \ldots, u_J) = \nabla \cdot (k \nabla P) + \boldsymbol{b} \cdot \nabla P - cP + N(P) + F(u_1, \ldots, u_J),$$

the brackets represent the $L^2(\Omega)$ inner product and ∇V^i denotes the Fréchet derivative of the function V^i with respect to P.

Furthermore, $\Lambda_i^*(P)$ is a maximizer of the right hand side of (11) and $V^i(P_0) = J_i(\boldsymbol{u}^*, P_0)$. We refer to Başar & Olsder (1999), Haurie et al. (2012), for a proof of this result in the finite dimensional case. See also Li & Yong (1995).

The Hamilton-Jacobi-Bellman system (11) is a non-linear infinite-dimensional system. It is well known that even in the finite dimensional the solutions of (11) can fail to have enough regularity and one has to resort to generalized solutions. Let us remark that, even in the finite dimensional case, problem (11) with the transversality condition (12) can have multiple solutions. We refer to Cannarsa & Da Prato (1990), Barbu (1993), Li & Yong (1995) for a deep analysis in the case $J = 1$. The analysis of (11) is out of the scope of this paper and it will be the subject of further research.

Appendix B: The discrete-space model

We present in this appendix the details of the derivation of the discrete-space model, which can be seen as a space discretization of the continuous space model state in Section 2.

We will use aggregated variables. Let us denote by $m_i = \int_{\Omega_i} \mathrm{d}\boldsymbol{x}$ the area of region Ω_i, $i = 1, \ldots, J$. We consider the averaged stock of pollution over region Ω_i,

$$p_i(t) = \frac{1}{m_i} \int_{\Omega_i} P(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}, \quad i = 1, \ldots, J, \tag{13}$$

and the averaged emissions over Ω_i

$$v_i(t) = \frac{1}{m_i} \int_{\Omega_i} u_i(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}, \quad i = 1, \ldots, J. \tag{14}$$

Integrating Equation (3) over region Ω_i, $i = 1, \ldots, J$ and using (4) we have

$$\int_{\Omega_i} \frac{\partial P}{\partial t}\, \mathrm{d}\boldsymbol{x} = \int_{\Omega_i} \big(\nabla \cdot (k\nabla P) + \boldsymbol{b}^T \nabla P - cP + N(P) + F_i(u_i)\big)\, \mathrm{d}\boldsymbol{x}.$$

In order to simultaneously treat the cases $\partial\Omega_i \cap \partial\Omega = \emptyset$ (interior subdomains) and $\partial\Omega_i \cap \partial\Omega \neq \emptyset$ (boundary subdomains with a part of their boundary over $\partial\Omega$), we introduce a fictitious subdomain $\Omega_0 = \mathbb{R}^2 \backslash \overline{\Omega}$. In this way $\partial_{i,0} = \overline{\Omega}_i \cap \overline{\Omega}_0 = \partial\Omega_i \cap \partial\Omega$.

We use the divergence theorem in the first term on the right to arrive to

$$\begin{aligned} \int_{\Omega_i} \nabla \cdot (k\nabla P)\, \mathrm{d}\boldsymbol{x} &= \int_{\partial\Omega_i} k\nabla P^T \boldsymbol{n}_i\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{\substack{j=0 \\ \partial_{ij} \neq \emptyset}}^{J} \int_{\partial_{ij}} k\nabla P^T \boldsymbol{n}_i\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}) \\ &\approx \sum_{\substack{j=0 \\ \partial_{ij} \neq \emptyset}}^{J} k_{ij}(p_j - p_i), \end{aligned} \tag{15}$$

where $\boldsymbol{n}_i$ is the normal vector pointing outwards Ω_i, and

$$k_{ij} = \frac{L_{ij}}{\mathrm{length}(\partial_{ij})} \int_{\partial_{ij}} k\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}), \quad j = 1, \ldots, J.$$

Here L_{ij} is a scaling parameter proportional to the ratio between $\mathrm{length}(\partial_{ij})$ and the distance between the center of masses of the subdomains Ω_i and Ω_j. In the case of a boundary subdomain, $j = 0$, the term $\int_{\partial_{ij}} k\nabla P^T \boldsymbol{n}_i\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x})$ is computed using the boundary condition in (3), so that we approximate

$$\int_{\partial_{i0}} k\nabla P^T \boldsymbol{n}_i\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}) \approx \alpha \mathrm{length}(\partial_{i0})(p_{i0} - p_i),$$

with

$$p_{i0} = \frac{1}{\mathrm{length}(\partial_{i0})} \int_{\partial_{i0}} P_b(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}).$$

For simplicity in the presentation, we assume that $p_{i0} = p_0$, for all $i = 1, \ldots, J$, and write $k_{i0} = \alpha \mathrm{length}(\partial_{i0})$.

The second term is treated similarly:

$$\begin{aligned} \int_{\Omega_i} \boldsymbol{b}^T \nabla P\, \mathrm{d}\boldsymbol{x} &= \int_{\partial\Omega_i} P(\boldsymbol{b}^T \boldsymbol{n}_i)\, \mathrm{d}\boldsymbol{x} = \sum_{\substack{j=0 \\ \partial_{ij} \neq \emptyset}}^{J} \int_{\partial_{ij}} P(\boldsymbol{b}^T \boldsymbol{n}_i)\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}) \\ &\approx \sum_{\substack{j=0 \\ \partial_{ij} \neq \emptyset}}^{J} b_{ij}\varphi(p_i, p_j), \end{aligned}$$

with

$$b_{ij} = \int_{\partial_{ij}} (\boldsymbol{b}^{\boldsymbol{T}} \boldsymbol{n_i})\, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{x}).$$

Here we have used the condition $\nabla \cdot \boldsymbol{b} = 0$. There are several possible choices for the numerical flux $\varphi(p_i, p_j)$. The choice $\varphi(p_i, p_j) = (p_i + p_j)/2$ gives a term that is reminiscent of second-order centered finite difference approximations. This choice is reasonable in problems in which diffusion is dominant, that is, if $\max k(x)$ is large compared with $\max |\boldsymbol{b}(x)|$. An alternative is to use an upwind flux defined, in the simplest case, by $\varphi(p_i, p_j) = p_i$ if $b_{ij} \leq 0$ and $\varphi(p_i, p_j) = p_j$ if $b_{ij} > 0$. With this choice one can deal with convection dominated problems.

The rest of the terms are handle as follows

$$\int_{\Omega_i} cP\, \mathrm{d}\boldsymbol{x} \approx c_i p_i,$$

with $c_i = \int_{\Omega_i} c(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$;

$$\int_{\Omega_i} N(P)\, \mathrm{d}\boldsymbol{x} \approx m_i N(p_i)$$

and

$$\int_{\Omega_i} F_i(u_i)\, \mathrm{d}\boldsymbol{x} \approx m_i F_i(v_i).$$

The term with the partial derivative with respect time is approximated also using the aggregated stock $p_i(t)$ by

$$\int_{\Omega_i} \frac{\partial P}{\partial t}\, \mathrm{d}\boldsymbol{x} \approx m_i \frac{\mathrm{d}p_i}{\mathrm{d}t}.$$

All in all, the dynamics of the aggregated stock of pollution is

$$m_i \frac{\mathrm{d}p_i}{\mathrm{d}t} = \sum_{\substack{j=0 \\ j\neq i}}^{J} k_{ij}(p_j - p_i) + \sum_{\substack{j=0 \\ j\neq i}}^{J} b_{ij}\varphi(p_i, p_j) - m_i c_i p_i + m_i N(p_i) + m_i F_i(v_i), \quad i = 1, \dots, J. \tag{16}$$

We note that (16) is a system of ordinary differential equations. The system is supplemented with the initial conditions given by

$$p_i(0) = \frac{1}{m_i} \int_{\Omega_i} P_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} := p_i^0, \quad i = 1, \dots, J, \tag{17}$$

where $P_0(\boldsymbol{x})$ is the initial data in (3).

The objective of player i is approximated using that

$$\int_{\Omega_i} \big(B_i(u_i) - D_i(P)\big)\, \mathrm{d}\boldsymbol{x} \approx m_i\big(B_i(v_i) - D_i(p_i)\big) := \widetilde{G}_i(v_i, p_i).$$

We state, finally, the discrete-space model: The objective of Player i is to maximize the space averaged payoff

$$\widetilde{J}_i(v_1, \dots, v_J, \boldsymbol{p}^0) = \int_0^\infty e^{-\rho t} \widetilde{G}_i(v_i, p_i)\, \mathrm{d}t, \tag{18}$$

subject to (16), where $\boldsymbol{p}^0 = [p_1^0, \dots, p_J^0]^T$.

Appendix C: The numerical method

As it is well known there is little hope to find analytical solutions to the problem at hand, so we approximate the solution by a numerical method adapted from De Frutos & Martín-Herrán (2015) consisting mainly of two steps. In the first step we consider a time-discrete version of problem (16)–(18). In the second step we discretize the state space.

Let $h > 0$ a positive parameter and let $t_n = nh$, the discrete times defined for all positive integers n. We denote by $\delta_h = 1 - \rho h$ the discrete discount factor. In what follows $\bar{u}_i$, $i = 1, \dots, J$ denotes a sequence of real numbers $\bar{u}_i = \{u_{i,n}\}_{n=0}^{\infty}$ and $\mathcal{U}$ denotes the set of real sequences $\bar{v}$ with $v_n \geq 0$ for all $n \in \mathbb{N}$.

We introduce the notation $\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{u}) = [g_1(\boldsymbol{p}, \boldsymbol{u}), \dots, g_J(\boldsymbol{p}, \boldsymbol{u})]^T$ to denote the function collecting the right hand side of (16). More precisely, for $\boldsymbol{p} = [p_1, \dots, p_J]^T \in \mathbb{R}^J$ and $\boldsymbol{u} = [u_1, \dots, u_J]^T \in \mathbb{R}^J$ with $u_i \geq 0$, $i = 1, \dots, J$,

$$g_i(\boldsymbol{p}, \boldsymbol{u}) = \sum_{\substack{j=0 \\ j \neq i}}^{J} \frac{k_{ij}}{m_i}(p_j - p_i) + \sum_{\substack{j=0 \\ j \neq i}}^{J} \frac{b_{ij}}{m_i}\varphi(p_i, p_j) - c_i p_i + N(p_i) + F_i(u_i).$$

We consider the time-discrete infinite horizon game in which player $i = 1, \dots, J$ aims to maximize

$$W_i(\bar{u}_i, \boldsymbol{p}_0) = h \sum_{n=1}^{\infty} \delta_h^n \widetilde{G}_i(u_{i,n}, p_{i,n}), \quad \bar{u}_i \in \mathcal{U}, \tag{19}$$

subject to

$$\boldsymbol{p}_{n+1} = \boldsymbol{p}_n + h\boldsymbol{g}(\boldsymbol{p}_n, \boldsymbol{u}_n), \quad n \geq 0, \tag{20}$$

where $\boldsymbol{p}_n = [p_{1,n}, \dots, p_{J,n}]^T$, $\boldsymbol{u}_n = [u_{1,n}, \dots, u_{J,n}]^T$ and $\boldsymbol{p}_0$ is a given initial state.

It is worth noting that the time-discrete problem (19)–(20) can be seen as a discretization of the functional (18) by means of the rectangle rule followed by a forward Euler discretization of the dynamics in (16).

The time-discrete value function $V_{h,i}(\boldsymbol{p})$, $i = 1, \dots, J$, is computed solving the Bellman equations

$$V_{h,i}(\boldsymbol{p}) = \max_{u_i \geq 0} \left\{ h\widetilde{G}_i(p_i, u_i) + \delta_h V_{h,i}\big(\boldsymbol{p} + h\boldsymbol{g}(\boldsymbol{p}, [u_i, \boldsymbol{u}^*_{-i}])\big) \right\}, \tag{21}$$

where, for $i = 1, \dots, J$,

$$u_i^* = \arg\max_{u_i \geq 0} \left\{ h\widetilde{G}_i(p_i, u_i) + \delta_h V_{h,i}\big(\boldsymbol{p} + h\boldsymbol{g}(\boldsymbol{p}, [u_i, \boldsymbol{u}^*_{-i}])\big) \right\}. \tag{22}$$

Here, and in the rest of this section, we are using the notation

$$[u_i, \boldsymbol{v}_{-i}] = [v_1, \dots, v_{i-1}, u_i, v_{i+1}, \dots, v_J]^T, \quad u_i \in \mathbb{R}, \boldsymbol{v} \in \mathbb{R}^J.$$

The solution of system (21) is approximated using a collocation method based on tensorial product of linear splines. Of course other type of discretizations are advisable. In De Frutos & Martín-Herrán (2015) a state discretization based on monotone cubic splines has been used. In Hager (2000) Runge-Kutta time discretizations for optimal control problems have been analyzed.

Let us introduce a positive parameter $P_M > 0$ big enough and consider, for a given integer $N > 0$ a partition of the interval $[0, P_M] \subset \mathbb{R}$, $0 = q_0 < q_1 < \cdots < q_N = P_M$. Let ϕ_k be the piecewise linear spline defined by $\phi_k(q_l) = \delta_{kl}$ where δ_{kl} is the Kronecker delta. Let us consider now the J-interval, $I = [0, P_M] \times \cdots \times [0, P_M] \subset \mathbb{R}^J$. We consider $\mathbb{S}_1^0$ the space of J-linear splines in I defined by

$$s(p_1, \dots, p_J) = \sum_{\nu_1, \dots, \nu_J = 0}^{N} \hat{s}_{\nu_1, \dots, \nu_J} \phi_{\nu_1}(p_1) \dots \phi_{\nu_J}(p_J).$$

Note that by construction the coefficients $\hat{s}_{\nu_1, \dots, \nu_J}$ are determined by

$$\hat{s}_{\nu_1, \dots, \nu_J} = s(q_{\nu_1}, \dots, q_{\nu_J}).$$

We introduce the notation $\boldsymbol{q}_{\nu_1,\ldots,\nu_J} = [q_{\nu_1}, \ldots, q_{\nu_J}]^J$, $\nu_i = 0, \ldots, N$, $i = 1, \ldots, J$.

The approximation $V_{h,i}^N \in \mathbb{S}_1^0$ to the time-discrete value function $V_{h,i}$ in (21) is computed by a fixed-point iteration solving for $r \geq 0$, and $i = 1, \ldots, J$,

$$\begin{aligned} V_{h,i}^{N,[r+1]}(\boldsymbol{q}_{\nu_1,\ldots,\nu_J}) = \max_{u_i \geq 0} \big\{ & h\widetilde{G}_i(q_{\nu_i}, u_i) \\ & + \delta V_{h,i}^{N,[r]}\big(\boldsymbol{q}_{\nu_1,\ldots,\nu_J} + h\boldsymbol{g}(\boldsymbol{q}_{\nu_1,\ldots,\nu_J}, [u_i, \boldsymbol{u}_{-i,\nu_1,\ldots,\nu_J,}^{[r]}])\big)\big\}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} u_{i,\nu_1,\ldots,\nu_J}^{[r+1]} = \arg\max_{u_i \geq 0} \big\{ & h\widetilde{G}_i(q_{\nu_i}, u_i) \\ & + \delta V_{h,i}^{N,[r]}\big(\boldsymbol{q}_{\nu_1,\ldots,\nu_J} + h\boldsymbol{g}(\boldsymbol{q}_{\nu_1,\ldots,\nu_J}, [u_i, \boldsymbol{u}_{-i,\nu_1,\ldots,\nu_J,}^{[r]}])\big)\big\}. \end{aligned}$$

The iteration is initialized with some given $V_{h,i}^{N,[0]}(\boldsymbol{q}_{\nu_1,\ldots,\nu_J})$ and $u_{i,\nu_1,\ldots,\nu_J}^{[0]}$, for $\nu_i = 0, \ldots, N$, $i = 1, \ldots, J$ and stopped when

$$\big|V_{h,i}^{N,[r+1]}(\boldsymbol{q}_{\nu_1,\ldots,\nu_J}) - V_{h,i}^{N,[r]}(\boldsymbol{q}_{\nu_1,\ldots,\nu_J})\big| < \mathrm{TOL},$$

for all possible values of the indices $\nu_1, \ldots, \nu_J$ and i. The parameter TOL is a prescribed positive tolerance.

When the iteration is stopped the approximated time-discrete value function is defined as the last iterant $V_{h,i}^N := V_{h,i}^{N,[r+1]}$. The approximated time-discrete optimal policies are defined as the unique J-linear spline $u_{h,N,i}^*$ interpolating the values $u_{i,\nu_1,\ldots,\nu_J}^{[r+1]}$, $\nu_i = 0, \ldots, N$, $i = 1, \ldots, J$. The approximated time-discrete optimal state trajectory is computed from

$$\boldsymbol{p}_{n+1}^* = \boldsymbol{p}_n^* + h\boldsymbol{g}(\boldsymbol{p}_n^*, \boldsymbol{u}_n^*), \quad n \geq 0,$$

where $\boldsymbol{u}_n^* = [u_{h,M,1}^*(\boldsymbol{p}_n^*), \ldots, u_{h,M,J}^*(\boldsymbol{p}_n^*)]^T$ and $\boldsymbol{p}_0^* = \boldsymbol{p}_0$.

References

[1] Anita, S., Capasso, V., Kunze, H., La Torre, D.: Optimal Control and Long-Run Dynamics for a Spatial Economic Growth Model with Physical Capital Accumulation and Pollution Diffusion. Applied Mathematics Letters 26, 908–912 (2013).

[2] Barbu, V.: Analysis and Control of Nonlinear Infinite Dimensional Systems, Academic Press, Boston. 1993.

[3] Başar, T., Olsder, G.J.: Dynamic Noncooperative Game Theory, SIAM, Philadelpia, 1999.

[4] Brito, P.: The Dynamics of Growth and Distribution in a Spatially Heterogenous World. WP13/2004/DE/UECE, Technical University of Lisbon, 2004.

[5] Brock, W., Xepapadeas, A.: General Pattern Formation in Recursive Dynamic Systems Models in Economics. MPRA PaperNo. 12305, University of Munich (2008).

[6] Brock, W., Xepapadeas, A.: Diffusion-Induced Instability and Pattern Formation in Infinite Horizon Recursive Optimal Control. Journal of Economic Dynamics & Control 32, 2745–2787 (2008).

[7] Brock, W., Xepapadeas, A.: Pattern Formation, Spatial Externalities and Regulation in Coupled Economic-Ecological Systems. Journal of Environmental Economics and Management 59, 149–164 (2010).

[8] Brock, W., Xepapadeas, A., Yannacopoulos, A.N.: Optimal Agglomerations in Dynamic Economics. Journal of Mathematical Economics 53, 1–15 (2014).

[9] Brock, W., Xepapadeas, A., Yannacopoulos, A.N.: Optimal Control in Space and Time and the Management of Environmental Resources. Annual Reviews of Resource Economics 6, 33–68 (2014).

[10] Boucekkine, R., Camacho, C., Zou, B.: Bridging the Gap Between Growth Theory and the New Economic Geography: The Spatial Ramsey Model. Macroeconomic Dynamics 13, 20–45 (2009).

[11] Boucekkine, R., Camacho, C., Fabbri, G.: Spatial Dynamics and Convergence: The Spatial Ak Model. Journal of Economic Theory 148, 2719–2736 (2013).

[12] Boucekkine, R., Camacho, C., Fabbri, G.: On the Optimal Control of Some Parabolic Partial Differential Equations Arising in Economics. Serdica Mathematical Journal 39, 331–354 (2013).

[13] Camacho, C., Pérez-Barahona, A.: Land Use Dynamics and the Environment. Journal of Economic Dynamics & Control 52, 96–118 (2015).

[14] Camacho, C., Zou, B., Briani, M.: On the Dynamics of Capital Accumulation Across Space. European Journal of Operational Research 186, 451–465 (2008).

[15] Cannarsa, P., Da Prato, G.: Some Results on Nonlinear Optimal Control Problems and Hamilton-Jacobi Equations in Infinite Dimensions. Journal of Functional Analysis 90, 27–47 (1990).

[16] De Frutos, J., Martín-Herrán, G.: Does Flexibility Facilitate Sustainability of Cooperation Over Time? A Case Study from Environmental Economics. Journal of Optimization Theory and Applications 165(2), 657–677 (2015).

[17] Desmet, K., Rossi-Hansberg, E.: On Spatial Dynamics. Journal of Regional Science 50, 43–63 (2010).

[18] Desmet, K., Rossi-Hansberg, E.: On Spatial Economic Impact of Global Warming. Journal of Urban Economics 88, 16–37 (2015).

[19] Dockner, E.J., Long, N.V.: International Pollution Control: Cooperative versus Noncooperative Strategies. Journal of Environmental Economics and Management 25, 13–29 (1993).

[20] Fabbri, G.: Geographical Structure and Convergence: A Note on Geometry in Spatial Growth Models. Journal of Economic Theory 162, 114–136 (2016).

[21] Graß, D., Uecker, H.: Optimal Management and Spatial Patterns in a Distributed Shalow Lake Model. Mimeo, 2015.

[22] Hager, W. W.: Runge-Kutta Methods in Optimal Control and the Transformed Adjoint System. Numerische Mathematik 87, 247–282 (2000).

[23] Haurie, H, Krawczyk, J.B, Zaccour, G.: Games and Dynamic Games, World Scientific, Singapore, 2012.

[24] Jørgensen, S., Martín-Herrán, G., Zaccour, G.: Dynamic Games in the Economics and Management of Pollution. Environmental Modeling and Assessment 15, 433–467 (2010).

[25] Kossioris, G., Plexousakis, M., Xepapadeas, A., de Zeeuw, A., Maler, K.-G.: Feedback Nash Equilibria for Non-Linear Differential Games in Pollution Control. Journal of Economic Dynamics and Control 32, 1312–1331 (2008).

[26] Li, X., Yong, J.: Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Boston, 1995.

[27] La Torre, D., Liuzzi, D., Marsiglio, S.: Pollution Diffusion and Abatement Activities across Space and over Time. Mathematical Social Sciences 78, 48–63 (2015).

[28] Mäler, K-G., De Zeeuw, A.: The Acid Rain Differential Game. Environmental and Resource Economics 12, 167–184 (1998).

[29] Scheffer, M.: Ecology of Shallow Lakes, Chapman & Hall, New York, 1997.

[30] Tröltzsch, F.: Optimal Control of Partial Differential Equations. Theory, Methods and Applications, AMS, Providence, 2009.

[31] Van der Ploeg, F., De Zeeuw, A.J.: International Aspects of Pollution Control. Environmental and Resource Economics 2, 117–139 (1992).

[32] Wagener, F.O.O.: Skiba Points and Heteroclinic Bifurcations, with Applications to the Shallow Lake System. Journal of Economic Dynamics & Control 27, 1533–1561 (2003).

[33] Xepapadeas, A.: The Spatial Dimension in Environmental and Resource Economics. Environment and Development Economics 15, 747–758 (2010).

[34] Yamaguchi, R.: Sustainability and Spatial Externality in a Capital-Resource Economy. Mimeo, 2014.